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SIMPLY CONNECTED SPIN MANIFOLDS WITH POSITIVE SCALAR CURVATURE

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ABSTRACT. Let l_n be equal to 2 or 4 according as n is congruent mod 8 to zero or not. If the $\hat{A}$-genus of a compact simply connected spin manifold of dimension $n \geqslant 5$ vanishes, then the connected sum of l_n copies of the manifold admits a metric of positive scalar curvature. This supports a conjecture of Gromov and Lawson.

Introduction. In relation to the works of Kazdan and Warner and others (see [**2**]), it is an important problem to describe the condition for a given manifold to admit a metric of positive scalar curvature.

Lichnerowicz' theorem [**6**] states that the vanishing of the $\hat{A}$-genus is one of the necessary conditions for spin manifolds to have such metrics. Moreover, Hitchin [**5**] proved a vanishing theorem for the KO-characteristic number $\hat{\mathscr{A}}$: $\Omega_*^{\text{Spin}} \to \text{KO}_*(\text{pt.})$, which essentially coincides with the $\hat{A}$-genus in dimensions congruent to zero mod 4.

On the other hand, a cobordism-theoretic approach was achieved in one of the papers of Gromov and Lawson [**4**]. They showed that every compact simply connected manifold which is not spin and whose dimension is not less than 5 carries a metric of positive scalar curvature. Also they showed that if the $\hat{A}$-genus of a compact simply connected manifold of dimension not less than 5 vanishes, the connected sum of some copies of the manifold carries a metric of positive scalar curvature. They conjectured that for a compact simply connected manifold of dimension not less than 5, the KO-characteristic number $\hat{\mathscr{A}}$ is the complete obstruction to the existence of a metric of positive scalar curvature.

Let us put it in another way. Let $P \subset \Omega_*^{\text{Spin}}$ be the ideal consisting of the set of classes containing representatives with positive scalar curvature. We consider the homomorphism

$$\Pi\colon \Omega_*^{\text{Spin}} \to \Omega_*^{\text{Spin}}/P.$$

Gromov and Lawson proved that a simply connected spin manifold which is spin cobordant to a spin manifold of positive scalar curvature also admits a metric of positive scalar curvature. Therefore, the conjecture of Gromov and Lawson means that $\ker \hat{\mathscr{A}}$ coincides with P and, as a result, Π and Ω_*^{Spin}/P coincide with $\hat{\mathscr{A}}$ and $\text{KO}_*(\text{pt.})$, respectively. They proved that $\Pi \otimes Q$ is exactly the $\hat{A}$-genus.

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In this paper, we show a more refined result concerning this conjecture. Note that the theorem of Hitchin guarantees that P is contained in $\ker \hat{\mathscr{A}}$. What we want is its converse. Let l_k be equal to 4 or 2 according as k is congruent mod 8 to zero or not. We denote $\ker\hat{\mathscr{A}} \cap \Omega_k^{\mathrm{Spin}}$ and $P \cap \Omega_k^{\mathrm{Spin}}$ by $\ker\hat{\mathscr{A}}_k$ and P_k respectively. Our result is the following.

THEOREM. *$l_k \ker\hat{\mathscr{A}}_k$ is contained in P_k. In particular, let M be a compact simply connected manifold of dimension $k \geqslant 5$ whose $\hat{A}$-genus vanishes. Then the connected sum of l_k copies of M admits a metric of positive scalar curvature.*

Note that this implies that the conjecture is true mod any odd number p, i.e. $\Pi \otimes \mathbf{Z}_p$ coincides with $\hat{\mathscr{A}} \otimes \mathbf{Z}_p$.

The conjecture of Gromov and Lawson has a close relation to the following problem. If some multiple $M \# \cdots \# M$ carries a metric of positive scalar curvature, then does M itself carry such a metric? In general, the answer is no even in the case of simply connected spin manifolds. In fact, there exists a 9-dimensional simply connected spin manifold M such that the KO-characteristic number $\hat{\mathscr{A}}$ does not vanish and such that $M \# M$ carries positive scalar curvature (it is cobordant to S^9). However, our Theorem asserts that the answer is yes for odd numbers.

COROLLARY. *Let M be a compact simply connected spin manifold such that the odd multiple $M \# \cdots \# M$ carries a metric of positive scalar curvature. Then M itself carries such a metric.*

REMARK. One can show that each class of $\ker\hat{\mathscr{A}}_k$ ($k \leqslant 12$) or $\ker\hat{\mathscr{A}}_k/\mathrm{Tor}$ ($k \leqslant 20$) is represented by a spin manifold of positive scalar curvature using $\mathbf{HP}^4$ and manifolds in our proof and in [7]. This means that the conjecture is true in low dimensions.

Concluding this Introduction, the author wants to propose the following problem.

