

**SIMPLY CONNECTED CLOSED SMOOTH 5-MANIFOLDS
WITH EFFECTIVE SMOOTH $U(2)$ -ACTIONS**

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0. Introduction. In [1] Asoh classified connected closed smooth manifolds of dimension less than 5, which admit a non-trivial smooth $SU(2)$ -action up to $SU(2)$ -equivariant diffeomorphisms. In [5] Nakanishi investigated an equivariant classification of smooth $SO(3)$ -actions on closed connected orientable smooth 5-manifolds such that the orbit space is an orientable surface. In [4] Hudson classified simply connected closed 5-manifolds with $SO(3)$ -actions admitting at least one singular orbit up to $SO(3)$ -equivariant diffeomorphisms. The purpose of this note is to prove the following theorem.

THEOREM. *Suppose that a compact simply connected smooth 5-manifold M^5 without boundary admits an effective smooth $U(2)$ -action $\phi: U(2) \times M^5 \rightarrow M^5$. Then M^5 is $U(2)$ -equivariantly diffeomorphic to one of the following $U(2)$ -manifolds.*

(I) S^3 on which $U(2)$ acts by

$$\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} \zeta \\ z \\ w \end{pmatrix} \right) \mapsto \begin{pmatrix} \zeta(ad - bc)^k \\ (az + cw)(ad - bc)^{-m} \\ (bz + dw)(ad - bc)^{-m} \end{pmatrix}$$

where $(k, 2m - 1) = 1$.

(II) $M_k^5 = U(2) \times_{O(2)} D^2 \cup_{f_k} S^3 \times D^2$,

where the $O(2)$ -action on 2-disk D^2 is the natural one, $U(2)$ acts on $S^3 \times D^2$ by

$$\left(A, \left(\begin{pmatrix} z \\ w \end{pmatrix}, \zeta \right) \right) \mapsto \left(A \begin{pmatrix} z \\ w \end{pmatrix} (\det A)^{-2k}, (\det A)^2 \zeta \right).$$

$U(2)/O(1)$ is $U(2)$ -equivariantly diffeomorphic to $\partial(U(2) \times_{O(2)} D^2)$ by $[A] \mapsto [A, 1]$ and the attaching map f_k is given by

$$f_k([A]) = \left(A \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\det A)^{-2k}, (\det A)^2 \right).$$

$$(III) \quad M \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\} = U(2) \times_{r^2} L(b, d),$$

where a, b, c, d, A, B, C, D are integers satisfying the conditions: $ad - bc = \pm 1$ and $(A + B, b) = 1 = (A - B, C - D)$ and where $L(b, d)$ is a 3-dimensional lens space for $b \neq 0$, while we put

$$L(0, \pm 1) = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{C}^2; (|z| - R)^2 + |w|^2 = r^2 \right\}$$

for fixed r, R with $0 < r < R$. The T^2 -action on $L(b, d)$ is as follows. Put

$$\begin{pmatrix} X & U \\ Y & V \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

If $b \neq 0$, then

$$\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} \right) \mapsto \begin{bmatrix} zx^y y^{-x} \\ w(x^b y^{-d})^\delta \end{bmatrix},$$

where $x^b = \xi, y^b = \eta$ and $\delta = ad - bc$. If $b = 0$, then

$$\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \right) \mapsto \begin{pmatrix} z\xi^{-y}\eta^x \\ w\xi^{-y}\eta^u \end{pmatrix},$$

where $(|z| - R)^2 + |w|^2 = r^2$.

REMARK. (i) In (II), M_k^s is diffeomorphic to the Wu-manifold $SU(3)/SO(3)$ by [2, Theorem 2.3] and in particular M_0^s is $U(2)$ -equivariantly diffeomorphic to $SU(3)/SO(3)$ admitting the following $U(2)$ -action:

$$(A, [X]) \mapsto \left[\begin{pmatrix} (\det A)^{-1} & 0 \\ 0 & A \end{pmatrix} X \right],$$

where $X \in SU(3)$ and $[X] \in SU(3)/SO(3)$.

(ii) In (II), denote ϕ by ϕ_k in case $M^s = M_k^s$. If $j \neq k$, then ϕ_j is not weakly equivariant to ϕ_k .

The remainder of this note is divided into three sections. In Section 1, we state necessary lemmas and show that the principal orbits of the $U(2)$ -action are of codimension one or two. In Section 2, we show that if the codimension of the principal orbits is one, then M^s is $U(2)$ -equivariantly diffeomorphic to either (I) with $k \neq 0$, (II) or (III) with $AD - BC \neq 0$. In Section 3, we show that if the codimension of the principal orbits is two, then M^s is $U(2)$ -equivariantly diffeomorphic to either (I) with $k = 0$ or (III) with $AD - BC = 0$.

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1. Preliminaries. In the first place, we define symbols and notations.

$$G_k^3 = \{A \in U(2); (\det A)^k = 1\} \quad (k \neq 0).$$

$$T^2 = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \in U(2); |\xi| = 1 = |\eta| \right\}.$$

$$N^2 = T^2 \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T^2.$$

$$G_{m,n}^1 = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \in U(2); \xi^n = \eta^m \right\} \quad (m \neq 0 \text{ or } n \neq 0).$$

$$C_{p,q,k} = \left\{ \begin{pmatrix} \omega^p & 0 \\ 0 & \omega^q \end{pmatrix} \in U(2); \omega^k = 1 \right\} \quad (k \neq 0 \text{ and } (p, q, k) = 1).$$

$$A_k = C_{1,-1,2k+1} \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} C_{1,-1,2k+1} \quad (k \geq 1).$$

$S^{2m-1} = \{\mathbf{a} \in \mathbf{C}^m; \|\mathbf{a}\| = 1\}$ ($m \geq 1$), where \mathbf{C}^m is the complex vector space of m -dimensional complex column vectors.

$\mathbf{Z}_k = \mathbf{Z}/k\mathbf{Z}$ or $\{\omega \in S^1; \omega^k = 1\}$ ($k \neq 0$), where \mathbf{Z} is the additive group of all rational integers.

[] denotes an equivalence class of a certain equivalence relation.

For integers k, q with $(k, q) = 1$ and $k \neq 0$, a free \mathbf{Z}_k -action $\phi_{k,q}: S^3 \times \mathbf{Z}_k \rightarrow S^3$ is defined by

$$\phi_{k,q} \left(\begin{pmatrix} z \\ w \end{pmatrix}, \omega \right) = \begin{pmatrix} z & \omega \\ w & \omega^q \end{pmatrix}.$$

We denote the orbit space of the action $\phi_{k,q}$ by $L(k, q)$. It is called a 3-dimensional lens space. For convenience, we put

$$L(0, q) = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbf{C}^2; (|z| - R)^2 + |w|^2 = r^2 \right\},$$

where $q = \pm 1$ and $0 < r < R$. Then $L(0, q)$ is diffeomorphic to $S^1 \times S^2$.

Let G be a compact Lie group and let K be a closed subgroup. Let ρ be a smooth K -action on a smooth manifold X . Then K acts on $G \times X$ by $(h, (g, x)) \mapsto (gh^{-1}, \rho(h, x))$. We denote the orbit space of this K -action by $G \times_K X$ or $(G \times X)/K$. Define a canonical G -action on $G \times_K X$ by $(g',$

$[g, x] \mapsto [g'g, x]$. This smooth G -manifold $G \times_{\kappa} X$ is called a twisted product. By (H) we denote the type of the principal isotropy subgroup of ϕ , that is, every principal isotropy subgroup of ϕ is conjugate to H . The $U(2)$ -action ϕ is effective if and only if each principal isotropy subgroup does not contain any proper normal subgroup of $U(2)$. The proper normal subgroups of $U(2)$ are G_k^3 ($k \neq 0$) or subgroups of $G_{1,1}^1$, where $G_{1,1}^1$ is the center of $U(2)$. Thus ϕ is effective if and only if $H \cap G_{1,1}^1 = \{E_2\}$, where E_2 is the unit matrix of $U(2)$.

LEMMA 1.1. *Suppose that a closed subgroup H of $U(2)$ satisfies $H \cap G_{1,1}^1 = \{E_2\}$. Then H is conjugate to one of the following subgroups:*

$$C_{p,q,k} \quad ((p - q, k) = 1), \quad \Delta_k, \quad G_{r,r+1}^1,$$

where k, p, q, r are some integers.

PROOF. Since the closed subgroup of $U(2)$ whose dimension is greater than one contains a non-trivial subgroup of $G_{1,1}^1$ except $\{E_2\}$, H must be a finite subgroup or a 1-dimensional closed subgroup of $U(2)$. It is easy to see that if H is a 1-dimensional closed subgroup, then H is conjugate to $G_{r,r+1}^1$ for some r . We also see easily that if H is a finite cyclic group of $U(2)$, then H is conjugate to $C_{p,q,k}$ for some p, q, k with $(p - q, k) = 1$.

Thus we have only to prove that non-cyclic H is conjugate to Δ_k for some k . Now let H be a non-cyclic finite subgroup of $U(2)$. Moreover suppose that $H \cap G_{1,1}^1 = \{E_2\}$. We define two homomorphisms $\omega: S^1 \times SU(2) \rightarrow U(2)$, $\pi: S^1 \times SU(2) \rightarrow SU(2)$ by $\omega(\alpha, A) = \alpha A$, $\pi(\alpha, A) = A$. The homomorphism ω is surjective and its kernel $\omega^{-1}(E_2)$ is equal to $\{(1, E_2), (-1, -E_2)\}$. For a subgroup G of $U(2)$ we put $\tilde{G} = \omega^{-1}(G)$. The following facts are clear or well known:

- (a) $\pi|_{\tilde{H}}$ is injective, i.e., $\tilde{H} \cong \pi(\tilde{H})$,
- (b) $g \in \tilde{H}$ and $\pi(g) = -E_2$ implies $g = (-1, -E_2)$.
- (c) Up to conjugacy, a non-cyclic finite subgroup of $SU(2)$ is isomorphic to one of the following:

$$\begin{aligned} D_{4n}^* &= \{x, y \mid x^2 = (xy)^2 = y^n, x^4 = 1\} \quad (\text{binary dihedral group, } n \geq 2), \\ T^* &= \{x, y \mid x^2 = (xy)^3 = y^3, x^4 = 1\} \quad (\text{binary tetrahedral group}), \\ O^* &= \{x, y \mid x^2 = (xy)^3 = y^4, x^4 = 1\} \quad (\text{binary octahedral group}), \\ I^* &= \{x, y \mid x^2 = (xy)^5 = y^5, x^4 = 1\} \quad (\text{binary icosahedral group}). \end{aligned}$$

By (a), (b), (c), the subgroup H is isomorphic to one of D_{2n} (dihedral group), T (tetrahedral group), O (octahedral group), I (icosahedral group).

