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Simply connected K-contact and Sasakian manifolds of dimension 7

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Abstract We construct a compact simply connected 7-dimensional manifold admitting a K-contact structure but not a Sasakian structure. We also study rational homotopy properties of such manifolds, proving in particular that a simply connected 7-dimensional Sasakian manifold has vanishing cup product $H^2 \times H^2 \to H^4$ and that it is formal if and only if all its triple Massey products vanish.

Keywords Sasakian manifold · Contact structure · Symplectic manifold · Formality

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1 Introduction

Sasakian geometry has become an important and active subject, especially after the appearance of the fundamental treatise of Boyer and Galicki [3]. Chapter 7 of this book contains an extended discussion of the topological problems in the theory of Sasakian, and, more generally, K-contact manifolds. These are odd-dimensional analogues to Kähler and symplectic manifolds, respectively.

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The precise definition is as follows. Let (M, η) be a co-oriented contact manifold with a contact form $\eta \in \Omega^1(M)$, that is $\eta \wedge (d\eta)^n > 0$ everywhere, with $\dim M = 2n+1$. We say that (M, η) is *K-contact* if there is an endomorphism Φ of TM such that:

- $\Phi^2 = -\operatorname{Id} + \xi \otimes \eta$, where ξ is the Reeb vector field of η (that is $i_\xi \eta = 1$, $i_\xi(d\eta) = 0$),
- the contact form η is compatible with Φ in the sense that $d\eta(\Phi X, \Phi Y) = d\eta(X, Y)$, for all vector fields X, Y,
- $d\eta(\Phi X, X) > 0$ for all nonzero $X \in \ker \eta$, and
- the Reeb field ξ is Killing with respect to the Riemannian metric defined by the formula $g(X, Y) = d\eta(\Phi X, Y) + \eta(X)\eta(Y)$.

In other words, the endomorphism Φ defines a complex structure on $\mathcal{D} = \ker \eta$ compatible with $d\eta$, hence Φ is orthogonal with respect to the metric $g|_{\mathcal{D}}$. By definition, the Reeb vector field ξ is orthogonal to $\ker \eta$, and it is a Killing vector field.

Let (M, η, g, Φ) be a K-contact manifold. Consider the contact cone as the Riemannian manifold $C(M) = (M \times \mathbb{R}^{>0}, t^2 g + dt^2)$. One defines the almost complex structure I on $C(M)$ by:

- $I(X) = \Phi(X)$ on $\ker \eta$,
- $I(\xi) = t\frac{\partial}{\partial t}$, $I\left(t\frac{\partial}{\partial t}\right) = -\xi$, for the Killing vector field ξ of η.

We say that (M, η, Φ, g, I) is *Sasakian* if I is integrable. Thus, by definition, any Sasakian manifold is K-contact.

There are several topological obstructions to the existence of the aforementioned structures on a compact manifold M of dimension $2n+1$, for example:

(1) the evenness of the pth Betti number for p odd with $1 \le p \le n$, of a Sasakian manifold,
(2) some torsion obstructions in dimension 5 discovered by Kollár [17],
(3) the fundamental group of Sasakian manifolds are special,
(4) the cohomology algebra of a Sasakian manifold satisfies the hard Lefschetz property,
(5) formality properties of the rational homotopy type.

An early result [13] establishes that the odd Betti numbers up to the middle dimension of Sasakian manifolds must be even. The parity of b_1 was used to produce the first examples of K-contact manifolds with no Sasakian structure [3, example 7.4.16]. More refined tools are needed in the case of even Betti numbers. The cohomology algebra of a Sasakian manifold satisfies a hard Lefschetz property [4]. Using it examples of K-contact non-Sasakian manifolds are produced in [5] in dimensions 5 and 7. These examples are nilmanifolds with even Betti numbers, so in particular they are not simply connected.

The fundamental group can also be used to construct K-contact non-Sasakian manifolds. Fundamental groups of Sasakian manifolds are called Sasaki groups, and satisfy strong restrictions. Using this it is possible to construct (non-simply connected) compact manifolds which are K-contact but not Sasakian [8].

When one moves to the case of simply connected manifolds, K-contact non-Sasakian examples of any dimension ≥ 9 were constructed in [16] using the evenness of the third Betti number of a compact Sasakian manifold. Alternatively, using the hard Lefschetz property for Sasakian manifolds there are examples [19] of simply connected K-contact non-Sasakian manifolds of any dimension ≥ 9.

In [24] and in [2] the rational homotopy type of Sasakian manifolds is studied. In [2] it is proved that all higher order Massey products for simply connected Sasakian manifolds vanish, although there are Sasakian manifolds with non-vanishing triple Massey products. This yields examples of simply connected K-contact non-Sasakian manifolds in dimensions ≥ 17. However, Massey products are not suitable for the analysis of lower dimensional manifolds.

Hence, the problem of the existence of simply connected K-contact non-Sasakian compact manifolds (open problem 7.4.1 in [3]) is still open in dimensions 5 and 7. Dimension 5 is the most difficult one, and it is treated in [3] separately. Here one has to use the obstructions of [17] which are very subtle torsion obstructions associated to the classification of Kähler surfaces. By definition, a simply connected compact oriented 5-manifold is called a *Smale–Barden manifold*. These manifolds are classified topologically by $H_2(M, \mathbb{Z})$ and the second Stiefel–Whitney class. Chapter 10 of the book by Boyer and Galicki is devoted to a description of some Smale–Barden manifolds which carry Sasakian structures. The following problem is still open (open problem 10.2.1 in [3]).

Do there exist Smale–Barden manifolds which carry K-contact but do not carry Sasakian structures?

In this note we solve the described problem in the easier case of dimension 7 (the solution is still possible by means of homotopy theory combined with symplectic surgery).

