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SIMPLY CONNECTED OPEN 3-MANIFOLDS WITH RIGID GENUS ONE ENDS

DENNIS GARITY, DUŠAN REPOVŠ, AND DAVID WRIGHT

Abstract. We construct uncountably many simply connected open 3-manifolds with genus one ends homeomorphic to the Cantor set. Each constructed manifold has the property that any self homeomorphism of the manifold (which necessarily extends to a homeomorphism of the ends) fixes the ends pointwise. These manifolds are complements of rigid generalized Bing-Whitehead (BW) Cantor sets. Previous examples of rigid Cantor sets with simply connected complement in R^3 had infinite genus and it was an open question as to whether finite genus examples existed. The examples here exhibit the minimum possible genus, genus one. These rigid generalized BW Cantor sets are constructed using variable numbers of Bing and Whitehead links. Our previous result with Željko determining when BW Cantor sets are equivalently embedded in R^3 extends to the generalized construction. This characterization is used to prove rigidity and to distinguish the uncountably many examples.

1. Introduction

Each Cantor set C in S^3 has complement an open 3-manifold with end set C. Properties of the embedding of the Cantor set give rise to properties of the corresponding complementary 3-manifold. See [SS12] for an example of this and [BC87] for another setting in which wild Cantor sets in S^3 arise.

In [GRŽ06] new examples were constructed of Cantor sets that were both rigidly embedded and had simply connected complement. Prior to that, only examples having one of these properties were known. The

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Antoine type [Ant20] rigid Cantor sets [Shi74] failed to have simply connected complement. Their rigidity was a consequence of Sher's result [She68] that if two Antoine Cantor sets are equivalently embedded, then the stages in the defining sequences must match up exactly.

The new examples [GRŽ06] had points in the Cantor set of arbitrarily large local genus and thus the complementary 3-manifold did not have finite genus at infinity. The rigidity here was a consequence of a dense set of points in the Cantor set having distinct local genera. The simple connectivity of the complement was a consequence of a Skora type [Sko86] construction for the stages. The question arose as to whether finite genus examples possessing both properties were possible. For any such examples, rigidity would have to be determined in some other manner since local genus could no longer be used.

In this paper, we construct examples of Cantor sets that are rigid, have simply connected complement, and have local (and global) genus one (See Theorem 4.2). These examples exhibit the minimal possible genus. For the corresponding statement concerning simply connected open 3-manifolds of genus one at infinity with rigid end structure, see Theorem 5.1.

In Section 2, we give definitions and the basic results needed for working with generalized Bing-Whitehead (BW) Cantor sets. In the following section, Section 3, we generalize the results from our previous paper with Željko [GRWŽ11] that are needed in the new setting. In Section 4 we state and prove the main results about Cantor sets. In Section 5 we state the corresponding results for the complementary 3-manifolds. Section 6 lists some remaining questions.

2. Preliminaries

Background information. Refer to [DO74], [Wri89] and [GRWŽ11] for results about BW Cantor sets, to [GRŽ05, GRŽ06] for a discussion of Cantor sets in general and rigid Cantor sets, and to [Žel05] for results about local genus of points in Cantor sets and defining sequences for Cantor sets. See [CMZ] for results about dynamics of self-maps of the Cantor set. The bibliographies in these papers contain additional references to results about Cantor sets. For background on Freudenthal compactifications and theory of ends, see [Fre42], [Dic68], and [Sie65].

Two Cantor sets X and Y in S^3 are *equivalent* if there is a self homeomorphism of S^3 taking X to Y. If there is no such homeomorphism,

the Cantor sets are said to be *inequivalent*, or *inequivalently embedded*. A Cantor set C is *rigidly embedded* in S^3 if the only self homeomorphism of C that extends to a homeomorphism of S^3 is the identity. This is in marked contrast to the standard embedding of a Cantor set in S^3 for which every self homeomorphism extends. See Daverman [Dav79] for nonstandard embeddings of Cantor sets with this strong homeomorphism extension property.

Let T be a solid torus. Throughout this paper, we assume that the tori we are working with are *unknotted* in S^3. (Our results and constructions also work in R^3.) A *Bing link* in T is a union of 2 linked solid tori $F_1 \cup F_2$ embedded in T as shown in Figure 1. A *Whitehead link* in T is a solid torus W embedded in T as shown in Figure 1. For background details and terminology, see Wright's paper [Wri89].

