Simplified equivalent multiple baseline solutions with elevation dependent weights

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Abstract. Since the assumption of all stations tracking the same satellites with identical weights was previously employed by Shen and Xu (2008) to derive the simplified GNSS single- and double-differenced equivalent equations, this supplementary paper expands these simplified equations in the case of each station tracking different satellites with elevation-dependent weights. In addition, the numerical experiments are performed to demonstrate the computational efficiency of simplified equivalent algorithm in the different scenarios of multi-baseline solutions with tracking different satellites. The results show that faster computational speed is always assigned to the simplified equivalent algorithm, comparing with the traditional method, which will potentially benefit the local, regional and even global GNSS multi-baseline solutions as well as the combined GNSS application.

Keywords. GNSS data processing, Multi-baseline solutions, Equivalent representation, Combined GNSS application

1. Introduction

The GNSS (Global Navigation Satellite Systems) single- and double-differenced simplified equivalent observation equations were previously derived by Shen and Xu (2008) through adding the pseudo-observations, and their corresponding unbiased variance estimators of unit weight were also derived according to the theorem proposed by Schaffrin and Grafarend (1986) and Xu (2002). Although the stochastic model for the GPS measurements could be more complicated to reflect the reality in the real application (Li et al., 2008; Wang et al., 1998, 2008), if the original observables are assumed to be independent, the covariance matrix is no longer necessary to be transformed in their simplified equivalent equations. However, all developed formulae of the simplified representations are subject to the assumption of all stations tracking the same satellites with identical weights. Nevertheless, it is not the case in real GNSS application. For example, each satellite can only cover the area less than hemisphere in global networks whereas different station can track the different part of the total satellites because of obstruction in regional station networks, as well as the varying variances should be set up for the observables of tracked satellites with different elevation angles. Therefore, these simplified observation equations must be expanded in the case of different station tracking different satellites with elevation-dependent weights to benefit the local, regional and even global GNSS multi-baseline solutions as well as combined GNSS application, which is the right motivation of this paper.

The next content of this paper is arranged as follows. Section 2 and section 3 will develop the single- and double-differenced simplified equivalent observation equations in the case of different station tracking the different satellites with elevation-dependent weights, respectively. In section 4, the numerical experiments are performed to evaluate the computational efficiency of the proposed simplified equivalent algorithm. In addition, its potential applications for future multiple satellite systems with multiple frequencies are also discussed. In the last section, the research findings are summarized to conclude the paper.

2. Single-differenced simplified equivalent observation equations

The GNSS observation equations for one epoch can be symbolically expressed as,

$$\boldsymbol{\varepsilon} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} + \boldsymbol{C}\boldsymbol{z} - \boldsymbol{l}\,, \qquad \boldsymbol{P} \tag{1}$$

where $\boldsymbol{y}$ and $\boldsymbol{z}$ are the vectors of station- and satellite-specific biases respectively, and $\boldsymbol{B}$ and $\boldsymbol{C}$ denote their coefficient matrices with full column rank; $\boldsymbol{x}$ is a column vector with t parameters, and $\boldsymbol{A}$ is its coefficient matrix also with full column rank; $\boldsymbol{l}$ and $\boldsymbol{\varepsilon}$ are the column vectors of observables and normally distributed observation errors; $\boldsymbol{P}$ is weight matrix of observations. In this paper, the elevation-dependent weights are involved in addition to the case of different station tracking the different satellites, but all correlations among the observables, such as temporal correlation, cross correlation and channel correlation, are still free of consideration. In other words, the weight matrix $\boldsymbol{P}$ is diagonal with unequal elements. One can be referred to Leick (2004) for the detailed interpretation of these parameters. If there are total of k stations and each station only tracks the part of total n satellites, then $\boldsymbol{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_k \end{bmatrix}^T$ and $\boldsymbol{z} = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^T$. The coefficient matrices, vector of observables and weight matrix are grouped with the following sub-blocks in the order of satellite as,

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}^1 \\ \boldsymbol{A}^2 \\ \vdots \\ \boldsymbol{A}^n \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}^1 \\ \boldsymbol{B}^2 \\ \vdots \\ \boldsymbol{B}^n \end{pmatrix}, \ \boldsymbol{C} = \begin{pmatrix} \boldsymbol{e}_{k_1} & & & \\ & \boldsymbol{e}_{k_2} & & \\ & \cdots & \vdots & \\ & & & \boldsymbol{e}_{k_n} \end{pmatrix}, \ \boldsymbol{l} = \begin{pmatrix} \boldsymbol{l}^1 \\ \boldsymbol{l}^2 \\ \vdots \\ \boldsymbol{l}^n \end{pmatrix}, \ \boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}^1 & & & \\ & \boldsymbol{P}^2 & & \\ & & \ddots & \\ & & & \boldsymbol{P}^n \end{pmatrix} \tag{2}$$

