COMPOSITIO MATHEMATICA

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Compositio Math. **151** (2015), 377–396.

doi:10.1112/S0010437X14007799

FOUNDATION COMPOSITIO MATHEMATICA

LONDON MATHEMATICAL SOCIETY 150 YEARS

Compositio Math. 151 (2015) 377–396
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Simplicity of heads and socles of tensor products

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Abstract

We prove that, for simple modules M and N over a quantum affine algebra, their tensor product $M \otimes N$ has a simple head and a simple socle if $M \otimes M$ is simple. A similar result is proved for the convolution product of simple modules over quiver Hecke algebras.

Introduction

Let g be a complex simple Lie algebra and $U_q(\mathsf{g})$ the associated quantum group. The multiplicative property of the upper global basis $\mathbf{B}$ of the negative half $U_q^-(\mathsf{g})$ was investigated in [BZ93, Lec03]. Set $q^{\mathbb{Z}}\mathbf{B} = \{q^n b \mid b \in \mathbf{B}, n \in \mathbb{Z}\}$. In [BZ93], Berenstein and Zelevinsky conjectured that, for b_1, $b_2 \in \mathbf{B}$, the product $b_1 b_2$ belongs to $q^{\mathbb{Z}}\mathbf{B}$ if and only if b_1 and b_2 q-commute (i.e. $b_2 b_1 = q^n b_1 b_2$ for some $n \in \mathbb{Z}$). However, Leclerc found examples of $b \in \mathbf{B}$ such that $b^2 \notin q^{\mathbb{Z}}\mathbf{B}$ [Lec03].

On the other hand, the algebra $U_q^-(\mathsf{g})$ is categorified by quiver Hecke algebras [KL09, KL11, Rou08] and also by quantum affine algebras [HL10, HL13, KKK13a, KKK13b]. In this context, the products in $U_q^-(\mathsf{g})$ correspond to the convolution or the tensor products in quiver Hecke algebras or quantum affine algebras. The upper global basis corresponds to the set of isomorphism classes of simple modules over the quiver Hecke algebras or the quantum affine algebras [Ari96, Rou12, VV11] under suitable conditions. Then Leclerc conjectured several properties of products of upper global bases and also convolutions and tensor products of simple modules. The purpose of this paper is to give an affirmative answer to some of his conjectures.

In this introduction, we state our results in the case of modules over quantum affine algebras. Similar results hold also for quiver Hecke algebras (see § 3.1).

Let $\mathfrak{g}$ be an affine Lie algebra and $U'_q(\mathfrak{g})$ the associated quantum affine algebra. A simple $U'_q(\mathfrak{g})$-module M is called *real* if $M \otimes M$ is also simple.

Conjecture [Lec03, Conjecture 3]. Let M and N be finite-dimensional simple $U'_q(\mathfrak{g})$-modules. We assume, further, that M is real. Then $M \otimes N$ has a simple socle S and a simple head H. Moreover, if S and H are isomorphic, then $M \otimes N$ is simple.

In this paper we shall give an affirmative answer to this conjecture (Theorem 3.12 and Corollary 3.16). In the course of the proof, R-matrices play an important role. Indeed, the simple socle of $M \otimes N$ coincides with the image of the renormalized R-matrix $\mathbf{r}_{N,M} : N \otimes M \to M \otimes N$ and the simple head of $M \otimes N$ coincides with the image of the renormalized R-matrix $\mathbf{r}_{M,N} : M \otimes N \to N \otimes M$.

Received 16 April 2014, accepted in final form 22 August 2014, published online 26 November 2014.
2010 Mathematics Subject Classification 81R50, 16GXX, 17B37 (primary).
Keywords: quantum affine algebra, Khovanov–Lauda–Rouquier algebra, R-matrix.
The first author's work was supported by NRF grant #2014021261 and NRF grant #2013055408. The second author's work was supported by Grant-in-Aid for Scientific Research (B) 22340005, Japan Society for the Promotion of Science. The third author's work was supported by BK21 PLUS SNU Mathematical Sciences Division.

Denoting by $M \diamond N$ the head of $M \otimes N$, we also prove that $N \mapsto M \diamond N$ is an automorphism of the set of the isomorphism classes of simple $U_q'(\mathfrak{g})$-modules (Corollary 3.14). The inverse is given by $N \mapsto N \diamond {}^*M$, where ${}^*M$ is the right dual of M. It is an analogue of [Lec03, Conjecture 2] originally stated for global bases.

1. Quiver Hecke algebras

In this section, we briefly recall the basic facts on quiver Hecke algebras and R-matrices following [KKK13a]. Since the grading of quiver Hecke algebras is not important in this paper, we ignore the grading. Throughout the paper, *modules mean left modules.*

1.1 Convolutions

We recall the definition of quiver Hecke algebras. Let $\mathbf{k}$ be a field. Let I be an index set. Let Q be the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i\in I}$. Set $\mathsf{Q}^+ = \sum_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$. For $\beta = \sum_{k=1}^n \alpha_{i_k} \in \mathsf{Q}^+$, we set $\mathrm{ht}(\beta) = n$. For $n \in \mathbb{Z}_{\geqslant 0}$ and $\beta \in \mathsf{Q}^+$ such that $\mathrm{ht}(\beta) = n$, we set

$$I^\beta = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Let us take a family of polynomials $(Q_{ij})_{i,j\in I}$ in $\mathbf{k}[u, v]$ which satisfy

$$Q_{ij}(u, v) = Q_{ji}(v, u) \quad \text{for any } i,\ j \in I,$$
$$Q_{ii}(u, v) = 0 \quad \text{for any } i \in I.$$

For $i, j \in I$, we set

$$\overline{Q}_{ij}(u, v, w) = \frac{Q_{ij}(u, v) - Q_{ij}(w, v)}{u - w} \in \mathbf{k}[u, v, w].$$

We denote by $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1}\rangle$ the symmetric group on n letters, where $s_i := (i, i+1)$ is the transposition of i and $i+1$. Then $\mathfrak{S}_n$ acts on I^n by place permutations.

DEFINITION 1.1. For $\beta \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = n$, the *quiver Hecke algebra* $R(\beta)$ at β associated with a matrix $(Q_{ij})_{i,j\in I}$ is the $\mathbf{k}$-algebra generated by the elements $\{e(\nu)\}_{\nu\in I^\beta}$, $\{x_k\}_{1\leqslant k\leqslant n}$, $\{\tau_k\}_{1\leqslant k\leqslant n-1}$ satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \qquad \sum_{\nu\in I^\beta} e(\nu) = 1,$$
$$x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu)x_k,$$
$$\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k\tau_m = \tau_m\tau_k \quad \text{if } |k - m| > 1,$$
$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$
$$(\tau_k x_m - x_{s_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } m = k \text{ and } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } m = k+1 \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$
$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \overline{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2}) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}. \end{cases}$$

For an element w of the symmetric group $\mathfrak{S}_n$, let us choose a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$, and set

$$\tau_w = \tau_{i_1} \cdots \tau_{i_\ell}.$$

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In general, this depends on the choice of reduced expressions w. Then we have the PBW decomposition

$$R(\beta) = \bigoplus_{\nu\in I^\beta, w\in\mathfrak{S}_n} \mathbf{k}[x_1,\dots,x_n]e(\nu)\tau_w. \tag{1.1}$$

We denote by $R(\beta)$-mod the category of $R(\beta)$-modules M such that M is finite-dimensional over $\mathbf{k}$ and the action of x_k on M is nilpotent for any k.

For an $R(\beta)$-module M, the dual space

$$M^* := \operatorname{Hom}_{\mathbf{k}}(M, \mathbf{k})$$

is endowed with the $R(\beta)$-module structure given by

$$(r\cdot f)(u) := f(\psi(r)u) \quad \text{for } f\in M^*, r\in R(\beta), u\in M,$$

where ψ denotes the $\mathbf{k}$-algebra anti-involution on $R(\beta)$ which fixes the generators $\{e(\nu)\}_{\nu\in I^\beta}$, $\{x_k\}_{1\leqslant k\leqslant n}$, $\{\tau_k\}_{1\leqslant k\leqslant n-1}$.

