Simulation of hitting times for Bessel processes with non-integer dimension

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In this paper, we complete and improve the study of the simulation of the hitting times of some given boundaries for Bessel processes. These problems are of great interest in many application fields as finance and neurosciences. In a previous work (*Ann. Appl. Probab.* **23** (2013) 2259–2289), the authors introduced a new method for the simulation of hitting times for Bessel processes with integer dimension. The method, called walk on moving spheres algorithm (WoMS), was based mainly on the explicit formula for the distribution of the hitting time and on the connection between the Bessel process and the Euclidean norm of the Brownian motion. This method does not apply anymore for a non-integer dimension. In this paper we consider the simulation of the hitting time of Bessel processes with non-integer dimension $\delta \geq 1$ and provide a new algorithm by using the additivity property of the laws of squared Bessel processes. We split each simulation step of the algorithm in two parts: one is using the integer dimension case and the other one considers hitting time of a Bessel process starting from zero.

Keywords: Bessel processes with non-integer dimension; hitting time; numerical algorithm

1. Introduction

This paper aims at constructing new and efficient methods for approximating hitting times of a given threshold for Bessel processes with non-integer dimension $\delta \geq 1$. Diffusion hitting times are important quantities in many fields such as mathematical science, finance, geophysics or neurosciences.

In neurosciences for example, the interspike interval is identified with the first-passage time of an Ornstein–Uhlenbeck process through some threshold and the spike train forms a renewal process. The choice of the Ornstein–Uhlenbeck process is developed in Tuckwell [38] and in Burkitt's survey [4], see also the book [15] or the survey [33]. In Feller's model, generalized Bessel processes appear as a more realistic alternative to the Ornstein–Uhlenbeck process, see for instance [33] for a description of this new model and [26] for a comparison of these models. That is why, Feller's model plays an important role in neuroscience even if it is hard to compute the first passage problem to general boundaries. A modification of the Ornstein–Uhlenbeck model and the Feller model also permits to deal with refractoriness [2].

Another typical example is the study of path dependent exotic options as barrier options in finance. The study of a large class of options pricing in finance is based on the numerical approx-

imation of relevant hitting times or passage times for specific stochastic processes, for instance the Cox–Ingersoll–Ross process which can be expressed as a Bessel process after a change of time transformation [21]. In particular, many studies consider the pricing of barrier options within this scope. In Gobet [17,18] a numerical Monte Carlo based scheme is proposed for the approximation of the price of barrier options. The method is based on the law of a real diffusion killed as it goes out of a domain (interval of $\mathbb{R}$), when the diffusion is approximated by its continuous Euler scheme.

On one hand, analytic expressions for hitting time densities are well known and studied only in some very particular situations. On the other hand, the study of the approximation of the hitting times for stochastic differential equations is an active area of research since very few results exist up to now. For the Brownian motion, we can approach this quantity simply by using Gaussian random variables [23].

Several alternatives for dealing with the characterization of hitting times for the Brownian motion and general Gaussian Markov processes exist. These methods aim to approximate the probability distribution function or the hitting probabilities by using:

- Volterra integral equations, more precisely to obtain the approximate solutions for Brownian motion or Gaussian Markov processes crossing continuously differentiable boundaries, the tangent approximation and other image methods have been used by Strassen [37]; Daniels [8]; Ferebee [13]; Lerche [27]. This method does not work in the general diffusion framework as it relies on the explicit form of the transition probabilities;
- series expansions, as performed by Durbin [11,12]; Ferebee [14]; Ricciardi *et al.* [32]; Giorno *et al.* [16]. The numerical approach proposed in [3] seems to be particularly efficient. This method can be adapted to more general diffusion processes as in Sacerdote and Tomassetti [34];
- partial differential equation approaches, which are based on the explicit form of the probability distribution function of the Brownian motion. This method can be generalized for a large class of diffusions as in Patie and Winter [29];
- or Monte Carlo methods as in the paper by Ichiba and Kardaras [24] that handles the representation of the passage density as the mean of a three dimensional Brownian motion or in the works of Pötzelberger and Wang [30,39] for the Brownian motion and some particular transformations of the Brownian motion.

These methods are useful for approximating the distribution of the hitting time. If we have to approximate directly the hitting time one needs to develop other methods.

For the general diffusion case, very few studies in this direction exist. The only methods that can be used are the Monte Carlo methods and time splitting method like the Euler schemes. An approximation of the hitting time can be obtained by using the hitting time of the boundary for the Euler scheme. This will give an upper bound for the diffusion hitting time. In order to improve this approximation we can use penalisation methods, that means construct boundaries ε-close to the real boundary, with ε a small parameter. Some works have already been done in the context of smooth drift and diffusion coefficient by Gobet and Menozzi [19,20].

Our study focuses on the numerical approach of the hitting time of a Bessel process. Bessel processes and their hitting times are important issues either for theoretical research or for many practical applications. Some issues on these subjects are: Alili and Patie [1] investigate the Bessel

processes via some boundary crossing identities for diffusions having the time inversion property; Salminen and Yor consider the hitting time of affine boundaries for the 3-dimensional Bessel process [35]; Hamana and Matsumoto [22] give explicit expressions for the distribution functions and the densities of the first hitting time of a given level for the Bessel process; Byczkowski, Malecki and Ryznar [5] obtain uniform estimates of the hitting time density function by using an integral formula for the density of the first hitting time of a given level for the Bessel process.

In all these papers, the formulas are explicit functions, often given as a series expansion, and are thus hard to use for a numerical purpose as they include in their formulation Bessel functions. The main idea of our present work, which completes and improves the results of our previous work in Deaconu and Herrmann [9], is to get rid of this difficulty by using the properties of Bessel processes.

The main result of this study is the construction of an iterative procedure for approximating the hitting time by using the structure and the particular properties of the Bessel process. In particular, our approach avoids splitting time methods. More precisely, we consider the simulation of the hitting time of Bessel processes with non-integer dimension and construct a new algorithm by using the additivity property of the laws of squared Bessel processes. Each simulation step is split in two parts: the first one uses the already known algorithm for the integer dimension case and the second one constructs the hitting time of a Bessel process starting from zero.