where $\boldsymbol{A}^j = \begin{pmatrix} \boldsymbol{a}^j_{S^j(1)} \\ \boldsymbol{a}^j_{S^j(2)} \\ \vdots \\ \boldsymbol{a}^j_{S^j(k_j)} \end{pmatrix}$, $\boldsymbol{l}^j = \begin{pmatrix} l^j_{S^j(1)} \\ l^j_{S^j(2)} \\ \vdots \\ l^j_{S^j(k_j)} \end{pmatrix}$, $\boldsymbol{P}^j = \begin{pmatrix} p^j_{S^j(1)} & & & \\ & p^j_{S^j(2)} & & \\ & & \ddots & \\ & & & p^j_{S^j(k_j)} \end{pmatrix}$. $\boldsymbol{a}^j_{S^j(i)}$ and $l^j_{S^j(i)}$ are respectively the coefficient row vector and observable of the satellite j tracked by the station $S^j(i)$, and $p^j_{S^j(i)}$ is its weight. Here, S^j is the set of all stations that instantaneously track the satellite j and $S^j(i)$ the order of its ith station in the total stations. k_j is the number of stations that track the satellite j, $\boldsymbol{e}_{k_j} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ is a k_j vector. The coefficient for the jth satellite $\boldsymbol{B}^j$ is a $k_j \times k$ matrix consisting of k_j canonical row vectors, and in each canonical row vector all elements are zeros expect only element with respect to its tracking receiver being equal to one. For example, if there are 5 stations and the 2nd station does not track the 3rd satellite, then the matrix $\boldsymbol{B}^3$ reads $\boldsymbol{B}^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

The satellite-specific parameter vector $\boldsymbol{z}$ can be eliminated by the operation of single difference in station domain or applying the transformation matrix $\boldsymbol{R}$ right to original observation equation immediately. The transformation matrix is symbolized (Teunissen 1997)

$$\boldsymbol{R} = \boldsymbol{I}_{\Sigma k} - \boldsymbol{C}\left(\boldsymbol{C}^T\boldsymbol{P}\boldsymbol{C}\right)^{-1}\boldsymbol{C}^T\boldsymbol{P} = \begin{pmatrix} \boldsymbol{R}_{k_1} & & & \\ & \boldsymbol{R}_{k_2} & & \\ & & \ddots & \\ & & & \boldsymbol{R}_{k_n} \end{pmatrix} \tag{3}$$

where the dimension of identity matrix $\boldsymbol{I}_{\Sigma k}$ is $\Sigma k = \sum_{j=1}^{n} k_j$, and

$$\boldsymbol{R}_{k_j} = \boldsymbol{I}_{k_j} - \frac{1}{\sum_{i \in S^j} p_i^j} \boldsymbol{e}_{k_j}\boldsymbol{e}_{k_j}^T \boldsymbol{P}^j = \boldsymbol{I}_{k_j} - \frac{1}{p^{\Sigma j}} \boldsymbol{P}^j \otimes \boldsymbol{e}_{k_j}^T \tag{4}$$

with $p^{\Sigma j} = \sum_{i \in S^j} p_i^j$ being the sum of weights of observables for all stations that tack the jth satellites.

Multiplying the transformation matrix $\boldsymbol{R}$ by (1), we obtain the equivalently transformed observation equations,

$$\tilde{\boldsymbol{\varepsilon}} = \tilde{\boldsymbol{A}}\boldsymbol{x} + \tilde{\boldsymbol{B}}\boldsymbol{y} - \tilde{\boldsymbol{l}}, \qquad \boldsymbol{P} \tag{5}$$

where $\tilde{\boldsymbol{A}} = \boldsymbol{R}\boldsymbol{A}$, $\tilde{\boldsymbol{B}} = \boldsymbol{R}\boldsymbol{B}$, $\tilde{\boldsymbol{l}} = \boldsymbol{R}\boldsymbol{l}$ and $\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{R}\boldsymbol{\varepsilon}$. As shown in (3), $\boldsymbol{R}$ is a block-wise matrix with the sub-matrix $\boldsymbol{R}_{k_j}$. Therefore, (5) can be further simplified to be,

$$\tilde{\boldsymbol{\varepsilon}}^j = \tilde{\boldsymbol{A}}^j\boldsymbol{x} + \tilde{\boldsymbol{B}}^j\boldsymbol{y} - \tilde{\boldsymbol{l}}^j, \quad \boldsymbol{P}^j, \quad j = 1, 2, \cdots, n \tag{6}$$

with

$$\tilde{\boldsymbol{A}}^j = \boldsymbol{R}_{k_j}\boldsymbol{A}^j, \ \tilde{\boldsymbol{B}}^j = \boldsymbol{R}_{k_j}\boldsymbol{B}^j, \ \tilde{\boldsymbol{l}}^j = \boldsymbol{R}_{k_j}\boldsymbol{l}^j \tag{7}$$

It is obvious that $\boldsymbol{R}_{k_j}$ is of rank defect with number of one. This means that one station-specific parameter can be linearly represented with the others. In other words, only k-1 station-specific parameters can be independently parameterized.