Theorem 1 *There exist* 7*-dimensional compact simply connected K-contact manifolds which do not admit a Sasakian structure.*

We then turn around to the study of the rational homotopy type of K-contact and Sasakian simply connected manifolds of dimension 7. In particular, we prove:

Corollary 2 *Let M be a simply connected compact K-contact* 7*-dimensional manifold. Suppose that the cup product map $H^2(M) \times H^2(M) \longrightarrow H^4(M)$ is non-zero. Then M does not admit a Sasakian structure.*

Formality is a very useful rational homotopy property that has been widely used to distinguish between symplectic and Kähler manifolds [21] (see Sect. 6 for definitions and details). Simply connected compact manifolds of dimension ≤ 6 are always formal, so formality becomes interesting in dimension 7. We study this property in detail giving a precise characterisation for Sasakian manifolds (see Theorem 15). In particular, we have the following:

Corollary 3 *Let M be a simply connected compact Sasakian* 7*-dimensional manifold. Then M is formal if and only if all triple Massey products are zero.*

2 Gompf–Cavalcanti manifold

Let (M, ω) be a symplectic manifold of dimension $2n$. For every $0 \leq k \leq n$, we define the Lefschetz map as $L_\omega : H^{n-k}(M) \to H^{n+k}(M)$, $L_\omega([\beta]) = [\beta \wedge \omega^{n-k}]$. We say that M satisfies the hard Lefschetz property if L_ω is an isomorphism for every $0 \leq k \leq n$.

Proposition 4 *There exists a simply connected* 6*-dimensional symplectic manifold* (M, ω) *such that* $\dim \ker (L_\omega : H^2(M) \to H^4(M))$ *is odd.*

Proof Gompf constructs in [14, Theorem 7.1] an example of a simply connected 6-dimensional symplectic manifold (M, ω) which does not satisfy the hard Lefschetz property, that is, the Lefschetz map $L_\omega : H^2(M) \to H^4(M)$ is not an isomorphism. If $\dim \ker L_\omega$ is already odd then we have finished.

So let us suppose that $\dim \ker L_\omega$ is even. Take a cohomology class $a \in H^2(M)$ which belongs to the kernel of L_ω. In [7, Lemma 2.4] Cavalcanti proves that given a symplectic manifold (M, ω) as above satisfying that there exists a symplectic surface $S \hookrightarrow M$ with

$\langle a, [S]\rangle \neq 0$, then there is another 6-dimensional symplectic manifold (M', ω') (the symplectic blow-up of M along S) satisfying

$$\dim\ker\left(L_{\omega'} : H^2\left(M'\right) \to H^2\left(M'\right)\right) = \dim\ker\left(L_{\omega} : H^2\left(M\right) \to H^2\left(M\right)\right) - 1.$$

The symplectic blow-up of M along S is constructed in [20], where it is proved that the fundamental groups $\pi_1(M') \cong \pi_1(M)$, hence M' is simply connected. This means that the simply connected 6-dimensional symplectic manifold M' satisfies that $\dim\ker(L_{\omega'} : H^2(M') \to H^4(M'))$ is odd, as required.

It remains to find $S \hookrightarrow M$ as required. The cohomology class a is non-zero, so there is some $b \in H^4(M, \mathbb{Z})$ such that $a \cup b \neq 0$. It is easy to see that there is a rank 2 complex vector bundle $E \to M$ with $c_1(E) = 0$, $c_2(E) = 2b$. This corresponds to the fact that the map $[M, B\,\mathrm{SU}(2)] \to H^4(M, \mathbb{Z})$ given by the second Chern class exhausts $2\,H^4(M, \mathbb{Z})$. A short proof runs as follows: $B\,\mathrm{SU}(2)$ has trivial 3-skeleton and it has $\pi_4(B\,\mathrm{SU}(2)) = \mathbb{Z}$ and $\pi_5(B\,\mathrm{SU}(2)) = \mathbb{Z}_2$. Represent the cohomology class b by a cocycle $\varphi_b : C_4(M) \to \mathbb{Z}$, where $C_4(M)$ is the space of cellular chains. Given b, we define $f : M \to B\,\mathrm{SU}(2)$ inductively on the skeleta (in what follows we denote by $X[k]$ the k-skeleton of a space X). It is trivial on the 3-skeleton of M. For every 4-cell c, we define $f : c \to B\,\mathrm{SU}(2)[4] = S^4$ to have degree $\varphi_b(c) \in \mathbb{Z}$. As M is simply connected there are no 5-cells, so it only remains to attach the 6-cell c_6 to the 4-skeleton $M[4]$. The attaching map is given by some $g : S^5 \to M[4]$. When composed with f, we have a map $f \circ g : S^5 \to B\,\mathrm{SU}(2)$, which gives an *obstruction* element $o_f \in \pi_5(B\,\mathrm{SU}(2)) = \mathbb{Z}_2$. If we multiply b by two, then the map φ_b gets multiplied by 2. The corresponding f is given by composing f with a double cover of S^4, hence the obstruction element is $2o_f = 0$. This means that the map f associated to $2b$ can be extended to $M \to B\,\mathrm{SU}(2)$.