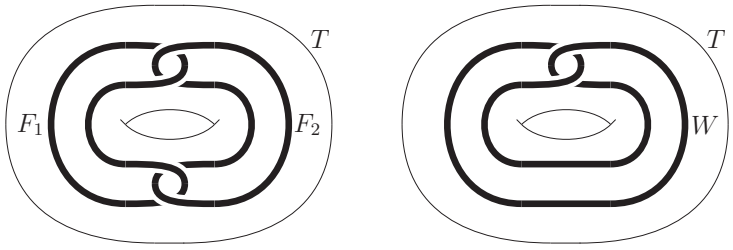

FIGURE 1. Bing and Whitehead links

Construction of (generalized) Bing-Whitehead compacta. For completeness and consistency of notation, we outline the steps in the construction of standard Bing-Whitehead (BW) compacta. Let M_0 be an unknotted torus in S^3, and M_1 be obtained from M_0 by placing a Bing link in M_0. Inductively obtain M_k from M_{k-1} *either* by placing a Bing link in each component of M_{k-1} *or* by placing a Whitehead link in each component of M_{k-1}. Let m_1 denote the number of consecutive Whitehead links placed in M_1 before the second Bing link occurs, and let m_k denote the number of consecutive Whitehead links that occur between the k-th and $(k+1)$st Bing links.

Definition 2.1. The *standard Bing-Whitehead compactum* associated with this construction is defined to be

$$X = \bigcap_{i=0}^{\infty} M_i \quad \text{and is denoted } X = BW(m_1, m_2, \ldots).$$

We also define $M_i, i < 0$, so that M_i is a Whitehead link in M_{i-1} and let X_M^∞ be $\bigcap_i (S^3 \setminus M_i)$. The set X_M^∞ is called *the compactum at infinity* associated with X corresponding to the defining sequence $M = (M_i)$.

This approach leads to the association of a BW compactum with a BW (infinite) labeled binary tree where the vertices correspond to one of the two torus components of a Bing link placed in a torus and the labels on the vertices correspond to the number of Whitehead constructions to perform in the torus associated with that vertex before performing another Bing construction. See Figure 2.2 where $m_1 = w_{1,1} = w_{1,2}$, $m_2 = w_{2,1} = \cdots = w_{2,4}$, and so on.

Note that in this standard BW construction we require that all tori in M_k are obtained *by the same* (Bing or Whitehead) *construction* from the respective tori in M_{k-1}. We also assume that infinitely many of the $M_i, i > 0$, arise from Bing constructions and that infinitely many of them arise from Whitehead constructions.

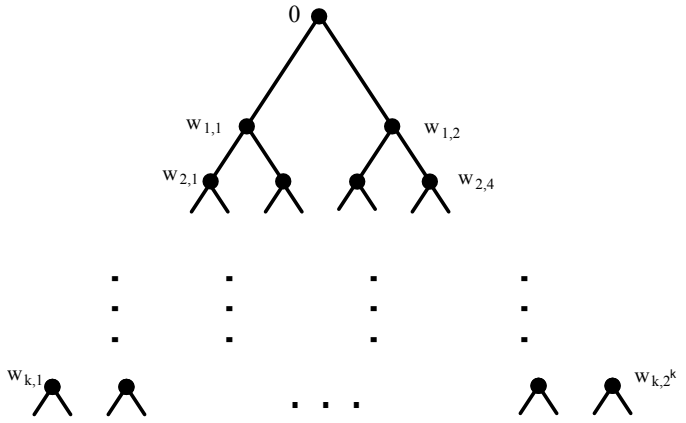

FIGURE 2. Bing-Whitehead tree

In a generalized BW construction we require that all tori in M_k are either obtained by Bing or Whitehead construction from the respective tori in M_{k-1}, *but not necessarily all of the same kind.*

The construction can also be associated with a labeled binary tree (see Figure 2). The top node represents an unknotted solid torus in S^3 denoted by M_0 or $T_{0,1}$. Each of the successive nodes at depth i represents an unknotted torus $T_{i,j}, j = 1 \ldots 2^i$, which is one component of a Bing link in the immediately preceding stage. The numbers $w_{i,j} \in \{0, 1, 2, \ldots\}$ on nodes denote the number of consecutive Whitehead

constructions placed in the torus associated with that node before the next Bing construction occurs. Define $M_i = \bigcup_{j=1}^{2^i} T_{i,j}$.

Definition 2.2. The *generalized BW compactum* associated with this construction is defined to be

$$X = \bigcap_{i=0}^{\infty} M_i \text{ and is denoted } X = BW(z_1, z_2, \ldots)$$

where $z_i = [w_{i,1}, w_{i,2}, \ldots, w_{i,2^i}]$.