For $\beta,\gamma\in \mathsf{Q}^+$ with $\operatorname{ht}(\beta)=m$ and $\operatorname{ht}(\gamma)=n$, set

$$e(\beta,\gamma) = \sum_{\substack{\nu\in I^{m+n},\\ (\nu_1,\dots,\nu_m)\in I^\beta,\\ (\nu_{m+1},\dots,\nu_{m+n})\in I^\gamma}} e(\nu)\in R(\beta+\gamma).$$

Then $e(\beta,\gamma)$ is an idempotent. Let

$$R(\beta)\otimes R(\gamma)\to e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma)$$

be the $\mathbf{k}$-algebra homomorphism given by

$$\begin{aligned}
e(\mu)\otimes e(\nu) &\mapsto e(\mu*\nu) \quad (\mu\in I^\beta, \nu\in I^\gamma),\\
x_k\otimes 1 &\mapsto x_k e(\beta,\gamma) \quad (1\leqslant k\leqslant m),\\
1\otimes x_k &\mapsto x_{m+k} e(\beta,\gamma) \quad (1\leqslant k\leqslant n),\\
\tau_k\otimes 1 &\mapsto \tau_k e(\beta,\gamma) \quad (1\leqslant k< m),\\
1\otimes \tau_k &\mapsto \tau_{m+k} e(\beta,\gamma) \quad (1\leqslant k< n).
\end{aligned}$$

Here $\mu*\nu$ is the concatenation of μ and ν, i.e.

$$\mu*\nu = (\mu_1,\dots,\mu_m,\nu_1,\dots,\nu_n).$$

For an $R(\beta)$-module M and an $R(\gamma)$-module N, we define their *convolution product* $M\circ N$ by

$$M\circ N = R(\beta+\gamma)e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M\otimes N). \tag{1.2}$$

Set $m=\operatorname{ht}(\beta)$ and $n=\operatorname{ht}(\gamma)$. Set

$$\mathfrak{S}_{m,n} := \{w\in\mathfrak{S}_{m+n} \mid w|_{[1,m]} \text{ and } w|_{[m+1,m+n]} \text{ are increasing}\}.$$

Here $[a,b] := \{k\in\mathbb{Z}\mid a\leqslant k\leqslant b\}$. Then we have

$$M\circ N = \bigoplus_{w\in\mathfrak{S}_{m,n}} \tau_w(M\otimes N). \tag{1.3}$$

We also have (see [LV11, Theorem 2.2(2)])

$$(M\circ N)^* \simeq N^*\circ M^*. \tag{1.4}$$

1.2 *R*-matrices for quiver Hecke algebras

1.2.1 *Intertwiners.* For $\mathrm{ht}(\beta) = n$ and $1 \leqslant a < n$, we define $\varphi_a \in R(\beta)$ by

$$\varphi_a e(\nu) = \begin{cases} (\tau_a x_a - x_a \tau_a) e(\nu) \\ \quad = (x_{a+1}\tau_a - \tau_a x_{a+1}) e(\nu) \\ \quad = (\tau_a(x_a - x_{a+1}) + 1) e(\nu) \\ \quad = ((x_{a+1} - x_a)\tau_a - 1) e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ \tau_a e(\nu) & \text{otherwise.} \end{cases} \tag{1.5}$$

They are called the *intertwiners*.

Lemma 1.2.

(i) $\varphi_a^2 e(\nu) = (Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}) + \delta_{\nu_a,\nu_{a+1}}) e(\nu)$.

(ii) $\{\varphi_k\}_{1 \leqslant k < n}$ *satisfies the braid relation.*

(iii) *For $w \in \mathfrak{S}_n$, let $w = s_{a_1} \cdots s_{a_\ell}$ be a reduced expression for w and set $\varphi_w = \varphi_{a_1} \cdots \varphi_{a_\ell}$. Then φ_w does not depend on the choice of reduced expressions for w.*

(iv) *For $w \in \mathfrak{S}_n$ and $1 \leqslant k \leqslant n$, we have $\varphi_w x_k = x_{w(k)} \varphi_w$.*

(v) *For $w \in \mathfrak{S}_n$ and $1 \leqslant k < n$, if $w(k+1) = w(k) + 1$, then $\varphi_w \tau_k = \tau_{w(k)} \varphi_w$.*

For $m, n \in \mathbb{Z}_{\geqslant 0}$, let us denote by $w[m, n]$ the element of $\mathfrak{S}_{m+n}$ defined by

$$w[m,n](k) = \begin{cases} k + n & \text{if } 1 \leqslant k \leqslant m, \\ k - m & \text{if } m < k \leqslant m + n. \end{cases} \tag{1.6}$$

Let $\beta, \gamma \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = m$, $\mathrm{ht}(\gamma) = n$, and let M be an $R(\beta)$-module and N an $R(\gamma)$-module. Then the map $M \otimes N \to N \circ M$ given by $u \otimes v \longmapsto \varphi_{w[n,m]}(v \otimes u)$ is $R(\beta) \otimes R(\gamma)$-linear by the above lemma, and it extends to an $R(\beta + \gamma)$-module homomorphism

$$R_{M,N} : M \circ N \longrightarrow N \circ M. \tag{1.7}$$

Then we obtain the following commutative diagrams:

$$\begin{tikzcd} L \circ M \circ N \arrow[r, "R_{L,M}"] \arrow[dr, "R_{L,M\circ N}"'] & M \circ L \circ N \arrow[d, "R_{L,N}"] \\ & M \circ N \circ L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \circ M \circ N \arrow[r, "R_{M,N}"] \arrow[dr, "R_{L\circ M,N}"'] & L \circ N \circ M \arrow[d, "R_{L,N}"] \\ & N \circ L \circ M \end{tikzcd} \tag{1.8}$$

1.2.2 *Spectral parameters.*

Definition 1.3. For $\beta \in \mathsf{Q}^+$, the quiver Hecke algebra $R(\beta)$ is called *symmetric* if $Q_{i,j}(u, v)$ is a polynomial in $u - v$ for all $i, j \in \mathrm{supp}(\beta)$. Here, we set $\mathrm{supp}(\beta) = \{i_k \mid 1 \leqslant k \leqslant n\}$ for $\beta = \sum_{k=1}^n \alpha_{i_k}$.

Assume that the quiver Hecke algebra $R(\beta)$ is symmetric. Let z be an indeterminate, and let ψ_z be the algebra homomorphism

$$\psi_z : R(\beta) \to \mathbf{k}[z] \otimes R(\beta)$$

given by

$$\psi_z(x_k) = x_k + z, \quad \psi_z(\tau_k) = \tau_k, \quad \psi_z(e(\nu)) = e(\nu).$$

For an $R(\beta)$-module M, we denote by M_z the $(\mathbf{k}[z]\otimes R(\beta))$-module $\mathbf{k}[z]\otimes M$ with the action of $R(\beta)$ twisted by ψ_z. Namely,

$$\begin{aligned} e(\nu)(a\otimes u) &= a\otimes e(\nu)u,\\ x_k(a\otimes u) &= (za)\otimes u + a\otimes(x_k u),\\ \tau_k(a\otimes u) &= a\otimes(\tau_k u) \end{aligned} \tag{1.9}$$

for $\nu\in I^\beta$, $a\in\mathbf{k}[z]$ and $u\in M$. For $u\in M$, we sometimes denote by u_z the corresponding element $1\otimes u$ of the $R(\beta)$-module M_z.

For a non-zero $M\in R(\beta)$-mod and a non-zero $N\in R(\gamma)$-mod,

let s be the order of zero of $R_{M_z,N}: M_z\circ N\longrightarrow N\circ M_z$, i.e. the largest non-negative integer such that the image of $R_{M_z,N}$ is contained in $z^s(N\circ M_z)$. (1.10)

Note that such an s exists because $R_{M_z,N}$ does not vanish [KKK13a, Proposition 1.4.4(iii)].

DEFINITION 1.4. Assume that $R(\beta)$ is symmetric. For a non-zero $M\in R(\beta)$-mod and a non-zero $N\in R(\gamma)$-mod, let s be an integer as in (1.10). We define

$$\mathbf{r}_{M,N}: M\circ N\to N\circ M$$

by

$$\mathbf{r}_{M,N} = (z^{-s}R_{M_z,N})|_{z=0},$$

and call it the *renormalized R-matrix*.

By the definition, the renormalized R-matrix $\mathbf{r}_{M,N}$ never vanishes.

We define also

$$\mathbf{r}_{N,M}: N\circ M\to M\circ N$$

by

$$\mathbf{r}_{N,M} = ((-z)^{-t}R_{N,M_z})|_{z=0},$$

where t is the multiplicity of zero of R_{N,M_z}.

Note that if $R(\beta)$ and $R(\gamma)$ are symmetric, then s coincides with the multiplicity of zero of R_{M,N_z}, and $(z^{-s}R_{M_z,N})|_{z=0} = ((-z)^{-s}R_{M,N_z})|_{z=0}$. Indeed, we have

$$\begin{aligned} R_{M_{z_1},N_{z_2}}((u)_{z_1}\otimes(v)_{z_2}) &= \varphi_{w[n,m]}((v)_{z_2}\otimes(u)_{z_1})\\ &\in \sum_{w,u',v'}\mathbf{k}[z_1-z_2]\tau_w((v')_{z_2}\otimes(u')_{z_1}) \end{aligned} \tag{1.11}$$

for $u\in M$ and $v\in N$. Here w ranges over

$$\mathfrak{S}_{n,m} := \{w\in\mathfrak{S}_{m+n} \mid w|_{[1,n]} \text{ and } w|_{[n+1,n+m]} \text{ are strictly increasing}\}$$

and $v'\in N$ and $u'\in M$. Hence, $\mathbf{r}_{M,N}$ is well defined whenever at least one of $R(\beta)$ and $R(\gamma)$ is symmetric.

The proof of (1.11) will be given later in § 4.

2. Quantum affine algebras

In this section, we briefly review the representation theory of quantum affine algebras following [AK97, Kas02]. When concerned with quantum affine algebras, we take the algebraic closure of $\mathbb{C}(q)$ in $\bigcup_{m>0}\mathbb{C}((q^{1/m}))$ as a base field $\mathbf{k}$.

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2.1 Integrable modules

Let I be an index set and $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix of affine type.

We choose $0 \in I$ as the leftmost vertices in the tables in [Kac90, pp. 54, 55] except in the $A_{2n}^{(2)}$ case where we take the longest simple root as α_0. Set $I_0 = I\setminus\{0\}$.