More precisely, we aim to approximate τ_L the first passage time through the threshold L for the Bessel process starting under the level L. Let us note that our study does not concern hitting times for Bessel processes starting above the threshold: in such a case, the finiteness of the hitting time is no longer ensured. We introduce a particular algorithm called (NI) based on a random process $(M(n), \Theta_n)_{n \geq 0}$ which satisfies the following properties:

1. The first coordinate $(M(n))_{n \geq 0}$ is a $[0, L]$-valued random walk.
2. The second coordinate represents an increasing sequence of stopping times.
3. If N^ε is defined by

$$N^\varepsilon = \inf\{n \geq 0 : M^2(n) \geq L^2 - \varepsilon\},$$

then the random variable Θ_{N^ε} gives a ε-precise approximation for the hitting time τ_L.

Due to the stopping criterion of (NI) and since $(\Theta_n)_{n \geq 0}$ is an increasing sequence, smaller is the parameter ε higher is the number of iterations of the algorithm.

In this study, the convergence and the efficiency of this new algorithm is emphasized namely by the main result.

Theorem 1.1. *Let us consider a Bessel process of dimension $\delta \geq 1$. If the starting position of the Bessel process is smaller than L, then the number of steps N^ε of the Algorithm (NI) is almost surely finite. Moreover, there exist constants $C > 0$ and $\varepsilon_0 > 0$, such that*

$$\mathbb{E}[N^\varepsilon] \leq C|\log \varepsilon|, \qquad \text{for all } \varepsilon \leq \varepsilon_0.$$

Furthermore Θ_{N^ε} converges in distribution towards τ_L, the hitting time of the level L for the δ dimensional Bessel process, as $\varepsilon \to 0$.

The paper is organized as follows. Section 2 introduces some important properties on Bessel processes needed in the paper. Section 3 gives the construction of the algorithm. In Section 4, we introduce some preliminary results in order to obtain the convergence of the algorithm in Section 5. Finally, Section 6 presents and discusses some numerical results.

2. Bessel processes: Definitions and main properties

Let $\delta \geq 0$. We define the *square of a δ-dimensional Bessel process* started at y_0, as the unique strong solution of the following stochastic differential equation

$$\begin{cases} Y_t^{\delta,y_0} = Y_0^{\delta,y_0} + 2 \int_0^t \sqrt{Y_s^{\delta,y_0}} \, dB_s + \delta t, \\ Y_0^{\delta,y_0} = y_0, \qquad y_0 \geq 0, \end{cases} \tag{2.1}$$

where $(B_t)_{t\geq 0}$ stands for the standard one-dimensional Brownian motion. The usual notation for the distribution of the process $(Y_t^{\delta,y_0})_{t\geq 0}$ is *BESQ$^\delta$*. This unique solution satisfies $Y_t^{\delta,y_0} \geq 0$ for any $t \geq 0$. It is therefore possible to consider its square root, the so-called *Bessel process of dimension δ* starting at $x_0 \geq 0$, defined by:

$$X_t^{\delta,x_0} = \sqrt{Y_t^{\delta,x_0^2}}, \qquad \text{for all } t \geq 0. \tag{2.2}$$

Let us recall that the Bessel process is characterized either by its dimension δ or, alternatively, by its index ν given by $\nu = \delta/2 - 1$.

For a fixed $L > 0$ let us denote by

$$\tau_L = \inf\{t \geq 0; X_t^{\delta,x_0} = L\}, \tag{2.3}$$

the first time that the process hits the threshold L. For the Bessel process case an explicit form of the Laplace transform of τ_L exists:

$$\mathbb{E}_{x_0}\big[e^{-\lambda \tau_L}\big] = \frac{(x_0)^{-\nu}}{L^{-\nu}} \frac{I_\nu(x_0\sqrt{2\lambda})}{I_\nu(L\sqrt{2\lambda})}, \qquad y > 0,$$

here $I_\nu(x)$ denotes the modified Bessel function. Ciesielski and Taylor [6] proved that for $\delta \in \mathbb{N}$ the tail distribution is given by, when starting from 0

$$\mathbb{P}(\tau_L > t) = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{\mathcal{J}_{\nu+1}(j_{\nu,k})} e^{-\frac{j_{\nu,k}^2}{2L^2}t},$$

where $\mathcal{J}.$ denotes the Bessel function of the first kind, and $j._{,k}$ is the associated sequence of its positive zeros. These formulas are restricted to the integer dimension case (see [22] for non-integer dimensions) and are obviously miss-adapted and not suited for numerical approaches. The Laplace transform of the first hitting time is also available for various generalizations of the

Bessel process with integer dimension, namely for radial and squared radial Ornstein–Uhlenbeck processes and for Bessel squared processes with negative dimensions [21].

Let us also recall some properties of Bessel processes (resp. squared Bessel processes) with respect to their dimension as follows from Revuz and Yor [31] or Jeanblanc, Yor and Chesney [25]:

- for $\delta > 2$ the process $BESQ^\delta$ is transient and, for $\delta \leq 2$, it is recurrent,
- for $\delta \geq 2$ the point 0 is polar and for $\delta \leq 1$ it is reached almost surely,
- for $\delta = 0$ the point 0 is absorbing,
- for $0 < \delta < 2$ the point 0 is instantaneously reflecting.

Let us first introduce some relations connecting Bessel processes of different dimensions. The first relation is based on Girsanov's transformation and the second one gives a decomposition of the squared Bessel process as a sum of two independent squared Bessel processes.

On the canonical space $\Omega = \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, let Z be the canonical map and $\mathcal{F}_t = \sigma\{Z_s, 0 \leq s \leq t\}$ be the canonical filtration. We denote by $\mathbb{P}^{\delta, x_0}$ the law of the Bessel process of dimension δ starting from x_0, $x_0 > 0$.

Let us state the following result from Jeanblanc, Yor and Chesney [25] (Proposition 6.1.5.1, page 364).