As in the standard construction, we also define $M_i, i < 0$, so that M_i is a Whitehead link in M_{i-1}, and let X^∞ be $\bigcap_i (S^3 \setminus M_i)$.

Consider now a component x in a generalized BW compactum X. This component corresponds to a unique ray starting at the root vertex in the associated binary tree. The vertex labels along this ray form a sequence $s(x) = (0, w_{1,x(1)}, w_{2,x(2)} \ldots)$.

Definition 2.3. The sequence $s(x) = (0, w_{1,x(1)}, w_{2,x(2)}, \ldots)$ is called the *Whitehead sequence associated with the component* x with respect to the generalized BW compactum $X = BW(z_1, z_2, \ldots)$.

This will be of interest in the next few sections, particularly when the components consist of single points so that the compactum X is a Cantor set.

Geometric Index. If K is a link in the interior of a solid torus T, then we denote the *geometric index* of K in T by $N(K, T)$. This is the minimum of the cardinality $|K \cap D|$ over all meridional disks D of T. If T is a solid torus and M is a finite union of disjoint solid tori so that $M \subset \mathrm{Int}(T_1)$, then the geometric index $N(M, T)$ of M in T is $N(K, T)$ where K is a core of M. The geometric index of a Bing link in a torus T is 2. The geometric index of a Whitehead link in a torus T is also 2. Let T_0 and T_1 be unknotted solid tori in S^3 with $T_0 \subset \mathrm{Int}(T_1)$. Then ∂T_0 and ∂T_1 are parallel if the manifold $T_1 - \mathrm{Int}(T_0)$ is homeomorphic to $\partial T_0 \times I$ where I is the closed unit interval $[0, 1]$.

Remark 2.4. See Schubert [Sch53] and [GRWŽ11] for the following needed results about geometric index.

- Let T_0 and T_1 be unknotted solid tori in S^3 with $T_0 \subset \mathrm{Int}(T_1)$ and $N(T_0, T_1) = 1$. Then ∂T_0 and ∂T_1 are parallel.

- Let T_0 be a finite union of disjoint tori. Let T_1 and T_2 be tori so that $T_0 \subset \text{int}T_1$ and $T_1 \subset \text{Int}T_2$. Then $N(T_0, T_2) = N(T_0, T_1) \cdot N(T_1, T_2)$.
- Let T be a torus in S^3 and let T_1, T_2 be unknotted tori in T, each of geometric index 0 in T. Then the geometric index of $\cup_{i=1}^{2} T_i$ in T is even.

3. Results on generalized Bing-Whitehead compacta

Constructions yielding Cantor sets.
We are primarily interested in generalized BW compacta that are Cantor sets. At one extreme, if no Whitehead links are used, then the construction can result in a Bing Cantor set [Bin52]. At the other extreme, if no Bing links are used, the resulting compactum has a single component which is the Whitehead continuum and the decomposition of S^3 that shrinks this to a point does not yield S^3.

For standard BW compacta, Ancel and Starbird [AS89], and independently Wright [Wri89], gave a complete characterization of when the construction can be done so as to yield a Cantor set in S^3. What is needed is sufficiently many Bing links, or equivalently sufficiently few Whitehead links in the process. For a generalized BW construction, let $w_i = \max(z_i) = \max[w_{i,1}, w_{i,2}, \ldots, w_{i,2^i}]$. An examination of Appendix A in [Wri89] shows that the argument there proves the following result:

Theorem 3.1. *If* $\sum_{i=1}^{\infty} \frac{1}{2^{\sigma_i}} = \infty$ *where* $\sigma_i = w_1 + w_2 + \cdots + w_i$*, then the decomposition given by points and components of the generalized BW compactum* X *is shrinkable.*

This result will guide us in the next section in the construction associated with Theorem 4.2.

Properties of BW Constructions and Compacta.

Many of the results in [Wri89] and [GRWŽ11] about standard BW constructions carry over directly with the same proofs to the setting of generalized BW constructions. We list some of the results from the 1989 paper, restated in the general setting, that will be useful for our main results.