If H is isomorphic to D_{2n} , then \tilde{H} is isomorphic to D_{4n}^* by π . Suppose

$x, y \in \pi(\tilde{H})$ satisfy $x^2 = (xy)^2 = y^n$, $x^4 = E_2$ and let $(u, x), (v, y) \in \tilde{H} (\subset S^1 \times SU(2))$. Then by (b), $u^2 = u^2v^2 = v^n = -1$ since $x^2 = (xy)^2 = y^n = -E_2$. Hence we see that $u = \pm\sqrt{-1}$, $v = -1$ and $n \equiv 1 \pmod{2}$. Up to conjugacy in $SU(2)$, we can put

$$y = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \quad \lambda = \exp(\pi\sqrt{-1}/n).$$

Let

$$X = \omega(u, x) \quad Y = \omega(v, y) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\bar{\lambda} \end{pmatrix}.$$

Then $X^2 = E_2 \neq X$ and $X \notin G_{1,1}^1$.

Hence we may put

$$X = \begin{pmatrix} a & \bar{z} \\ z & -a \end{pmatrix} \quad (a^2 + |z|^2 = 1, \quad a \in R).$$

Since $YXY = X$, we see that $a = 0$ and $|z| = 1$. Thus

$$X = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \bar{z} \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} Y \begin{pmatrix} 1 & 0 \\ 0 & \bar{z} \end{pmatrix}.$$

Hence H is conjugate to a subgroup of $U(2)$ which is generated by matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \quad (\alpha = \exp(2\pi\sqrt{-1}/(2k+1)) \text{ for some } k \geq 1).$$

Therefore if H is isomorphic to the dihedral group D_{2n} ($n \geq 2$), then H is conjugate to Δ_k .

Next suppose that H is isomorphic to T (tetrahedral group). Then $\pi(\tilde{H})$ is isomorphic to T^* . Suppose $x, y \in \pi(\tilde{H})$ satisfy $x^2 = (xy)^3 = y^3$, $x^4 = E_2$ and let $(u, x), (v, y) \in \tilde{H} (\subset S^1 \times SU(2))$. Then since $x^2 = (xy)^3 = y^3 = -E_2$, we have $u^2 = (uv)^3 = v^3 = -1$ by (b). This is a contradiction. Thus H is not isomorphic to the tetrahedral group.

In the same manner, we can show that H is isomorphic neither to the octahedral group O nor to the icosahedral group I . Hence if H is a non-cyclic finite subgroup, then H is conjugate to Δ_k for some $k \geq 1$.

q.e.d.

By Lemma 1.1, we have the following corollary.

COROLLARY 1.2. *The codimension of the principal orbits of ϕ is one or two.*

We omit the proof of the following lemma.

LEMMA 1.3. (i) A 3-dimensional closed subgroup of $U(2)$ is G_k^3 for some $k \neq 0$ and is normal in $U(2)$.

(ii) A 2-dimensional closed subgroup of $U(2)$ is conjugate to T^2 or N^2 .

(iii) The identity component of a 1-dimensional closed subgroup of $U(2)$ is conjugate to $G_{p,q}^1$ for some p, q with $(p, q) = 1$.

Define a smooth map $p_{j,k}: U(2) \rightarrow L(k, 1)$ for j, k with $j^2 + k^2 \neq 0$ by

$$p_{j,k} \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = \left[\begin{pmatrix} a \\ b \end{pmatrix} (ad - bc)^{-j/k} \right] \quad (k \neq 0),$$

$$p_{j,0} \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} \right) = \begin{pmatrix} (R + r(|a|^2 - |b|^2))(ad - bc)^j \\ r(2\bar{a}b) \end{pmatrix}.$$

Then $p_{j,k}$ induces the following $U(2)$ -diffeomorphism.

LEMMA 1.4. $U(2)/G_{j,k-j}^1 \cong L(k, 1)$ for j, k with $j^2 + k^2 \neq 0$.

The following lemmas are proved easily.

LEMMA 1.5. Assume that the integers m, n, k, p, q satisfy the conditions $k \neq 0$, $(p, q, k) = 1$ and $(m, n) = 1$. Then $G_{m,n}^1 \cdot C_{p,q,k} = G_{m,j,n,j}^1$, where $j = k / \left(k, \begin{vmatrix} m & p \\ n & q \end{vmatrix} \right)$, and there exists a complex representation $\rho: G_{m,j,n,j}^1 \rightarrow S^1$ whose kernel is $C_{p,q,k}$. Moreover, the representation ρ is unique up to complex-conjugation.

LEMMA 1.6. Let K be a 1-dimensional closed subgroup of $U(2)$ and let H be a finite cyclic subgroup of $K \cap T^2$. Suppose that the order of H is greater than one, $H \cap G_{1,1}^1 = \{E_2\}$ and K/H is connected. Then K is a subgroup of T^2 , or there exists $g \in N^2$ such that $H = gO(1)g^{-1}$ and $K = gO(2)g^{-1}$.

LEMMA 1.7. For $a, b, c, d \in \mathbf{Z}$ with $ad - bc = \pm 1$, define a diffeomorphism $f: S^1 \times S^1 \rightarrow S^1 \times S^1$ by $f(z, w) = (z^a w^b, z^c w^d)$. The closed smooth 3-manifold $S^1 \times D^2 \cup_f S^1 \times D^2$ formed by attaching $S^1 \times D^2$ to $S^1 \times D^2$ on their boundaries under f is diffeomorphic to the 3-dimensional lens space $L(b, d)$.

PROOF. In the first place we treat the case $b \neq 0$. Put

$$U_1 = \left\{ \begin{bmatrix} Z \\ W \end{bmatrix} \in L(b, d); |Z| \leq |W| \right\}, \quad U_2 = \left\{ \begin{bmatrix} Z \\ W \end{bmatrix} \in L(b, d); |Z| \geq |W| \right\}.$$

Define $\phi_s: S^1 \times D^2 \rightarrow U_s$ ($s = 1, 2$) by

$$\phi_1(z, w) = \left[(1 + |w|^2)^{-1/2} \begin{pmatrix} \bar{w}\zeta^{-a} \\ \zeta^{-\delta} \end{pmatrix} \right], \quad \phi_2(z, w) = \left[(1 + |w|^2)^{-1/2} \begin{pmatrix} \zeta^{-1} \\ w\zeta^{-d} \end{pmatrix} \right],$$

where $\zeta^b = z$ and $\delta = ad - bc$. Then since $\phi_1 = \phi_2 \circ f$ on $S^1 \times S^1$, $\phi_1 \cup_f \phi_2$ induces a diffeomorphism of $S^1 \times D^2 \cup_f S^1 \times D^2$ onto $L(b, d)$. Next we treat the case $b = 0$. Put

$$U_1 = \left\{ \begin{pmatrix} Z \\ W \end{pmatrix} \in \mathbb{C}^2; (|Z| - R)^2 + |W|^2 = r^2 \quad \text{and} \quad |Z| \geq R \right\},$$

$$U_2 = \left\{ \begin{pmatrix} Z \\ W \end{pmatrix} \in \mathbb{C}^2; (|Z| - R)^2 + |W|^2 = r^2 \quad \text{and} \quad |Z| \leq R \right\}.$$

Define $\phi_s: S^1 \times D^2 \rightarrow U_s$ ($s = 1, 2$) by

$$\phi_1(z, w) = \begin{pmatrix} (R + r(1 - |w|^2)^{1/2})z^a \\ rz^c w^d \end{pmatrix}, \quad \phi_2(z, w) = \begin{pmatrix} (R - r(1 - |w|^2)^{1/2})z \\ rw \end{pmatrix},$$

where for the convenience of notations, even if $|w| < 1$, we regard w^{-1} as \bar{w} since $d = \pm 1$. Then since $\phi_1 = \phi_2 \circ f$ on $S^1 \times S^2$, $\phi_1 \cup_f \phi_2$ induces a diffeomorphism of $S^1 \times D^2 \cup_f S^1 \times D^2$ onto $L(0, d)$. q.e.d.

We omit the proof of the following lemma.

LEMMA 1.8. *For $a, b, c, d, A, B, C, D \in \mathbb{Z}$ with $ad - bc = \pm 1$, define the same T^2 -action on $L(b, d)$ that is defined in Part (III) of the main theorem. Then*

$$M \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\} = U(2) \times_{r^2} L(b, d)$$

is simply connected if and only if $(A + B, b) = 1$. The canonical $U(2)$ -action on

$$M \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\} = U(2) \times_{r^2} L(b, d)$$

is effective if and only if $(A - B, C - D) = 1$.

Let G be a compact Lie group. Let X_1, X_2 be compact connected manifolds on which G acts smoothly. Assume that ∂X_1 is equivariantly diffeomorphic to ∂X_2 as G -manifolds. Denote by $M(f) = X_1 \cup_f X_2$ the compact connected G -manifold formed from X_1 and X_2 by the identification of points of ∂X_1 and ∂X_2 under a G -diffeomorphism $f: \partial X_1 \rightarrow \partial X_2$. The following lemma is described in [8, p. 161].