Proposition 2.1. *The following absolute continuity relation, between a Bessel process of dimension $\delta \geq 2$ and a Bessel process of dimension 2, holds*

$$\mathbb{P}^{\delta, x_0}|_{\mathcal{F}_t} = \left(\frac{Z_t}{x_0}\right)^{\delta/2-1} \exp\left\{-\frac{(\delta/2-1)^2}{2} \int_0^t \frac{\mathrm{d}s}{Z_s^2}\right\} \mathbb{P}^{2, x_0}|_{\mathcal{F}_t}, \qquad \forall t \geq 0. \tag{2.4}$$

For $\delta < 2$, a similar absolute continuity relation holds before the first hitting time of 0.

It seems difficult to use the expression (2.4) in order to simulate Bessel hitting times for an arbitrarily dimension δ even if it reduces to study only the 2-dimensional case. Thus, the use of Radon–Nikodym's derivative happens to be unuseful for numerical purposes.

An important property, due to Shiga and Watanabe [36], is the additivity property for the family of squared Bessel processes. Let us denote by $\mathbb{P}_1 * \mathbb{P}_2$ the convolution of $\mathbb{P}_1$ and $\mathbb{P}_2$, where $\mathbb{P}_1$ and $\mathbb{P}_2$ are probability measures. In the following, we denote by $\mathbb{Q}^{\delta, y_0}$ the law of the squared Bessel process of dimension δ starting from y_0.

Proposition 2.2. *For every $y_0, y_0' \geq 0$ and for every $\delta, \delta' \geq 0$ we have*

$$\mathbb{Q}^{\delta, y_0} * \mathbb{Q}^{\delta', y_0'} = \mathbb{Q}^{\delta+\delta', y_0+y_0'}. \tag{2.5}$$

3. Algorithm for approaching the hitting time

In a previous paper [9], the authors developed an algorithm in order to simulate, in just a few steps, the Bessel hitting time. This was done for integer dimensions $\delta \geq 1$. The particular connection between the Bessel process and the δ-dimensional Brownian motion gives in this case a

geometrical interpretation in terms of the exit problem from a disk for a δ-dimensional Brownian motion. This geometrical approach does not work any longer for non-integer dimensions. In order to overcome this difficulty, we construct a new algorithm which essentially uses that we are able to simulate, in just one step, each of the following stopping times:

- the first exit time of the multidimensional Brownian motion from a sphere with a particular time-dependent radius (moving sphere) [9],
- the first passage time of a Bessel process starting from the origin through a particular curved boundary (Section 4).

The main task is the choice of both the time-dependent radius and the curved boundary in such a way that the simulation of the stopping times remains extremely simple.

Then the additivity property expressed in Section 2 permits to point out a procedure of simulation for a general Bessel process whatever the starting point or the dimension $\delta \geq 1$. Each step of the algorithm is based on a couple of Bessel processes: one of dimension $\delta' := \delta - \lfloor \delta \rfloor$ starting in 0 and hitting a curved boundary and the other one of dimension $\lfloor \delta \rfloor$ starting away from the origin and hitting a moving sphere. An intelligent combination of these two boundaries permits to deal with the hitting problem of the general Bessel process. Before presenting the Algorithm (NI), let us start by making some notations.

Some notations: Set $a > 0$ and $\delta > 0$. Let us consider the following nonlinear boundary:

$$\psi_{a,\delta}(t) = \sqrt{2t \log \frac{a}{\Gamma(\frac{\delta}{2})t^{\frac{\delta}{2}}2^{\frac{\delta}{2}-1}}}, \qquad \text{for } t \in \mathrm{Supp}(\psi) := [0, T_{a,\delta}], \tag{3.1}$$

where $T_{a,\delta}$ is given by

$$T_{a,\delta} := \left(\frac{a}{\Gamma(\frac{\delta}{2})2^{\frac{\delta}{2}-1}} \right)^{\frac{2}{\delta}}. \tag{3.2}$$

For a given dimension $\delta > 0$ denote $\delta' := \delta - \lfloor \delta \rfloor$. Consider also $\gamma \in [0, 1)$ (close to 1) and $L > 0$. We define, for $x > 0$

$$I(\delta, x) := 2^{\lfloor \delta \rfloor/2 - 1} \Gamma\left(\frac{\lfloor \delta \rfloor}{2} \right) \left(\frac{\sqrt{e}\gamma(L^2 - x^2)}{\sqrt{(\lfloor \delta \rfloor - \delta\gamma)x^2 + \delta\gamma L^2} + \sqrt{\lfloor \delta \rfloor x^2}} \right)^{\lfloor \delta \rfloor} \tag{3.3}$$

and

$$N(\delta, x) := 2^{\delta'/2 - 1} \Gamma\left(\frac{\delta'}{2} \right) \left(\frac{\sqrt{e}\gamma(L^2 - x^2)}{\sqrt{(\delta' - \delta\gamma)x^2 + \delta\gamma L^2} + \sqrt{\delta'x^2}} \right)^{\delta'}. \tag{3.4}$$

Description of the algorithm for $\delta \geq 1$. Let us now introduce the algorithm.

Algorithm (NI) Simulation of $\tau_L = \inf\{t \geq 0 : X_t^{\delta, x_0} = L\}$

Initialization: $\Theta_0 = 0$, $M(0) = x_0$, $\gamma \in (0, 1)$ a chosen parameter, close to 1.

Step n ($n \geq 1$): The Bessel process starts at time Θ_{n-1} in $M(n-1)$.

While $L^2 - M^2(n-1) > \varepsilon$ do:

(n.1) Construct a Bessel process of dimension $\lfloor \delta \rfloor$ starting from $M(n-1)$ and stop this process at time $\theta_n^{(1)}$, the exit time of the $\lfloor \delta \rfloor$-dimensional Brownian motion B_t from the moving sphere centered in $\underline{M}(n-1) := (M(n-1), 0, 0, \ldots, 0) \in \mathbb{R}^{\lfloor \delta \rfloor}$ and with radius $\psi_{\alpha_n, \lfloor \delta \rfloor}(t)$, where

$$\alpha_n = I\big(\delta, M(n-1)\big),$$

following the definition given in (3.3). The exit time and the exit position are obtained by the moving sphere algorithm (WoMS) [9].