Now take the rank 2 bundle $E \to M$ just constructed. Assume that $[\omega]$ is a an integral cohomology class (which can always be done by perturbing ω slightly to make it rational and multiplying it by a large integer). Let $L \to M$ be the line bundle with first Chern class $c_1(L) = [\omega]$. We now use the asymptotically holomorphic techniques introduced by Donaldson [10]. Specifically, the result of [1] guarantees the existence of a suitable large $k \gg 0$ and a section of $E \otimes L^{\otimes k}$ whose zero locus is a symplectic manifold (an asymptotically holomorphic manifold in fact). This zero locus $S \subset M$ is a symplectic surface, and the cohomology class defined by S is $c_2(E \otimes L^{\otimes k}) = c_2(E) + 2kc_1(L) = 2b + 2k[\omega]$. Therefore $\langle a, [S]\rangle = \langle a, 2b + 2k[\omega]\rangle = 2\langle a, b\rangle \neq 0$, as required. □

We will call the manifold produced in Proposition 4 the *Gompf–Cavalcanti manifold*, because it is constructed by the surgery technique of Gompf [14] together with the symplectic blow-up of Cavalcanti [7]. Note however that this is not a unique one but a family of manifolds.

3 Simply-connected K-contact non-Sasakian manifolds in dimension 7

We show the existence of simply connected compact K-contact non-Sasakian manifolds in dimension 7 by proving that the Boothby–Wang fibration over the Gompf–Cavalcanti manifold is K-contact but non-Sasakian. The existence of a K-contact structure on such fibration is shown in [2] and [16]. For the convenience of the reader we briefly recall these constructions.

Let (B, ω) be a symplectic manifold such that the cohomology class $[\omega]$ is integral. Consider the principal S^1-bundle $\pi : M \to B$ given by the cohomology class $[\omega] \in H^2(B, \mathbb{Z})$. Fibrations of this kind were first considered by Boothby and Wang and are called

Boothby–Wang fibrations. By [25], the total space M carries an S^1-invariant contact form η such that η is a connection form whose curvature is $d\eta = \pi^*\omega$. We have the following result (which is known, compare Theorem 6.1.26 and Proposition 7.1.2 in [3]).

Theorem 5 *Any Boothby–Wang fibration admits a K-contact structure on the total space.*

Proof To prove this theorem we need to introduce a certain tool, called the universal contact moment map in the sense of Lerman [18]. Recall that by our assumption the given contact distribution $\mathcal{D}$ is determined by the contact form η, that is $\mathcal{D} = \ker \eta$. Consider its annihilator $\mathcal{D}^0 \subset T^*M$. Clearly, $\mathcal{D}^0$ is a line bundle, and, therefore, it has two components after the removal of the zero section,

$$\mathcal{D}^0 \backslash M = \mathcal{D}^0_+ \sqcup \mathcal{D}^0_-.$$

Single out one of these components, say $\mathcal{D}^0_+$. Consider the Lie algebra of contact vector fields $\chi(M, \eta)$ on M. It is known that this Lie algebra can be identified with a space of sections of the vector bundle $TM/\mathcal{D}$, that is $\chi(M, \eta) \cong \Gamma(M, TM/\mathcal{D})$. Because of that there is a natural pairing between points of the line bundle $\mathcal{D}^0$ and contact vector fields given by the formula

$$\mathcal{D}^0 \times \chi(M, \eta) \to \mathbb{R}, \quad ((p, \beta), X) \mapsto \langle \beta, X_p \rangle$$

where $\beta \in \mathcal{D}^0$, $X_p \in T_pM$, $p \in M$. Suppose that a Lie algebra $\mathfrak{g}$ acts on M by contact vector fields, that is, there exists a representation $\rho : \mathfrak{g} \to \chi(M, \eta)$. Define the *universal moment map* as the map

$$\psi : \mathcal{D}^0_+ \to \mathfrak{g}^*$$

by the formula

$$\langle \psi(p, \beta), X \rangle = \langle (p, \beta), \rho(X) \rangle = \langle \beta, \rho(X)_p \rangle,$$

where $(p, \beta) \in (\mathcal{D}^0_+)_p \subset T^*_pM$, $X \in \mathfrak{g}$. Now the proof becomes a consequence of the following criterion proved by Lerman [18].

Proposition 6 *A compact co-orientable contact manifold (M, η) admits a K-contact metric g if and only if there exists an action of a torus T on M preserving the contact structure $\mathcal{D}$ and a vector $X \in \mathfrak{t} = L(T)$ so that the function $\langle \psi, X \rangle : \mathcal{D}^0_+ \to \mathbb{R}$ is strictly positive.* □

We continue with the proof of Theorem 5. Consider the S^1-action on M given by the Reeb vector field. Let $\mathfrak{g} = L(S^1)$, and $\rho : \mathfrak{g} \to \chi(M, \eta)$ be the homomorphism of Lie algebras determined by this action (thus, $\mathfrak{g} = \mathfrak{t} = L(S^1)$ in this particular situation). Since the S^1-action is free, $\rho(X)_p \neq 0$ for any $p \in M$. Now,

$$\langle \psi, X \rangle(p, \beta) = \langle \psi(p, \beta), X \rangle = \langle \beta, \rho(X)_p \rangle.$$

Note that in the considered case $\beta \in (\mathcal{D}^0_+)_p \subset T^*_pM$, and, therefore, $\beta \neq 0$. Also (p, β) belongs to the annihilator of the distribution $\mathcal{D}$, while $\rho(X)$ is transversal to $\mathcal{D}$, since it is given by the Reeb vector field. Thus, for any point p, $\langle (p, \beta), \rho(X)_p \rangle \neq 0$. Hence, X may be chosen to yield positive sign everywhere, and we complete the proof by applying Proposition 6. □

Remark 7 Proposition 7.1.2 from [3] is due to Rukimbira. In this work we give a different proof based on Lerman's criterion given by Proposition 6.

The following gives a proof of Theorem 1.

Theorem 8 *The total space of the Boothby–Wang fibration over the Gompf–Cavalcanti manifold is a simply connected K-contact non-Sasakian manifold of dimension* 7.