LEMMA 1.9. *Let $f, f': \partial X_1 \rightarrow \partial X_2$ be G -invariant diffeomorphisms.*

Then $M(f)$ is equivariantly diffeomorphic to $M(f')$ as G -manifolds, if f is G -diffeotopic to f' .

In particular, if ∂X_1 and ∂X_2 are both equivariantly diffeomorphic to G/H as G -manifolds for a closed subgroup H of G , then every G -invariant diffeomorphism $\partial X_1 = G/H \rightarrow G/H = \partial X_2$ is the right translation by an element of $W_G(H) = N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G . We thus have the following corollary.

COROLLARY 1.10. *If $\alpha, \beta \in W_G(H)$ belong to the same component of $W_G(H)$, then $M(\alpha) = X_1 \cup_{\alpha} X_2$ is G -equivariantly diffeomorphic to $M(\beta) = X_1 \cup_{\beta} X_2$.*

Let K_1, K_2 be two closed subgroups of G such that $H \subset K_1 \cap K_2$ and let ρ_s ($s = 1, 2$) be a k_s -dimensional orthogonal representation of K_s such that K_s/H is diffeomorphic to $O(k_s)/O(k_s - 1)$ by ρ_s . Let L be a closed subgroup of G . Suppose that $K_1 \cup K_2 \subset L$. Then the following lemma is proved easily.

LEMMA 1.11. *For each $\alpha \in W_L(H) \subset W_G(H)$,*

$$G \times \left(L \times_{K_1} D^{k_1} \cup_{\alpha} L \times_{K_2} D^{k_2} \right)$$

is G -equivariantly diffeomorphic to $G \times_{K_1} D^{k_1} \cup_{\alpha} G \times_{K_2} D^{k_2}$.

2. The proof when ϕ admits principal orbits of codimension 1. Then the principal isotropy subgroup of ϕ is finite. We denote the type of the principal isotropy subgroup by (H) . Using some results due to Uchida [8, Sections 1 and 5] concerning manifolds which admit a Lie group action with codimension one orbits, we easily see the following facts:

Each principal orbit of ϕ is $U(2)$ -equivariantly diffeomorphic to $U(2)/H$ and there are only two singular orbits $U(2)(x_1) \cong U(2)/K_1$, $U(2)(x_2) \cong U(2)/K_2$ where K_1, K_2 are some closed subgroups of $U(2)$ such that $H \subset K_1 \cap K_2$. In fact, there are two slice representations ρ_1, ρ_2 of K_1, K_2 respectively and there is an element $\alpha \in W_{U(2)}(H)$ such that M^5 is $U(2)$ -equivariantly diffeomorphic to

$$M(\alpha, \rho_1, \rho_2) = U(2) \times_{K_1} D^{k_1} \cup_{\alpha} U(2) \times_{K_2} D^{k_2}.$$

(2.1) $K_s \neq U(2)$. In fact, $U(2)$ does not act transitively on $\partial D^5 = S^4$.

(2.2) K_s is not 2-dimensional. In fact, neither T^2 nor N^2 acts transitively on $\partial D^3 = S^2$.

(2.3) K_s is not finite, since $U(2)/K_s$ is a singular orbit.

(2.4) Both K_1 and K_2 are not 3-dimensional. For otherwise, $U(2)/K_1$

and $U(2)/K_2$ would be $U(2)$ -equivariantly diffeomorphic to S^1 . Since in the manifold M^5 the codimension of the orbit $U(2)/K_2$ is greater than 2, $U(2)/K_1$ is simply connected by [8, Lemma 2.2.3]. This is a contradiction.

(2.5) Suppose that K_1 is 3-dimensional and K_2 is 1-dimensional. Then $K_1 = G_k^3$ for some k with $k \neq 0$. Since $\partial D^4 = S^3$ is not homeomorphic to G_k^3/Δ_j for any $j \geq 1$, K is not conjugate to Δ_j for any $j \geq 1$. First we investigate the slice representation $\rho: G_k^3 \rightarrow O(4)$ of the isotropy subgroup G_k^3 . Since the identity component of G_k^3 is $SU(2)$ and $K_1 = G_k^3$ acts transitively on $\partial D^4 = S^3$ by ρ , we have $G_k^3/H = S^3$. Hence $SU(2) \cap H = \{E_2\}$. Thus the restriction $\rho|_{SU(2)}$ of ρ to $SU(2)$ is a real representation induced by the natural $SU(2)$ -action on C^2 . From this fact it follows that for some p with $|p| < |k|$ we have

$$\rho(A) \cdot \begin{pmatrix} z \\ w \end{pmatrix} = A \begin{pmatrix} z \\ w \end{pmatrix} (\det A)^{-p}, \quad \text{where } A \in G_k^3 \text{ and } \begin{pmatrix} z \\ w \end{pmatrix} \in C^2 = \mathbf{R}^4.$$

Hence $H = C_{p,1-p,k}$ with $(2p-1, k) = 1$. Since $X_1 = U(2) \times_{K_1} D^4$, $X_2 = U(2) \times_{K_2} D^2$, $X_1 \cap X_2 = U(2)/H$ and $X_1 \cup X_2 = M^5$, $H_1(X_1) = H_1(U(2)/G_k^3) = H_1(S^1) = \mathbf{Z}$, $H_1(X_2) = H_1(U(2)/K_2)$, $H_1(X_1 \cap X_2) = H_1(U(2)/C_{p,1-p,k}) = \mathbf{Z}$ and $H_1(X_1 \cup X_2) = H_1(M^5) = 0$. By Mayer-Vietoris homology sequence of X_1 and X_2 , $H_1(X_1 \cap X_2) \rightarrow H_1(X_1) \oplus H_1(X_2) \rightarrow H_1(X_1 \cup X_2)$ is an exact sequence. Hence $H_1(X_2) = H_1(U(2)/K_2) = 0$. Now we study K_2 . Since $H_1(U(2)/O(2)) \neq 0$, K_2 is not conjugate to $O(2)$. If $H = C_{p,1-p,k} = \{E_2\}$, then we may regard K_2 as a closed subgroup of T^2 . If $k \geq 2$, then by Lemma 1.6, $K_2 \subset T^2$. Hence $K_2 = G_{m,n}^1 \cdot C_{p,1-p,k}$ for some m, n with $(m, n) = 1$. By Lemma 1.5, $K_2 = G_{m,j,n,j}^1$ where $j = k/(k, m - (m+n)p)$. Since $U(2)/K_2 = L(j(m+n), 1)$ by Lemma 1.4, $H_1(X_2) = H_1(X_2) = H_1(L(j(m+n), 1)) = \mathbf{Z}_{j(m+n)} = 0$. Hence $|j(m+n)| = 1$. Thus $j = \pm 1$ and $m+n = \pm 1$. Since in general, $G_{\mu,\nu}^1 = G_{-\mu,-\nu}^1$ for $(\mu, \nu) = 1$, it is no loss of generality to suppose that $j = 1$ and $m+n = 1$. Then $m \equiv p \pmod{k}$. Since $(2p-1, k) = 1$, we have $(2m-1, k) = 1$. By Lemma 1.5, there is a unique slice representation $\sigma: G_{m,1-m}^1 \rightarrow O(2)$ whose kernel is $H = C_{p,1-p,k} = C_{m,1-m,k}$. By Lemma 1.10, there exists at most one $U(2)$ -equivariant diffeomorphism class of such M^5 for k, m with $(2m-1, k) = 1$, since $W_{U(2)}(H) = T^2$ (or $U(2)$) is connected. We see easily that in this case M^5 is $U(2)$ -equivariantly diffeomorphic to S^5 of (I) with $k \neq 0$ in the main theorem.

(2.6) Suppose that both K_1 and K_2 are 1-dimensional. Then the principal isotropy subgroup H is conjugate to $C_{p,q,k}$ for some p, q, k with $(p-q, k) = 1$ or to Δ_k for some $k \geq 1$. Now we consider the Mayer-Vietoris homology sequence of X_1 and X_2 . Since $H_1(X_1 \cap X_2) = H_1(U(2)/H)$, $H_1(X_s) = H_1(U(2) \times_{K_s} D^2) = H_1(U(2)/K_s)$ ($s = 1, 2$) and $H_1(X_1 \cup X_2) = H_1(M^5)$,

the sequence $H_1(U(2)/H) \xrightarrow{\mu} H_1(U(2)/K_1) \oplus H_1(U(2)/K_2) \xrightarrow{\nu} H_1(M^5)$ is exact. Since M^5 is simply connected, we have $H_1(M^5) = 0$. Therefore μ is surjective. The principal isotropy subgroup H is not conjugate to Δ_k for any $k \geq 1$. In fact, assume that H is conjugate to Δ_k for some $k \geq 1$. Then $H_1(U(2)/H) = \mathbf{Z}$ and K_2 must be conjugate to one of the following subgroups:

$$L_1 = G_{1,-1}^1 \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G_{1,-1}^1, \quad L_2 = G_{2,-2}^1 \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G_{2,-2}^1,$$

$$L_3 = G_{2k+1,2k+1}^1 \cup \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} G_{2k+1,2k+1}^1.$$

$H_1(U(2)/L_1) = \mathbf{Z}$, $H_1(U(2)/L_2) = \mathbf{Z} \oplus \mathbf{Z}_2$ and $H_1(U(2)/L_3) = \mathbf{Z}_4$. Thus $\mu: H_1(U(2)/H) \rightarrow H_1(U(2)/K_1) \oplus H_1(U(2)/K_2)$ is not surjective. This is a contradiction. Hence H is a finite cyclic subgroup of $U(2)$. We may regard H as a subgroup of $T^2 \cap K_1 \cap K_2$.