(n.2) Construct also a second Bessel process, independent with respect to the previous one, of dimension $\delta' := \delta - \lfloor \delta \rfloor$ starting from 0. Stop this process the first time $\theta_n^{(2)}$ it hits the curved boundary $\psi_{\beta_n, \delta'}(t)$, where

$$\beta_n = N\big(\delta, M(n-1)\big),$$

following the definition given in (3.4). This hitting time is constructed in Section 4.

(n.3) Define the stopping time (comparison of the two hitting times)

$$\theta_n = \inf\big\{\theta_n^{(1)}, \theta_n^{(2)}\big\}. \tag{3.5}$$

First notice that the additivity property of the Bessel processes ensures that $(X_t^{\delta, M(n-1)})^2$ has the same distribution as the sum of the two independent processes defined in steps $(n.1)$ and $(n.2)$. We denote by

$$\big(Z_t^{\delta, M(n-1)}\big)^2 = \big\|\underline{M}(n-1) + B_t\big\|^2 + \big(X_t^{\delta', 0}\big)^2.$$

The values of α_n and β_n have been chosen in order to ensure the following bound:

$$\sup_{t \leq \theta_n}\big(Z_t^{\delta, M(n-1)}\big)^2 \leq \sup_{t \leq \theta_n^{(1)}}\big\|\underline{M}(n-1) + B_t\big\|^2 + \sup_{t \leq \theta_n^{(2)}}\big(X_t^{\delta', 0}\big)^2$$
$$\leq M^2(n-1) + \gamma\big(L^2 - M^2(n-1)\big). \tag{3.6}$$

In particular, since $\gamma < 1$,

$$\sup_{t \leq \theta_n} Z_t^{\delta, M(n-1)} < L.$$

We lastly define $M(n) = Z_{\theta_n}^{\delta, M(n-1)} < L$ and $\Theta_n = \Theta_{n-1} + \theta_n$. This achieves the nth step.

Outcome: N^ε is then the number of steps entirely completed, that is the first time n in the algorithm such that $L^2 - (M(n))^2 \leq \varepsilon$, Θ_{N^ε} the approximate hitting time and $M(N^\varepsilon)$ the approximate exit position.

The realization of this algorithm and its convergence are developed in Section 5. The upper-bound (3.6) will be discussed in the proof of Theorem 5.1 (see the bound (5.10) in Step 2).

4. Preliminary results

We start by recalling results and notations introduced in [9] that will be needed in the sequel.

Consider the first hitting time of a curved boundary for the Bessel process of dimension δ, starting from the origin. Let $\psi(t)$ denote the boundary, and introduce the following hitting time:

$$\tau_\psi = \inf\{t > 0; X_t^{\delta,0} \geq \psi(t)\}. \tag{4.1}$$

For some suitable choice of the boundary, the distribution of τ_ψ can be explicitly computed. The result is based on the method of images (see, for instance, [7], for the origin of this method, and [27] for a complete presentation).

Proposition 4.1. *Set $a > 0$ and $\delta > 0$. Let us consider the nonlinear boundary $\psi_{a,\delta}$ given by (3.1). We can express explicitly the distribution of τ_ψ (simplified notation corresponding to $\tau_{\psi_{a,\delta}}$). It has its support in $\mathrm{Supp}(\psi)$ and is given by*

$$\mathbb{P}_0(\tau_\psi \in dt) = \frac{1}{2at}\left(2t \log\frac{a}{\Gamma(\frac{\delta}{2})t^{\frac{\delta}{2}}2^{\frac{\delta}{2}-1}}\right)^{\frac{\delta}{2}}\mathbb{1}_{\mathrm{Supp}(\psi)}(t)\,dt$$

$$= \frac{1}{2at}(\psi_{a,\delta})^\delta(t)\mathbb{1}_{\mathrm{Supp}(\psi)}(t)\,dt. \tag{4.2}$$

Remark 4.2 (Scaling property). Let us stress a scaling property which will be used in the sequel. By using relations (3.1) and (3.2), we obtain:

$$\psi_{a,\delta}^2(t) = 2t \log\left(\frac{T_{a,\delta}}{t}\right)^{\frac{\delta}{2}} = \delta T_{a,\delta}\left(\frac{t}{T_{a,\delta}}\log\left(\frac{T_{a,\delta}}{t}\right)\right) = \delta T_{a,\delta}\Phi^2\left(\frac{t}{T_{a,\delta}}\right), \tag{4.3}$$

where

$$\Phi(t) = \sqrt{t \log\left(\frac{1}{t}\right)}\mathbb{1}_{[0,1]}(t). \tag{4.4}$$

Remark 4.3. Let us note that the maximum of the function $\psi_{a,\delta}(t)$ is reached for $t_{\max}(a) = \frac{T_{a,\delta}}{e}$ and equals

$$W_{a,\delta} := \sup_{t \in \mathrm{Supp}(\psi)} \psi_{a,\delta}(t) = \sqrt{\frac{\delta}{e}\left(\frac{a}{\Gamma(\frac{\delta}{2})2^{\frac{\delta}{2}-1}}\right)^{\frac{2}{\delta}}} = \sqrt{\frac{\delta}{e}T_{a,\delta}}. \tag{4.5}$$

Moreover the distribution $u_{a,\delta}(t, x)\,dx := \mathbb{P}(X_t^{\delta,0} \in dx, \tau_\psi > t)$ has the form

$$u_{a,\delta}(t, x) = \left(\frac{1}{2^{\frac{\delta}{2}-1}\Gamma(\frac{\delta}{2})t^{\frac{\delta}{2}}} \exp\left(-\frac{x^2}{2t}\right) - \frac{1}{a}\right)x^{\delta-1}. \tag{4.6}$$

Proof of Proposition 4.1. The proof was already presented in [9] for integer dimensions $\delta \in \mathbb{N}^*$. Using completely different arguments based on stochastic tools (which are inspired by arguments developed for the Brownian motion by Lerche [27]), we extend the statement to any dimension $\delta > 0$.