In the single-differenced equivalent observation equations, the independent parameterized station-specific parameters are generally merged into $\boldsymbol{x}$, and (7) becomes,

$$\tilde{\boldsymbol{\varepsilon}}^j = \tilde{\boldsymbol{A}}^j\boldsymbol{x} - \tilde{\boldsymbol{l}}^j, \quad \boldsymbol{P}^j, \quad j = 1,2,\cdots,n \tag{8}$$

where the transformed coefficient matrix and observation vector can also be further simplified as,

$$\tilde{\boldsymbol{A}}^j = \boldsymbol{R}_{k_j}\boldsymbol{A}^j = \boldsymbol{A}^j - \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\boldsymbol{e}_{k_j}^T\boldsymbol{P}^j\boldsymbol{A}^j = \boldsymbol{A}^j - \delta\boldsymbol{A}^j \tag{9a}$$

$$\tilde{\boldsymbol{l}}^j = \boldsymbol{R}_{k_j}\boldsymbol{l}^j = \boldsymbol{l}^j - \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\boldsymbol{e}_{k_j}^T\boldsymbol{P}^j\boldsymbol{l}^j = \boldsymbol{l}^j - \delta\boldsymbol{l}^j \tag{9b}$$

with

$$\delta\boldsymbol{A}^j = \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\boldsymbol{e}_{k_j}^T\boldsymbol{P}^j\boldsymbol{A}^j = \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\sum_{i\in S^j} p_i^j\boldsymbol{a}_i^j \;= \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\left[\boldsymbol{a}^j\right] \tag{10a}$$

$$\delta\boldsymbol{l}^j = \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\boldsymbol{e}_{k_j}^T\boldsymbol{P}^j\boldsymbol{l}^j = \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\sum_{i\in S^j} p_i^j l_i^j \;= \frac{1}{p^{\Sigma j}}\boldsymbol{e}_{k_j}\left[l^j\right] \tag{10b}$$

where, $\left[\boldsymbol{a}^j\right] = \sum_{i\in S^j} p_i^j\boldsymbol{a}_i^j$ and $\left[l^j\right] = \sum_{i\in S^j} p_i^j l_i^j$, each element of the column vector $\delta\boldsymbol{l}^j$ and each column vector of $\delta\boldsymbol{A}^j$ are the weighted mean of their corresponding column vectors. Therefore, the transformed vector $\tilde{\boldsymbol{l}}^j$ is just the centrobaric vector of $\boldsymbol{l}^j$, and the transformed matrix $\tilde{\boldsymbol{A}}^j$ just the column centrobaric matrix of $\boldsymbol{A}^j$. In other words, the equivalent observation equations (8) can also be simply obtained through the centrobaric operation to the column vector of $\boldsymbol{A}^j$ and $\boldsymbol{l}^j$.

In addition, the expression (8) can be alternatively expanded with the same way as described by Shen and Xu (2008) in the form of pseudo-observations,

$$\tilde{\boldsymbol{\varepsilon}}^j = \boldsymbol{A}^j\boldsymbol{x} - \boldsymbol{l}^j, \quad \boldsymbol{P}^j, \quad j = 1,2,\cdots,n \tag{11a}$$

$$\left[\varepsilon^j\right] = \left[\boldsymbol{a}^j\right]\boldsymbol{x} - \left[l^j\right], \quad -1\big/p^{\Sigma j}, \quad j = 1,2,\cdots,n \tag{11b}$$

where $\left[\tilde{\varepsilon}^j\right]$ denotes the residual of the jth sum pseudo-observation. The same normal equations can be obtained by the equivalent observation equations (8) and (11), and the proof is released in Appendix A. Once the unknown parameter vector $\hat{\boldsymbol{x}}$ is solved, the residual vector is computed by

$$\boldsymbol{v}^j = \tilde{\boldsymbol{A}}^j\hat{\boldsymbol{x}} - \tilde{\boldsymbol{l}}^j, \qquad j = 1,2,\cdots,n \tag{12}$$

3. Double-differenced simplified equivalent observation equations

If there are more than two stations and each station may track the part of total n satellites, the double-differenced equivalent observation equations for multi-baseline solutions will be much more complicated than single-differenced ones. In order to derive the simplified double-differenced equivalent observation equations, we rearrange (5) with the sub-blocks in the order of receivers and use the same symbols as (5) to represent the rearranged single-differenced observation equations as,

$$\tilde{\boldsymbol{\varepsilon}} = \tilde{\boldsymbol{A}}\boldsymbol{x} + \tilde{\boldsymbol{B}}\boldsymbol{y} - \tilde{\boldsymbol{l}}, \qquad \boldsymbol{P} \tag{13}$$