PROBLEM. Can each class in the kernel of the map

$$\hat{\mathscr{A}} \otimes \mathbf{Z}_2 \colon \Omega_*^{\mathrm{Spin}} \otimes \mathbf{Z}_2 \to \mathrm{KO}_*(\mathrm{pt.}) \otimes \mathbf{Z}_2$$

be represented by a spin manifold with positive scalar curvature?

Our Theorem implies that if this problem is solved affirmatively, then the original conjecture is true.

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We have learned since writing this paper that Jonathan Rosenberg has independently proved a parallel result except that he needs a larger power of 2 in dimensions congruent to 4 mod 8.

Proof of the Theorem. In order to represent each class of $l_k \ker\hat{\mathscr{A}}_k$ by a spin manifold of positive scalar curvature, we need three types of manifolds. $P(E(M))$ denotes the complex or quaternion projective space bundle of the complex or quaternion vector bundle E over the manifold M. $c\theta^k$ (resp. $h\theta^k$) denotes a trivial k-dimensional complex (resp. quaternion) vector bundle.

First, we consider a $4k$-dimensional spin manifold ($k \geqslant 2$)

$$X_k = P\big(h\lambda \oplus h\theta^2(\mathbf{HP}^{k-2})\big),$$

where $h\lambda$ is the canonical line bundle. By some calculation (cf. [**9**]), we get

$$s_{(k)}(\mathscr{P}(\tau))[X_k] = \pm 2(2k+1)(k-1).$$

Secondly, we consider a $4k$-dimensional manifold ($k \geqslant 2$)

$$P\big(\zeta \oplus c\theta^3\big(P\big(\xi \oplus c\theta^1(\mathbf{CP}^{2k-4})\big)\big)\big),$$

where ζ and ξ are complex line bundles. Let a and b be the first Chern classes of the canonical line bundles over $\mathbf{CP}^{2k-4}$ and $P(\xi \oplus c\theta^1(\mathbf{CP}^{2k-4}))$ respectively, and let ξ and ζ have the first Chern classes la and $ma+nb$ ($l, m, n \in \mathbf{Z}$) respectively. For given numbers l, m, n, we denote the above manifold by $Y_{k,l,m,n}$. Then we get (cf. [**3**])

$$s_{(k)}(\mathscr{P}(\tau))[Y_{k,l,m,n}] = \pm(2k+1)(k-2)\big((m+nl)^{2k-3} - m^{2k-3}\big)/l.$$

The necessary and sufficient condition for $Y_{k,l,m,n}$ to be spin is that $l+m$ is odd and n is even.

Thirdly, we consider the $4k$-dimensional manifold $H_{m,n,p,q}$, where $m+n-1 = 2k$, $n, m \geqslant 2$. $H_{m,n,p,q}$ is the hypersurface of $\mathbf{CP}^m \times \mathbf{CP}^n$ which is dual to $pu+qv$, where u and v are the first Chern classes of the canonical line bundles over $\mathbf{CP}^m$ and $\mathbf{CP}^n$ respectively, and p and q are integers. We get (cf. [**10**])

$$s_{(k)}(\mathscr{P}(\tau))\big[H_{m,n,p,q}\big] = \pm\binom{2k+1}{m}p^m q^n.$$

Since $m+n$ is odd, we may assume that m is odd and n is even. Then the necessary and sufficient condition for $H_{m,n,p,q}$ to be spin is that p is even and q is odd.

LEMMA. (1) X_k ($k \geqslant 2$) *and* $Y_{k,l,m,n}$ ($k \geqslant 2$) *carry metrics of positive scalar curvature.*

(2) $H_{m,n,p,q}$ *carries a metric of positive scalar curvature if* $m < n$ *and* $p < 0$, $q = -1$.

PROOF. (1) X_k and $Y_{k,l,m,n}$ carry such metrics because, for each complex or quaternion projective space bundle of a vector bundle, a metric of positive scalar curvature is constructed as follows. In the direction along the fibers, it is constructed from the ordinary metric of a complex or quaternion projective space and in the direction vertical to the fibers, it is constructed from an arbitrary metric of the base space using a metric connection of the vector bundle. This defines a Riemannian submersion with totally geodesic fibers. Shrinking the metric in the fibers, it deforms to a metric of positive scalar curvature (cf. [**8**]).

(2) We use a homogeneous coordinate $([z_0,\dots,z_m],[w_0,\dots,w_n])$ of $\mathbf{CP}^m \times \mathbf{CP}^n$. It is straightforward to see $H_{m,n,p,q}$ ($m<n$, $p<0$, $q=-1$) is represented by the nonsingular hypersurface of degree $(p, 1)$ defined by $\sum_{i=0}^m z_i^p w_i = 0$ (cf. [**10**, p. 81, 131]). $H_{m,n,p,q}$ is a $\mathbf{CP}^{n-1}$-bundle over $\mathbf{CP}^m$ whose projection is inherited from the first projection $\mathbf{CP}^m \times \mathbf{CP}^n \to \mathbf{CP}^m$. Give $H_{m,n,p,q}$ the metric induced from $\mathbf{CP}^m \times \mathbf{CP}^n$. Then pull back the metric of the base space $\mathbf{CP}^m$ in the direction vertical to the fibers. This defines a Riemannian submersion with totally geodesic fibers. Shrinking the metric in the fibers, one has the desired metric.