(i) Assume that either K_1 or K_2 is conjugate to $O(2)$. We may put $K_1 = O(2)$. Then $H_1(X_1 \cap X_2) = H_1(U(2)/H) = \mathbf{Z}$, since $H = O(1)$, where $O(1) = C_{0,1,2}$ by Lemma 1.6. Since the homomorphism $\mu: H_1(U(2)/H) \rightarrow H_1(U(2)/K_1) \oplus H_1(U(2)/K_2)$ is surjective, we have $H_1(U(2)/K_2) = 0$. Hence K_2 is not conjugate to $O(2)$. By Lemma 1.6, we can regard K_2 as a closed subgroup of T^2 . Let the identity component of K_2 be $G_{a,b}^1$ with $(a, b) = 1$. Then $K_2 = G_{a,b}^1 \cdot H$. By Lemma 1.5, $G_{a,b}^1 \cdot C_{0,1,2} = G_{a,b}^1$ or $G_{2a,2b}^1$. Hence by Lemma 1.4, $U(2)/K_2 = L(a+b, 1)$ or $L(2(a+b), 1)$. Since in general $H_1(L(k, q)) = \mathbf{Z}_k$ and $H_1(U(2)/K_2) = 0$, we have $K_2 = G_{a,b}^1$ with $a+b = \pm 1$. Without loss of generality, we may assume that $b = 1 - a$ and a is even. Hence if $K_1 = O(2)$, then $H = O(1)$ and $K_2 = G_{2k,1-2k}^1$ for some integer k . Now the $O(2)$ -action on the 2-disk D^2 whose principal isotropy subgroup is conjugate to $O(1)$ is necessarily the $O(2)$ -action induced by the canonical 2-dimensional real representation and by Lemma 1.5, the $G_{2k,1-2k}^1$ -action on the 2-disk D^2 whose principal isotropy subgroup is conjugate to $O(1)$ is necessarily the $G_{2k,1-2k}^1$ -action induced by the following 1-dimensional complex representation

$$\left(\begin{pmatrix} \tau^{2k} & 0 \\ 0 & \tau^{1-2k} \end{pmatrix}, \zeta \right) \mapsto \tau^2 \zeta$$

where $\tau \in \mathbf{C}$ with $|\tau| = 1$ and $\zeta \in \mathbf{C}$. For $\alpha \in W_{U(2)}(O(1))$. Put

$$M_k^5(\alpha) = U(2) \times_{O(2)} D^2 \cup_{\alpha} (U(2) \times D^2) / G_{2k,1-2k}^1.$$

Since $W_{U(2)}(O(1)) = T^2/O(1)$, where $O(1) = C_{0,1,2}$, we have $M_k^5(\alpha) = M_k^5([E_2])$ by

Lemma 1.10. Now we shall show that $M_k^5([E_2])$ is simply connected. Let $i_s: X_1 \cap X_2 \rightarrow X_s$ ($s=1, 2$) be a natural inclusion, where $X_1 = U(2) \times D^2$ and $X_2 = (U(2) \times D^2)/G_{2k, 1-2k}^1$. Since the induced homomorphism $i_{1*}: \pi_1(X_1 \cap X_2) = \pi_1(U(2)/O(1)) \rightarrow \pi_1(X_1) = \pi_1(U(2)/O(2))$ is surjective and $\pi_1(X_2) = \pi_1(L(1, 1)) = 0$, we see that $M_k^5([E_2]) = X_1 \cup_{[E_2]} X_2$ is simply connected by van Kampen's theorem. Next we study the $U(2)$ -manifold X_2 . We have the following commutative diagram:

$$\begin{array}{ccc} [A] & \xrightarrow{\quad\quad\quad} & [A, 1] \\ U(2)/O(1) & \longrightarrow & (U(2) \times D^2)/G_{2k, 1-2k}^1 \\ \downarrow f_k & & \downarrow F_k \\ S^3 \times S^1 & \subset & S^3 \times D^2, \end{array}$$

where

$$F_k([A, \zeta]) = \left(A \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\det A)^{-2k}, (\det A)^2 \zeta \right),$$

$$f_k([A]) = \left(A \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\det A)^{-2k}, (\det A)^2 \right).$$

Define a $U(2)$ -action on $S^3 \times D^2$ by

$$\left(A, \left(\begin{pmatrix} z \\ w \end{pmatrix}, \zeta \right) \right) \mapsto \left(A \begin{pmatrix} z \\ w \end{pmatrix} (\det A)^{-2k}, (\det A)^2 \zeta \right).$$

Then F_k is a $U(2)$ -equivariant diffeomorphism. Therefore if one of the two singular isotropy subgroups of the $U(2)$ -action ϕ is conjugate to $O(2)$, then the manifold M^5 is $U(2)$ -equivariantly diffeomorphic to

$$M_k^5 = U(2) \times_{O(2)} D^2 \cup_{f_k} S^3 \times D^2.$$

By Mayer-Vietoris homology sequence, we have $H_2(M_k^5) = \mathbf{Z}_2$. Hence M_k^5 is diffeomorphic to the Wu-manifold $SU(3)/SO(3)$ by [2, Theorem 2.3]. Now put

$$SA(3) = \{L \in SU(3); {}^t L = L\}$$

and let $U(2)$ act on $SA(3)$ by

$$(A, X) \mapsto \begin{pmatrix} \delta^{-1} & 0 \\ 0 & A \end{pmatrix} X \begin{pmatrix} \delta^{-1} & 0 \\ 0 & {}^t A \end{pmatrix},$$

where $A \in U(2)$, $X \in SA(3)$ and $\delta = \det A$. Then $SU(3)/SO(3)$ admitting the $U(2)$ -action in Remark in Section 0 is $U(2)$ -equivariantly diffeomorphic to $SA(3)$ with the above $U(2)$ -action by the map sending $[U]$ to $U^t U$,

where $U \in SU(3)$ and $[U] \in SU(3)/SO(3)$. Denote the isotropy subgroup at $X \in SA(3)$ by $U(2)_X$. For $X \in SA(3)$, put

$$X = \begin{pmatrix} \lambda & \mu & \nu \\ \mu & \alpha & \gamma \\ \nu & \gamma & \beta \end{pmatrix}.$$

If $|\lambda| = 1$, then $U(2)_X$ is conjugate to $O(2)$. If $0 < |\lambda| < 1$, then $U(2)_X$ is conjugate to $C_{0,1,2} = O(1)$. If $\lambda = 0$, then $U(2)_X$ is conjugate to $G_{0,1}^1 = U(1)$. Hence $M_0^5 = SA(3) = SU(3)/SO(3)$.

Denote ϕ by ϕ_k in case $M^5 = M_k^5$. Next we show that if $j \neq k$, then ϕ_j is not weakly equivariant to ϕ_k . Suppose that ϕ_j is weakly equivariant to ϕ_k . Then there exists an automorphism α of $U(2)$ and there exists a diffeomorphism $f: M_j^5 \rightarrow M_k^5$ such that the following diagram is commutative:

$$\begin{array}{ccc} U(2) \times M_j^5 & \xrightarrow{\phi_j} & M_j^5 \\ \downarrow \alpha \times f & & \downarrow f \\ U(2) \times M_k^5 & \xrightarrow{\phi_k} & M_k^5 \end{array}.$$

The automorphism α maps the center of $U(2)$ into itself i.e., induces an automorphism of $G_{1,1}^1$. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} S^1 \times M_j & \xrightarrow{\psi_j} & M_j \\ \downarrow \beta \times f & & \downarrow f \\ S^1 \times M_k & \xrightarrow{\psi_k} & M_k \end{array},$$

where ψ_j, ψ_k are the S^1 -actions induced by the restriction of the $U(2)$ -actions ϕ_j, ϕ_k to $G_{1,1}^1$ respectively and β is the automorphism of S^1 induced by α . The isotropy types of ψ_j are $(Z_1), (Z_2), (Z_{4j-1})$ and the isotropy types of ψ_k are $(Z_1), (Z_2), (Z_{4k-1})$. Hence $|4j-1| = |4k-1|$. Thus $j = k$. q.e.d.

(ii) Suppose that neither K_1 nor K_2 is conjugate to $O(2)$. By Lemma 1.6, we may assume that $K_1, K_2 \subset T^2$. Let ρ_s ($s = 1, 2$) be a 2-dimensional real representation such that the induced K_s -action on D^2 is transitive on ∂D^2 and the kernel of ρ_s is equal to H . For $\alpha \in W_{U(2)}(H)$, let M^5 be $U(2)$ -equivariantly diffeomorphic to

$$M_\alpha^5 = M(\alpha, \rho_1, \rho_2) = U(2) \times_{K_1} D^2 \cup_\alpha U(2) \times_{K_2} D^2.$$

Now if $p + q \not\equiv 0 \pmod{k}$, then the normalizer of $H = C_{p,q,k}$ in $U(2)$ is T^2 .

If $p + q \equiv 0 \pmod{k}$ then the normalizer of $H = C_{p,q,k}$ in $U(2)$ is N^2 . Hence $W_{U(2)}(H)$ is connected or has two components. If α belongs to the identity component of $W_{U(2)}(H)$, then by Corollary 1.10 and Lemma 1.11, M_α^s is $U(2)$ -equivariantly diffeomorphic to

$$U(2) \times_{T^2} (T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_2} D^2).$$

If $W_{U(2)}(H) = N^2/H$ and α belongs to the component of $[\lambda]$, then $H = C_{1,-1,2j+1} = C_{j,j+1,2j+1}$ for some j . By the same corollary and lemma we have

$$M_\alpha^s \cong U(2) \times_{N^2} (N^2 \times_{K_1} D^2 \cup_{[\lambda]} N^2 \times_{K_2} D^2),$$

where $\lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N^2$.

Now we shall prove the following lemma.

LEMMA 2.1. $N^2 \times_{K_1} D^2 \cup_{[\lambda]} N^2 \times_{K_2} D^2$ is N^2 -equivariantly diffeomorphic to

$$N^2 \times_{T^2} (T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_\lambda} D^2),$$

where $K_\lambda = \lambda K_2 \lambda^{-1}$ and K_λ acts on D^2 by $\rho_\lambda(h) = \rho_2(\lambda^{-1} h \lambda)$ ($h \in K_\lambda$). Moreover $H = \lambda H \lambda^{-1} \subset K_\lambda \subset T^2$.