where $\tilde{\boldsymbol{A}} = \begin{pmatrix} \tilde{\boldsymbol{A}}_1 \\ \tilde{\boldsymbol{A}}_2 \\ \vdots \\ \tilde{\boldsymbol{A}}_k \end{pmatrix}$ with $\tilde{\boldsymbol{A}}_i = \begin{pmatrix} \tilde{\boldsymbol{a}}_i^{S_i\ 1} \\ \tilde{\boldsymbol{a}}_i^{S_i\ 2} \\ \vdots \\ \tilde{\boldsymbol{a}}_i^{S_i\ n_i} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_i^{S_i\ 1} - \left[\boldsymbol{a}^{S_i\ 1}\right]\big/p^{\Sigma S_i\ 1} \\ \boldsymbol{a}_i^{S_i\ 2} - \left[\boldsymbol{a}^{S_i\ 2}\right]\big/p^{\Sigma S_i\ 2} \\ \vdots \\ \boldsymbol{a}_i^{S_i\ n_i} - \left[\boldsymbol{a}^{S_i\ n_i}\right]\big/p^{\Sigma S_i\ n_i} \end{pmatrix}$, n_i is the number of satellites tracked by the station i and S_i denotes a set comprised of these n_i satellites. $S_i\ l$ is the order of the lth satellite in the total satellites; $\tilde{\boldsymbol{l}} = \begin{pmatrix} \tilde{\boldsymbol{l}}_1 \\ \tilde{\boldsymbol{l}}_2 \\ \vdots \\ \tilde{\boldsymbol{l}}_k \end{pmatrix}$ with $\tilde{\boldsymbol{l}}_i = \begin{pmatrix} \tilde{l}_i^{S_i 1} \\ \tilde{l}_i^{S_i 2} \\ \vdots \\ \tilde{l}_i^{S_i n_i} \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} \boldsymbol{P}_1 & & & \\ & \boldsymbol{P}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{P}_k \end{pmatrix}$ with $\boldsymbol{P}_i = \begin{pmatrix} p_i^{S_i 1} & & & \\ & p_i^{S_i 2} & & \\ & & \ddots & \\ & & & p_i^{S_i n_i} \end{pmatrix}$ and $\tilde{\boldsymbol{B}} = \begin{bmatrix} \tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_3 & \cdots & \tilde{\boldsymbol{b}}_k \end{bmatrix}$. The first element in $\boldsymbol{y}$ is fixed to zero for independent parameterization. According to (6) and (7), we can determine the rearranged column vector $\tilde{\boldsymbol{b}}_i$ as follows

$$\tilde{\boldsymbol{b}}_i = \begin{bmatrix} -(\boldsymbol{Q}_1\boldsymbol{G}_i\boldsymbol{\alpha}_i)^T & \cdots & -(\boldsymbol{Q}_{i-1}\boldsymbol{G}_i\boldsymbol{\alpha}_i)^T & (\boldsymbol{Q}_i(\boldsymbol{e}_n-\boldsymbol{\alpha}_i))^T & -(\boldsymbol{Q}_{i+1}\boldsymbol{G}_i\boldsymbol{\alpha}_i)^T & \cdots & -(\boldsymbol{Q}_k\boldsymbol{G}_i\boldsymbol{\alpha}_i)^T \end{bmatrix}^T \tag{14}$$

where $\boldsymbol{\alpha}_i = \begin{pmatrix} \frac{p_i^1}{p^{\Sigma 1}} & \frac{p_i^2}{p^{\Sigma 2}} & \cdots & \frac{p_i^n}{p^{\Sigma n}} \end{pmatrix}^T$, $\boldsymbol{G}_i$ is a $n \times n$ diagonal matrix and its diagonal element is equal to either one (corresponding to tracked satellite) or zero (corresponding to non-tracked satellite). The n_i non-zero row vectors of $\boldsymbol{G}_i$ construct the $n_i \times n$ matrix $\boldsymbol{Q}_i$. If there are 6 satellites and the 3rd station does not track the 2nd and 5th satellites, the matrices $\boldsymbol{G}_3$ and $\boldsymbol{Q}_3$ are expressed as

$$\boldsymbol{G}_3 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 1 \end{pmatrix}, \quad \boldsymbol{Q}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{15}$$

It is obvious that the matrices $\boldsymbol{G}_i$ and $\boldsymbol{Q}_i$ hold true for the following properties,

$$\boldsymbol{G}_i = \boldsymbol{Q}_i^T\boldsymbol{Q}_i, \quad \boldsymbol{Q}_i\boldsymbol{G}_i = \boldsymbol{Q}_i, \quad \boldsymbol{G}_i = \boldsymbol{G}_i^T, \quad \boldsymbol{G}_i\boldsymbol{G}_i = \boldsymbol{G}_i \tag{16}$$

In order to determine the transformation matrix $\tilde{\boldsymbol{R}}$ for eliminating station-specific parameters, the following matrix should be primarily computed,

$$\tilde{\boldsymbol{B}}^T\boldsymbol{P}\tilde{\boldsymbol{B}} = \begin{pmatrix} \tilde{\boldsymbol{b}}_2^T\boldsymbol{P}\tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_2^T\boldsymbol{P}\tilde{\boldsymbol{b}}_3 & & \tilde{\boldsymbol{b}}_2^T\boldsymbol{P}\tilde{\boldsymbol{b}}_k \\ \tilde{\boldsymbol{b}}_3^T\boldsymbol{P}\tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_3^T\boldsymbol{P}\tilde{\boldsymbol{b}}_3 & & \tilde{\boldsymbol{b}}_3^T\boldsymbol{P}\tilde{\boldsymbol{b}}_k \\ & \cdots & \vdots & \\ \tilde{\boldsymbol{b}}_k^T\boldsymbol{P}\tilde{\boldsymbol{b}}_2 & \tilde{\boldsymbol{b}}_k^T\boldsymbol{P}\tilde{\boldsymbol{b}}_3 & & \tilde{\boldsymbol{b}}_k^T\boldsymbol{P}\tilde{\boldsymbol{b}}_k \end{pmatrix} \tag{17}$$