We define first the squared Bessel process $Y_t = (X_t^{\delta,x_0})^2$ by (2.2). Let us denote the transition probabilities by $p_{y_0}(t, dx) := \mathbb{P}((X_t^{\delta,x_0})^2 \in dx) = \mathbb{P}_{y_0}(Y_t \in dx)$ with $y_0 = x_0^2$. The density is given (see, for instance, Corollary 1.4, page 441 in [31]) by:

$$p_y(t, x) = \frac{1}{2t}\left(\frac{x}{y}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right)I_\nu\left(\frac{\sqrt{xy}}{t}\right), \qquad \text{for } t > 0, y > 0, x \geq 0, \tag{4.7}$$

where $\nu = \delta/2 - 1$ and $I_\nu(z)$ is the Bessel function:

$$I_\nu(z) = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{\nu+2n} \frac{1}{n!\Gamma(\nu+n+1)}.$$

Moreover, for $y = 0$, we get

$$p_0(t, x) = \frac{x^{\frac{\delta}{2}-1}}{(2t)^{\delta/2}\Gamma(\delta/2)}e^{-\frac{x}{2t}}, \qquad x \geq 0.$$

Step 1. Let us denote by $\mathbb{P}_x^{t,y}$ the distribution of the squared Bessel bridge starting at x and hitting y at time t. Denote also by $\mathbb{P}_x$ the distribution of the squared Bessel process starting from x and by $(\mathcal{F}_t)_{t\geq 0}$ the filtration associated to the Bessel process.

Let $t_0 > 0$. Simple computations (using the transition probabilities and the Markov property of the Bessel process) allow to obtain the following Radon–Nikodym derivative for $t < t_0$ (see, for instance, Exercice 3.11 from [31], page 468):

$$\left.\frac{d\mathbb{P}_x^{t_0,y}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{p_{Y_t}(t_0 - t, y)}{p_x(t_0, y)}, \qquad y > 0, x \geq 0. \tag{4.8}$$

Observe that this formula cannot be extended to $t = t_0$ since $\mathbb{P}_x^{t_0,y}$ is not absolutely continuous with respect to $\mathbb{P}_x$ on $\mathcal{F}_{t_0}$. For $y = 0$, the result can be obtained by continuity:

$$\left.\frac{d\mathbb{P}_x^{t_0,0}}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \lim_{y\to 0}\frac{p_{Y_t}(t_0 - t, y)}{p_x(t_0, y)}, \qquad x \geq 0. \tag{4.9}$$

Let us compute explicitly the r.h.s. of (4.9). By (4.7) and for $x > 0$, we get

$$\frac{\mathrm{d}\mathbb{P}_x^{t_0,0}}{\mathrm{d}\mathbb{P}_x}\bigg|_{\mathcal{F}_t} = \lim_{y \to 0} \frac{\frac{1}{2(t_0-t)}(\frac{y}{Y_t})^{\nu/2}\exp(-\frac{Y_t+y}{2(t_0-t)})I_\nu(\frac{\sqrt{Y_t y}}{t_0-t})}{\frac{1}{2t_0}(\frac{y}{x})^{\nu/2}\exp(-\frac{x+y}{2t_0})I_\nu(\frac{\sqrt{xy}}{t_0})}$$

$$= \left(\frac{t_0}{t_0-t}\right)^{\nu+1}\exp\left(-\frac{Y_t}{2(t_0-t)}+\frac{x}{2t_0}\right).$$

This result can be expressed both with respect to the transition probability and to the invariant measure μ satisfying $\mu(x)p_x(t, y) = \mu(y)p_y(t, x)$ that is $\mu(x) = x^\nu$. Defining

$$\xi(t, x) = \frac{p_0(t, x)}{\mu(x)} = \frac{1}{(2t)^{\delta/2}\Gamma(\delta/2)}e^{-\frac{x}{2t}}, \qquad \text{for } x > 0, t > 0, \tag{4.10}$$

we obtain finally for any $t < t_0$:

$$D_t := \frac{\mathrm{d}\mathbb{P}_x^{t_0,0}}{\mathrm{d}\mathbb{P}_x}\bigg|_{\mathcal{F}_t} = \frac{\xi(t_0-t, Y_t)}{\xi(t_0, x)}. \tag{4.11}$$

Moreover, this result can be extended continuously to the case $x = 0$ by defining $\xi(t, 0) = \lim_{x\to 0}\xi(t, x) = (2t)^{-\delta/2}\Gamma(\delta/2)^{-1}$. We can also notice that $(D_s)_{s\le t}$ is a martingale with respect to $\mathbb{P}_x$ for $t < t_0$.

Step 2. We prove now that $U(t_0, x)$ defined by

$$U(t_0, x)\,\mathrm{d}x := \mathbb{P}_0\big(Y_{t_0} \in \mathrm{d}x, \tau_{\psi^2}(Y) > t_0\big)$$

satisfies

$$U(t_0, x) = \left(\frac{1}{(2t_0)^{\delta/2}\Gamma(\delta/2)}e^{-\frac{x}{2t_0}} - \frac{1}{2a}\right)x^\nu,$$

which directly implies (4.6).

By conditioning with respect to Y_{t_0}, we obtain

$$U(t_0, x) = \mathbb{P}_0\big(\tau_{\psi^2}(Y) > t_0|Y_{t_0} = x\big)p_0(t_0, x).$$

We employ now a time inversion transformation by using the Radon–Nikodym derivative given in Step 1 in order to express the following equation, for $x < \psi^2(t_0)$:

$$U(t_0, x) = \lim_{t\to t_0, t<t_0} \mathbb{P}_x^{t_0,0}\big(\hat{\tau}(Y) > t\big)p_0(t_0, x)$$

$$= \lim_{t\to t_0, t<t_0} \mathbb{E}_x[\mathbb{1}_{\{\hat{\tau}>t\}}D_t]p_0(t_0, x)$$

$$= \lim_{t\to t_0, t<t_0} \mathbb{E}_x[D_{t\wedge\hat{\tau}}]p_0(t_0, x) - \lim_{t\to t_0, t<t_0} \mathbb{E}_x[\mathbb{1}_{\{\hat{\tau}\le t\}}D_{\hat{\tau}}]p_0(t_0, x),$$

where

$$\hat{\tau} = \tau_{\psi^2(t_0-\cdot)}. \tag{4.12}$$

Since D_t is a continuous martingale, the optimal stopping theorem (in the time inverse filtration) leads to $\mathbb{E}_x[D_{t \wedge \hat{\tau}}] = D_0 = 1$.