PROPOSITION. *For each integer $k \geqslant 2$, there is a $4k$-dimensional spin manifold M_k which admits a metric of positive scalar curvature and*

$$s_{(k)}(\mathscr{P}(\tau))[M_k] = \begin{cases} 2 & \text{if } 2k+1 \neq r^s \text{ for any prime } r \text{ and integer } s, \\ 2r & \text{if } 2k+1 = r^s \text{ for some prime } r \text{ and integer } s. \end{cases}$$

PROOF. The s-numbers of X_k, $Y_{k,1,0,2}$ and $Y_{k,0,1,2}$ are $\pm 2(2k+1)(k-1)$, $\pm 2^{2k-3}(2k+1)(k-2)$ and $\pm 2(2k+1)(k-2)(2k-3)$ respectively. Taking the greatest common divisor of these three numbers, one can construct a spin manifold N_k with its s-number $2(2k+1)$.

Let r be the odd prime that divides $2k+1$. If $2k+1$ is not a power of r, then one may write $2k+1 = r^t d$ with $d \not\equiv 0 \bmod r$. Then the s-number of $H_{r^t, r^t(d-1), -2, -1}$ is

$$\pm \binom{r^t d}{r^t} 2^{r^t}$$

and this is not congruent to 0 mod r. If $2k+1 = r^s$ for some $s \geqslant 2$, then the s-number of $H_{r^{s-1}, r^2 - r^{s-1}, -2, -1}$ is

$$\pm \binom{r^s}{r^{s-1}} 2^{r^{s-1}}$$

and this is not congruent to 0 mod r^2. Again taking the greatest common divisor, we construct M_k.

Now we can prove the Theorem. Since all torsion in the spin cobordism ring has order two, we restrict ourselves to $\Omega_*^{\mathrm{Spin}}/\mathrm{Tor}$. Let $B^{\mathrm{SO}} \subset \Omega_*^{\mathrm{SO}}/\mathrm{Tor}$ be the polynomial algebra generated by the classes y_i ($i \geqslant 1$) characterized by

$$s_{(i)}(\mathscr{P}(\tau))[y_i] = \pm m_{2i} m_{2i-1} \quad \text{for } i \geqslant 2,$$

$$s_{(1)}(\mathscr{P}(\tau))[y_1] = \pm 2^3 3,$$

where $m_i = r$ if $i+1 = r^s$ for some prime r and positive integer s and $m_i = 1$ otherwise (cf. [**10**, p. 280]). Let M_k be the manifold given in the Proposition. Since the s-number of M_k is $2m_{2k}$, we may assume $M_i = y_i$ if $i = 2^s$ and $M_i = 2y_i$ if $i \neq 2^s$ ($i \geqslant 2$). Therefore $2y_i$ ($i \geqslant 2$) is represented by a $4k$-dimensional spin manifold with positive scalar curvature.

Note that $\Omega_{8i}^{\mathrm{Spin}}/\mathrm{Tor} = B_{8i}$ and $\Omega_{8i+4}^{\mathrm{Spin}}/\mathrm{Tor} = 2B_{8i+4}$ (cf. [**10**, p. 340]). Additive generators of $\Omega_{8i+4}^{\mathrm{Spin}}/\mathrm{Tor}$ are grouped into two types:

(i) $2y_1^{2i+1}$. The $\hat{A}$-genus is not zero and this cannot carry positive scalar curvature.

(ii) $2y_{i_1} \cdots y_{i_j}$, $i_1 + \cdots + i_j = 2i+1$ and $i_j \neq 1$. $2(2y_{i_1} \cdots y_{i_j})$ is spin cobordant to $(2y_{i_1} \cdots y_{i_{j-1}})(2y_{i_j})$, which is represented by a product manifold of a spin manifold and a spin manifold of positive scalar curvature. Similarly, generators of $\Omega_{8i}^{\mathrm{Spin}}$ are grouped into two types.

(i) y_1^{2i}. This cannot carry positive scalar curvature.

(ii) $y_{i_1} \cdots y_{i_j}$, $i_1 + \cdots + i_j = 2i$ and $i_j \neq 1$. $4y_{i_1} \cdots y_{i_j}$ is spin cobordant to $(2y_{i_1} \cdots y_{i_{j-1}})(2y_{i_j})$ and carries positive scalar curvature.

The Theorem is the immediate consequence of these.

Proof of the Corollary. By the Theorem, $4M$ carries positive scalar curvature. Then take the greatest common divisor.

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