Lemma 3.2. *Let* M *be a Bing or a Whitehead link in a torus* T*. Let* X *be a generalized BW compactum with defining sequence* $M = (M_i)$ *and* X_M^{∞} *the associated continuum at infinity. Then the following holds:*

- [Wri89, Theorem 4.6] *No sphere in the complement of $X \cup X^\infty$ separates $X \cup X^\infty$.*
- [Wri89, Theorem 4.3] *A loop on the boundary of M_i is essential in the boundary of M_i if and only if it is essential in the complement of $X \cup X^\infty$.*
- [Wri89, Theorem 4.4] *If loops ℓ_1 and ℓ_2 in ∂M_i and ∂M_j respectively, $i \neq j$, are homotopic in the complement of $X \cup X^\infty$, then they are inessential in $X \cup X^\infty$.*
- [DO74], [Wri89, Appendix B]) *The complement of a generalized BW compactum in S^3 is simply connected.*

Matching up Stages. The key to proving rigidity for certain generalized BW Cantor sets is to show that if such a Cantor set C has two defining sequences (M_i) and (N_i) then the components can be made to match up. For completeness, we provide outlines of the proofs of key results: Lemma 3.3, Lemma 3.4, Corollary 3.5 and Theorem 3.6. For more complete details, see [GRWŽ11].

Lemma 3.3. *Let X be a generalized BW compactum with two defining BW sequences (M_i) and (N_j). Let T be a component of some M_i and let T' be the component of M_{i-1} containing T. If T lies in the interior of a component S' of some N_j, and $S' \subset M_0$, then there is a homeomorphism h of S^3, fixed on $X \cup (S^3 - (T' \cup S'))$, so that $h(\partial(T)) \cap \partial(N_{j+1} \cap S') = \emptyset$.*

Proof. After a general position adjustment, we may assume that $\partial(T) \cap \partial(N_{j+1} \cap S')$ consists of a finite collection of simple closed curves. Since a curve on the boundary of T is essential if and only if it is essential in the complement of $X \cup X_M^\infty$ and a curve in the boundary of N_{j+1} is essential if and only if it is essential in the complement of $X \cup X_N^\infty$, we see that a curve on ∂T is trivial on $\partial(T) \cap N_{j+1}$ is trivial on ∂T if and only if it is trivial on ∂N_{j+1}.

Removing Trivial Curves of Intersection: If there are any trivial curves, choose a component S of $S' \cap N_{j+1}$ that contains such a curve in ∂S. Choose an innermost trivial simple closed curve α on ∂S. Since α is innermost, it bounds a disk D' with interior missing ∂T. The curve α also bounds a disk D in ∂T.

The 2-sphere $D \cup D'$ bounds a three-cell in $S' \cap T'$ that contains no points of X. Use this three-cell to push D onto D' and then a little past D' into a collar on the cell by a homeomorphism h of S^3. This

homeomorphism can be chosen to fix X, $S^3 - S'$, and $S^3 - T'$. This has the result that $h(\partial T) \cap \partial(S' \cap N_{j+1})$ has fewer curves of intersection than $\partial T \cap \partial(S' \cap N_{j+1})$, and so that no new curves of intersection with $\partial(N_{j+1})$ are introduced. Continuing this process eventually removes all trivial curves of intersection of $\partial T \cap \partial(S' \cap N_{j+1})$.

Removing Nontrivial Curves of Intersection: At this point, there is at most one component S of $N_{j+1} \cap S'$ for which $\partial T \cap \partial S \neq \emptyset$. These remaining curves of intersection on ∂T must be parallel (p, q) torus curves and the corresponding curves on ∂S must be parallel (s, t) curves. If both p and q are greater than 1, so that the torus curve is a nontrivial knot, then $(s, t) = (p, q)$ or $(s, t) = (q, p)$.

To simplify notation in what remains, we will refer to $h(T)$ as (the new) T. We work towards removing these remaining curves of intersection of the boundaries, so that T is either contained in a component $N_{j+1} \cap S$ or contains the components of $(N_{j+1} \cap S)$. Consider an annulus A on the boundary of S bounded by two adjacent curves of the intersection of ∂S and ∂T. Choose this annulus so that its interior lies in the interior of T. We consider separately all possibilities for how the boundary curves of A lie on S.

I. Curves of intersection on T that are (p, q) curves for $p \geq 2$.

Consider a meridional disk D for T in general position with respect to A so that $D \cap A$ consists of p arcs intersecting the boundary of D in endpoints and of simple closed curves. Figure 3 illustrates a possible situation when $p = 5$ and $q = 3$ and trivial curves of intersection have been removed. The solid unlabeled disks indicate the intersection of the next stage M_{i+1} with D.

The intersection of A with D can be adjusted using cut and paste techniques similar to that used in the trivial curve case so that the end result is intersections as in one of the two cases in Figure 3.