The weight lattice P is given by

$$P = \left(\bigoplus_{i\in I} \mathbb{Z}\Lambda_i\right) \oplus \mathbb{Z}\delta,$$

and the simple roots are given by

$$\alpha_i = \sum_{j\in I} a_{ji}\Lambda_j + \delta(i=0)\delta.$$

The weight δ is called the imaginary root. There exist $d_i \in \mathbb{Z}_{>0}$ such that

$$\delta = \sum_{i\in I} d_i\alpha_i.$$

Note that $d_i = 1$ for $i = 0$. The simple coroots $h_i \in P^\vee := \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ are given by

$$\langle h_i, \Lambda_j\rangle = \delta_{ij}, \quad \langle h_i, \delta\rangle = 0.$$

Hence we have $\langle h_i, \alpha_j\rangle = a_{ij}$.

Let $c = \sum_{i\in I} c_i h_i$ be a unique element such that $c_i \in \mathbb{Z}_{>0}$ and

$$\mathbb{Z}\, c = \left\{ h \in \bigoplus_{i\in I} \mathbb{Z}h_i \;\middle|\; \langle h, \alpha_i\rangle = 0 \text{ for any } i \in I \right\}.$$

Let us take a $\mathbb{Q}$-valued symmetric bilinear form $(\bullet\,,\bullet)$ on P such that

$$\langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad (\delta, \lambda) = \langle c, \lambda\rangle \quad \text{for any } \lambda \in P.$$

Let q be an indeterminate. For each $i \in I$, set $q_i = q^{(\alpha_i,\alpha_i)/2}$.

Definition 2.1. The *quantum group* $U_q(\mathfrak{g})$ associated with (A, P) is the $\mathbf{k}$-algebra generated by e_i, f_i $(i \in I)$ and q^λ $(\lambda \in P)$ satisfying the following relations:

$$\begin{gathered}
q^0 = 1, \quad q^\lambda q^{\lambda'} = q^{\lambda+\lambda'} \quad \text{for } \lambda, \lambda' \in P, \\
q^\lambda e_i q^{-\lambda} = q^{(\lambda,\alpha_i)} e_i, \quad q^\lambda f_i q^{-\lambda} = q^{-(\lambda,\alpha_i)} f_i \quad \text{for } \lambda \in P, i \in I, \\
e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \quad \text{where } K_i = q^{\alpha_i}, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i e_i^{1-a_{ij}-r} e_j e_i^r = 0 \quad \text{if } i \neq j, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i f_i^{1-a_{ij}-r} f_j f_i^r = 0 \quad \text{if } i \neq j.
\end{gathered}$$

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Here, we set $[n]_i = q_i^n - q_i^{-n}/q_i - q_i^{-1}, [n]_i! = \prod_{k=1}^n [k]_i$ and $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_i = [m]_i!/[m-n]_i![n]_i!$ for each $n \in \mathbb{Z}_{\geqslant 0}$, $i \in I$ and $m \geqslant n$.

We denote by $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1} (i \in I)$, and call it a *quantum affine algebra*. The algebra $U_q'(\mathfrak{g})$ has a Hopf algebra structure with the coproduct:

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, \\ \Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i, \\ \Delta(f_i) &= f_i \otimes 1 + K_i \otimes f_i. \end{aligned} \tag{2.1}$$

Set

$$P_{\text{cl}} = P/\mathbb{Z}\delta$$

and call it the *classical weight lattice*. Let $\text{cl} : P \to P_{\text{cl}}$ be the projection. Then $P_{\text{cl}} = \bigoplus_{i \in I} \mathbb{Z}\,\text{cl}(\Lambda_i)$. Set $P_{\text{cl}}^0 = \{\lambda \in P_{\text{cl}} \mid \langle c, \lambda \rangle = 0\} \subset P_{\text{cl}}$.

A $U_q'(\mathfrak{g})$-module M is called an *integrable module* if:

(a) M has a weight space decomposition

$$M = \bigoplus_{\lambda \in P_{\text{cl}}} M_\lambda,$$

where $M_\lambda = \{u \in M \mid K_i u = q_i^{\langle h_i, \lambda \rangle} u \text{ for all } i \in I\}$;

(b) the actions of e_i and f_i on M are locally nilpotent for any $i \in I$.

Let us denote by $U_q'(\mathfrak{g})$-mod the abelian tensor category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules.

If M is a simple module in $U_q'(\mathfrak{g})$-mod, then there exists a non-zero vector $u \in M$ of weight $\lambda \in P_{\text{cl}}^0$ such that λ is dominant (i.e. $\langle h_i, \lambda \rangle \geqslant 0$ for any $i \in I_0$) and all the weights of M lie in $\lambda - \sum_{i \in I_0} \mathbb{Z}_{\geqslant 0} \alpha_i$. We say that λ is the *dominant extremal weight* of M and u is a *dominant extremal vector* of M. Note that a dominant extremal vector of M is unique up to a constant multiple.

Let z be an indeterminate. For a $U_q'(\mathfrak{g})$-module M, let us denote by M_z the module $\mathbf{k}[z, z^{-1}] \otimes M$ with the action of $U_q'(\mathfrak{g})$ given by

$$e_i(u_z) = z^{\delta_{i,0}}(e_i u)_z, \quad f_i(u_z) = z^{-\delta_{i,0}}(f_i u)_z, \quad K_i(u_z) = (K_i u)_z.$$

Here, for $u \in M$, we denote by u_z the element $1 \otimes u \in \mathbf{k}[z, z^{-1}] \otimes M$.

2.2 *R*-matrices

We recall the notion of R-matrices [Kas02, § 8]. Let us choose the following *universal R-matrix*. Let us take a basis $\{P_\nu\}_\nu$ of $U_q^+(\mathfrak{g})$ and a basis $\{Q_\nu\}_\nu$ of $U_q^-(\mathfrak{g})$ dual to each other with respect to a suitable coupling between $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$. Then for $U_q'(\mathfrak{g})$-modules M and N define

$$R_{MN}^{\text{univ}}(u \otimes v) = q^{(\text{wt}(u), \text{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u, \tag{2.2}$$

so that R_{MN}^{univ} gives a $U_q'(\mathfrak{g})$-linear homomorphism from $M \otimes N$ to $N \otimes M$ provided that the infinite sum has a meaning.

Let M and N be $U_q'(\mathfrak{g})$-modules in $U_q'(\mathfrak{g})$-mod, and let z_1 and z_2 be indeterminates. Then $R_{M_{z_1}, N_{z_2}}^{\text{univ}}$ converges in the (z_2/z_1)-adic topology. Hence we obtain a morphism of

$\mathbf{k}[[z_2/z_1]] \bigotimes_{\mathbf{k}[z_2/z_1]} \mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}] \otimes U_q'(\mathfrak{g})$-modules

$$R^{\mathrm{univ}}_{M_{z_1},N_{z_2}} : \mathbf{k}[[z_2/z_1]] \underset{\mathbf{k}[z_2/z_1]}{\otimes} (M_{z_1} \otimes N_{z_2}) \to \mathbf{k}[[z_2/z_1]] \underset{\mathbf{k}[z_2/z_1]}{\otimes} (N_{z_2} \otimes M_{z_1}).$$

If there exist $a \in \mathbf{k}((z_2/z_1))$ and a $\mathbf{k}[z_1^{\pm 1}, z_2^{\pm 1}] \otimes U_q'(\mathfrak{g})$-linear homomorphism

$$R : M_{z_1} \otimes N_{z_2} \to N_{z_2} \otimes M_{z_1}$$

such that $R^{\mathrm{univ}}_{M_{z_1},N_{z_2}} = aR$, then we say that $R^{\mathrm{univ}}_{M_{z_1},N_{z_2}}$ is *rationally renormalizable.*

Now assume further that M and N are non-zero. Then we can choose R so that, for any $c_1, c_2 \in \mathbf{k}^\times$, the specialization of R at $z_1 = c_1$, $z_2 = c_2$,

$$R|_{z_1=c_1,z_2=c_2} : M_{c_1} \otimes N_{c_2} \to N_{c_2} \otimes M_{c_1},$$

does not vanish. Such an R is unique up to a multiple of $\mathbf{k}[(z_1/z_2)^{\pm 1}]^\times = \bigsqcup_{n\in\mathbb{Z}} \mathbf{k}^\times z_1^n z_2^{-n}$. We write

$$\mathbf{r}_{M,N} := R|_{z_1=z_2=1} : M \otimes N \to N \otimes M,$$

and call it the *renormalized R-matrix.* The renormalized R-matrix $\mathbf{r}_{M,N}$ is well defined up to a constant multiple when $R^{\mathrm{univ}}_{M_{z_1},N_{z_2}}$ is rationally renormalizable. By the definition, $\mathbf{r}_{M,N}$ never vanishes.