PROOF. Identify N^2/H with $\partial(N^2 \times_{K_1} D^2)$ by the map $[A] \mapsto [A, 1]$, where $K = K_1, K_2$ or K_λ . Define an N^2 -diffeomorphism $\chi: N^2 \times D^2 \rightarrow N^2 \times_{K_\lambda} D^2$ by $([A, \zeta]) = [A\lambda^{-1}, \zeta]$. Then $(N^2 \times_{K_2} D^2, N^2/H)$ is N^2 -equivariantly diffeomorphic to $(N^2 \times_{K_\lambda} D^2, N^2/H)$ by χ , where $(\chi|_{N^2/H})([A]) = [A\lambda^{-1}]$. Hence χ induces an N^2 -diffeomorphism of $N^2 \times D^2 \cup_{[\lambda]} N^2 \times_{K_2} D^2$ onto $N^2 \times_{K_1} D^2 \cup_{[E_2]} N^2 \times_{K_\lambda} D^2$. Thus $N^2 \times_{K_1} D^2 \cup_{[\lambda]} N^2 \times_{K_2} D^2$ is N^2 -equivariantly diffeomorphic to $N^2 \times_{T^2} (T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_\lambda} D^2)$ by Lemma 1.11. q.e.d.

By this lemma, $U(2) \times_{N^2} (N^2 \times_{K_1} D^2 \cup_{[\lambda]} N^2 \times_{K_2} D^2)$ is $U(2)$ -equivariantly diffeomorphic to $U(2) \times_{N^2} (N^2 \times_{T^2} (T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_\lambda} D^2)) = U(2) \times_{T^2} (T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_\lambda} D^2)$. Thus if neither K_1 nor K_2 is conjugate to $O(2)$, then

$$M^s \cong M(\alpha, \rho_1, \rho_2) \cong U(2) \times_{T^2} (T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_2} D^2).$$

Now we investigate $L = T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_2} D^2$. Since K_1, K_2 are the 1-dimensional closed subgroups of T^2 , we have $K_1 = G_{A,B}^1, K_2 = G_{X,Y}^1$ for some $A, B, X, Y \in \mathbb{Z}$ with $A^2 + B^2 \neq 0, X^2 + Y^2 \neq 0$. Then we can put

$$\rho_1 \left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \right) = \xi^{-D} \eta^C, \quad \rho_2 \left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \right) = \xi^{-V} \eta^U,$$

where C, D and U, V must satisfy $AD - BC \neq 0$ and $XV - YU \neq 0$, respectively, since the K_s -action on ∂D^2 induced by ρ_s is transitive. Define the T^2 -action $\tilde{\rho}_s$ on $S^1 \times D^2$ by

$$\begin{aligned}\tilde{\rho}_1\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, (z, w)\right) &= (\xi^{-B}\eta^A z, \xi^{-D}\eta^C w) \\ \tilde{\rho}_2\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, (z, w)\right) &= (\xi^{-Y}\eta^X z, \xi^{-V}\eta^U w).\end{aligned}$$

Moreover, define the map $\hat{\rho}_s$ of $T^2 \times_{K_s} D^2$ onto $S^1 \times D^2$ by

$$\begin{aligned}\hat{\rho}_1\left(\left[\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \zeta\right]\right) &= (\xi^{-B}\eta^A, \xi^{-D}\eta^C \zeta) \\ \hat{\rho}_2\left(\left[\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \zeta\right]\right) &= (\xi^{-Y}\eta^X, \xi^{-V}\eta^U \zeta).\end{aligned}$$

Then $(T^2 \times_{K_s} D^2, T^2/H)$ is T^2 -equivariantly diffeomorphic to $(S^1 \times D^2, S^1 \times S^1)$ by $\hat{\rho}_s$, where

$$\begin{aligned}(\hat{\rho}_1 | T^2/H)\left(\left[\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\right]\right) &= \hat{\rho}_1\left(\left[\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, 1\right]\right) = (\xi^{-B}\eta^A, \xi^{-D}\eta^C) \\ (\hat{\rho}_2 | T^2/H)\left(\left[\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}\right]\right) &= \hat{\rho}_2\left(\left[\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, 1\right]\right) = (\xi^{-Y}\eta^X, \xi^{-V}\eta^U).\end{aligned}$$

Furthermore, we have the following commutative diagram

$$\begin{array}{ccccccc} T^2 \times_{K_1} D^2 & \longleftarrow & T^2/H & \xrightarrow{[E_2]} & T^2/H & \longrightarrow & T^2 \times_{K_2} D^2 \\ & & & \text{identity} & & & \\ \downarrow \hat{\rho}_1 & & \downarrow \hat{\rho}_1 | T^2/H & & \downarrow \hat{\rho}_2 | T^2/H & & \downarrow \hat{\rho}_2 \\ S^1 \times D^2 & \supset & S^1 \times S^1 & \xrightarrow{f} & S^1 \times S^1 & \subset & S^1 \times D^2 \end{array}$$

where $f = (\hat{\rho}_2 | T^2/H) \cdot (\hat{\rho}_1 | T^2/H)^{-1}$ is a T^2 -equivariant diffeomorphism. The map f is an automorphism of the topological group $S^1 \times S^1$. Hence for some $a, b, c, d \in \mathbf{Z}$ with $ad - bc = \pm 1$, we have $f(z, w) = (z^a w^b, z^c w^d)$. On the other hand, $f(\xi^{-B}\eta^A, \xi^{-D}\eta^C) = \hat{\rho}_2 \hat{\rho}_1^{-1}(\xi^{-B}\eta^A, \xi^{-D}\eta^C) = (\xi^{-Y}\eta^X, \xi^{-V}\eta^U)$ for each $\xi, \eta \in \mathbf{C}$ with $|\xi| = 1$ and $|\eta| = 1$. Hence $\xi^{-Y}\eta^X = \xi^{-Ba-Db}\eta^{Aa+Cb}$, $\xi^{-V}\eta^U = \xi^{-Bc-Dd}\eta^{Ac+Cd}$ for arbitrary $\xi, \eta \in \mathbf{C}$ with $|\xi| = 1, |\eta| = 1$. Therefore

$$\begin{pmatrix} X & U \\ Y & V \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Moreover $L = T^2 \times_{K_1} D^2 \cup_{[E_2]} T^2 \times_{K_2} D^2$ is T^2 -equivariantly diffeomorphic to $S^1 \times D^2 \cup_f S^1 \times D^2$. By Lemma 1.7, as a T^2 -manifold we have $S^1 \times D^2$

$U_r S^1 \times D^2 \cong L(b, d)$, where the T^2 -action on $L(b, d)$ is the same action that is defined in Part (III) of the main theorem described in the introduction. Hence $T^2 \times D^2 \cup_{[E_2]} T^2 \times D^2 \cong L(b, d)$ as a T^2 -manifold. Thus as a $U(2)$ -manifold $M^5 = M(\alpha, \rho_1, \rho_2) \cong U(2) \times_{T^2} L(b, d)$ for some $a, b, c, d, A, B, C, D \in \mathbf{Z}$ with $ad - bc = \pm 1$ and $AD - BC \neq 0$. Moreover by Lemma 1.8, we have $(A + B, b) = 1$ and $(A - B, C - D) = 1$, since $M^5 \cong M_\alpha^5 = M(\alpha, \rho_1, \rho_2) \cong U(2) \times_{T^2} L(b, d)$ is simply connected and the $U(2)$ -action on M^5 is effective. Then $H = G_{A,B}^1 \cap G_{C,D}^1 = G_{X,Y}^1 \cap G_{U,V}^1 = C_{r,r+1,AD-BC}$, where $\begin{pmatrix} r & \\ r+1 & \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for some $\alpha, \beta \in \mathbf{Z}$. Therefore in this case M^5 is $U(2)$ -equivariantly diffeomorphic to the $U(2)$ -manifold

$$M \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\} = U(2) \times_{T^2} L(b, d)$$

of Part (III) with $AD - BC \neq 0$ in the main theorem.

3. The proof when ϕ admits principal orbits of codimension 2. Then the principal isotropy subgroup of ϕ is 1-dimensional. We denote the type of the principal isotropy subgroup by (H) . Then we may regard H as $G_{r,r+1}^1$ for some $r \in \mathbf{Z}$ by Lemma 1.1.

(3.1) Suppose that $U(2)$ appears as an isotropy subgroup. We investigate 5-dimensional real representations of $U(2)$. Let V be the 5-dimensional real vector space of all symmetric 3×3 real matrices with trace 0. Let τ be the 5-dimensional real representation of $SO(3)$ on V defined by $\tau(A, X) = AXA^{-1}$ for $A \in SO(3)$, $X \in V$. We denote by λ_1 the canonical 2-dimensional complex representation of $U(2)$ or $SU(2)$. Denote the determinant representation of $U(2)$ by λ_2 . Let ρ be the natural homomorphism of $U(2)$ onto $SO(3) \cong U(2)/G_{1,1}^1$. There are only the following three possibilities of irreducible real representations of $SU(2)$ with dimension less than six: $\rho_0: SU(2) \rightarrow SU(2)/\{\pm E_2\} \cong SO(3)$, $r(\lambda_1): SU(2) \rightarrow SO(4)$, $\sigma_0: SU(2) \rightarrow SO(5)$, where ρ_0 is the restriction of the above ρ to $SU(2)$, $r(\lambda_1)$ is the underlying real representation of the complex representation λ_1 and $\sigma_0 = \tau \circ \rho_0$ (composition of τ and ρ_0 as maps). These representations can be uniquely extended, respectively, to the following representations: $\rho: U(2) \rightarrow SO(3)$, $r(\lambda_1 \lambda_2^m): U(2) \rightarrow SO(4)$, $\sigma: U(2) \rightarrow SO(5)$, where $r(\lambda_1 \lambda_2^m)$ is the underlying real representation of the complex representation $\lambda_1 \lambda_2^m(A) = A(\det A)^m$ for $A \in U(2)$ and $\sigma = \tau \circ \rho$. Thus the following are all the 5-dimensional real representations of $U(2)$: $\rho + r(\lambda_2^3): U(2) \rightarrow SO(3) \times SO(2) \subset O(5)$, $r(\lambda_1 \lambda_2^3) + 1: U(2) \rightarrow SO(4) \times SO(1) \subset O(5)$, $\sigma: U(2) \rightarrow SO(5) \subset O(5)$, where $r(\lambda_2^3)$ is the underlying real representation