Proof Let (M,ω) be a Gompf–Cavalcanti manifold as given by Proposition 4. We can assume that $[\omega]$ is an integral cohomology class. Let

$$S^1 \to E \to M \tag{1}$$

be the associated Boothby–Wang fibration. By Theorem 5, E has a K-contact structure. Now we need to prove that E cannot carry Sasakian structures.

There is an exact sequence

$$H_2(M) \to H_1\left(S^1\right) = \mathbb{Z} \to H_1(E) \to 0$$

from the Serre spectral sequence. The map $H_2(M) \to \mathbb{Z}$ is cupping with $[\omega] \in H^2(M)$. Taking $[\omega]$ integral cohomology class and primitive, we have that $H_2(M) \to \mathbb{Z}$ is surjective and hence $H_1(E) = 0$. The long homotopy exact sequence gives $\pi_1(S^1) = \mathbb{Z} \to \pi_1(E) \to \pi_1(M) = 0$, hence $\pi_1(E)$ is abelian. Therefore E is simply connected.

The Gysin exact sequence associated to (1) is

$$H^1(M) = 0 \xrightarrow{\wedge\omega} H^3(M) \longrightarrow H^3(E) \longrightarrow H^2(M) \xrightarrow{\wedge\omega} H^4(M).$$

Thus

$$b^3(E) = b^3(M) + \dim\left(\ker L_\omega : H^2(M) \to H^4(M)\right).$$

As M is a 6-manifold, we have that $b^3(M)$ is even (by Poincaré duality, the intersection pairing on $H^3(M)$ is an antisymmetric non-degenerate bilinear form, hence the dimension of $H^3(M)$ is even). By construction, $\dim(\ker L_\omega : H^2(M) \to H^4(M))$ is odd, so $b^3(E)$ is odd. As the third Betti number of a 7-dimensional Sasakian manifold has to be even [13], we have that E cannot admit a Sasakian structure. □

4 Regularity and quasi-regularity

A Sasakian or a K-contact structure on a compact manifold M is called *quasi-regular* if there is a positive integer δ satisfying the condition that each point of M has a foliated coordinate chart (U,t) with respect to ξ (the coordinate t is in the direction of ξ) such that each leaf for ξ passes through U at most δ times. If $\delta = 1$, then the Sasakian or K-contact structure is called *regular* (see [3, p. 188]).

If N is a Kähler manifold whose Kähler form ω defines an integral cohomology class, then the total space of the circle bundle $S^1 \hookrightarrow M \xrightarrow{\pi} N$ with Euler class $[\omega] \in H^2(M,\mathbb{Z})$ is a regular Sasakian manifold with contact form η such that $d\eta = \pi^*(\omega)$. The converse also holds: if M is a regular Sasakian structure then the space of leaves N is a Kähler manifold, and we have a circle bundle $S^1 \to M \to N$ as above. If M has a quasi-regular Sasakian structure, then the space of leaves N is a Kähler orbifold with cyclic quotient singularities, and there is an orbifold circle bundle $S^1 \to M \to N$ such that the contact form η satisfies $d\eta = \pi^*(\omega)$, where ω is the orbifold Kähler form.

Similar properties hold in the K-contact case, substituting Kähler by symplectic (actually almost Kähler). If M has a regular K-contact structure, then it is the total space of a circle bundle $S^1 \hookrightarrow M \xrightarrow{\pi} N$, where (N,ω) is a symplectic manifold, with Euler class $[\omega] \in H^2(M,\mathbb{Z})$ and $d\eta = \pi^*(\omega)$. If M has a quasi-regular K-contact structure, then it is the total space of an orbifold circle bundle $S^1 \to M \to N$ over a symplectic orbifold N with cyclic quotient singularities and Euler class $[\omega] \in H^2(M, \mathbb{Z})$, where ω is the orbifold symplectic form.

A result of [22] says that if M admits a Sasakian structure, then it admits also a quasi-regular Sasakian structure. This also extends to the case of K-contact structures (Rukimbira [23], see also Theorem 7.1.10 in [3]).

Proposition 9 *If a compact manifold M admits a K-contact structure, it admits a quasi-regular contact structure.*

Proof Assume that there is a K-contact structure on M. By Proposition 6, there exists a torus action $T \times M \to M$ preserving the contact distribution and a vector $X \in \mathfrak{t}$ such that $\langle \psi, X\rangle > 0$. Choose a vector $Y \in \mathfrak{t}$ with the property that it is tangent to an embedding $T' = S^1 \hookrightarrow T$. Clearly, the corresponding fundamental vector field Y_M has the property that the leaves of the corresponding foliations are compact. The set of such Y is dense in $\mathfrak{t}$. Therefore, for vectors Y which are sufficiently close to X, the condition $\langle \psi, Y\rangle > 0$ is still satisfied.

So it remains to see that there is K-contact structure whose Reeb vector field is Y_M, since this will be quasi-regular because the leaves of the characteristic foliation are all compact. We follow the notations of the proof of Theorem 5. The action of the circle T' on M preserves $\mathcal{D}$, hence the lifted action of T' on T^*M preserves $\mathcal{D}^0$. Since T' is connected, the lifted action preserves the connected component $\mathcal{D}^0_+$ as well. It follows that for any 1-form β on M with $\ker \beta = \mathcal{D}$, the average $\bar\beta$ of β over T' still satisfies $\ker \bar\beta = \mathcal{D}$. So $\bar\beta \in \mathcal{D}^0$. Now use the formula [derived in [18], formulae (3.4) and (3.5)],

$$i_{Y_M}\bar\beta = \left\langle \psi \circ \bar\beta, Y\right\rangle > 0.$$

Now let

$$\eta = \left(\langle \psi \circ \bar\beta, Y\rangle\right)^{-1} \bar\beta,$$

which satisfies $i_{Y_M}\eta = 1$. Hence η defines the contact structure and Y_M is its Killing vector field. Then $TM = \mathcal{D} \oplus \langle Y_M\rangle$, and the splitting is T'-invariant. We use the splitting to define the desired Riemannian metric g. Declare $\mathcal{D}$ and $\langle Y_M\rangle$ to be orthogonal and define $g(Y_M, Y_M) = 1$, thus Y_M becomes a unit normal to $\mathcal{D}$. On $\mathcal{D}$ we choose a T'-invariant complex structure compatible with $d\eta|_{\mathcal{D}}$ and define $g|_{\mathcal{D}}(\cdot,\cdot) = d\eta|_{\mathcal{D}}(\cdot, \Phi\cdot)$. Then g is T'-invariant and hence $L_{Y_M} g = 0$. Thus we have obtained a K-contact structure on M. □