Each of the regions labeled M_i can be shown to be a meridional disk of a torus T_1 that is contained in T. This torus will then be used to push across and remove the intersections of A with T. The intersection of T with S corresponding to A can now be removed by a homeomorphism of S^3 fixed on X and on the complement of a small neighborhood of T that takes A through T_1 to an annulus parallel to A_1 and just outside of T. Inductively, all curves of intersection of T with S can be removed by a homeomorphism of S^3 fixed on X and the complement of a small neighborhood of T.

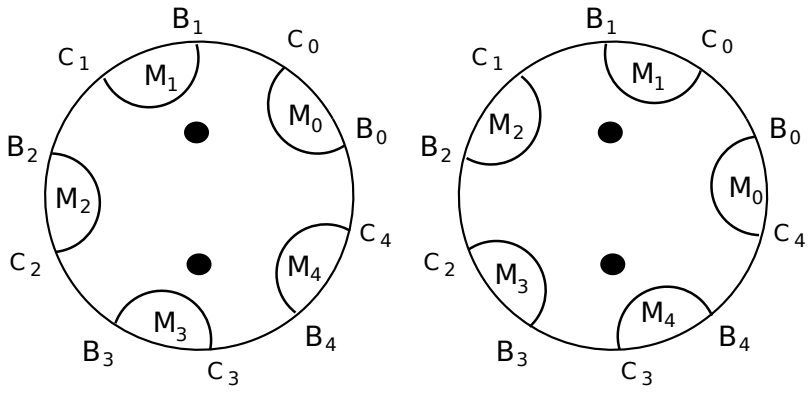

FIGURE 3. Meridional Disk D of T after Adjustment

II. Curves of intersection on S' that are (p, q) curves for $p = 1$.

An argument similar to that above can be used.

III. Curves of intersection on S' that are (p, q) curves for $p = 0$.

In this case the curve is a $(0, q)$ curve for the torus T, but it is a $(q, 0)$ curve for the complementary torus with $q \neq 0$. In this case there is an annulus A on the boundary of T that has its interior in the exterior of T, so that the intersection of A with the boundary of T consists of curves in the intersection of the boundaries of T and S'. This essentially turns the problem inside out, and we can use the previous methods to push A to the interior of T fixed on a slightly shrunken T, all the other components of M_i, and the complement of M_{i-1}.

□

Lemma 3.4. *Assume that X is a generalized BW compactum with two defining BW sequences (M_i) and (N_j). Choose N so that the components of M_N are in the interior of N_0. Let T be a component of M_N. Then there is an integer K and a homeomorphism h of S^3, fixed on $X \cup (S^3 - (N_0 \cup M_{N-1}))$ so that $h(T)$ is interior to components of $N_i, i < K$ and so that $h(T)$ is a component of N_K.*

Proof. Choose an integer L so that the components of N_L are interior to M_N. Inductively apply Lemma 3.3 producing successive homeomorphisms so that the image of $\partial(T)$ is disjoint from $\partial(N_i)$ until a first stage $K \leq L$ is reached where a component of N_K is interior to the image of T.

At this point, there is a homeomorphism h of S^3, fixed on $X \cup (S^3 - N_0) \cup (S^3 - M_{N-1})$ and a component S' of N_{K-1} so that $h(T) \subset \text{Int}(S')$ and so that a component S of $N_K \cap S'$ is interior to $h(T)$.

Case 1. *S is a Whitehead link in S':*

The geometric index of S in S' is two. Thus the geometric index of $h(T)$ in S' is non-zero and the geometric index of S in $h(T)$ is non-zero. So the geometric index of $h(T)$ in S' must be one or two. If it is one, then there is a further homeomorphism of h' of S^3 that is fixed on X and the complement of a small neighborhood of S' that takes $h(T)$ to S'. If the geometric index of $h(T)$ in S' is two, then the geometric index of S in $h(T)$ is one and there is a further homeomorphism of h' of S^3 that is fixed on X and the complement of S' that takes $h(T)$ to S.

Case 2. *S is one component of a Bing link in S'*:

Let R be the other component of the Bing link. If $h(T)$ contains $S \cup R$, the geometric index of $S \cup R$ in S' is two, and so the geometric index of $S \cup R$ in T and the geometric index of $h(T)$ in S' must be nonzero and ≤ 2. The geometric index of each of R and S in $h(T)$ must then be zero since the geometric index of R and S in S' is zero. So the geometric index of $S \cup R$ in $h(T)$ is even and must be two, and the geometric index of $h(T)$ in S' must be one. Then there is a further homeomorphism of h' of S^3 that is fixed on X and the complement of a small neighborhood of S' that takes $h(T)$ to S'.