of the complex representation $\lambda_2^n(A) = (\det A)^n$ for $A \in U(2)$ and 1 is the trivial 1-dimensional real representation. Hence the 5-dimensional real representation of $U(2)$ which induces an effective action with principal orbits of codimension 2 is $r(\lambda_1 \lambda_2^m) + 1$ for $m = 0, -1$. Therefore if $U(2)$ appears as an isotropy subgroup, then such action is of two isotropy types $(G_{m,m+1}^1, U(2))$ for $m = 0, -1$. We denote the set of fixed points of this action by $F(U(2), M^5)$ or F . It follows from [3, IV 8.6. Theorem] that the orbit space M^* of this action is a 2-disk D^2 and $F(U(2), M^5) = \partial D^2 = S^1$. Denote by U the $U(2)$ -invariant closed tubular neighborhood of $F = S^1$ in M^5 and let X be the closure of $M^5 - U$ in M^5 . Then X is also $U(2)$ -invariant. Since $U(2)/G_{m,m+1}^1 \cong S^3$ for $m = 0, -1$ and $W_{U(2)}(G_{m,m+1}^1) = S^1$, we have

$$X \cong S^3 \times_{S^1} F(G_{m,m+1}^1, X)$$

by [7, Lemma 4.2], where $F(G_{m,m+1}^1, X) = \{x \in X; G_{m,m+1}^1 \subset U(2)_x\}$ and $W_{U(2)}(G_{m,m+1}^1) = S^1$ acts freely on $F(G_{m,m+1}^1, X)$. Moreover we have the S^3 -bundle $X \rightarrow X/U(2)$ with a $U(2)$ -action. Now the orbit space $X/U(2)$ is the 2-dimensional disk D^2 . Thus X is $U(2)$ -equivariantly diffeomorphic to $S^3 \times D^2$. Moreover, ∂X is $U(2)$ -equivariantly diffeomorphic to $S^3 \times S^1$, hence so is ∂U . On the other hand, $U \rightarrow F = S^1$ is a D^4 -bundle with a $U(2)$ -action. Thus U is $U(2)$ -equivariantly diffeomorphic to $D^4 \times S^1$. Consequently, there exists a $U(2)$ -equivariant diffeomorphism $f: S^3 \times S^1 \rightarrow S^3 \times S^1$, so that M^5 is $U(2)$ -equivariantly diffeomorphic to the manifold $M(f) = D^4 \times S^1 \cup_f S^3 \times D^2$. Now for such f there exist a smooth map $\alpha: S^1 \rightarrow S^1$ and a diffeomorphism $\beta: S^1 \rightarrow S^1$ such that $f(q, \zeta) = (q\alpha(\zeta), \beta(\zeta))$ for $(q, \zeta) \in S^3 \times S^1$. Extend f to the $U(2)$ -equivariant diffeomorphism $F: D^4 \times S^1 \rightarrow D^4 \times S^1$ defined by $F(tq, \zeta) = (tq\alpha(\zeta), \beta(\zeta))$ ($0 \leq t \leq 1$). Then F induces a $U(2)$ -equivariant diffeomorphism $S^5 = D^4 \times S^1 \cup_{\text{id}} S^3 \times D^2 \rightarrow M(f) = D^4 \times S^1 \cup_f S^3 \times D^2$, where id is the identity map of $S^3 \times S^1$. Consequently, M^5 is $U(2)$ -equivariantly diffeomorphic to S^5 of (I) with $k = 0$ in the main theorem.

(3.2) G_k^3 does not appear as an isotropy subgroup of ϕ . Indeed, the identity component of a 1-dimensional closed subgroup of G_k^3 is $G_{1,-1}^1$.

(3.3) N^2 does not appear as an isotropy subgroup of ϕ . Indeed, suppose that N^2 is an isotropy subgroup and $\rho: N^2 \rightarrow O(3)$ is the slice representation of ρ . Then the identity component of its principal isotropy subgroup is $G_{1,-1}^1$. This is a contradiction.

(3.4) Suppose that T^2 appears as an isotropy subgroup of ϕ . For $r \in \mathbf{Z}$, let $\zeta_r: T^2 \rightarrow S^1$ be a complex representation defined by

$$\zeta_r \left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix} \right) = \xi^{r+1} \eta^{-r}$$

and let $r(\zeta_r)$ be the real representation induced by ζ_r . The slice representation of T^2 whose principal isotropy subgroup is $G_{r,r+1}^1$ is necessarily $\rho_r = r(\zeta_r) + 1: T^2 \rightarrow SO(2) \times SO(1) \subset O(3)$, where 1 is the 1-dimensional trivial representation. The isotropy type of the T^2 -action induced by the slice representation ζ_r is $(G_{r,r+1}^1, T^2)$. Thus if T^2 appears as an isotropy subgroup of ϕ , then by (3.1), (3.2), (3.3) and this fact, ϕ is of two isotropy types $(G_{r,r+1}^1, T^2)$ for some $r \in \mathbf{Z}$.

Denote by $M_{(T^2)}$ the set of all points whose isotropy groups are conjugate to T^2 . Since the isotropy type (T^2) is maximal, $M_{(T^2)}$ is a $U(2)$ -invariant closed submanifold of M^5 . By [3, IV 8.6 Theorem], the orbit spaces $M^5/U(2)$ and $M_{(T^2)}/U(2)$ are homeomorphic to D^2 and $\partial D^2 = S^1$, respectively.

Denote by $F(T^2, M_{(T^2)})$ or F the set of all points of $M_{(T^2)}$ whose isotropy subgroup contains T^2 . We identify $U(2)/T^2$ with S^2 as $U(2)$ -spaces. By [7, Lemma 4.2] we have $M_{(T^2)} = U(2) \times_{N^2} F = (U(2)/T^2) \times_{W(T^2)} F$, where $W(T^2) = W_{U(2)}(T^2) = N^2/T^2$ and $W(T^2)$ acts freely on F . We may identify $W(T^2)$ with $S^0 = \{\pm 1\}$. S^0 acts on S^2 by $(\pm 1, \mathbf{a}) \mapsto \pm \mathbf{a}$, where $\mathbf{a} \in S^2$ and $\pm 1 \in S^0$. Thus $M_{(T^2)} = S^2 \times_{S^0} F$ as a $U(2)$ -manifold. Moreover, $F/S^0 = M_{(T^2)}/U(2) = S^1$. Since S^0 acts freely on F , we see that $F \rightarrow S^1$ is a principal S^0 -bundle over S^1 . Hence $F = S^1$ or $S^1 \times S^0$.

Denote the normal bundle of $M_{(T^2)}$ in M^5 by ν . First we show that ν has a $U(2)$ -invariant complex structure, so that ν is an orientable real plane bundle with a $U(2)$ -action. Next we show that $F = S^1 \times S^0$ by means of the Gysin sequence. Consider the following commutative diagram:

$$\begin{array}{ccc} \mu = j^* \nu & \longrightarrow & \nu \\ \downarrow & & \downarrow \\ F & \xrightarrow{j} & M_{(T^2)} = U(2) \times_{N^2} F \end{array}$$

where j is the inclusion map and $\mu = j^* \nu$ is the induced bundle. Then μ is a real plane bundle with N^2 -action and $\nu = U(2) \times_{N^2} \mu$. Thus if μ has an N^2 -invariant complex structure, then it naturally induces a $U(2)$ -invariant complex structure on ν . Now we introduce a canonical complex structure on μ .

Since the T^2 -action which is the restriction of the N^2 -action on μ leaves F fixed, each element of T^2 induces an automorphism of every

fiber of μ . In particular, consider the $G_{1,1}^1$ -action which is the restriction of such a T^2 -action. Since the above N^2 -action is induced by the $U(2)$ -action ϕ and the isotropy type of ϕ is $(G_{r,r+1}^1, T^2)$, such a $G_{1,1}^1$ -action is free on the associated sphere bundle $S(\mu)$ of μ . Thus we can define the complex structure on μ by means of the action of $\sqrt{-1}E_2 \in G_{1,1}^1$. Then since $G_{1,1}^1$ is the center of $U(2)$, such a complex structure is compatible with the N^2 -action on μ , i.e., N^2 -invariant. Hence ν has a $U(2)$ -invariant complex structure and the normal bundle ν is an orientable plane bundle. In order to prove that $F = S^1 \times S^0$ let us assume $F = S^1$ and derive a contradiction. Then in the principal bundle $F = S^1 \xrightarrow{p} S^1$ the projection p is the map $p(z) = z^2$ for $z \in F = S^1$. Consider the bundle $M_{(T^2)} = S^2 \times_{S^0} F \rightarrow S^2/S^0 = P_2$ (real projective plane). This is the sphere bundle associated to the complex line bundle $\xi = S^2 \times_{S^0} C \rightarrow P_2$, where the S^0 -action $S^0 \times C \rightarrow C$ is defined by $(\pm 1, z) \mapsto \pm z$. Since the bundle ξ can be regarded as a real orientable plane bundle, we can apply the Gysin sequence of the sphere bundle $M_{(T^2)} \rightarrow P_2 = S^2/S_0$. Thus the following sequence is exact:

$$0 = H_3(P_2) \rightarrow H_1(P_2) \rightarrow H_2(M_{(T^2)}) \rightarrow H_2(P_2) = 0 .$$