According to (14) and (16), the expression of each component of matrix $\tilde{\boldsymbol{B}}^T\boldsymbol{P}\tilde{\boldsymbol{B}}$ derived in Appendix B can be finalized as follows,

$$\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_i = p_{\Sigma i} - \sum_{l \in S_i} \frac{p_i^l p_i^l}{p^{\Sigma l}} \tag{18a}$$

$$\tilde{\boldsymbol{b}}_i^T\boldsymbol{P}\tilde{\boldsymbol{b}}_j = -\sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}} \tag{18b}$$

where $p_{\Sigma i} = \sum_{j \in S_i} p_i^j$ is the sum of weights of observables for all satellites tracked by the ith station. S_{ij} is a intersection set of S_i and S_j, namely $S_{ij} = S_i \cap S_j$ with symbol '$\cap$' denoting the operator of intersection of two sets, which refers to the set of satellites instantaneously tracked by both station *i* and station *j*. The matrix $\tilde{\boldsymbol{B}}^T\boldsymbol{P}\tilde{\boldsymbol{B}}$

can be efficiently computed by (18), but its inverse is rather complicated and no longer symbolically expressed. Therefore, the transformation matrix is numerically computed by

$$\tilde{\boldsymbol{R}} = \boldsymbol{I}_{\Sigma k} - \tilde{\boldsymbol{B}}\left(\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}\right)^{-1} \tilde{\boldsymbol{B}}^T \boldsymbol{P} = \boldsymbol{I}_{\Sigma k} - \tilde{\boldsymbol{J}} \tag{19}$$

Analogously, multiplying the transformation matrix $\tilde{\boldsymbol{R}}$ by (13), the double-differenced equivalent equations are obtained as,

$$\bar{\bar{\boldsymbol{\varepsilon}}} = \bar{\bar{\boldsymbol{A}}}\boldsymbol{x} - \bar{\bar{\boldsymbol{l}}}, \quad \boldsymbol{P} \tag{20}$$

with

$$\bar{\bar{\boldsymbol{A}}} = \tilde{\boldsymbol{R}}\tilde{\boldsymbol{A}} = \tilde{\boldsymbol{A}} - \tilde{\boldsymbol{J}}\tilde{\boldsymbol{A}} \tag{21a}$$

$$\bar{\bar{\boldsymbol{l}}} = \tilde{\boldsymbol{R}}\tilde{\boldsymbol{l}} = \tilde{\boldsymbol{l}} - \tilde{\boldsymbol{J}}\tilde{\boldsymbol{l}} \tag{21b}$$

As mentioned in section 2, $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{l}}$ are comprised of all sub-matrices $\tilde{\boldsymbol{A}}^j$ and sub-vectors $\tilde{\boldsymbol{l}}^j$ respectively, and can be very efficiently computed by centrobaric operation to their column vectors. The $(k-1) \times (k-1)$ square matrix $\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}$ and its inverse matrix must be primarily compute to determine the transformation matrix $\tilde{\boldsymbol{R}}$. The matrix $\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}$ can be efficiently implemented by (18), although its inverse matrix can be trivially computed, which is certainly more efficient than that to directly compute the weight matrix of double-differenced observables for multi-baseline solutions. Once the least squares solution to parameter vector $\hat{\boldsymbol{x}}$ is obtained, the residuals can be exactly computed by,

$$\boldsymbol{v} = \bar{\bar{\boldsymbol{A}}}\hat{\boldsymbol{x}} - \bar{\bar{\boldsymbol{l}}} \tag{22}$$

4. Numerical experiments

The numerical experiments are performed to compare the computational efficiency in forming the double-differenced normal equations respectively using the proposed simplified equivalent algorithm and traditional method. Here, the traditional method is to form the weight matrix for double differenced observables and thus to form the normal equations, as described above. All computations are performed with Matlab7.3.0 programs on a Pentium D, 3.2GHz PC with 1GB memory running Windows XP professional. As we know, most of the commercial GNSS softwares currently employ the simple model of single baseline solution due to the complicated transformation of weight matrix and inefficient computation in the model of multiple baseline solution. In addition, we are about to enter the multi-GNSS era and soon there might be 20 satellites or more in common view at all times and, consequently those softwares with traditional method would be right necessary to be updated.

Considering the situation of more satellites tracked by multiple stations, we set up the experiments with four scenarios, elaborating the efficiency of the simplified equivalent algorithm mainly under the different satellites and stations, respectively. All experiments are carried out over 1000 epochs to distinctly illustrate the time difference between two methods. In Figures 1 and 2, the efficiency in satellite domain is evaluated. Figure 1 illustrates the computational time by the simplified equivalent algorithm and traditional method under the different satellites tracked by 10 stations. It will consume more and more computational time with the increase of tracked satellites, which is more significant for traditional method than for simplified equivalent algorithm. Figure 2 presents a similar comparison expect 20 tracking stations. Obviously, when 23 satellites are tracked by total 20 stations, the traditional method needs more than 30 minutes but just about 1 minute for the simplified equivalent algorithm. The Figures 3 and 4 demonstrate the efficiency in station domain with 11 and 19 tracked satellites, respectively. Similarly, the computational time becomes more and more with the increase of the tracking stations. It needs about 16 minutes to form the normal equations by traditional method and only about 37 seconds for simplified equivalent algorithm to obtain the identical results in the case of 11 satellites tracked by 25 stations, but about 43 minutes and 97 seconds respectively for the case of 19 satellites tracked by 25 stations. We have noticed that the influence of number of stations on the consumed time is more serious than that of number of satellites. It is because the elimination of satellite-specific biased by pseudo-observation equation (11) is so efficient that the consumed time is much less sensitive to the number of satellites, compared with the elimination of the station-specific biases by the equations (17-21). Therefore, if there are more stations than satellites, we had better first eliminate station-specific biases and then satellite-specific biases for the efficient implementation. On all accounts, the results are very promising, especially for the situation of more satellites tracked by more stations.