Moreover the function $\psi^2(t)$ has the following property:

- if $0 \le x < \psi^2(t)$, then $\xi(t, x) < \frac{1}{2a}$,
- if $x > \psi^2(t)$, then $\xi(t, x) > \frac{1}{2a}$ and $\xi(t, \psi^2(t)) = \frac{1}{2a}$.

In other words, the stopping time $\hat{\tau}$ can be defined as follows:

$$\hat{\tau} = \inf\left\{ t > 0 : \xi(t_0 - t, Y_t) \ge \frac{1}{2a} \right\} = \inf\left\{ t > 0 : D_t \ge \frac{1}{2a\xi(t_0, x)} \right\}.$$

We deduce that

$$\mathbb{E}_x[\mathbb{1}_{\{\hat{\tau} \le t\}} D_{\hat{\tau}}] = \frac{1}{2a\xi(t_0, x)} \mathbb{P}_x(\hat{\tau} \le t).$$

Therefore,

$$U(t_0, x) = p_0(t_0, x)\left(1 - \frac{1}{2a\xi(t_0, x)} \lim_{t \to t_0} \mathbb{P}_x(\hat{\tau} \le t) \right) = p_0(t_0, x)\left(1 - \frac{\mathbb{P}_x(\hat{\tau} \le t_0)}{2a\xi(t_0, x)} \right).$$

Since ψ is a continuous function and $\psi(0) = 0$, we deduce that $\mathbb{P}_x(\hat{\tau} \le t_0) = 1$. Thus, we obtain the result

$$U(t_0, x) = p_0(t_0, x)\left(1 - \frac{1}{2a\xi(t_0, x)} \right).$$

Step 3. As an immediate consequence, the expression of $u_{a,\delta}(t, x)$ defined by (4.6) leads to

$$\mathbb{P}_0\big(\tau_{\psi^2}(Y) > t\big) = \mathbb{P}_0\big(\tau_\psi(X) > t\big)$$

$$= \int_0^{\psi_{a,\delta}^2(t)} U(t, x)\,dx$$

$$= \int_0^{\psi_{a,\delta}(t)} 2w U\big(t, w^2\big)\,dw$$

$$= \int_0^{\psi_{a,\delta}(t)} u_{a,\delta}(t, w)\,dw.$$

The density of the hitting time can be easily obtained by derivation, see Proposition 2.2 in [9]. $\square$

5. Convergence of Algorithm (NI)

Let us now describe the realization and the proof of the convergence of the Algorithm (NI) presented in Section 3.

Realization of the algorithm

One particular important task in this procedure is the simulation of $Z_{\theta_n}^{\delta, M(n-1)}$ in the nth step. The method we use is the following:

- If $\theta_n = \theta_n^{(1)}$, then

$$
\begin{aligned}
\left\| \underline{M}(n-1) + B_{\theta_n} \right\|^2 &= M^2(n-1) + 2M(n-1)\pi_1(U)\psi_{\alpha_n, \lfloor \delta \rfloor}(\theta_n) + \psi_{\alpha_n, \lfloor \delta \rfloor}^2(\theta_n) \\
&= \left(M(n-1) + \pi_1(U)\psi_{\alpha_n, \lfloor \delta \rfloor}(\theta_n) \right)^2 + \left(1 - \pi_1^2(U) \right)\psi_{\alpha_n, \lfloor \delta \rfloor}^2(\theta_n),
\end{aligned}
\tag{5.1}
$$

where π_1 is the projection on the first coordinate and U is a random variable in $\mathbb{R}^{\lfloor \delta \rfloor}$ uniformly distributed on the sphere of radius 1. It suffices now to simulate $X_{\theta_n}^{\delta', 0}$. Since $\theta_n^{(1)}$ and $\theta_n^{(2)}$ are independent, we get

$$
\mathbb{E}\left[f\left(X_{\theta_n}^{\delta', 0} \right) | \theta_n^{(2)} > \theta_n^{(1)} \right] = \int_{\mathbb{R}_+} f(x) w(\theta_n, x) \, \mathrm{d}x,
$$

where

$$
w(t, x) \, \mathrm{d}x = \mathbb{P}\left(X_t^{\delta', 0} \in \mathrm{d}x | \tau_\psi > t \right) = \frac{\mathbb{P}(X_t^{\delta', 0} \in \mathrm{d}x, \tau_\psi > t)}{\mathbb{P}(\tau_\psi > t)} = \frac{u(t, x)}{\int_0^{\psi(t)} u(t, x) \, \mathrm{d}x}
$$

and $u(t, x)$ stands here for $u_{\beta_n, \delta'}(t, x)$ which was already defined in the previous paper [9] and was restated in (4.6). More precisely,

$$
u(t, x) = \left(\frac{1}{2^{\delta'/2-1}} \frac{1}{t^{\delta'/2}} \frac{1}{\Gamma(\delta'/2)} \exp\left(-\frac{x^2}{2t} \right) - \frac{1}{\beta_n} \right) x^{\delta'-1}.
$$

Here for notational simplicity the index ψ of τ_ψ stands for $\psi_{\beta_n, \delta'}$. Let us just notice that the support of $w(\theta_n, \cdot)$ is $[0, \psi_{\beta_n, \delta'}(\theta_n)]$. In order to simulate $X_{\theta_n}^{\delta', 0}$, given $\tau_\psi > \theta_n$, we employ a rejection sampling method. Let S be a random variable defined on the interval $[0, \psi_{\beta_n, \delta'}(\theta_n)]$ with probability density function:
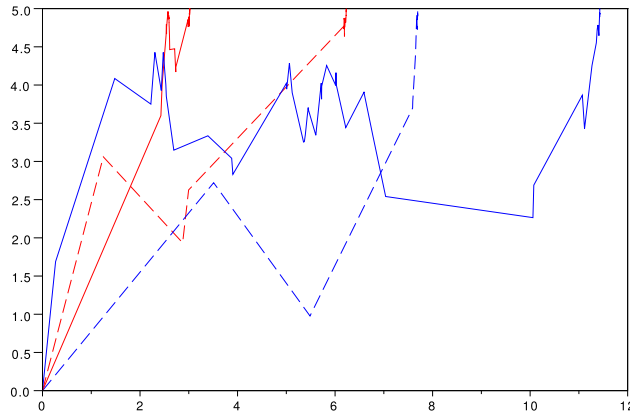
$$
r(x) = \frac{\delta' x^{\delta'-1}}{(\psi_{\beta_n, \delta'}(\theta_n))^{\delta'}}, \qquad \text{for } 0 \le x \le \psi_{\beta_n, \delta'}(\theta_n).
$$