5 Minimal models and formality

Now we want to analyse the rational homotopy type of K-contact and Sasakian simply connected 7-manifolds, in particular the property of formality. Simply connected compact manifolds of dimension ≤ 6 are always formal [12], so dimension 7 is the first instance in which formality is an issue.

We start by reviewing concepts about minimal models and formality from [11,12,15]. A *differential graded algebra* (or DGA) over the real numbers $\mathbb{R}$, is a pair (A, d) consisting of a graded commutative algebra $A = \oplus_{k\geq 0} A^k$ over $\mathbb{R}$, and a differential d satisfying the Leibnitz rule $d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db)$, where $|a|$ is the degree of a. Given a differential graded commutative algebra (A, d), we denote its cohomology by $H^*(A)$. The cohomology of a differential graded algebra $H^*(A)$ is naturally a DGA with the product inherited from that on A and with the differential being identically zero. The DGA (A, d) is *connected* if $H^0(A) = \mathbb{R}$, and A is *1-connected* if, in addition, $H^1(A) = 0$. Henceforth we shall assume that all our DGAs are connected. In our context, the main example of DGA is the de Rham complex $(\Omega^*(M), d)$ of a connected differentiable manifold M, where d is the exterior differential.

Morphisms between DGAs are required to preserve the degree and to commute with the differential. A morphism $f : (A, d) \to (B, d)$ is a quasi-isomorphism if the map induced in cohomology $f^* : H^*(A, d) \to H^*(B, d)$ is an isomorphism. Quasi-isomorphism produces an equivalence relation in the category of DGAs.

A DGA $(\mathcal{M}, d)$ is *minimal* if

(1) $\mathcal{M}$ is free as an algebra, that is, $\mathcal{M}$ is the free algebra $\bigwedge V$ over a graded vector space $V = \bigoplus_i V^i$, and
(2) there is a collection of generators $\{x_\tau\}_{\tau \in I}$ indexed by some well ordered set I, such that $|x_\mu| \le |x_\tau|$ if $\mu < \tau$ and each dx_τ is expressed in terms of preceding x_μ, $\mu < \tau$.

We say that $(\bigwedge V, d)$ is a *minimal model* of the differential graded commutative algebra (A, d) if $(\bigwedge V, d)$ is minimal and there exists a quasi-isomorphism $\rho\colon (\bigwedge V, d) \longrightarrow (A, d)$. A connected DGA (A, d) has a minimal model unique up to isomorphism. For 1-connected DGAs, this is proved in [9]. In this case, the minimal model satisfies that $V^1 = 0$ and the condition (2) above is equivalent to dx_τ not having a linear part.

A *minimal model* of a connected differentiable manifold M is a minimal model $(\bigwedge V, d)$ for the de Rham complex $(\Omega^*(M), d)$ of differential forms on M. If M is a simply connected manifold, then the dual of the real homotopy vector space $\pi_i(M) \otimes \mathbb{R}$ is isomorphic to V^i for any i (see [9]).

A *model* of a DGA (A, d) is any DGA (B, d) with the same minimal model (that is, they are equivalent with respect to the equivalence relation determined by the quasi-isomorphisms).

A minimal algebra $(\bigwedge V, d)$ is called *formal* if there exists a morphism of differential algebras $\psi\colon (\bigwedge V, d) \longrightarrow (H^*(\bigwedge V), 0)$ inducing the identity map on cohomology. Also a differentiable manifold M is called formal if its minimal model is formal. The formality of a minimal algebra is characterized as follows.

Proposition 10 [9] *A minimal algebra $(\bigwedge V, d)$ is formal if and only if the space V can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$ and d injective on N, such that every closed element in the ideal $I(N)$ in $\bigwedge V$ generated by N is exact.*

This characterization of formality can be weakened using the concept of s-formality introduced in [12].

Definition 11 A minimal algebra $\left(\bigwedge V, d\right)$ is s-formal ($s > 0$) if for each $i \le s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

(1) $d(C^i) = 0$,
(2) the differential map $d\colon N^i \longrightarrow \bigwedge V$ is injective, and
(3) any closed element in the ideal $I_s = I\left(\bigoplus_{i\le s} N^i\right)$, generated by the space $\bigoplus_{i\le s} N^i$ in the free algebra $\bigwedge\left(\bigoplus_{i\le s} V^i\right)$, is exact in $\bigwedge V$.

A differentiable manifold M is s-formal if its minimal model is s-formal. Clearly, if M is formal then M is s-formal, for any $s > 0$. The main result of [12] shows that sometimes the weaker condition of s-formality implies formality.

Theorem 12 [12] *Let M be a connected and orientable compact manifold of dimension $2n$ or $(2n-1)$. Then M is formal if and only if it is $(n-1)$-formal.*

By Corollary 3.3 in [12] a simply connected compact manifold is always 2-formal. Therefore, Theorem 12 implies that any simply connected compact manifold of dimension not more than six is formal. For simply connected 7-dimensional compact manifolds, we have that M is formal if and only if M is 3-formal.