We would also like to highlight the benefits of the simplified equivalent algorithm and its potential applications. First of all, the final formulae for forming the equivalent normal equations are very simple and thus ease the realization of computer program in real applications, although the derivation process is so complicated. Another crucial and promising benefit is its highly efficient performance, especially for the multiple baseline solutions with more satellites and more stations. Moreover, the multiple GNSS systems would become available in the recent future and we would face to their integrated applications. In this situation, a scale factor can be introduced to balance the observables from the different systems and the derived formulae can still work well. Therefore, it is expected that the proposed algorithm can provide a theoretical and technical cornerstone for developing the future efficient GNSS softwares, serving the real-time large scale regional and global multiple baseline solutions. To say the least, it can be applied to improve the performance efficiency of current commercial softwares.

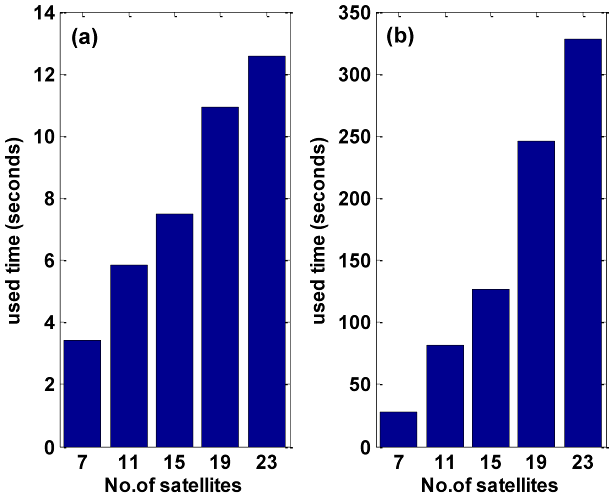

Figure 1: computational time by simplified equivalent algorithm and traditional method under the different satellites tracked by 10 stations (a: simplified equivalent algorithm; b: traditional method)

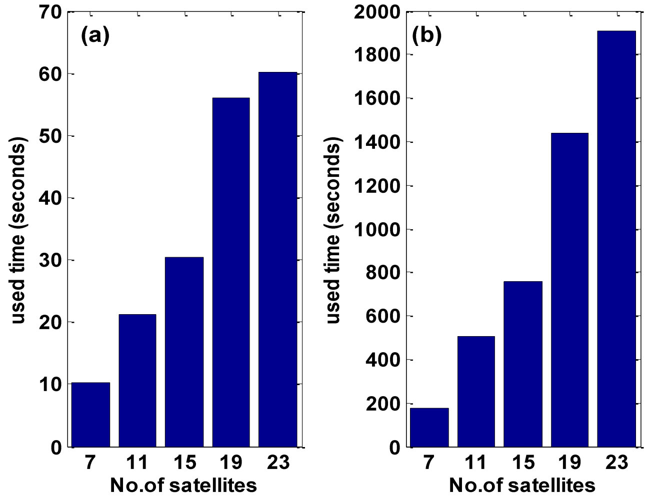

Figure 2: computational time by simplified equivalent algorithm and traditional method under the different satellites tracked by 20 stations (a: simplified equivalent algorithm; b: traditional method)

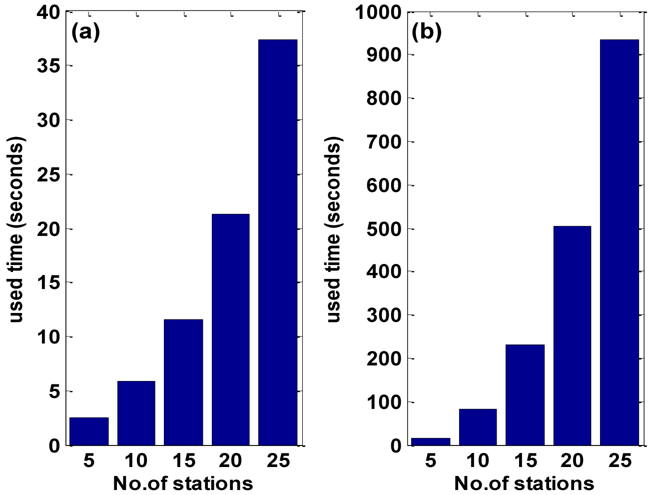

Figure 3: computational time by simplified equivalent algorithm and traditional method under the different stations that track total 11 satellites (a: simplified equivalent algorithm; b: traditional method)

