Bol. Soc. Paran. Mat. (3s.) **v. 39** 4 (2021): 73–82.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.41758

Simplifying Coefficients in Differential Equations Related to Generating Functions of Reverse Bessel and Partially Degenerate Bell Polynomials

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ABSTRACT: In the paper, by virtue of the Faá di Bruno formula and identities for the Bell polynomials of the second kind, the author simplifies coefficients in a family of ordinary differential equations related to generating functions of reverse Bessel and partially degenerate Bell polynomials.

Key Words: Ordinary differential equation, Coefficient, Generating function, Simplification, Partially degenerate Bell polynomial, Reverse Bessel polynomial, Bell polynomials of the second kind, Faá di Bruno formula.

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1. Motivation and main results

In [3, Theorem 1], it was established inductively and recursively that the family of differential equations

$$G^{(n)}(t) = G(t) \sum_{i=n}^{2n-1} a_{i-n}(n,x)(1-2t)^{-i/2}, \quad n \in \mathbb{N} \tag{1.1}$$

has the same solution

$$G(t) = e^{x(1-\sqrt{1-2t}\,)}, \tag{1.2}$$

where $a_0(n,x) = x^n$, $a_{n-1}(n,x) = (2n-3)!!x$, and

$$a_j(n,x) = x^{n-j} \sum_{i_j=0}^{n-j-1} \sum_{i_{j-1}=0}^{n-j-1-i_j} \cdots \sum_{i_1=0}^{n-j-1-i_j-\cdots-i_2} \prod_{k=1}^{j} (n - i_j - i_{j-1} - \cdots - i_k - (j-(2k-2))) \tag{1.3}$$

2010 *Mathematics Subject Classification:* Primary 34A05; Secondary 05A15, 11A25, 11B73, 11B83, 11C08, 16A24, 42C10.

Submitted February 19, 2018. Published June 03, 2018

for $1 \le i \le n-2$. The function $G(t)$ in (1.2) can be used to generate the reverse Bessel polynomials $p_k(x)$ by

$$G(t) = e^{x(1-\sqrt{1-2t}\,)} = \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!}.$$

The expression (1.3) was used in [3, Theorem 2].

In [4, Theorem 2.2], it was also established inductively and recursively that the family of differential equations

$$F_\lambda^{(n)}(t) = F_\lambda(t) \sum_{i=1}^{n} b_i(n,\lambda) x^i (1+\lambda t)^{i/\lambda - n}, \quad n \in \mathbb{N} \tag{1.4}$$

has a solution

$$F_\lambda(t) = e^{x[(1+\lambda t)^{1/\lambda} - 1]}, \tag{1.5}$$

where $b_1(n,\lambda) = (1-(n-1)\lambda|\lambda)_{n-1}$,

$$\begin{aligned} b_i(n,\lambda) = \sum_{k_{i-1}=0}^{n-i} \sum_{k_{i-2}=0}^{n-i-k_{i-1}} \cdots \sum_{k_1=0}^{n-i-k_{i-1}-\cdots-k_2} & \\ \prod_{\ell=2}^{i} \left(\ell - \left(n - \sum_{j=\ell}^{i-1} k_j - i - 1 + \ell \right) \lambda \middle| \lambda \right)_{k_{\ell-1}} & \\ \times \left(1 - \left(n - \sum_{j=1}^{i-1} k_j - i \right) \lambda \middle| \lambda \right)_{n - \sum_{j=1}^{i-1} k_j - i} & \end{aligned} \tag{1.6}$$

for $2 \le i \le n$,

$$(x|\alpha)_n = \prod_{k=0}^{n-1} (x + k\alpha) = \begin{cases} x(x+\alpha)\cdots[x+(n-1)\alpha], & n \ge 1, \\ 1, & n = 0, \end{cases}$$

and the function $F_\lambda(t)$ in (1.5) can be used [42] to generate partially degenerate Bell polynomials $B_{n,\lambda}(x)$ by

$$F_\lambda(t) = e^{x[(1+\lambda t)^{1/\lambda} - 1]} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}.$$

For more information about the Bell numbers and polynomials, please refer to [2, 7,9,12,19,20,31,37,39,41] and closely related references.

It is obvious to see that

1. the expression (1.6) is too complicated to be remembered, understood, and computed easily;
2. the original proof of [4, Theorem 2.2] is long and tedious,

3. the generating functions $G(t)$ and $F_\lambda(t)$ are connected by

$$F_2(-t)=\frac{1}{G(t)}.$$

In this paper, we will provide nice and standard proofs for [3, Theorem 1] and [4, Theorem 2.2] and, more importantly, discover simple, meaningful, and significant form for $a_j(n,x)$ and $b_i(n,\lambda)$.