Theorem 12 also holds for compact connected orientable orbifolds, since the proof of [12] only uses that the cohomology $H^*(M)$ is a Poincaré duality algebra.

6 Homotopy properties of simply connected Sasakian 7-manifolds

Proposition 13 *Let M be a simply connected compact K-contact 7-dimensional manifold. Then a model for M is $(H \otimes \bigwedge(x), d)$, where H is the cohomology algebra of a simply connected symplectic 6-dimensional orbifold and $dx = \omega \in H^2$ is the class of the symplectic form.*

If M is Sasakian, then H is the cohomology algebra of a simply connected 6-dimensional Kähler orbifold.

Proof Suppose M admits a Sasakian structure. Then M admits a quasi-regular Sasakian structure [22]. Therefore, there is an orbifold circle bundle $S^1 \to M \to B$, where B is a compact Kähler orbifold of dimension 6, with Euler class given by the Kähler form $\omega \in H^2(B)$. We note that B is simply connected because M is so (see [3, Theorem 4.3.18]). In particular, $S^1 \to M \to B$ is a rational fibration, hence if $\mathcal{M}$ is a model for B, then $\mathcal{M} \otimes \bigwedge(x)$, with $|x| = 1$, $dx = \omega$, is a model for M.

Now B is a simply connected compact orbifold of dimension 6. So it is 2-formal. Theorem 12 also holds for orbifolds, hence B is formal. Therefore $\mathcal{M} \sim (H, 0)$, where $H = H^*(B)$ is the cohomology algebra of B. So a model for M is of the form $(H \otimes \bigwedge(x), d)$, $dx = \omega \in H^2$.

The case where M admits a K-contact structure is similar. By Proposition 9, it admits a quasi-regular K-contact structure. Therefore, M is an orbifold S^1-bundle over a symplectic orbifold $S^1 \to M \to B$, with Euler class given by the orbifold symplectic form $\omega \in H^2(B)$. As above, a model for M is $(H \otimes \bigwedge(x), d)$, $dx = \omega \in H^2$, where $H = H^*(B)$. □

We prove now Corollary 2.

Corollary 14 *Let M be a simply connected compact K-contact 7-dimensional manifold. Suppose that the cup product map $H^2(M) \times H^2(M) \longrightarrow H^4(M)$ is non-zero. Then M does not admit a Sasakian structure.*

Proof Let us compute the cohomology of M from its model $(\mathcal{M}, d) = (H \otimes \bigwedge(x), d)$, $dx = \omega$, where $H = H^*(B)$ is the cohomology algebra of a 6-dimensional simply connected symplectic manifold. Note that $\omega \in H^2$ is a non-zero element with $\omega^3 \in H^6$ generating the top cohomology.

Consider the Lefschetz map $L_\omega : H^* \to H^{*+2}$, and let $K^* = \ker L_\omega$, $Q^* = \operatorname{coker} L_\omega$. We have a (non-canonical) isomorphism $H^i(M) \cong Q^i \oplus K^{i-1}x$. Note that $Q^3 = K^3 = H^3$ and $H^6 = \mathbb{R}$. Also $Q^2 = H^2/\langle\omega\rangle$, and $K^4 = \ker(L_\omega : H^4 \to \mathbb{R})$ are vector spaces of codimension one. We have the following:

$$
\begin{aligned}
H^0(M) &= \mathbb{R}, \\
H^1(M) &= 0, \\
H^2(M) &= Q^2, \\
H^3(M) &= H^3 \oplus K^2 x, \\
H^4(M) &= Q^4 \oplus H^3 x, \\
H^5(M) &= K^4 x, \\
H^6(M) &= 0, \\
H^7(M) &= \langle \omega^3 x \rangle.
\end{aligned}
$$

The map $H^2(M) \times H^2(M) \to H^4(M)$ factors through $Q^2 \times Q^2 \to Q^4$. Hence if it is non-zero then $Q^4 \neq 0$. In particular, the Lefschetz map $L_\omega : H^2 \to H^4$ is not an isomorphism, so B is not hard Lefschetz.

If M admits a Sasakian structure, then there is a quasi-regular fibration $S^1 \to M \to B$ with B satisfying the hard Lefschetz property (it is a Kähler orbifold, so [26] is applicable). This contradicts the above. □

Now we shall study the case of Sasakian 7-manifolds in more detail. Let M be a simply connected compact Sasakian 7-dimensional manifold. Then

$$
\mathcal{M} = \left(H \otimes \bigwedge (x), d \right)
$$

is a model for M, by Proposition 13, where $H = H^*(B)$ is the cohomology algebra of a simply connected compact 6-dimensional Kähler orbifold. This algebra H has a very rich structure:

(1) there is a canonical isomorphism $H^6 \cong \mathbb{R}$, which is given by integration $\int_M : H^6 \to \mathbb{R}$;
(2) H is a Poincaré duality algebra, hence $H^3 \otimes H^3 \to \mathbb{R}$ is an antisymmetric bilinear pairing;
(3) there is a scalar product on each H^j. This is given by the Hodge star operator $* : H^j \to H^{6-j}$ combined with wedge and integration;
(4) H has a Hodge structure, that is, $H \otimes \mathbb{C}$ has a bigrading such that $H^k \otimes \mathbb{C} = \oplus_{p+q=k} H^{p,q}$, where $H^{p,q} = \overline{H^{q,p}}$, and the wedge product respects the bigrading;
(5) there is a distinguished element $\omega \in H^2$ which is in $H^{1,1}$. This defines the space of primitive forms $P = \langle \omega \rangle^\perp \subset H^2$. Hence $H^2 = \langle \omega \rangle \oplus P$. Moreover $P = P^{1,1} \oplus P^{2,0}$, where $P^{1,1} = P \cap H^{1,1}$ and $P^{2,0} = P \cap (H^{2,0} \oplus H^{0,2})$;
(6) the Lefschetz map $L_\omega : H^2 \to H^4$ is an isomorphism. Therefore $H^4 = \langle \omega^2 \rangle \oplus \omega P^{1,1} \oplus \omega P^{2,0}$. By Theorem 3.16 of Chapter V of [27], for $\alpha_1 \in P^{1,1}$ we have $*\alpha_1 = -\alpha_1 \wedge \omega$, for $\alpha_2 \in P^{2,0}$ we have $*\alpha_2 = \alpha_2 \wedge \omega$, and $*\omega = \frac{1}{2}\omega^2$. This implies that $L_\omega : \langle \omega \rangle \oplus P^{1,1} \oplus P^{2,0} \to \langle \omega^2 \rangle \oplus \omega P^{1,1} \oplus \omega P^{2,0}$ is of the form $L_\omega(\alpha) = L_\omega(\alpha_0 + \alpha_1 + \alpha_2) = \frac{1}{2} * \alpha_0 - *\alpha_1 + *\alpha_2$, where $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ is the decomposition according to $H^2 = \langle \omega \rangle \oplus P^{1,1} \oplus P^{2,0}$.

The Lefschetz map $L_\omega : H^2 \to H^4$ is an isomorphism so there is an inverse $L_\omega^{-1} : H^4 \to H^2$. Using it, we can define a map $\mathcal{F} : P \times P \times P \times P \to \mathbb{R}$ by

$$
\mathcal{F}(\alpha, \beta, \gamma, \delta) = \int_M L_\omega^{-1}(\alpha \wedge \beta) \wedge \gamma \wedge \delta.
$$

This clearly factors through $\mathrm{Sym}^2 P \times \mathrm{Sym}^2 P$. Using (5) above, we have the alternative description

$$\mathcal{F}(\alpha, \beta, \gamma, \delta) = 2\langle(\alpha \wedge \beta)_0 , (\gamma \wedge \delta)_0\rangle - \langle(\alpha \wedge \beta)_1 , (\gamma \wedge \delta)_1\rangle + \langle(\alpha \wedge \beta)_2 , (\gamma \wedge \delta)_2\rangle,$$

from where it follows that $\mathcal{F}$ factors as a map $\mathrm{Sym}^2(\mathrm{Sym}^2 P) \to \mathbb{R}$.

Let $\mathcal{K}_M$ be the kernel of the map $\mathrm{Sym}^2(\mathrm{Sym}^2 P) \to \mathrm{Sym}^4 P$. Then we define a map

$$\mathcal{F}_M = \mathcal{F}|_{\mathcal{K}_M} : \mathcal{K}_M \to \mathbb{R}.$$

We have the following result.

Theorem 15 *Let M be a simply connected compact Sasakian 7-dimensional manifold. Then M is formal if and only if $\mathcal{F}_M = 0$.*

Proof Using Theorem 12, we only have to check whether M is 3-formal. For this we have to construct the minimal model $\rho : (\bigwedge V, d) \to \mathcal{M} = (H \otimes \bigwedge(x), d)$ up to degree 3. This is easy:

$$V^1 = 0,$$
$$V^2 = P,$$
$$V^3 = H^3 \oplus N^3, \qquad \text{where } N^3 = \mathrm{Sym}^2 P,$$

where the differential is given by $d = 0$ on P and H^3, and $d : N^3 \to \bigwedge V^2$ is the isomorphism $\mathrm{Sym}^2 P \to \bigwedge^2 P$. The map ρ is given as follows. $\rho : V^2 = P \to \mathcal{M}^2 = H^2$ is defined as the obvious (inclusion) map, $\rho : H^3 \to \mathcal{M}^3 = H^3 \oplus H^2 x$ is the inclusion on the first summand, and $\rho : N^3 = \mathrm{Sym}^2 P \to \mathcal{M}^3 = H^3 \oplus H^2 x$ is defined as $\rho(\alpha \cdot \beta) = L_\omega^{-1}(\alpha \wedge \beta)\, x$. Note that

$$\begin{aligned} d(\rho(\alpha \cdot \beta)) &= L_\omega^{-1}(\alpha \wedge \beta)\, dx = L_\omega^{-1}(\alpha \wedge \beta)\omega \\ &= \alpha \wedge \beta = \rho(\alpha) \wedge \rho(\beta) = \rho(\alpha \wedge \beta) = \rho\left(d(\alpha \cdot \beta)\right), \end{aligned}$$

so ρ is a DGA map. Clearly it is a 3-equivalence (it induces an isomorphism on cohomology up to degree 3 and an inclusion on degree 4).

The space of closed elements is $C^3 = H^3$. Now let us check when the elements $z \in I(N^3)$ with $dz = 0$ satisfy $[\rho(z)] = 0 \in H^*(M)$. The only cases to check is when z has degree 5 or 7. If z has degree 5, then $[\rho(z)] \neq 0$ if and only if there exists some $\beta \in P$, $[\rho(\beta)] \in H^2(M)$, such that $[\rho(z)] \wedge [\rho(\beta)] \neq 0$, by Poincaré duality. Hence $[\rho(z\beta)] \neq 0$. This means that we can restrict to elements z of degree 7, that is $z \in N^3 \cdot \bigwedge^2 P$.