Now assume that M_1 and M_2 are simple $U_q'(\mathfrak{g})$-modules in $U_q'(\mathfrak{g})$-mod. Then the universal R-matrix $R^{\mathrm{univ}}_{(M_1)_{z_1},(M_2)_{z_2}}$ is rationally renormalizable. More precisely, letting u_1 and u_2 be dominant extremal weight vectors of M_1 and M_2, respectively, there exists $a(z_2/z_1) \in \mathbf{k}[[z_2/z_1]]^\times$ such that

$$R^{\mathrm{univ}}_{(M_1)_{z_1},(M_2)_{z_2}}((u_1)_{z_1} \otimes (u_2)_{z_2}) = a(z_2/z_1)((u_2)_{z_2} \otimes (u_1)_{z_1}).$$

Then $R^{\mathrm{norm}}_{M_1,M_2} := a(z_2/z_1)^{-1} R^{\mathrm{univ}}_{(M_1)_{z_1},(M_2)_{z_2}}$ is a unique $\mathbf{k}(z_1, z_2) \otimes U_q'(\mathfrak{g})$-module homomorphism

$$R^{\mathrm{norm}}_{M_1,M_2} : \mathbf{k}(z_1, z_2) \underset{\mathbf{k}[z_1^{\pm 1},z_2^{\pm 1}]}{\otimes} ((M_1)_{z_1} \otimes (M_2)_{z_2}) \longrightarrow \mathbf{k}(z_1, z_2) \underset{\mathbf{k}[z_1^{\pm 1},z_2^{\pm 1}]}{\otimes} ((M_2)_{z_2} \otimes (M_1)_{z_1}) \quad (2.3)$$

satisfying

$$R^{\mathrm{norm}}_{M_1,M_2}((u_1)_{z_1} \otimes (u_2)_{z_2}) = (u_2)_{z_2} \otimes (u_1)_{z_1}. \quad (2.4)$$

Note that $\mathbf{k}(z_1, z_2) \otimes_{\mathbf{k}[z_1^{\pm 1},z_2^{\pm 1}]} ((M_1)_{z_1} \otimes (M_2)_{z_2})$ is a simple $\mathbf{k}(z_1, z_2) \otimes U_q'(\mathfrak{g})$-module [Kas02, Proposition 9.5]. We call $R^{\mathrm{norm}}_{M_1,M_2}$ the *normalized R-matrix.*

Let $d_{M_1,M_2}(u) \in \mathbf{k}[u]$ be a monic polynomial of the smallest degree such that the image of $d_{M_1,M_2}(z_2/z_1)R^{\mathrm{norm}}_{M_1,M_2}$ is contained in $(M_2)_{z_2} \otimes (M_1)_{z_1}$. We call $d_{M_1,M_2}(u)$ the *denominator of* $R^{\mathrm{norm}}_{M_1,M_2}$. Then we have

$$d_{M_1,M_2}(z_2/z_1)R^{\mathrm{norm}}_{M_1,M_2} : (M_1)_{z_1} \otimes (M_2)_{z_2} \longrightarrow (M_2)_{z_2} \otimes (M_1)_{z_1}, \quad (2.5)$$

and the renormalized R-matrix

$$\mathbf{r}_{M_1,M_2} : M_1 \otimes M_2 \longrightarrow M_2 \otimes M_1$$

is equal to the specialization of $d_{M_1,M_2}(z_2/z_1)R^{\mathrm{norm}}_{M_1,M_2}$ at $z_1 = z_2 = 1$ up to a constant multiple.

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Note that R^{univ} satisfies the following properties. For M, M_1, M_2, N, N_1, N_2 in $U_q'(\mathfrak{g})$-mod, the diagrams

$$\begin{tikzcd} M_1 \otimes M_2 \otimes N \arrow[r, "M_1\otimes R^{\mathrm{univ}}_{M_2,N}"'] \arrow[rr, bend left, "R^{\mathrm{univ}}_{M_1\otimes M_2,N}"] & M_1 \otimes N \otimes M_2 \arrow[r, "R^{\mathrm{univ}}_{M_1,N}\otimes M_2"'] & N \otimes M_1 \otimes M_2, \end{tikzcd}$$

$$\begin{tikzcd} M \otimes N_1 \otimes N_2 \arrow[r, "R^{\mathrm{univ}}_{M,N_1}\otimes N_2"'] \arrow[rr, bend left, "R^{\mathrm{univ}}_{M,N_1\otimes N_2}"] & N_1 \otimes M \otimes N_2 \arrow[r, "N_1\otimes R^{\mathrm{univ}}_{M,N_2}"'] & N_1 \otimes N_2 \otimes M \end{tikzcd}$$

commute. Hence, if $R^{\mathrm{univ}}_{(M_1)_{z_1},N_{z_2}}$ and $R^{\mathrm{univ}}_{(M_2)_{z_1},N_{z_2}}$ are rationally renormalizable, then $R^{\mathrm{univ}}_{(M_1\otimes M_2)_{z_1},N_{z_2}}$ is also rationally renormalizable. Moreover, we have

$$(\mathbf{r}_{M_1,N} \otimes M_2) \circ (M_1 \otimes \mathbf{r}_{M_2,N}) = c\,\mathbf{r}_{M_1\otimes M_2,N} \quad \text{for some } c \in \mathbf{k}. \tag{2.6}$$

Note that c may vanish. In particular, if M_1, M_2 and N are simple modules in $U_q'(\mathfrak{g})$-mod, then $R^{\mathrm{univ}}_{(M_1\otimes M_2)_{z_1},N_{z_2}}$ is rationally renormalizable.

3. Simple heads and socles of tensor products

In this section we give a proof of the conjecture in the Introduction for the quiver Hecke algebra case and the quantum affine algebra case.

3.1 Quiver Hecke algebra case

We first discuss the quiver Hecke algebra case.

Lemma 3.1. *Let $\beta_k \in \mathsf{Q}^+$ and $M_k \in R(\beta_k)$-mod $(k = 1, 2, 3)$. Let X be an $R(\beta_1+\beta_2)$-submodule of $M_1 \circ M_2$ and Y an $R(\beta_2+\beta_3)$-submodule of $M_2 \circ M_3$ such that $X \circ M_3 \subset M_1 \circ Y$ as submodules of $M_1 \circ M_2 \circ M_3$. Then there exists an $R(\beta_2)$-submodule N of M_2 such that $X \subset M_1 \circ N$ and $N \circ M_3 \subset Y$.*

Proof. Set $n_k = \mathrm{ht}(\beta_k)$. Set $N = \{u \in M_2 \mid u \otimes M_3 \subset Y\}$. Then N is the largest $R(\beta_2)$-submodule of M_2 such that $N \circ M_3 \subset Y$. Let us show that $X \subset M_1 \circ N$. Let us take a basis $\{v_a\}_{a\in A}$ of M_1.

By (1.3), we have

$$M_1 \circ M_2 = \bigoplus_{w\in\mathfrak{S}_{n_1,n_2}} \tau_w(M_1 \otimes M_2).$$

Hence, any $u \in X$ can be uniquely written as

$$u = \sum_{w\in\mathfrak{S}_{n_1,n_2},a\in A} \tau_w(v_a \otimes u_{a,w})$$

with $u_{a,w} \in M_2$. Then, for any $s \in M_3$, we have

$$u \otimes s = \sum_{w\in\mathfrak{S}_{n_1,n_2},\,a\in A} \tau_w(v_a \otimes u_{a,w} \otimes s) \in X \circ M_3 \subset M_1 \circ Y.$$

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Since
$$M_1 \circ Y = \bigoplus_{w \in \mathfrak{S}_{n_1, n_2+n_3}} \tau_w(M_1 \otimes Y)$$
and $\mathfrak{S}_{n_1,n_2} \subset \mathfrak{S}_{n_1,n_2+n_3}$, we have
$$u_{a,w} \otimes s \in Y \quad \text{for any } a \in A \text{ and } w \in \mathfrak{S}_{n_1,n_2}.$$
Therefore we have $u_{a,w} \in N$. $\square$

Theorem 3.2. *Let $\beta, \gamma \in \mathsf{Q}^+$ and $M \in R(\beta)$-mod and $N \in R(\gamma)$-mod. We assume, further, the following condition:*

$$\begin{aligned} &\text{(a)} \ \ R(\beta) \text{ is symmetric and } \mathbf{r}_{M,M} \in \mathbf{k}\,\mathrm{id}_{M\circ M};\\ &\text{(b)} \ \ M \text{ is non-zero;}\\ &\text{(c)} \ \ N \text{ is a simple } R(\gamma)\text{-module.}\end{aligned} \tag{3.1}$$

Then:

(i) *$M \circ N$ has a simple socle and a simple head. Similarly, $N \circ M$ has a simple socle and a simple head;*

(ii) *moreover, $\mathrm{Im}(\mathbf{r}_{N,M})$ is equal to the socle of $M \circ N$ and also equal to the head of $N \circ M$. Similarly, $\mathrm{Im}(\mathbf{r}_{M,N})$ is equal to the socle of $N \circ M$ and to the head of $M \circ N$.*

In particular, M is a simple module.

Proof. Let us show that $\mathrm{Im}(\mathbf{r}_{N,M})$ is a unique simple submodule of $M \circ N$. Let $S \subset M \circ N$ be an arbitrary non-zero $R(\beta+\gamma)$-submodule. Let m and m' be the multiplicity of zero of $R_{N,(M)_z} : N \circ (M)_z \to (M)_z \circ N$ and $R_{M,(M)_z} : M \circ (M)_z \to (M)_z \circ M$ at $z = 0$, respectively. Then by the definition, $\mathbf{r}_{N,M} = (z^{-m} R_{N,(M)_z})|_{z=0} : N \circ M \to M \circ N$ and $\mathbf{r}_{M,M} = (z^{-m'} R_{M,(M)_z})|_{z=0} : M \circ M \to M \circ M$. Now we have a commutative diagram

$$\begin{tikzcd} S \circ (M)_z \arrow[rr, "z^{-m-m'} R_{S,(M)_z}"] \arrow[d, hook] & & (M)_z \circ S \arrow[d, hook] \\ M \circ N \circ (M)_z \arrow[r, "M \circ z^{-m} R_{N,(M)_z}"] & M \circ (M)_z \circ N \arrow[r, "z^{-m'} R_{M,(M)_z} \circ N"] & (M)_z \circ M \circ N \end{tikzcd}$$

Therefore $z^{-m-m'} R_{S,(M)_z} : S \circ (M)_z \to (M)_z \circ S$ is well defined, and we obtain the following commutative diagram by specializing the above diagram at $z = 0$:

$$\begin{tikzcd} S \circ M \arrow[rr] \arrow[d, hook] & & M \circ S \arrow[d, hook] \\ M \circ N \circ M \arrow[r, "M \circ \mathbf{r}_{N,M}"] & M \circ M \circ N \arrow[r, "\mathrm{id}_{M\circ M\circ N}"] & M \circ M \circ N \end{tikzcd}$$

Here, we have used the assumption that $\mathbf{r}_{M,M}$ is equal to $\mathrm{id}_{M\circ M}$ up to a constant multiple.