Our main results can be stated as the following theorem.

Theorem 1.1. *For $n\ge 0$, the function $F_\lambda(t)$ defined by (1.5) satisfies*

$$F_\lambda^{(n)}(t)=\frac{F_\lambda(t)}{(1+\lambda t)^n}\sum_{k=0}^{n}\frac{(-1)^k}{k!}\Biggl[\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-q\lambda)\Biggr]\bigl[x(1+\lambda t)^{1/\lambda}\bigr]^k \tag{1.7}$$

and the function $G(t)$ defined by (1.2) satisfies

$$G^{(n)}(t)=\frac{(-1)^nG(t)}{(1-2t)^n}\sum_{k=0}^{n}\frac{1}{k!}\Biggl[\sum_{\ell=0}^{k}(-1)^\ell\binom{k}{\ell}\prod_{m=0}^{n-1}(\ell-2m)\Biggr]\bigl(x\sqrt{1-2t}\,\bigr)^k \tag{1.8}$$

and

$$G^{(n)}(t)=\frac{G(t)}{(1-2t)^n}\sum_{k=0}^{n}\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!\bigl(x\sqrt{1-2t}\,\bigr)^k, \tag{1.9}$$

where the empty product means 1 as usual.

2. Proof of Theorem 1.1

The famous Faà di Bruno formula reads that

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\,\mathrm{B}_{n,k}\bigl(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\bigr) \tag{2.1}$$

for $n\ge 0$, where the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ for $n\ge k\ge 0$ are defined [1, p. 134, Theorem A] and [1, p. 139, Theorem C] by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

Applying $u = h(t) = (1 + \lambda t)^{1/\lambda}$ and $f(u) = e^{x(u-1)}$ to (2.1) gives

$$\begin{aligned}
F_\lambda^{(n)}(t) &= \sum_{k=0}^{n} \frac{\mathrm{d}^k\, e^{x(u-1)}}{\mathrm{d}\, u^k} \mathrm{B}_{n,k}\Biggl((1+\lambda t)^{1/\lambda-1}, (1+\lambda t)^{1/\lambda-2}(1-\lambda), \\
&\quad (\lambda t+1)^{1/\lambda-3}(1-\lambda)(1-2\lambda), \dots, (1+\lambda t)^{1/\lambda-(n-k+1)} \prod_{\ell=1}^{n-k}(1-\ell\lambda)\Biggr) \\
&= \sum_{k=0}^{n} \frac{x^k e^{x(u-1)}}{(1+\lambda t)^{n-k/\lambda}} \mathrm{B}_{n,k}\Biggl(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\Biggr) \\
&= F_\lambda(t) \sum_{k=0}^{n} \frac{x^k}{(1+\lambda t)^{n-k/\lambda}} \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell - q\lambda),
\end{aligned}$$

where we used the identities

$$\mathrm{B}_{n,k}\bigl(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\bigr) = a^k b^n\, \mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2.2}$$

and

$$\begin{aligned}
\mathrm{B}_{n,k}&\Biggl(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k}(1-\ell\lambda)\Biggr) \\
&= \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell - q\lambda), \quad \lambda \in \mathbb{C},
\end{aligned} \tag{2.3}$$

which is equivalent to

$$\mathrm{B}_{n,k}(\langle\alpha\rangle_1, \langle\alpha\rangle_2, \dots, \langle\alpha\rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \langle\alpha\ell\rangle_n, \quad \alpha \in \mathbb{C}, \tag{2.4}$$

in [1, p. 135], [21, First proof of Theorem 2], [22, Lemma 2.2], [25, Remark 6.1], [26, Lemma 4], [29, Remark 1], [37, Lemma 2.6], and [38, Theorems 2.1 and 4.1]. The formula (1.7) is thus proved.

Similarly, applying $u = h(t) = \sqrt{1-2t}$ and $f(u) = e^{x(1-u)}$ to (2.1) yields

$$\begin{aligned}
G^{(n)}(t) &= \sum_{k=0}^{n} \frac{\mathrm{d}^k\, e^{x(1-u)}}{\mathrm{d}\, u^k} \mathrm{B}_{n,k}\biggl(-\frac{1}{(1-2t)^{1/2}}, -\frac{1}{(1-2t)^{3/2}}, \\
&\quad -\frac{3}{(1-2t)^{5/2}}, \dots, -\frac{[2(n-k+1)-3]!!}{(1-2t)^{[2(n-k+1)-1]/2}}\biggr) \\
&= \sum_{k=0}^{n} (-x)^k e^{x(1-u)} \frac{(-1)^k}{(1-2t)^{n-k/2}} \mathrm{B}_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!) \\
&= G(t) \sum_{k=0}^{n} \frac{x^k}{(1-2t)^{n-k/2}} \frac{(-1)^n}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1}(\ell - 2q),
\end{aligned}$$

where we used the identity

$$\mathrm{B}_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!) = \frac{(-1)^n}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - 2q) \quad (2.5)$$

in [25,27,29,46,42]. The formula (2.5) can also be derived from taking $\lambda = 2$ in (2.3) or taking $\alpha = \frac{1}{2}$ in (2.4) and utilizing the identity (2.2). The formula (1.8) is thus proved.