Let $z \in N^3 \cdot \bigwedge^2 P \cong \mathrm{Sym}^2 P \times \mathrm{Sym}^2 P$. Then the map $d : N^3 \cdot \bigwedge^2 P \to \bigwedge^4 P$ coincides the full symmetrization map $\mathrm{Sym}^2 P \times \mathrm{Sym}^2 P \to \mathrm{Sym}^4 P$. So

$$Z = \ker d|_{I(N^3)^7} = \mathcal{K}_M \oplus \mathrm{Ant}^2\left(\mathrm{Sym}^2 P\right),$$

where $\mathrm{Ant}^2(W)$ denotes the antisymmetric 2-power of a vector space W.

Now we have to study the map

$$\rho : Z \to H^7(M) = H^6 x,$$

and see if this is non-zero. This is given (on the basis elements) by

$$\rho\left((\alpha \cdot \beta) \cdot (\gamma \cdot \delta)\right) = \left(L_\omega^{-1}(\alpha \wedge \beta) \wedge \gamma \wedge \delta\right) x,$$

so $\mathcal{F}_M = \rho|_{\mathcal{K}_M}$. Note that ρ automatically vanishes on $\mathrm{Ant}^2(\mathrm{Sym}^2 P)$, hence M is formal if and only if ρ vanishes on $\mathcal{K}_M$ if and only if $\mathcal{F}_M = 0$.

According to Theorem 12, to check non-formality we have to test the relevant property (2) on *any* splitting $V^3 = C^3 + N'^3$. If we take another splitting $V^3 = C^3 + N'^3$, then the projection $\pi : V^3 \to N^3$ gives an isomorphism $\pi : N'^3 \to N^3$, and so an isomorphism $N'^3 \cdot \mathrm{Sym}^2 P \cong N^3 \cdot \mathrm{Sym}^2 P$. Clearly, $d \circ \pi = d$ on N'^3, so the spaces of cycles correspond $\mathcal{K}' \cong \mathcal{K}$. On the other hand $H^3 \cdot H^2 \cdot H^2 = 0$, so the maps $\rho : \mathcal{K} \to H^6 x$ and $\rho : \mathcal{K}' \to H^6 x$ also correspond. This means that the corresponding $\mathcal{F}$ and $\mathcal{F}'$ coincide under the isomorphism $\mathcal{K} \cong \mathcal{K}'$. This means that the choice of splitting is not relevant. □

This result means that the formality or non-formality of M only depends on the cohomology algebra H. Theorem 15 can be applied to the examples in Section 5.3 of [2]. For instance for $B = \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, we have a simply connected Sasakian 7-manifold which is non-formal (Theorem 12 of [2]). For $B = \mathbb{C}P^3$, we have obviously $P = 0$ and hence M is formal.

The element $\mathcal{F}_M$ of Theorem 15 is the *principal Massey product* defined by Crowley and Nordström [6] for simply connected compact 7-manifolds in general. The principal Massey product is the full obstruction to formality for simply connected compact 7-manifolds.

Now we deduce Corollary 3.

Corollary 16 *Let M be a simply connected compact Sasakian 7-dimensional manifold. Then M is formal if and only if all triple Massey products are zero.*

Proof Suppose that $\mathcal{F}_M \neq 0$. We choose an orthonormal basis for $H^2 = \langle e_0, e_1, \ldots, e_m \rangle$, where $e_0 = \frac{1}{\sqrt{3}}\omega$, $P^{1,1} = \langle e_1, \ldots, e_s \rangle$, $P^{2,0} = \langle e_{s+1}, \ldots, e_m \rangle$. The vector space $\mathcal{K}_M$ is generated by elements of the form

$$a_{ijkl} = \left(e_i \cdot e_j\right) \cdot (e_k \cdot e_l) - (e_k \cdot e_j) \cdot (e_i \cdot e_l),$$

for $0 \le i, j, k, l \le m$ (here, as usual, the dot product means symmetric product). Now define the numbers

$$\lambda_{ijk} = \int_M e_i \wedge e_j \wedge e_k \in \mathbb{R},$$

for $0 \le i, j, k \le m$. Note that these numbers are fully symmetric on i, j, k. Also $\lambda_{000} = \frac{2}{\sqrt{3}}$ and $\lambda_{ij0} = \frac{1}{\sqrt{3}}\varepsilon_i \delta_{ij}$, for $(i, j) \neq (0, 0)$, where $\varepsilon_i = -1$ for $1 \le i \le s$ and $\varepsilon_i = 1$ for $s + 1 \le i \le m$. Then

$$L_\omega^{-1}(e_i \wedge e_j) = 2 * (e_i \wedge e_j)_0 - (e_i \wedge e_j)_1 + (e_i \wedge e_j)_2 = 2\lambda_{ij0} e_0 + \sum_{t>0} \varepsilon_t \lambda_{ijt} e_t.$$

So

$$\mathcal{F}_M((e_i \cdot e_j) \cdot (e_k \cdot e_l)) = 2\lambda_{ij0}\lambda_{kl0} + \sum_{t>0} \varepsilon_t \lambda_{ijt} \lambda_{klt}.$$

Evaluating $\mathcal{F}_M$ on a_{ijkl} gives a set of equations to determine the formality of M. M is non-formal when there exists some a_{ijkl} with $\mathcal{F}_M(a_{ijkl}) \neq 0$. By [6], we have that the triple Massey product $\langle e_i, e_j, e_k \rangle$ is a well-defined element of $H^5(M)$ and it satisfies

$$\mathcal{F}_M(a_{ijkl}) = \langle e_i, e_j, e_k \rangle \cup e_l.$$

So $\langle e_i, e_j, e_k \rangle \neq 0$, as required. □

This result is of relevance since it is not known if for general simply connected compact 7-dimensional manifolds there are obstructions to formality different from triple Bianchi-Massey tensor, as remarked in [6]. It is true that for higher dimensional manifolds, there are obstructions to formality even when all Massey products (triple and higher order) can be zero.

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