Hence we obtain $(M \circ \mathbf{r}_{N,M})(S \circ M) \subset M \circ S$, or equivalently
$$S \circ M \subset M \circ (\mathbf{r}_{N,M})^{-1}(S).$$
By the preceding lemma, there exists an $R(\gamma)$-submodule K of N such that $S \subset M \circ K$ and $K \circ M \subset (\mathbf{r}_{N,M})^{-1}(S)$. By the first inclusion, we have $K \neq 0$. Since N is simple, we have $K = N$

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and we obtain $N \circ M \subset (\mathbf{r}_{N,M})^{-1}(S)$, or equivalently, $\operatorname{Im}(\mathbf{r}_{N,M}) \subset S$. Noting that S is an arbitrary non-zero submodule of $M \circ N$, we conclude that $\operatorname{Im}(\mathbf{r}_{N,M})$ is a unique simple submodule of $M \circ N$.

The proof of the other statements in (i) and (ii) is similar.

The simplicity of M follows from (i) and (ii) by taking the one-dimensional $R(0)$-module $\mathbf{k}$ as N. Note that $\mathbf{r}_{M,\mathbf{k}}$ and $\mathbf{r}_{\mathbf{k},M}$ coincide with the identity morphism id_M. □

A simple $R(\beta)$-module M is called *real* if $M \circ M$ is simple. Then the following corollary is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Assume that $R(\beta)$ is symmetric and M is a non-zero $R(\beta)$-module in $R(\beta)$-mod. Then the following conditions are equivalent:*

(a) *M is a real simple $R(\beta)$-module;*

(b) $\mathbf{r}_{M,M} \in \mathbf{k}\,\mathrm{id}_{M\circ M}$*;*

(c) $\operatorname{End}_{R(2\beta)}(M \circ M) \simeq \mathbf{k}\,\mathrm{id}_{M\circ M}$.

We have also the following corollary.

Corollary 3.4. *If $R(\beta)$ is symmetric and M is a real simple $R(\beta)$-module, then $M^{\circ n} := \overbrace{M \circ \cdots \circ M}^{n}$ is a simple $R(n\beta)$-module for any $n \geqslant 1$.*

Proof. The quiver Hecke algebra version of (2.6) implies that $\mathbf{r}_{M^{\circ m},M^{\circ n}}$ is equal to $\mathrm{id}_{M^{\circ(m+n)}}$ up to a constant multiple. □

Thus we have established the first statement of the conjecture in the Introduction in the quiver Hecke algebra case.

Lemma 3.5. *Let $\beta, \gamma \in \mathsf{Q}^+$, and let $M \in R(\beta)\text{-mod}$ and $L \in R(\beta+\gamma)\text{-mod}$. Then there exist X, $Y \in R(\gamma)\text{-mod}$ satisfying the following universal properties:*

$$\operatorname{Hom}_{R(\beta+\gamma)}(M \circ Z, L) \simeq \operatorname{Hom}_{R(\gamma)}(Z, X), \tag{3.2}$$
$$\operatorname{Hom}_{R(\beta+\gamma)}(L, Z \circ M) \simeq \operatorname{Hom}_{R(\gamma)}(Y, Z) \tag{3.3}$$

functorially in $Z \in R(\gamma)\text{-mod}$.

Proof. Set $X = \operatorname{Hom}_{R(\beta+\gamma)}(M \circ R(\gamma), L)$. Then

$$\begin{aligned}\operatorname{Hom}_{R(\beta+\gamma)}(M \circ Z, L) &\simeq \operatorname{Hom}_{R(\beta)\otimes R(\gamma)}(M \otimes Z, L)\\ &\simeq \operatorname{Hom}_{R(\gamma)}(Z, \operatorname{Hom}_{R(\beta)}(M, L)).\end{aligned}$$

Similarly, set $Y = (\operatorname{Hom}_{R(\beta+\gamma)}(M^* \circ R(\gamma), L^*))^*$. Then, by using (1.4), we have

$$\begin{aligned}\operatorname{Hom}_{R(\beta+\gamma)}(L, Z \circ M) &\simeq \operatorname{Hom}_{R(\beta+\gamma)}(M^* \circ Z^*, L^*)\\ &\simeq \operatorname{Hom}_{R(\beta)\otimes R(\gamma)}(M^* \otimes Z^*, L^*)\\ &\simeq \operatorname{Hom}_{R(\gamma)}(Z^*, Y^*) \simeq \operatorname{Hom}_{R(\gamma)}(Y, Z).\end{aligned}$$ □

Proposition 3.6. *Let $\beta, \gamma \in \mathsf{Q}^+$. Assume that $R(\beta)$ is symmetric, and let M be a real simple module in $R(\beta)$-mod, and L a simple module in $R(\beta+\gamma)$-mod. Then the $R(\gamma)$-module $X := \operatorname{Hom}_{R(\beta+\gamma)}(M \circ R(\gamma), L)$ is either zero or has a simple socle.*

Proof. The $R(\gamma)$-module X satisfies the functorial property (3.2). Assume that $X \neq 0$. Let $p : M \circ X \to L$ be the canonical morphism. Since L is simple, it is an epimorphism. Let Y be as in Lemma 3.5, and let $i : L \to Y \circ M$ be the canonical morphism. For an arbitrary simple $R(\gamma)$-submodule S of X, since $\mathrm{Hom}_{R(\beta+\gamma)}(M \circ S, L) \simeq \mathrm{Hom}_{R(\gamma)}(S, X)$, the composition $M \circ S \to M \circ X \xrightarrow{p} L$ does not vanish. Hence, by Theorem 3.2, L is the simple head of $M \circ S$ and is the simple socle of $S \circ M$. Moreover, $L \cong \mathrm{Im}(\mathbf{r}_{M,S})$. Since the monomorphism $L \to S \circ M$ factors through i by (3.3), the morphism $i : L \to Y \circ M$ is a monomorphism.

As in the proof of Theorem 3.2, we have a commutative diagram

$$\begin{tikzcd}
M \circ L \arrow[r] \arrow[d, tail, "M \circ i"] & L \circ M \arrow[d, tail, "i \circ M"] \\
M \circ Y \circ M \arrow[r, "\mathbf{r}_{M,Y} \circ M"] & Y \circ M \circ M
\end{tikzcd}$$

Then we obtain $M \circ i(L) \subset (\mathbf{r}_{M,Y})^{-1}(i(L)) \circ M$. Hence, by Lemma 3.1, there exists an $R(\gamma)$-submodule Z of Y such that $\mathbf{r}_{M,Y}(M \circ Z) \subset i(L)$ and $i(L) \subset Z \circ M$. The last inclusion induces a morphism $L \to Z \circ M$ and a morphism $Y \to Z$ by (3.3). Since the composition $Y \to Z \to Y$ is the identity again by (3.3), we have $Z = Y$. Hence $\mathrm{Im}(\mathbf{r}_{M,Y}) \subset i(L)$, which gives the commutative diagram

$$\begin{tikzcd}
M \circ Y \arrow[r] \arrow[rr, bend left, "\mathbf{r}_{M,Y}"] & L \arrow[r, tail, "i"'] & Y \circ M
\end{tikzcd}$$

By the argument dual to the above one (see also the proof of Proposition 3.8), we have a commutative diagram

$$\begin{tikzcd}
M \circ X \arrow[r, two heads, "p"'] \arrow[rr, bend left, "\mathbf{r}_{M,X}"] & L \arrow[r, "\xi"'] & X \circ M
\end{tikzcd}$$

Hence $\xi : L \to X \circ M$ is a monomorphism, and $\mathrm{Im}(\mathbf{r}_{M,X})$ is isomorphic to L. By (3.3), there exists a unique morphism $\varphi : Y \to X$ such that ξ factors as

$$\begin{tikzcd}
L \arrow[r, tail, "i"'] \arrow[rr, bend left, "\xi"] & Y \circ M \arrow[r, "\varphi \circ M"'] & X \circ M
\end{tikzcd}$$

Let us show that $\mathrm{Im}(\varphi)$ is a unique simple submodule of X. In order to see this, let S be an arbitrary simple $R(\beta)$-submodule of X. We have seen that L is isomorphic to the head of $M \circ S$ and isomorphic to $\mathrm{Im}(\mathbf{r}_{M,S})$. Since the composition $M \circ S \to M \circ X \xrightarrow{\mathbf{r}_{M,X}} X \circ M$ does not vanish, we have a commutative diagram by [KKK13a, Lemma 1.4.8]:

$$\begin{tikzcd}
M \circ S \arrow[r, "\mathbf{r}_{M,S}"] \arrow[d, tail] & S \circ M \arrow[d, tail] \\
M \circ X \arrow[r, "\mathbf{r}_{M,X}"] & X \circ M
\end{tikzcd}$$

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Since $\mathrm{Im}(\mathbf{r}_{M,S}) \simeq \mathrm{Im}(\mathbf{r}_{M,X}) \simeq L$, the morphism $\xi : L \to X \circ M$ factors as $L \to S \circ M \to X \circ M$. Hence (3.3) implies that $\varphi : Y \to X$ factors through $Y \to S \to X$. Thus we obtain $\mathrm{Im}(\varphi) \subset S$. Since S is an arbitrary simple submodule of X, we conclude that $\mathrm{Im}(\varphi)$ is a unique simple submodule of X. $\square$

Let $\beta, \gamma \in \mathsf{Q}^+$. For a simple $R(\beta)$-module M and a simple $R(\gamma)$-module N, let us denote by $M \diamond N$ the head of $M \circ N$.