The other case where $h(T)$ contains only S but not R follows from the fact that there is a homeomorphism of S^3 to itself that takes $T - \text{Int}(S \cup R)$ to itself and takes ∂S to ∂T. This homeomorphism follows from the fact that $S \cup R \cup (S^3 - \text{Int}(T))$ are Borromean Rings.

□

The next result shows that once stages of defining sequences match up, all further stages can be made to match up.

Corollary 3.5. *If x is a component of the generalized BW compactum with two defining sequences (M_i) and (N_j) and with $M_0 = N_0$, then the Whitehead sequence of x with respect to (M_i) is the same as the Whitehead sequence of x with respect to (N_j).*

Proof. It suffices to show that there is a self homeomorphism of $M_0 = N_0$ that is fixed on $\partial M_0 = \partial N_0$ and on X that takes M_i onto N_i for any specified finite number of stages.

Suppose that such a homeomorphism h_n exists that matches the components up through n stages. Let T be a component of N_n. Let M equal $h_n(M_{n+1}) \cap T$ and N equal $N_{n+1} \cap T$. By Lemma 3.3 we may assume that the boundaries of M and N are disjoint.

A geometric index argument shows that M and N both have the same number of components. If M and N both have one component, a geometric index argument similar to Case 1 of Lemma 3.4 shows ∂N is parallel to ∂T or ∂M. But the geometric index of N in T is 2 so ∂M and ∂N are parallel and the boundaries can be matched up with a homeomorphism of T taking ∂M to ∂N fixed on X and ∂T. The same argument works if N lies in M.

Suppose now that M and N both have two components. Then one component of M contains or is contained in one component of N and the other component of M contains or is contained in the other component of N. A geometric index argument similar to Case 2 of Lemma 3.4 can be used to show that ∂M and ∂N are parallel and as before we can get a homeomorphism fixed on X and ∂T taking M to N.

Repeating this argument in each component of N_n gives the homeomorphism h_{n+1}. □

The previous lemmas and corollary were used in [GRWŽ11] to show that if a standard BW Cantor set X in S^3 has two defining sequences (M_i) and (N_j), then past some finite stage N, the BW pattern for X from (M_i) must be identical to the BW pattern for X from (N_j). That is, the BW patterns for the two defining sequences can differ only in a finite number of Whitehead constructions. This is a consequence of the fact that the lemmas above, together with the characterization of when a Cantor set is obtained, show that if the BW patterns do not match up at some same finite stage, then the BW patterns are repeating which contradicts the fact that a Cantor set is obtained.

The situation is not quite so simple for generalized BW constructions. The pattern illustrated in Figure 4 can occur. If A, B, C, and D represent four generalized BW compacta, they can be assembled into a single BW compactum in either of the two ways (and in other ways) indicated in this Figure. On the left, the constructions for A, B, C, and D are performed after doing two initial Bing constructions. On the right, the construction for A is done in one of the two tori of an initial Bing construction. The construction for B is done in one of the two tori

after another Bing construction in the component not corresponding to A. The constructions for C and D are done at the next stage.

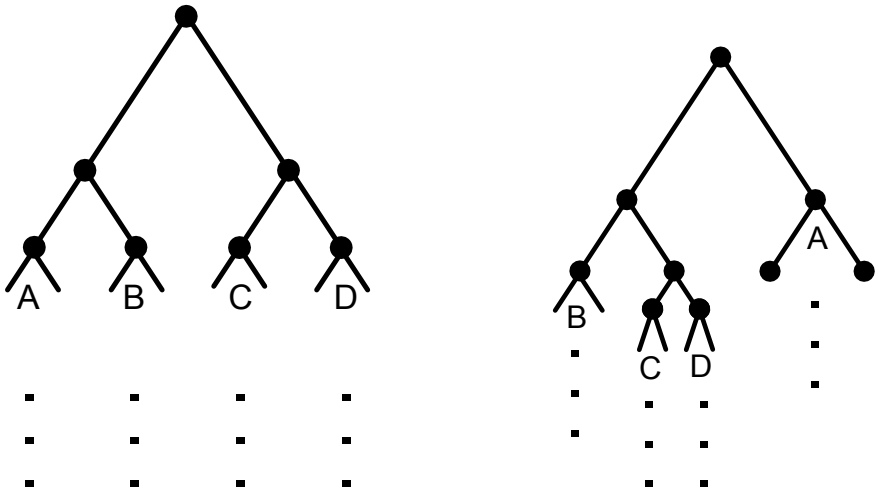

FIGURE 4. Equivalent Generalized Constructions

We can however analyze the following situation that is needed in proving rigidity (the case $X = Y$ of the following).