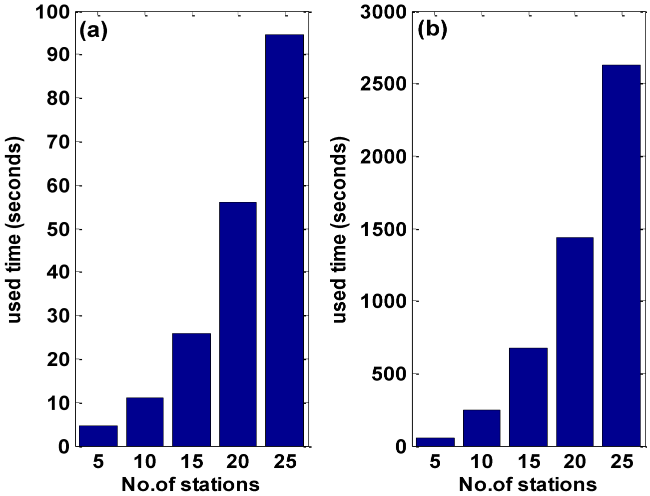

Figure 4: computational time by simplified equivalent algorithm and traditional method under the different stations that track total 19 satellites (a: simplified equivalent algorithm; b: traditional method)

5 Concluding Remarks

In this paper, all formulae of simplified equivalent algorithm previously derived by Shen and Xu (2008) have been expanded in the case of all stations tracking the different satellites with elevation-dependent weights in order to cater for the real application. The whole derivation procedure is rather complicated, but the final formulae remain its simpleness. In addition, all experiment results have shown that the faster computational speed is always assigned to the simplified equivalent algorithm, comparing with the traditional method, especially in the scenario of more satellites tracked by multiple stations. This promising finding will promote the development of efficient GNSS softwares, potentially benefiting the local, regional and even global GNSS multi-baseline solutions as well as the combined GNSS applications.

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References

Leick A. (2004) GPS satellite surveying, 3rd edition, *New York, John Wiley & Sons*.

Li B., Shen Y., Xu P. (2008) Assessment of stochastic models for GPS measurements with different types of receivers, *Chinese Science Bulletin*, doi:10.1007/s11434- 008-0293-6

Schaffrin B., Grafarend E. (1986) Generating classes of equivalent linear models by nuisance parameter elimination, applications to GPS observations, *Manuscript Geodesy*, 11:262–271.

Shen Y., Xu G. (2008) Simplified equivalent representation of GPS observation equations, *GPS Solutions*, 12:99-108

Teunissen P. (1997) GPS double difference statistics: with and without using satellite geometry, *Journal of Geodesy*, 71: 137-148.

Wang J., Stewart M., Tsakiri M. (1998) Stochastic modelling for GPS static baseline data processing, *Journal Journal of Surveying Engineering*, USA, 124:171-181

Wang J., Satirapod C., Rizos C. (2002) Stochastic assessment of GPS carrier phase measurements for precise static relative positioning, *Journal of Geodesy*, 76(2): 95-104

Xu G. (2002) GPS data processing with equivalent observation equations, *GPS Solutions*, 6 (1-2): 28-33.

Appendix A:

The proof that the single-differenced observation equations can be equivalently achieved by (8) and (11) is released as follows. The single-differenced normal equations from (8) and (9) are obtained,

$$\tilde{\boldsymbol{A}}^{j\ T}\boldsymbol{P}^j\tilde{\boldsymbol{A}}^j = \boldsymbol{A}^{j\ T}\boldsymbol{P}^j\boldsymbol{A}^j - \boldsymbol{A}^{j\ T}\boldsymbol{P}^j\delta\boldsymbol{A}^j - \delta\boldsymbol{A}^{j\ T}\boldsymbol{P}^j\boldsymbol{A}^j + \delta\boldsymbol{A}^{j\ T}\boldsymbol{P}^j\delta\boldsymbol{A}^j \tag{A1}$$

$$\tilde{\boldsymbol{A}}^{j\ T}\boldsymbol{P}^j\tilde{\boldsymbol{l}}^j = \boldsymbol{A}^{j\ T}\boldsymbol{P}^j\boldsymbol{l}^j - \boldsymbol{A}^{j\ T}\boldsymbol{P}^j\delta\boldsymbol{l}^j - \delta\boldsymbol{A}^{j\ T}\boldsymbol{P}^j\boldsymbol{l}^j + \delta\boldsymbol{A}^{j\ T}\boldsymbol{P}^j\delta\boldsymbol{l}^j \tag{A2}$$

Inserting (10a) into (A1),

$$\left(\tilde{\boldsymbol{A}}^j\right)^T \boldsymbol{P}^j \tilde{\boldsymbol{A}}^j = \left(\boldsymbol{A}^j\right)^T \boldsymbol{P}^j \boldsymbol{A}^j - \frac{2}{p^{\Sigma j}} \left[\boldsymbol{a}^j\right]^T \left[\boldsymbol{a}^j\right] + \frac{1}{\left(p^{\Sigma j}\right)^2} \left[\boldsymbol{a}^j\right]^T \boldsymbol{e}_{k_j}^T \boldsymbol{P}^j \boldsymbol{e}_{k_j} \left[\boldsymbol{a}^j\right] \tag{A3}$$