Corollary 3.7. *Let $\beta, \gamma \in \mathsf{Q}^+$. Assume that $R(\beta)$ is symmetric, and let M be a real simple module in $R(\beta)$-mod. Then the map $N \mapsto M \diamond N$ is injective from the set of the isomorphism classes of simple objects of $R(\gamma)$-mod to the set of the isomorphism classes of simple objects of $R(\beta+\gamma)$-mod.*

Proof. Indeed, for a simple $R(\gamma)$-module N, $M \diamond N$ is a simple $R(\beta+\gamma)$-module by Theorem 3.2, and $N \subset X := \mathrm{Hom}_{R(\beta+\gamma)}(M \circ R(\gamma), M \diamond N)$ is the socle of X by the preceding proposition. $\square$

If $L(i)$ is the one-dimensional simple $R(\alpha_i)$-module, then $L(i)$ is real and $M \diamond L(i)$ corresponds to the crystal operator $\tilde{f}_i M$ and $L(i) \diamond M$ to the dual crystal operator $\tilde{f}_i^{\vee} M$ in [LV11]. Hence, $\diamond$ is a generalization of the crystal operator as suggested in [Lec03].

Proposition 3.8. *Let $\beta, \gamma \in \mathsf{Q}^+$. Assume that $R(\beta)$ is symmetric, and let M be a real simple module in $R(\beta)$-mod, and N a simple module in $R(\gamma)$-mod. Then $\mathrm{End}_{R(\beta+\gamma)}(M \circ N) \simeq \mathbf{k}\,\mathrm{id}_{M \circ N}$.*

Proof. Set $L = M \circ N$. Let $X, Y \in R(\gamma)$-mod be as in Lemma 3.5. Let $p : M \circ X \to L$ and $i : L \to Y \circ M$ be the canonical morphisms. Then the isomorphism $M \circ N \to L$ induces a morphism $j : N \to X$ such that the composition $M \circ N \xrightarrow{M \circ j} M \circ X \xrightarrow{p} L$ is that isomorphism. Hence $p : M \circ X \to L$ is an epimorphism. Since N is simple and j does not vanish, the morphism $j : N \to X$ is a monomorphism.

We have a commutative diagram

$$\begin{tikzcd} M \circ M \circ X \arrow[r, "\mathbf{r}_{M,M} \circ X"] \arrow[d, "M \circ p"] & M \circ M \circ X \arrow[r, "M \circ \mathbf{r}_{M,X}"] & M \circ X \circ M \arrow[d, "p \circ M"] \\ M \circ L \arrow[rr] & & L \circ M \end{tikzcd}$$

Since $\mathbf{r}_{M,M}$ is $\mathrm{id}_{M \circ M}$ up to a constant multiple, we obtain the commutative diagram

$$\begin{tikzcd} M \circ (M \circ X) \arrow[r, "M \circ \mathbf{r}_{M,X}"] \arrow[d, "M \circ p"] & M \circ X \circ M \arrow[d, "p \circ M"] \\ M \circ L \arrow[r] & L \circ M \end{tikzcd}$$

Therefore

$$M \circ (\mathbf{r}_{M,X}(\mathrm{Ker}\, p)) \subset (\mathrm{Ker}\, p) \circ M.$$

Hence Lemma 3.1 implies that there exists $Z \subset X$ such that $\mathbf{r}_{M,X}(\mathrm{Ker}\, p) \subset Z \circ M$ and $M \circ Z \subset \mathrm{Ker}\, p$. The last inclusion shows that $M \circ Z \to M \circ X \to L$ vanishes. Hence by (3.2), the morphism

$Z \to X$ vanishes, or equivalently, $Z = 0$. Hence we have $\mathbf{r}_{M,X}(\operatorname{Ker} p) = 0$. Therefore $\mathbf{r}_{M,X}$ factors through p:

$$\begin{tikzcd} M \circ X \arrow[r, two heads, "p"'] \arrow[rr, bend left, "\mathbf{r}_{M,X}"] & L \arrow[r, "\xi"'] & X \circ M \end{tikzcd}$$

Since $\mathbf{r}_{M,X} \neq 0$, the morphism ξ does not vanish. By (3.3), there exists $\varphi : Y \to X$ such that $\xi : L \to X \circ M$ coincides with the composition $L \xrightarrow{i} Y \circ M \xrightarrow{\varphi \circ M} X \circ M$. Then we have a commutative diagram with the solid arrows:

$$\begin{tikzcd} M \circ N \arrow[rr, "\mathbf{r}_{M,N}"] \arrow[d, tail, "M\circ j"] \arrow[dr, dotted, "\sim"] & & N \circ M \arrow[d, tail, "j\circ M"] \\ M \circ X \arrow[rr, "\mathbf{r}_{M,X}"'] \arrow[r, dotted, two heads, "p"] & L \arrow[r, dotted, "\xi"] & X \circ M \end{tikzcd}$$

Indeed, the commutativity follows from [KKK13a, Lemma 1.4.8] and the fact that the composition $M \circ N \xrightarrow{M\circ j} M \circ X \xrightarrow{\mathbf{r}_{M,X}} X \circ M$ does not vanish because it coincides with $M \circ N \xrightarrow{\sim} L \xrightarrow{\xi} X \circ M$.

Thus $\xi : L \to X \circ M$ coincides with the composition

$$L \simeq M \circ N \xrightarrow{\mathbf{r}_{M,N}} N \circ M \xrightarrow{j\circ M} X \circ M.$$

Hence (3.3) implies that $\varphi : Y \to X$ decomposes as

$$\begin{tikzcd} Y \arrow[r, "\psi"] & N \arrow[r, tail, "j"] & X \end{tikzcd}$$

Since N is simple, ψ is an epimorphism, and we conclude that N is the image of $\varphi : Y \to X$.

Now let us prove that any $f \in \operatorname{End}_{R(\beta+\gamma)}(L)$ satisfies $f \in \mathbf{k}\,\mathrm{id}_L$. By the universal properties (3.2) and (3.3), the endomorphism f induces endomorphisms $f_X \in \operatorname{End}_{R(\gamma)}(X)$ and $f_Y \in \operatorname{End}_{R(\gamma)}(Y)$ such that the following diagrams with the solid arrows commute:

$$\begin{tikzcd} M \circ X \arrow[r, two heads, "p"'] \arrow[d, "M\circ f_X"] \arrow[rr, dotted, bend left, "\mathbf{r}_{M,X}"] & L \arrow[r, dotted, "\xi"'] \arrow[d, "f"] & X \circ M \arrow[d, "f_X\circ M"] \\ M \circ X \arrow[r, two heads, "p"] \arrow[rr, dotted, bend right, "\mathbf{r}_{M,X}"'] & L \arrow[r, dotted, "\xi"] & X \circ M \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \arrow[r, "i"] \arrow[d, "f"] & Y \circ M \arrow[d, "f_Y\circ M"] \\ L \arrow[r, "i"] & Y \circ M \end{tikzcd} \tag{3.4}$$

Since $\mathbf{r}_{M,X}$ commutes with f, the left diagram with dotted arrows commutes. Hence, the following diagram with the solid arrows commutes:

$$\begin{tikzcd} Y \arrow[r, two heads, "\psi"'] \arrow[d, "f_Y"] \arrow[rr, bend left, "\varphi"] & N \arrow[r, tail, "j"'] \arrow[d, dotted, "f_N"] & X \arrow[d, "f_X"] \\ Y \arrow[r, two heads, "\psi"] \arrow[rr, bend right, "\varphi"'] & N \arrow[r, tail, "j"] & X \end{tikzcd} \tag{3.5}$$

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Then we can add the dotted arrow f_N so that the whole diagram (3.5) commutes. Since N is simple, we have $f_N = c\,\mathrm{id}_N$ for some $c \in \mathbf{k}$. By replacing f with $f - c\,\mathrm{id}_L$, we may assume from the beginning that $f_N = 0$. Then $f_X \circ j = 0$. Now $f = 0$ follows from the commutativity of the diagram

$$\begin{tikzcd} M \circ N \arrow[r, "M\circ j"] \arrow[rr, bend left, "\sim"] \arrow[dr, dotted, "0"'] & M \circ X \arrow[r, "p"] \arrow[d, "M\circ f_X"] & L \arrow[d, "f"] \\ & M \circ X \arrow[r, "p"] & L \end{tikzcd}$$

□

Corollary 3.9. *Let $\beta, \gamma \in \mathsf{Q}^+$, and assume that $R(\beta)$ is symmetric. Let M be a real simple module in $R(\beta)$-mod, and N a simple module in $R(\gamma)$-mod.*

(i) *If the head of $M \circ N$ and the socle of $M \circ N$ are isomorphic, then $M \circ N$ is simple and $M \circ N \simeq N \circ M$.*

(ii) *If $M \circ N \simeq N \circ M$, then $M \circ N$ is simple. Conversely, if $M \circ N$ is simple, then $M \circ N \simeq N \circ M$.*

Proof. (i) Let S be the head of $M \circ N$ and the socle of $M \circ N$. Then S is simple. Now we have the morphisms

$$M \circ N \twoheadrightarrow S \rightarrowtail M \circ N.$$

By the previous proposition, the composition is equal to $\mathrm{id}_{M\circ N}$ up to a constant multiple. Hence $M \circ N$ and $N \circ M$ are isomorphic to S.