This variable can be easily sampled by using a standard uniform random variable V: S has the same distribution as

$$
\psi_{\beta_n, \delta'}(\theta_n) V^{1/\delta'}.
$$

Considering the following constant:

$$
C = \frac{(\psi_{\beta_n, \delta'}(\theta_n))^{\delta'}}{\delta'} \left(\frac{1}{2^{\delta'/2-1} \theta_n^{\delta'/2} \Gamma(\delta'/2)} - \frac{1}{\beta_n} \right),
$$

Figure 1. A sample of 4 paths of the random walk $(M(n), n \geq 0)$ for $\delta = 2.6$, $\gamma = 0.9$ and $L = 5$.

we observe that $u(\theta_n, x) \leq Cr(x)$ for all x. Then the procedure is the following

1. Sample two independent r.v. U_* and S on respectively, $[0, 1]$ and $[0, \psi_{\beta_n, \delta'}(\theta_n)]$. The first one is uniformly distributed and the p.d.f. of the second one is given by $r(x)$.
2. If $U_* \leq \frac{u(\theta_n, S)}{Cr(S)}$ define $\xi' = S$ otherwise return to the first step.

With this algorithm, the p.d.f. of ξ' is equal to $w(\theta_n, x)$, it has the same distribution as $X^{\delta', 0}_{\theta_n}$ given $\theta_n = \theta_n^{(1)}$.

Finally, we obtain

$$Z^{\delta, M(n-1)}_{\theta_n} = \sqrt{\left(\xi'\right)^2 + M^2(n-1) + 2M(n-1)\pi_1(U)\psi_{\alpha_n, \lfloor \delta \rfloor}(\theta_n) + \psi^2_{\alpha_n, \lfloor \delta \rfloor}(\theta_n)}.$$

• If $\theta_n = \theta_n^{(2)}$ the result is quite similar. We obtain

$$Z^{\delta, M(n-1)}_{\theta_n} = \sqrt{\psi^2_{\beta_n, \delta'}(\theta_n) + M^2(n-1) + 2M(n-1)\pi_1(U)\xi + \xi^2},$$

where ξ is obtained in a similar way as ξ'. We just have to replace δ' by $\lfloor \delta \rfloor$ and β_n by α_n.

Figure 1 presents several paths of the random walk $(M(n), n \geq 0)$ defined by the Algorithm (NI) for $\delta = 2.7$, $\gamma = 0.9$ and for the level $L = 5$.

Theorem 5.1. *Set $\delta \geq 1$. Let us assume that the starting point of the Bessel process is such that $x_0 < L$. The number of steps N^ε of the Algorithm (NI) is almost surely finite. Moreover, there exist constants $C_\delta > 0$ and $\varepsilon_0(\delta) > 0$, such that*

$$\mathbb{E}\big[N^\varepsilon\big] \leq C_\delta |\log \varepsilon|, \qquad \text{for all } \varepsilon \leq \varepsilon_0(\delta).$$

Furthermore Θ_{N^ε} converges in distribution towards τ_L, the hitting time of the threshold L for the δ dimensional Bessel process, as $\varepsilon \to 0$.
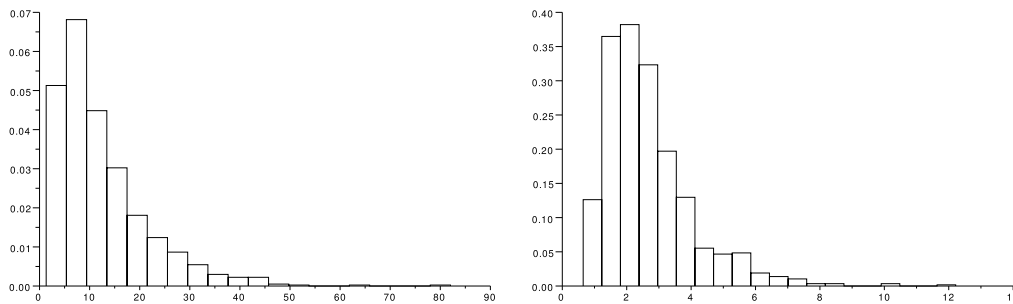
Figure 2. Distribution of the hitting time Θ_{N^ε}. Histograms based on 1000 simulations and performed with the following choice of parameters: $\gamma = 0.9$, $L = 5$ and $\varepsilon = 0.001$, $\delta = 1.5$ on the left and $\delta = 7.5$ on the right.

The following histograms (Figure 2) give the distribution of the hitting times Θ_{N^ε} for $\delta = 1.5$ and $\delta = 7.5$. Higher is the dimension of the Bessel process, smaller is the hitting time.

Proof of Theorem 5.1. Instead of considering the Markov chain $(M(n), n \geq 0)$, we focus our attention on the squared process $R(n) = (M(n))^2$ and we stop the algorithm as soon as $R(n)$ becomes larger than $L^2 - \varepsilon$.

Step 1. *Definition and decomposition of the operator P*. We estimate first the number of steps. Let us remark that by definition $(R(n), n \geq 0)$ is an homogeneous Markov chain, as at each step n we use the same construction which depends only on $M(n-1)$.