Inserting (10a) and (10b) into (A2),

$$\left(\tilde{\boldsymbol{A}}^j\right)^T \boldsymbol{P}^j \tilde{\boldsymbol{l}}^j = \left(\boldsymbol{A}^j\right)^T \boldsymbol{P}^j \boldsymbol{l}^j - \frac{2}{p^{\Sigma j}} \left[\boldsymbol{a}^j\right]^T \left[l^j\right] + \frac{1}{\left(p^{\Sigma j}\right)^2} \left[\boldsymbol{a}^j\right]^T \boldsymbol{e}_{k_j}^T \boldsymbol{P}^j \boldsymbol{e}_{k_j} \left[l^j\right] \tag{A4}$$

Substituting $\boldsymbol{e}_{k_j}^T \boldsymbol{P}^j \boldsymbol{e}_{k_j} = p^{\Sigma j}$ individually into (A3) and (A4), we have

$$\left(\tilde{\boldsymbol{A}}^j\right)^T \boldsymbol{P}^j \tilde{\boldsymbol{A}}^j = \left(\boldsymbol{A}^j\right)^T \boldsymbol{P}^j \boldsymbol{A}^j - \frac{1}{p^{\Sigma j}} \left[\boldsymbol{a}^j\right]^T \left[\boldsymbol{a}^j\right] \tag{A5}$$

$$\left(\tilde{\boldsymbol{A}}^j\right)^T \boldsymbol{P}^j \tilde{\boldsymbol{l}}^j = \left(\boldsymbol{A}^j\right)^T \boldsymbol{P}^j \boldsymbol{l}^j - \frac{1}{p^{\Sigma j}} \left[\boldsymbol{a}^j\right]^T \left[l^j\right] \tag{A6}$$

It is obvious that the single-differenced normal equations (A5) and (A6) are exactly equivalent to those from (11a) and (11b).

Appendix B:

According to the definition of $\tilde{\boldsymbol{b}}_i$ in (14), the entries $\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_i$ and $\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_j$ of matrix $\tilde{\boldsymbol{B}}^T \boldsymbol{P} \tilde{\boldsymbol{B}}$ can be expanded as follows,

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_i = \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l\right) \boldsymbol{G}_i \boldsymbol{\alpha}_i - \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{G}_i \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_i^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{\alpha}_i - 2\boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{\alpha}_i + \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{e}_n \tag{B1}$$

$$\begin{aligned} \tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_j = {} & \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l\right) \boldsymbol{G}_j \boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_j^T \boldsymbol{P}_j \boldsymbol{Q}_j \boldsymbol{G}_j \boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{G}_j \boldsymbol{\alpha}_j - \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{G}_j \boldsymbol{\alpha}_j \\ & + \boldsymbol{\alpha}_i^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{G}_j \boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_j^T \boldsymbol{P}_j \boldsymbol{Q}_j \boldsymbol{e}_n + \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_j^T \boldsymbol{P}_j \boldsymbol{Q}_j \boldsymbol{\alpha}_j \end{aligned} \tag{B2}$$

Considering the properties (16) of matrices $\boldsymbol{G}_i$ and $\boldsymbol{Q}_i$, (B1) and (B2) can be further simplified as,

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_i = \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l\right) \boldsymbol{G}_i \boldsymbol{\alpha}_i - 2\boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{\alpha}_i + \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{e}_n \tag{B3}$$

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_j = \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l\right) \boldsymbol{G}_j \boldsymbol{\alpha}_j - \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{G}_j \boldsymbol{\alpha}_j - \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_j^T \boldsymbol{P}_j \boldsymbol{Q}_j \boldsymbol{e}_n \tag{B4}$$

Apparently,

$$\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l = \begin{pmatrix} p^{\Sigma 1} & & & \\ & p^{\Sigma 2} & & \\ & & \ddots & \\ & & & p^{\Sigma n} \end{pmatrix} \tag{B5}$$

Therefore,

$$\boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l \right) \boldsymbol{G}_i \boldsymbol{\alpha}_i = \sum_{j \in S_i} \frac{\left(p_i^j\right)^2}{p^{\Sigma j}}, \quad \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{\alpha}_i = \sum_{l \in S_i} \frac{\left(p_i^l\right)^2}{p^{\Sigma l}}, \quad \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{e}_n = p_{\Sigma i} \tag{B6}$$

$$\boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \left(\sum_{l=1}^{k} \boldsymbol{Q}_l^T \boldsymbol{P}_l \boldsymbol{Q}_l \right) \boldsymbol{G}_j \boldsymbol{\alpha}_j = \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}}, \quad \boldsymbol{e}_n^T \boldsymbol{Q}_i^T \boldsymbol{P}_i \boldsymbol{Q}_i \boldsymbol{G}_j \boldsymbol{\alpha}_j = \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}}, \quad \boldsymbol{\alpha}_i^T \boldsymbol{G}_i^T \boldsymbol{Q}_j^T \boldsymbol{P}_j \boldsymbol{Q}_j \boldsymbol{e}_n = \sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}} \tag{B7}$$

Inserting (B6) and (B7) into (B3) and (B4), respectively, the final expressions are symbolized as,

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_i = p_{\Sigma i} - \sum_{l \in S_i} \frac{\left(p_i^l\right)^2}{p^{\Sigma l}} \tag{B8}$$

$$\tilde{\boldsymbol{b}}_i^T \boldsymbol{P} \tilde{\boldsymbol{b}}_j = -\sum_{l \in S_{ij}} \frac{p_i^l p_j^l}{p^{\Sigma l}} \tag{B9}$$

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