Theorem 3.6. *Assume that X and Y are generalized BW compacta with defining sequences (M_i) and (N_j). Let c be a component of X. Assume there is a self homeomorphism h of S^3 that takes X to Y and takes c to a component d of Y. Let $s(c)$ be the Whitehead sequence associated with c from (M_i) and let $s(d) = s(h(c))$ be the Whitehead sequence associated with d from (N_j) as in Definition 2.3. Then these two sequences have the same tails. That is, there is a stage n in the first sequence and a stage m in the second sequences such that for all k, $s(c)_{n+k} = s(d)_{m+k}$.*

Proof. By Corollary 3.5, there is a homeomorphism of S^3, fixed on Y that takes a component of some stage in the defining sequence $(h(M_i))$ for $h(c)$ to a component of some stage in the defining sequence (N_j) for d. By viewing these stages as the initial stages of generalized BW constructions, and by applying Corollary 3.5, one sees that the Whitehead sequence $s(h(c))$ with respect to $(h(M_i))$ matches up with the Whitehead sequence $s(d)$ with respect to (N_j) past this stage. But the Whitehead sequence for $s(h(c))$ with respect to $(h(M_i))$ is the same

as the Whitehead sequence $s(c)$ with respect to (M_i). The assertion follows. □

4. Main Results

Construction 4.1. For every increasing sequence $s = (s_1.s_2, s_3, \dots)$ of positive integers, we describe a construction of a BW compactum $C(s)$. The compactum $C(s)$ will be $BW(z_1, z_2, \dots)$ for a specific sequence $(z_1, z_2, \dots)$ associated with the sequence s. A subsequence $(n_1, n_2 \dots)$ of the sequence of positive integers will be described inductively. For i not in this subsequence, z_i is an ordered 2^i-tuple of zeros. For $i = n_j$ in this subsequence, z_{n_1} consists of the first 2^{n_1} terms in s, z_{n_2} consists of the next 2^{n_2} terms in s, and so forth. As before, let $w_i = \max(z_i)$ and $\sigma_i = w_1 + w_2 + \cdots w_i$. To make sure that we can choose $C(s)$ to be a Cantor set, we just choose the n_i appropriately as in Theorem 3.1. Choose $n_1 = 1$, and for $i > 1$ choose $n_{i+1} = 2^{\sigma_i} + n_i$. See Figure 5.

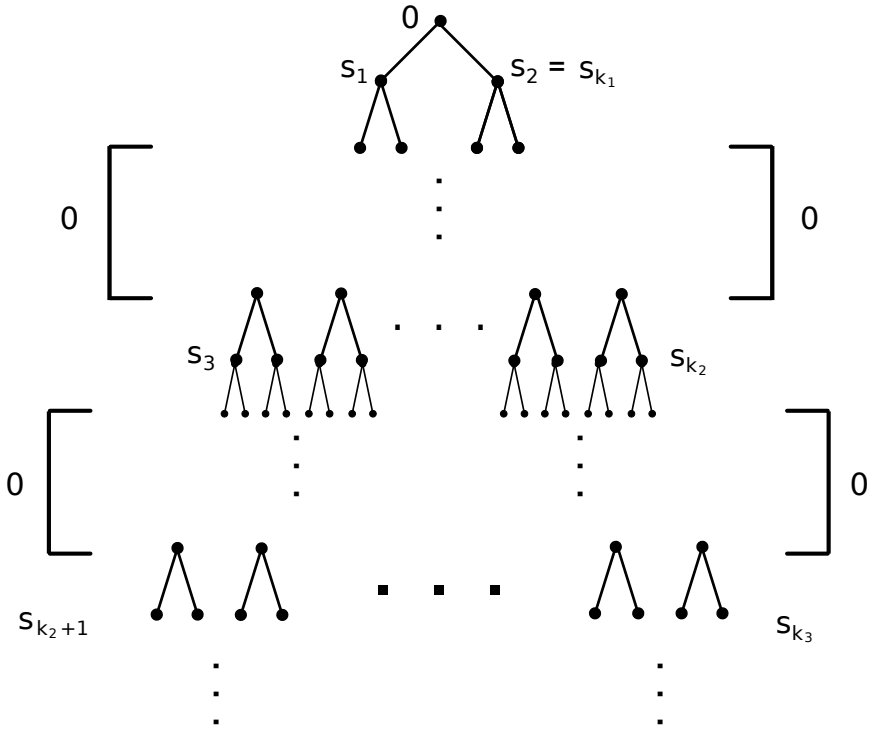

Figure 5. Stages in Main Construction

Theorem 4.2. *For every increasing sequence $s = (s_1.s_2, s_3, \ldots)$, the BW compactum $C(s)$ above can be constructed so as to be a rigid generalized BW Cantor set $C(s) \subset S^3$ with simply connected complement.*