(ii) Assume first that $M \circ N \simeq N \circ M$. Then the simplicity of $M \circ N$ immediately follows from (i) because the socle of $M \circ N$ is isomorphic to the head of $N \circ M$ by Theorem 3.2.

If $M \circ N$ is simple, then $\mathbf{r}_{M,N}$ is injective. Since $\dim(M \circ N) = \dim(N \circ M)$, $\mathbf{r}_{M,N} : M \circ N \to N \circ M$ is an isomorphism. □

Note that, when $R(\beta)$ and $R(\gamma)$ are symmetric, for a real simple $R(\beta)$-module M and a real simple $R(\gamma)$-module N, their convolution $M \circ N$ is real simple if $M \circ N \simeq N \circ M$.

3.2 Quantum affine algebra case

Similar results to Theorem 3.2 and Corollaries 3.7 and 3.9 hold also for quantum affine algebras. Let $U_q'(\mathfrak{g})$ be the quantum affine algebra as in § 2. Recall that $U_q'(\mathfrak{g})$-mod denotes the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules.

First note that the following lemma, an analogue of Lemma 3.1 in the quantum affine algebra case, is almost trivial. Indeed, a similar result holds for any rigid monoidal category which is abelian and the tensor functor is additive.

Lemma 3.10. *Let M_k be a module in $U_q'(\mathfrak{g})$-mod $(k = 1, 2, 3)$. Let X be a $U_q'(\mathfrak{g})$-submodule of $M_1 \otimes M_2$ and Y a $U_q'(\mathfrak{g})$-submodule of $M_2 \otimes M_3$ such that $X \otimes M_3 \subset M_1 \otimes Y$ as submodules of $M_1 \otimes M_2 \otimes M_3$. Then there exists a $U_q'(\mathfrak{g})$-submodule N of M_2 such that $X \subset M_1 \otimes N$ and $N \otimes M_3 \subset Y$.*

Corollary 3.11. (i) *Let M_k be a module in $U_q'(\mathfrak{g})$-mod $(k = 1, 2, 3)$, and let $\varphi_1 : L \to M_1 \otimes M_2$ and $\varphi_2 : M_2 \otimes M_3 \to L'$ be* non-zero *morphisms. Assume, further, that M_2 is a simple module.*

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Then the composition

$$L \otimes M_3 \xrightarrow{\varphi_1 \otimes M_3} M_1 \otimes M_2 \otimes M_3 \xrightarrow{M_1 \otimes \varphi_2} M_1 \otimes L' \tag{3.6}$$

does not vanish.

(ii) *Let M, N_1 and N_2 be simple modules in $U_q'(\mathfrak{g})$-mod. Then the following diagram commutes up to a constant multiple*:

$$\begin{tikzcd} M \otimes N_1 \otimes N_2 \arrow[r, "\mathbf{r}_{M,N_1} \otimes N_2"'] \arrow[rr, bend left, "\mathbf{r}_{M,\, N_1 \otimes N_2}"] & N_1 \otimes M \otimes N_2 \arrow[r, "N_1 \otimes \mathbf{r}_{M,N_2}"'] & N_1 \otimes N_2 \otimes M \end{tikzcd}$$

Proof. (i) Assume that the composition (3.6) vanishes. Then we have $\operatorname{Im} \varphi_1 \otimes M_3 \subset M_1 \otimes \operatorname{Ker} \varphi_2$. Hence, by the preceding lemma, there exists $N \subset M_2$ such that $\operatorname{Im} \varphi_1 \subset M_1 \otimes N$ and $N \otimes M_3 \subset \operatorname{Ker} \varphi_2$. The first inclusion implies $N \neq 0$ and the last inclusion implies $N \neq M_2$. This contradicts the simplicity of M_2.

(ii) By (i), $(N_1 \otimes \mathbf{r}_{M,N_2}) \circ (\mathbf{r}_{M,N_1} \otimes N_2)$ does not vanish. Hence it is equal to $\mathbf{r}_{M,\, N_1 \otimes N_2}$ up to a constant multiple by (2.6). □

Since the proof of the following theorem is similar to the quiver Hecke algebra case, we just state the result, omitting its proof.

Theorem 3.12. *Let M and N be simple modules in $U_q'(\mathfrak{g})$-mod. We assume, further, that*

$$\mathbf{r}_{M,M} \in \mathbf{k}\, \mathrm{id}_{M \otimes M}. \tag{3.7}$$

Then we have:

(i) *$M \otimes N$ has a simple socle and a simple head;*

(ii) *moreover, $\operatorname{Im}(\mathbf{r}_{M,N})$ is equal to the head of $M \otimes N$ and is also equal to the socle of $N \otimes M$.*

Recall that a simple $U_q'(\mathfrak{g})$-module M is called *real* if $M \otimes M$ is simple. Hence M in Theorem 3.12 is real.

For a module M in $U_q'(\mathfrak{g})$-mod, let us denote by ${}^*M$ and M^* the right dual and the left dual of M, respectively. Hence we have isomorphisms

$$\begin{aligned} \operatorname{Hom}_{U_q'(\mathfrak{g})}(M \otimes X, Y) &\simeq \operatorname{Hom}_{U_q'(\mathfrak{g})}(X, {}^*M \otimes Y), \\ \operatorname{Hom}_{U_q'(\mathfrak{g})}(X \otimes {}^*M, Y) &\simeq \operatorname{Hom}_{U_q'(\mathfrak{g})}(X, Y \otimes M), \\ \operatorname{Hom}_{U_q'(\mathfrak{g})}(M^* \otimes X, Y) &\simeq \operatorname{Hom}_{U_q'(\mathfrak{g})}(X, M \otimes Y), \\ \operatorname{Hom}_{U_q'(\mathfrak{g})}(X \otimes M, Y) &\simeq \operatorname{Hom}_{U_q'(\mathfrak{g})}(X, Y \otimes M^*) \end{aligned} \tag{3.8}$$

functorial in X, $Y \in U_q'(\mathfrak{g})$-mod.

Corollary 3.13. *Under the assumption of the theorem above, the head of $\operatorname{Im} \mathbf{r}_{M,N} \otimes {}^*M$ is isomorphic to N.*

Proof. Set $S = \operatorname{Im} \mathbf{r}_{M,N}$. Since $\operatorname{Hom}_{U_q'(\mathfrak{g})}(S, N \otimes M) \simeq \operatorname{Hom}_{U_q'(\mathfrak{g})}(S \otimes {}^*M, N)$, there exists a non-trivial morphism $S \otimes {}^*M \to N$. Since N is simple, we have an epimorphism

$$S \otimes {}^*M \twoheadrightarrow N.$$

Since ${}^*M \otimes {}^*M \simeq {}^*(M \otimes M)$ is a simple module, the tensor product $S \otimes {}^*M$ has a simple head by the preceding theorem. Hence, we obtain the desired result. □

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For simple $U_q'(\mathfrak{g})$-modules M and N, let us denote by $M \diamond N$ the head of $M \otimes N$.

Corollary 3.14. *Let M be a real simple module in $U_q'(\mathfrak{g})$-mod. Then the map $N \mapsto M \diamond N$ is bijective on the set of the isomorphism classes of simple $U_q'(\mathfrak{g})$-modules in $U_q'(\mathfrak{g})$-mod, and its inverse is given by $N \mapsto N \diamond {}^*M$.*

Lemma 3.15. *Let M be a real simple module in $U_q'(\mathfrak{g})$-mod and N a simple module in $U_q'(\mathfrak{g})$-mod. Then we have* $\mathrm{End}_{U_q'(\mathfrak{g})}(M \otimes N) \simeq \mathbf{k}\,\mathrm{id}_{M\otimes N}$.