Hence $H_2(M_{(T^2)}) \cong H_1(P_2) \cong \mathbf{Z}_2$. Now denote by U a $U(2)$ -invariant closed tubular neighborhood of $M_{(T^2)}$ in M^5 and let E be the closure of $M^5 - U$ in M^5 . Then E is also $U(2)$ -invariant. Moreover, we have the bundle $E \rightarrow E/U(2)$ whose typical fiber is $U(2)/G_{r,r+1}^1$. The orbit space $E/U(2)$ is diffeomorphic to the 2-disk D^2 . By Lemma 1.4, $U(2)/G_{r,r+1}^1 = L(2r+1, 1)$. Thus $E = L(2r+1, 1) \times D^2$. Hence $\partial U = \partial E = L(2r+1, 1) \times S^1$. On the other hand, the bundle $\partial U \rightarrow M_{(T^2)}$ can be regarded as the sphere bundle associated with the normal bundle ν of $M_{(T^2)}$ in M^5 . Since it has been already proved that ν is orientable, we can apply the Gysin sequence of the above sphere bundle and get the exact sequence

$$0 = H_4(M_{(T^2)}) \rightarrow H_2(M_{(T^2)}) \rightarrow H_3(\partial U) .$$

Here $H_2(M_{(T^2)}) \cong \mathbf{Z}_2$, $H_3(\partial U) = H_3(L(2r+1, 1) \times S^1) \cong \mathbf{Z}$. This is a contradiction. Therefore $F \neq S^1$. Hence $F = S^0 \times S^1$. Since $M_{(T^2)} = (U(2)/T^2) \times_{W(T^2)} F = S^2 \times_{S^0} F$ as a $U(2)$ -space, $M_{(T^2)} = (U(2)/T^2) \times S^1 = S^2 \times S^1$ as a $U(2)$ -space.

In the following commutative diagram of bundles

$$\begin{array}{ccc} \mu & \longrightarrow & \nu \\ \downarrow & & \downarrow \\ F & \xrightarrow{j} & M_{(T^2)} \end{array}$$

we have $\mu = j^*\nu$ and $\nu = U(2) \times_{N^2} \mu$. Since an orientable plane bundle over S^1 is trivial, $\mu = F \times C = (S^0 \times S^1) \times C$ as a bundle with an N^2 -action. Thus $\nu = U(2) \times_{N^2} (F \times C)$ as a bundle with a $U(2)$ -action. The T^2 -action, which is the restriction of such a $U(2)$ -action, induces a T^2 -action on the fiber C . Since the $U(2)$ -action on the associated D^2 -bundle of ν coincides with the $U(2)$ -action on the $U(2)$ -invariant closed tubular neighborhood U by ϕ , the principal isotropy type of such T^2 -action on C is $(G_{r,r+1}^1)$. Now we consider the plane bundle $\pi: U(2) \times_{T^2} (S^1 \times C) \rightarrow M_{(T^2)} = (U(2)/T^2) \times S^1 = S^2 \times S^1$ with a $U(2)$ -action, where T^2 acts on $S^1 \times C$ by

$$\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, (\tau, \zeta) \right) \mapsto (\tau, \alpha^{r+1}\beta^{-r}\zeta) \quad \text{and} \quad \pi([A, (\tau, \zeta)]) = ([A], \tau).$$

Define a map $h: U(2) \times_{T^2} (S^1 \times C) \rightarrow U(2) \times_{N^2} (S^0 \times S^1 \times C)$ by $h([A, (\tau, \zeta)]) = [A, (1, \tau, \zeta)]$. Then h is a $U(2)$ -equivariant isomorphism of vector bundles with $U(2)$ -actions. We consider the plane bundle $\pi: L(2r+1, 1) \times_{S^1} (S^1 \times C) \rightarrow S^2 \times S^1 = M_{(T^2)}$, where S^1 acts on $L(2r+1, 1)$ and $S^1 \times C$ by

$$\left(\tau, \begin{bmatrix} z \\ w \end{bmatrix} \right) \mapsto \left[\begin{bmatrix} z \\ w \end{bmatrix} \tau^{1/(2r+1)} \right] \quad \text{and} \quad (\tau, (\xi, \eta)) \mapsto (\xi, \tau\eta)$$

respectively. $U(2)$ acts on $L(2r+1, 1)$ by

$$\left(A, \begin{bmatrix} z \\ w \end{bmatrix} \right) \mapsto A \cdot \begin{bmatrix} z \\ w \end{bmatrix} = \left[A \begin{bmatrix} z \\ w \end{bmatrix} (\det A)^{-r/(2r+1)} \right]$$

and the above projection π is defined by

$$\pi \left(\left[\begin{bmatrix} z \\ w \end{bmatrix}, (\xi, \eta) \right] \right) = \left(\begin{pmatrix} |z|^2 - |w|^2 \\ 2 \operatorname{Re}(\bar{z}w) \\ 2 \operatorname{Im}(\bar{z}w) \end{pmatrix}, \xi \right).$$

$U(2) \times_{T^2} (S^1 \times C)$ is $U(2)$ -equivariantly isomorphic to $L(2r+1, 1) \times_{S^1} (S^1 \times C)$ by the map

$$[A, (\xi, \eta)] \mapsto \left[A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, (\xi, \eta) \right].$$

Thus we may regard $L(2r+1, 1) \times_{S^1} (S^1 \times C)$ as the normal bundle ν . Hence $U = L(2r+1, 1) \times_{S^1} (S^1 \times D^2)$. On the other hand, $E = L(2r+1, 1) \times_{S^1} D^2 = L(2r+1, 1) \times_{S^1} (D^2 \times S^1)$, where S^1 acts on $D^2 \times S^1$ by $(\tau, (t\xi, \eta)) \mapsto (t\xi, \tau\eta)$ ($0 \leq t \leq 1, |\xi| = |\eta| = 1$). Now as $U(2)$ -manifolds

$$\begin{aligned} \partial(L(2r+1, 1) \times_{S^1} (S^1 \times D^2)) &= \partial(L(2r+1, 1) \times_{S^1} (D^2 \times S^1)) \\ &= L(2r+1, 1) \times S^1. \end{aligned}$$

Denote by $M(f) = L(2r+1, 1) \times_{S^1} (S^1 \times D^2) \cup_f L(2r+1, 1) \times_{S^1} (D^2 \times S^1)$ the manifold which we obtain from $L(2r+1, 1) \times_{S^1} (S^1 \times D^2)$ and $L(2r+1, 1) \times_{S^1} (D^2 \times S^1)$ by identifying their boundaries under a $U(2)$ -equivariant diffeomorphism $f: L(2r+1, 1) \times S^1 \rightarrow L(2r+1, 1) \times S^1$. For any $U(2)$ -equivariant diffeomorphism $f: L(2r+1, 1) \times S^1 \rightarrow L(2r+1, 1) \times S^1$, $M(f)$ is $U(2)$ -equivariantly diffeomorphic to $M(\text{id})$ where id is the identity map of $L(2r+1, 1) \times S^1$. In fact, for every $U(2)$ -equivariant diffeomorphism f of $L(2r+1, 1) \times S^1$, there exist a smooth map $\alpha: S^1 \rightarrow L(2r+1, 1)$ and a diffeomorphism $\beta: S^1 \rightarrow S^1$ such that

$$f\left(A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \zeta\right) = (A \cdot \alpha(\zeta), \beta(\zeta)),$$

where $A \in U(2)$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in L(2r+1, 1)$ and $\zeta \in S^1$. By means of f we define a $U(2)$ -equivariant diffeomorphism $F: L(2r+1, 1) \times_{S^1} (S^1 \times D^2) \rightarrow L(2r+1, 1) \times_{S^1} (S^1 \times D^2)$ by

$$F\left(\left[A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}, (\xi, t\eta)\right]\right) = [A \cdot \alpha(\xi), (\beta(\xi), t\eta)],$$

where $A \in U(2)$, $0 \leq t \leq 1$ and $|\xi| = 1 = |\eta|$. F induces a $U(2)$ -equivariant diffeomorphism of $M(\text{id})$ onto $M(f)$.

Therefore for any $U(2)$ -equivariant diffeomorphism f of $L(2r+1, 1) \times S^1$, $M(f)$ is $U(2)$ -equivariantly diffeomorphic to $L(2r+1, 1) \times_{S^1} S^3$. Consequently, $M^s = U \cup E$ is $U(2)$ -equivariantly diffeomorphic to $L(2r+1, 1) \times_{S^1} S^3$. Now suppose that $a, b, c, d, A, B, C, D \in \mathbf{Z}$ satisfy the condition of (III) and $AD - BC = 0 = (A - B)(X - Y)$. Then $b = \pm 1$. If $X - Y = 0$ (resp. $A - B = 0$), then $X = Y = 0$ (resp. $A = B = 0$) and for some $r \in \mathbf{Z}$ we have

$$\begin{pmatrix} A \\ B \end{pmatrix} = \pm \begin{pmatrix} r \\ r+1 \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} X \\ Y \end{pmatrix} = \pm \begin{pmatrix} r \\ r+1 \end{pmatrix}\right).$$

Hence $L(b, d)$ is T^2 -equivariantly diffeomorphic to S^3 admitting the following T^2 -action

$$\left(\begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\right) \mapsto \begin{pmatrix} z \\ w\xi^{r+1}\eta^{-r} \end{pmatrix}.$$

Consequently under this situation

$$M \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\} = U(2) \times_{T^2} S^3.$$

On the other hand, $U(2) \times_{T^2} S^3$ is $U(2)$ -equivariantly diffeomorphic to $L(2r+1, 1) \times_{S^1} S^3$ by

$$[A, q] \mapsto \left[A \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\det A)^{-\tau/(2r+1)}, q \right],$$

where $A \in U(2)$ and $q \in S^3$. Therefore M^5 is $U(2)$ -equivariantly diffeomorphic to the $U(2)$ -manifold

$$M \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\} = U(2) \times_{T^2} L(b, d)$$

of (III) with $AD - BC = 0 = (A - B)(X - Y)$ for some $a, b, c, d, A, B, C, D \in \mathbf{Z}$.