Proof. The complement of the compactum $C(s)$ is simply connected by Lemma 3.2. It is an easy matter to check that $\sum_{i=1}^{\infty} \frac{1}{2^{\sigma_i}} = \infty$ and so it is possible to construct $C(s)$ to be a Cantor set by Theorem 3.1. It remains to be shown that $C(s)$ is rigidly embedded.

Let h be a self homeomorphism of S^3 that takes $C(s)$ to itself. If h restricted to $C(s)$ is not the identity, then there are points $x \neq y$ in $C(s)$ with $h(x) = y$. The construction guarantees that the Whitehead sequences $s(x)$ and $s(y)$ do not have the same tail, whereas Theorem 3.6 requires that they do. This contradiction establishes the result. □

Lemma 4.3. *If two sequences s and t with only finitely many terms in common are used to construct Cantor sets $C(s)$ and $C(t)$ as in Theorem 4.2, then $C(s)$ and $C(t)$ are inequivalently embedded.*

Proof. Assume there is a homeomorphism of S^3 taking $C(s)$ to $C(t)$. The construction guarantees that no point in $C(s)$ has Whitehead sequence with the same tail as a point in $C(t)$, contradicting Lemma 3.6. □

Theorem 4.4. *There are uncountably many inequivalent BW Cantor sets in S^3 with simply connected complement.*

Proof. This follows from Lemma 4.3 and the fact that there are uncountably many increasing sequences of positive integers, any two of which have only finitely many terms in common. One easy way to see this is to sequentially label the vertices of an infinite binary tree with the positive integers, and choose the sequences corresponding to descending rays through the tree. □

5. Application to 3-manifolds

Theorem 5.1. *For every generalized BW Cantor set $C(s)$ as in Theorem 4.2, there is an open 3-manifold $M(s)$ with end set $C(s)$ with the following properties:*

- *$M(s)$ is simply connected,*
- *The Freudenthal (endpoint) compactification of $M(s)$ is S^3,*

- *Any self homeomorphism h of $M(s)$ extends to a unique homeomorphism $\overline{h}$ of S^3,*
- *The homeomorphism $\overline{h}$ restricted to the ends is the identity, and*
- *$M(s)$ has genus one at infinity.*

Proof. Given $C(s)$ as in Theorem 4.2, let $M(s)$ be the complement of $C(s)$ in S^3. Theorem 4.2 shows that $M(s)$ is simply connected. The definition of $M(s)$ shows that the end set is $C(s)$ and that the Freudenthal (endpoint) compactification of $M(s)$ is S^3. Standard results about endpoint compactifications show that any self homeomorphism h of $M(s)$ extends to a unique homeomorphism $\overline{h}$ of S^3. The extension homeomorphism $\overline{h}$ must be the identity when restricted to the ends since $C(s)$ is rigidly embedded. The genus at infinity of $M(s)$ is the same as the genus of $C(s)$ which is one since $C(s)$ has a defining sequence with genus one components, and since no point of $C(s)$ is tamely embedded. □

Corollary 5.2. *There are uncountably many distinct 3-manifolds having the properties listed in Theorem 5.1.*

Proof. This follows directly from Theorem 4.4 since a homeomorphism between $M(s)$ and $M(t)$ implies equivalence of embeddings of $C(s)$ and $C(t)$. □

6. Questions

The following questions arise from a consideration of the results in this paper.

Question 6.1. *For the simply connected 3-manifolds constructed in this paper, each with end set a Cantor set, is every self homeomorphism isotopic either to an involution fixing the ends or to the identity?*

Question 6.2. *Does there exist a simply connected 3-manifold with only one end so that every self homeomorphism is isotopic either to an involution fixing the ends or to the identity?*

Question 6.3. *Can the techniques of this paper be used to show there exist rigid embeddings of compacta in S^3 with simply connected complement?*

Question 6.4. *What is the group of isotopy classes of self homeomorphisms of the Whitehead 3-manifold? The expected answer would be*

$\mathbb{Z} \times \mathbb{Z}_2$, where the first factor comes from ratcheting [Wri92] *and the second factor from flipping.*

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