Proof. By Corollary 3.11, we have a commutative diagram up to a constant multiple

$$\begin{tikzcd} M^* \otimes M \otimes N \arrow[r, "\mathbf{r}_{M^*,M}\otimes N"'] \arrow[rr, bend left, "\mathbf{r}_{M^*,M\otimes N}"] & M \otimes M^* \otimes N \arrow[r, "M\otimes \mathbf{r}_{M^*,N}"'] & M \otimes N \otimes M^* \end{tikzcd}$$

By Theorem 3.12, $\mathrm{Im}(\mathbf{r}_{M^*,M})$ is the simple socle of $M \otimes M^*$, and hence $\mathbf{r}_{M^*,M}$ is equal to the composition

$$M^* \otimes M \xrightarrow{\varepsilon} \mathbf{1} \longrightarrow M \otimes M^*$$

up to a constant multiple. Here $\mathbf{1}$ denotes the trivial representation of $U_q'(\mathfrak{g})$. Hence we have a commutative diagram up to a constant multiple

$$\begin{tikzcd} M^* \otimes M \otimes N \arrow[r, two heads, "\varepsilon\otimes N"'] \arrow[rr, bend left, "\mathbf{r}_{M^*,M\otimes N}"] & N \arrow[r, tail] & M \otimes N \otimes M^* \end{tikzcd}$$

Let $f \in \mathrm{End}_{U_q'(\mathfrak{g})}(M \otimes N)$. Let us show that $f \in \mathbf{k}\,\mathrm{id}_{M\otimes N}$. Since $\mathbf{r}_{M^*,M\otimes N}$ commutes with f, the following diagram with the solid arrows is commutative:

$$\begin{tikzcd} M^* \otimes M \otimes N \arrow[r, two heads] \arrow[d, "M^*\otimes f"] & N \arrow[r, tail] \arrow[d, dotted, "f_N"] & M \otimes N \otimes M^* \arrow[d, "f\otimes M^*"] \\ M^* \otimes M \otimes N \arrow[r, two heads, "\varepsilon\otimes N"] & N \arrow[r, tail] & M \otimes N \otimes M^* \end{tikzcd} \tag{3.9}$$

Hence we can add the dotted arrow f_N so that the whole diagram (3.9) commutes. Since N is simple, we have $f_N = c\,\mathrm{id}_N$ for some $c \in \mathbf{k}$. Then, by replacing f with $f - c\,\mathrm{id}_{M\otimes N}$, we may assume from the beginning that $f_N = 0$. Hence the composition

$$M^* \otimes M \otimes N \xrightarrow{M^*\otimes f} M^* \otimes M \otimes N \xrightarrow{\varepsilon\otimes N} N$$

vanishes. Therefore (3.8) implies that $M \otimes N \xrightarrow{f} M \otimes N$ vanishes. □

Corollary 3.16. *Let M be a real simple module in $U_q'(\mathfrak{g})$-mod, and N a simple module in $U_q'(\mathfrak{g})$-mod.*

(i) *If the head of $M \otimes N$ and the socle of $M \otimes N$ are isomorphic, then $M \otimes N$ is simple and $M \otimes N \simeq N \otimes M$.*

(ii) *If $M \otimes N \simeq N \otimes M$, then $M \otimes N$ is simple.*

This corollary follows from the preceding lemma by an argument similar to that in the proof of Corollary 3.9.

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4. Proof of (1.11)

We shall show (1.11). We retain the notation in § 1. We set

$$\tilde{x}_{a,b} = \sum_{\substack{\nu \in I^{\beta+\gamma}, \\ \nu_a, \nu_b \in \mathrm{supp}(\beta)\cap\mathrm{supp}(\gamma)}} (x_a - x_b)e(\nu) \quad \text{and} \quad \widetilde{\tau}_c = \sum_{\substack{\nu \in I^{\beta+\gamma}, \\ \nu_c \in \mathrm{supp}(\gamma),\, \nu_{c+1}\in\mathrm{supp}(\beta)}} \tau_c e(\nu)$$

for $1 \leqslant a, b \leqslant m+n$ and $1 \leqslant c < m+n$. They are elements of $R(\beta+\gamma)$.

We denote by A the commutative subalgebra of $R(\beta+\gamma)$ generated by $\tilde{x}_{a,b}$ and $e(\nu)$ where $1 \leqslant a < b \leqslant m+n$ and $\nu \in I^{\beta+\gamma}$. Let us denote by $\widetilde{R}_{\gamma,\beta}$ the subalgebra of $R(\beta+\gamma)$ generated by A and $\widetilde{\tau}_c$ where $1 \leqslant c < m+n$.

Then $\varphi_{w[n,m]}e(\gamma,\beta)$ belongs to $\widetilde{R}_{\gamma,\beta}$.

These generators satisfy the following commutation relations:

$$\begin{cases} \tilde{x}_{a,b}\widetilde{\tau}_c - \widetilde{\tau}_c\tilde{x}_{s_c(a),s_c(b)} \\ \quad = \displaystyle\sum_{\nu_c=\nu_{c+1}\in\mathrm{supp}(\beta)\cap\mathrm{supp}(\gamma)} (\delta(a=c+1)-\delta(a=c)-\delta(b=c+1)+\delta(b=c))e(\nu), \\ \widetilde{\tau}_a^2 = \displaystyle\sum_{\nu_a,\nu_{a+1}\in\mathrm{supp}(\beta)\cap\mathrm{supp}(\gamma)} Q_{\nu_a,\nu_{a+1}}(x_a,x_{a+1})e(\nu), \\ \widetilde{\tau}_a\widetilde{\tau}_b - \widetilde{\tau}_b\widetilde{\tau}_a = 0 \quad \text{if } |a-b|>1, \\ \widetilde{\tau}_{a+1}\widetilde{\tau}_a\widetilde{\tau}_{a+1} - \widetilde{\tau}_a\widetilde{\tau}_{a+1}\widetilde{\tau}_a \\ \quad = \displaystyle\sum_{\nu_a,\nu_{a+1}\in\mathrm{supp}(\beta)\cap\mathrm{supp}(\gamma),\, \nu_a=\nu_{a+2}} \overline{Q}_{\nu_a,\nu_{a+1}}(x_a,x_{a+1},x_{a+2})e(\nu). \end{cases} \tag{4.1}$$

Indeed, the last equality follows from

$$\widetilde{\tau}_{a+1}\widetilde{\tau}_a\widetilde{\tau}_{a+1} = \sum_{\nu} \tau_{a+1}\tau_a\tau_{a+1}\, e(\nu),$$
$$\widetilde{\tau}_a\widetilde{\tau}_{a+1}\widetilde{\tau}_a = \sum_{\nu} \tau_a\tau_{a+1}\tau_a\, e(\nu).$$

Here the sums in both formulas range over $\nu \in I^{\beta+\gamma}$, satisfying the conditions $\nu_a \in \mathrm{supp}(\gamma)$, $\nu_{a+1} \in \mathrm{supp}(\beta)\cap\mathrm{supp}(\gamma)$, and $\nu_{a+2}\in\mathrm{supp}(\beta)$.

Note that the error terms (i.e. the right-hand sides of the equalities in (4.1)) belong to the algebra A because we assume that $R(\beta)$ and $R(\gamma)$ are symmetric. Hence we have

$$\begin{cases} \tilde{x}_{a,b}\widetilde{\tau}_c - \widetilde{\tau}_c\tilde{x}_{s_c(a),s_c(b)} \in A \\ \widetilde{\tau}_a^2 \in A, \\ \widetilde{\tau}_a\widetilde{\tau}_b = \widetilde{\tau}_b\widetilde{\tau}_a \quad \text{if } |a-b|>1, \\ \widetilde{\tau}_{a+1}\widetilde{\tau}_a\widetilde{\tau}_{a+1} - \widetilde{\tau}_a\widetilde{\tau}_{a+1}\widetilde{\tau}_a \in A. \end{cases} \tag{4.2}$$

Now for each element $w \in \mathfrak{S}_{m+n}$ let us choose a reduced expression $w = s_{a_1}\cdots s_{a_\ell}$. We then set

$$\widetilde{\tau}_w = \widetilde{\tau}_{a_1}\cdots\widetilde{\tau}_{a_\ell}.$$

Then, similarly to a proof of the PBW decomposition (1.1) (see, for example, [KL09, Rou08]), the commutation relations (4.2) imply

$$\widetilde{R}_{\gamma,\beta} = \sum_{w\in\mathfrak{S}_{m+n}} \widetilde{\tau}_w A.$$

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In particular, we obtain

$$\widetilde{R}_{\gamma,\beta} \subset \bigoplus_{\substack{w\in\mathfrak{S}_{n,m},\\ w_1\in\mathfrak{S}_n, w_2\in\mathfrak{S}_m}} \tau_w(\tau_{w_1}\otimes\tau_{w_2})A.$$

Thus immediately implies (1.11), because we have, for $1 \leqslant a < b \leqslant m+n$, $\nu \in I^\gamma$, $\mu \in I^\beta$, $v \in e(\nu)N$ and $u \in e(\mu)M$,

$$\tilde{x}_{a,b}((v)_{z_2}\otimes(u)_{z_1}) = \begin{cases} ((x_a-x_b)v)_{z_2}\otimes(u)_{z_1} & \text{if } 1\leqslant a<b\leqslant n \text{ and } \nu_a,\nu_b\in\operatorname{supp}(\beta),\\ (z_2-z_1)((v)_{z_2}\otimes(u)_{z_1}) & \\ \quad +(x_a v)_{z_2}\otimes(u)_{z_1}-(v)_{z_2}\otimes(x_{b-n}u)_{z_1} & \text{if } 1\leqslant a\leqslant n<b\leqslant m+n \text{ and}\\ & \quad \nu_a\in\operatorname{supp}(\beta),\mu_{b-n}\in\operatorname{supp}(\gamma),\\ (v)_{z_2}\otimes((x_{a-n}-x_{b-n})u)_{z_1} & \text{if } n<a<b\leqslant m+n \text{ and } \mu_{a-n},\mu_{b-n}\in\operatorname{supp}(\gamma),\\ 0 & \text{otherwise,}\end{cases}$$

and $(\tau_{w_1}\otimes\tau_{w_2})((v)_{z_2}\otimes(u)_{z_1}) = (\tau_{w_1}v)_{z_2}\otimes(\tau_{w_2}u)_{z_1}$.

Acknowledgement

We would like to thank Bernard Leclerc for many fruitful discussions and his kind explanations of his work.

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