In [42, Theorem 1.2] and [40, Eq. (1.10)], it was derived that

$$\mathrm{B}_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!) = \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!!. \quad (2.6)$$

Substituting (2.6) into (1.8) concludes (1.9). The proof of Theorem 1.1 is complete.

3. Remarks

Finally, we list several remarks on our main results and closely related things.

Remark 3.1. *The equation (1.1) can be reformulated as*

$$G^{(n)}(t) = G(t) \sum_{i=n}^{2n-1} \frac{a_{i-n}(n,x)}{(1-2t)^{i/2}}, \quad n \in \mathbb{N}.$$

This means that the function $\frac{G^{(n)}(t)}{G(t)}$ is a linear combination of the base

$$\left(\frac{1}{\sqrt{1-2t}}\right)^n, \qquad \left(\frac{1}{\sqrt{1-2t}}\right)^{n+1}, \qquad \dots, \qquad \left(\frac{1}{\sqrt{1-2t}}\right)^{2n-1}.$$

The equation (1.8) shows that the function $\frac{G^{(n)}(t)}{G(t)}$ is a linear combination of the base

$$\left(\frac{1}{\sqrt{1-2t}}\right)^{2n}, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{2n-1}, \quad \dots, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{n+1}, \quad \left(\frac{1}{\sqrt{1-2t}}\right)^{n}.$$

These two bases are not equivalent to each other. Therefore, we surely disclose that the equation (1.1) is wrong and, consequently, main results in [3] are all wrong.

Remark 3.2. *Comparing (1.4) with (1.7) reveals that*

$$b_k(n,\lambda) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda).$$

This form for $b_k(n,\lambda)$ is apparently simpler, more meaningful, and more significant than the expression (1.6).

From (2.5) and (2.6), it follows that

$$\sum_{\ell=0}^{k}(-1)^{\ell}\binom{k}{\ell}\prod_{q=0}^{n-1}(\ell-2q)=(-1)^{n}k!\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!. \tag{3.1}$$

Consequently, we obtain

$$b_k(n,2)=(-1)^{n-k}\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!.$$

Remark 3.3. *The equation (1.7) can be rewritten as*

$$\begin{aligned}F_{\lambda}^{(n)}(t)&=\frac{F_{\lambda}(t)}{(1+\lambda t)^{n}}\sum_{\ell=0}^{n}\frac{[x(1+\lambda t)^{1/\lambda}]^{\ell}}{\ell!}\\&\quad\times\left[\prod_{q=0}^{n-1}(\ell-q\lambda)\right]\sum_{m=0}^{n-\ell}(-1)^{m}\frac{[x(1+\lambda t)^{1/\lambda}]^{m}}{m!}\\&=\frac{e^{-x}}{(1+\lambda t)^{n}}\sum_{\ell=0}^{n}\frac{[x(1+\lambda t)^{1/\lambda}]^{\ell}}{\ell!(n-\ell)!}\left[\prod_{q=0}^{n-1}(\ell-q\lambda)\right]\Gamma\bigl(n-\ell+1,-x(1+\lambda t)^{1/\lambda}\bigr),\end{aligned}$$

where $\Gamma(a,x)=\int_{x}^{\infty}t^{a-1}e^{-t}\,\mathrm{d}\,t$ denotes the incomplete gamma function which has been investigated in [10,30,34] and closely related references.

Remark 3.4. *Applying (3.1) to (1.7) results in*

$$F_{2}^{(n)}(t)=\frac{F_{2}(t)}{(1+2t)^{n}}\sum_{k=0}^{n}(-1)^{n-k}\binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!\bigl[x(1+2t)^{1/2}\bigr]^{k}.$$

Remark 3.5. *We leave a question to readers: what are the inversion formulas of the equations from (1.7) to (1.9)?*

Remark 3.6. *The motivations in the papers [5,6,8,11,13,14,15,16,17,23,24,28,29, 31,32,33,35,36,37,46,43,44,45,47,48,49] are same as the one in this paper.*

Remark 3.7. *This paper is a slightly modified version of the preprint [18].*

Acknowledgments

The author is thankful to anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

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