(3.5) Suppose that each isotropy subgroup of ϕ is 1-dimensional, that is, for some r the identity component of each isotropy subgroup is conjugate to $G_{r,r+1}^1$. Then it follows from [7, Lemma 4.2] that

$$\begin{aligned} M^5 &\cong (U(2) \times F(G_{r,r+1}^1, M^5)) / N_{U(2)}(G_{r,r+1}^1) \cong U(2) \times_{T^2} F(G_{r,r+1}^1, M^5) \\ &\cong ((U(2)/G_{r,r+1}^1) \times F(G_{r,r+1}^1, M^5)) / (T^2/G_{r,r+1}^1) \cong L(2r+1, 1) \times_{S^2} F(G_{r,r+1}^1, M^5), \end{aligned}$$

where $F(G_{r,r+1}^1, M^5)$ is the closed 3-dimensional submanifold of all points of M^5 whose isotropy subgroups contain $G_{r,r+1}^1$, S^1 acts on $L(2r+1, 1)$ by

$$\left(\begin{pmatrix} z \\ w \end{pmatrix}, \tau \right) \mapsto \left[\begin{pmatrix} z \\ w \end{pmatrix} \tau^{1/(2r+1)} \right],$$

on $F(G_{r,r+1}^1, M^5)$ almost freely (i.e., each isotropy subgroup is discrete) and $U(2)$ acts on $L(2r+1, 1)$ by

$$\left(A, \begin{pmatrix} z \\ w \end{pmatrix} \right) \mapsto \left[A \begin{pmatrix} z \\ w \end{pmatrix} (\det A)^{-\tau/(2r+1)} \right].$$

Now we investigate $F^3 = F(G_{r,r+1}^1, M^5)$. The above S^1 -action on F^3 is without fixed points and effective since each principal isotropy subgroup of ϕ is conjugate to $G_{r,r+1}^1$. The orbit space $M^5/U(2)$ is homeomorphic to the orbit space F^3/S^1 . Since M^5 is simply connected, by [3, II 6.3. Corollary] $M^5/U(2) \cong F^3/S^1$ is a simply connected compact topological 2-manifold. Hence it is D^2 or S^2 . It follows from [3, IV 3.12. Theorem and IV 8.3. Proposition] that $F^3/S^1 \cong M^5/U(2) \cong S^2$. Therefore by [6,

Theorems 2 and 4], F^3 is S^1 -equivariantly diffeomorphic to a 3-dimensional lens space admitting an effective S^1 -action with at most two exceptional orbits. Let Z_{m_1}, Z_{m_2} ($m_1 \neq 0, m_2 \neq 0$) be the two exceptional isotropy subgroups, where $Z_{m_s} = \{\omega \in C; \omega^{m_s} = 1\}$ ($s = 1, 2$). For each exceptional orbit S^1/Z_{m_s} , $s = 1, 2$, there exists an invariant closed tubular neighborhood U_s such that $F^3 = U_1 \cup U_2$, $U_1 \cap U_2 = \partial U_1 = \partial U_2$. Moreover U_s is a compact connected smooth manifold on which S^1 acts smoothly and is S^1 -equivariantly diffeomorphic to a twisted product $S^1 \times D^2$, where Z_{m_s} acts on 2-disk D^2 by $\sigma_s(\omega, w) = \omega^{n_s} w$ ($(m_s, n_s) = 1$). Define an S^1 -action $\tilde{\sigma}_s$ on $S^1 \times D^2$ by

$$\tilde{\sigma}_s(\tau, (z, w)) = (\tau^{m_s} z, \tau^{n_s} w).$$

Moreover, define the map $\hat{\sigma}_s$ of $S^1 \times D^2$ onto $S^1 \times D^2$ by

$$\hat{\sigma}_s([\xi, \eta]) = (\xi^{m_s}, \xi^{n_s} \eta).$$

Then $\hat{\sigma}_s$ is an S^1 -equivariant diffeomorphism. Hence $(U_s, \partial U_s)$ is S^1 -equivariantly diffeomorphic to $(S^1 \times D^2, S^1 \times S^1)$. Moreover the manifold F^3 is S^1 -equivariantly diffeomorphic to $S^1 \times D^2 \cup_f S^1 \times D^2$ where $f: S^1 \times S^1 \rightarrow S^1 \times S^1$ is an S^1 -equivariant diffeomorphism such that the following diagram is commutative:

$$\begin{array}{ccc} S^1 \times (S^1 \times S^1) & \xrightarrow{\tilde{\sigma}_1} & S^1 \times S^1 \\ \downarrow 1_{S^1} \times f & & \downarrow f \\ S^1 \times (S^1 \times S^1) & \xrightarrow{\tilde{\sigma}_2} & S^1 \times S^1. \end{array}$$

Now we must study the map f . Define another S^1 -action ρ on $S^1 \times S^1$ by $\rho(\tau, (z, w)) = (\tau z, w)$. Then every S^1 -equivariant diffeomorphism of $S^1 \times S^1$ admitting the S^1 -action ρ onto itself is S^1 -diffeotopic to the map $(z, w) \mapsto (z w^k, w^b)$ for some k and $\delta = \pm 1$. Define a diffeomorphism $\bar{\sigma}_s: S^1 \times S^1 \rightarrow S^1 \times S^1$ by $\bar{\sigma}_s(z, w) = (z^{m_s} w^{p_s}, z^{n_s} w^{q_s})$, where $m_s q_s - n_s p_s = 1$. Then $\bar{\sigma}_s \circ (1_{S^1} \times \bar{\sigma}_s) = \bar{\sigma}_s \circ \rho$ and $f_0 = \bar{\sigma}_2 \circ f \circ \bar{\sigma}_1^{-1}$ is an S^1 -equivariant diffeomorphism of $S^1 \times S^1$ admitting the S^1 -action ρ onto itself. Hence for some k, f_0 is S^1 -diffeotopic to the S^1 -equivariant diffeomorphism $g_0(z, w) = (z w^k, w^b)$, where $\delta = \pm 1$. Therefore f is S^1 -diffeotopic to the S^1 -equivariant diffeomorphism g with $\tilde{\sigma}_2 \circ (1_{S^1} \times g) = g \circ \tilde{\sigma}_1$ defined by

$$g(z, w) = (z^a w^b, z^c w^d),$$

where $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} m_1 & n_1 \\ p_1 & q_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ k & \delta \end{pmatrix} \begin{pmatrix} m_2 & n_2 \\ p_2 & q_2 \end{pmatrix}$ and the following diagram is commutative

$$\begin{array}{ccc}
S^1 \times S^1 & \xrightarrow{f_0 \cong g_0} & S^1 \times S^1 \\
\downarrow \bar{\sigma}_1 & & \downarrow \bar{\sigma}_2 \\
S^1 \times D^2 \supset S^1 \times S^1 & \xrightarrow{f \cong g} & S^1 \times S^1 \subset S^1 \times D^2 .
\end{array}$$

By Lemma 1.9, $S^1 \times D^2 \cup_f S^1 \times D^2$ is S^1 -equivariantly diffeomorphic to $S^1 \times D^2 \cup_g S^1 \times D^2$. Therefore F^3 is the 3-dimensional lens space $L(b, d)$ admitting the following S^1 -action:

$$\begin{aligned}
\left(\zeta, \begin{bmatrix} Z \\ W \end{bmatrix} \right) &\mapsto \begin{bmatrix} Zz^{-m_2} \\ Wz^{-m_1\delta} \end{bmatrix} & (b \neq 0, z^b = \zeta, ad - bc = \delta) \\
\left(\zeta, \begin{pmatrix} z \\ w \end{pmatrix} \right) &\mapsto \begin{pmatrix} z\zeta^{m_2} \\ w\zeta^{n_2} \end{pmatrix} & (b = 0) .
\end{aligned}$$

Put

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} = - \begin{pmatrix} r \\ r+1 \end{pmatrix} (m_1, n_1), \quad \begin{pmatrix} X & U \\ Y & V \end{pmatrix} = - \begin{pmatrix} r \\ r+1 \end{pmatrix} (m_2, n_2) .$$

Then

$$\begin{pmatrix} X & U \\ Y & V \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad AD - BC = 0$$

and the above S^1 -action on $L(b, d)$ induces the T^2 -action on $L(b, d)$ in Part (III) of the main theorem described in the introduction. Therefore if each isotropy subgroup of ϕ is of dimension 1, then M^5 is $U(2)$ -equivariantly diffeomorphic to

$$M \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} A & C \\ B & D \end{pmatrix} \right\} = U(2) \times_{r^2} L(b, d)$$

for some $a, b, c, d, A, B, C, D \in \mathbf{Z}$ with $ad - bc = \pm 1$, $AD - BC = 0$, $A - B \neq 0$ and $(A - B)a + (C - D)b \neq 0$. Moreover by Lemma 1.8, $(A + B, b) = 1$ and $(A - B, C - D) = 1$, since $M^5 = U(2) \times_{r^2} L(b, d)$ is simply connected and the $U(2)$ -action on M^5 is effective. Then $H = G_{r, r+1}^1$ where

$$r = - \begin{vmatrix} A & C \\ p_1 & q_1 \end{vmatrix}, \quad \begin{vmatrix} A - B & C - D \\ p_1 & q_1 \end{vmatrix} = 1$$

for some $p_1, q_1 \in \mathbf{Z}$. Moreover there are at most two non-principal orbits and they are exceptional orbits. Consequently in this case, the $U(2)$ -manifold M^5 is $U(2)$ -equivariantly diffeomorphic to the $U(2)$ -manifold of (III) with $AD - BC = 0 \neq (A - B)(X - Y)$ for some $a, b, c, d, A, B, C, D \in \mathbf{Z}$.

Here we complete the proof of the main theorem.

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