Let us start by computing the transition probabilities associated to $R(n)$. We introduce the operator Pf defined, for any non-negative measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$:

$$Pf(x) := \mathbb{E}\big[f\big(R(n)\big)|R(n-1) = x\big] = \mathbb{E}\big[f\big(\|\underline{\sqrt{x}} + B_{\theta_n}\|^2 + \big(X_{\theta_n}^{\delta',0}\big)^2\big)\big], \qquad (5.2)$$

where θ_n is defined in (3.5). Since $R(n)$ is an homogeneous Markov chain, the transition Pf does not depend on the time n. For notational simplicity, we neglect some indexes: the step n; $X^{\delta',0}$ is replaced by X; $\theta_n^{(i)}$ by $\theta^{(i)}$, for $i = 1, 2$; α_n is replaced by α and β_n is replaced by β. We will express (5.2) by splitting it into two parts $P_i f(x)$, $i = 1, 2$ with

$$P_i f(x) = \mathbb{E}\big[f\big(\|\underline{\sqrt{x}} + B_{\theta^{(i)}}\|^2 + (X_{\theta^{(i)}})^2\big)\mathbb{1}_{\{\theta = \theta^{(i)}\}}\big].$$

Thus

$$Pf(x) = P_1 f(x) + P_2 f(x). \qquad (5.3)$$

The class of functions that will be considered in the following satisfies the hypothesis:

(H) The function f is such that

$$f^{(p)} \leq 0, \qquad \forall p \in \{1, 2, 3, 4\},$$

this means that the first four derivatives of f are negative. An example of such a function will be used later on.

Step 2. *Distributions associated with $\theta^{(i)}$.* Let us denote by $U_i(t) \, dt$ the distribution of $\theta^{(i)}$ and its support $[0, s_i]$. With the notation in (3.2) we have

$$s_1 = T_{\alpha, \lfloor \delta \rfloor} \quad \text{and} \quad s_2 = T_{\beta, \delta'}. \tag{5.4}$$

These distributions are those of stopping times corresponding to the function $\psi_{a, \delta}(t)$, see (3.1), with a suitable parameter a and a suitable dimension δ. By applying the scaling property (4.3) we get

$$\psi_{\alpha, \lfloor \delta \rfloor}(t) = \sqrt{\lfloor \delta \rfloor s_1} \, \Phi\left(\frac{t}{s_1}\right). \tag{5.5}$$

By using the same arguments, the scaling property associated to $\theta^{(2)}$ is

$$\psi_{\beta, \delta'}(t) = \sqrt{\delta' s_2} \, \Phi\left(\frac{t}{s_2}\right), \qquad \text{with } \delta' = \delta - \lfloor \delta \rfloor.$$

Since

$$U_1(t) \, dt = \mathbb{P}_0\left(\theta^{(1)} \in dt\right)$$

$$= \frac{1}{2\Gamma(\lfloor \delta \rfloor / 2) 2^{\lfloor \delta \rfloor / 2 - 1} s_1^{\lfloor \delta \rfloor / 2} t} \left(2t \log\left(\frac{s_1}{t}\right)^{\lfloor \delta \rfloor / 2}\right)^{\lfloor \delta \rfloor / 2} \mathbb{1}_{[0, s_1]}(t) \, dt$$

$$= \frac{c_{\lfloor \delta \rfloor}}{t} \Phi^{\lfloor \delta \rfloor}\left(\frac{t}{s_1}\right) dt,$$

with $c_d = \frac{1}{\Gamma(d/2)}(\frac{d}{2})^{d/2}$, we obtain the scaling property:

$$U_1(s_1 t) = \frac{c_{\lfloor \delta \rfloor}}{s_1 t} \Phi^{\lfloor \delta \rfloor}(t) \quad \text{and} \quad U_2(s_2 t) = \frac{c_{\delta'}}{s_2 t} \Phi^{\delta'}(t). \tag{5.6}$$

With our choice of α and β, we have

$$s_1 = T_{\alpha, \lfloor \delta \rfloor} = \left[\frac{\alpha}{\Gamma(\frac{\lfloor \delta \rfloor}{2}) 2^{\frac{\lfloor \delta \rfloor}{2} - 1}}\right]^{\frac{2}{\lfloor \delta \rfloor}}$$

$$= \frac{e\gamma^2 (L^2 - x)^2}{(\sqrt{(\lfloor \delta \rfloor - \delta \gamma) x + \delta \gamma L^2} + \sqrt{\lfloor \delta \rfloor x})^2} \tag{5.7}$$

$$= s_2.$$

By using (4.5), we get

$$W_{\alpha, \lfloor \delta \rfloor} = \sup_{t \in [0, s_1]} \psi_{\alpha, \lfloor \delta \rfloor}(t) = \sqrt{\frac{\lfloor \delta \rfloor}{e}} s_1 \tag{5.8}$$

and the same property holds for s_2

$$W_{\beta,\delta'} = \sup_{t \in [0,s_2]} \psi_{\beta,\delta'}(t) = \sqrt{\frac{\delta'}{e}} s_2. \tag{5.9}$$

Since we stop the Markov chain $R(n)$ in order not to hit L^2, we have

$$(\sqrt{x} + W_{\alpha,\lfloor\delta\rfloor})^2 + W_{\beta,\delta'}^2 = x + \frac{\lfloor\delta\rfloor}{e}s_1 + 2\sqrt{x}\sqrt{\frac{\lfloor\delta\rfloor}{e}}s_1 + \frac{\delta'}{e}s_2$$

$$= x + \frac{\delta}{e}s_1 + 2\sqrt{x}\sqrt{\frac{\lfloor\delta\rfloor}{e}}s_1 \tag{5.10}$$

$$= x + \gamma(L^2 - x),$$

by using the choice of $s_1 = s_2$ and the value of s_1 given in (5.7). We deduce that for x close to 0, s_1 is close to $\gamma e L^2/\delta$ and for x close to L^2, s_1 is of the same order as $e\gamma^2(L^2 - x)^2/(4\lfloor\delta\rfloor L^2)$.

Step 3. *Computation of $P_1 f(x)$*. Using the definition of $P_1 f$, the identity (5.1) and the distribution of $\theta^{(1)}$ denoted by $U_1(t)\,dt$ we get

$$P_1 f(x) = \mathbb{E}\big[f\big(\|\sqrt{x} + B_\theta\|^2 + (X_\theta)^2\big)\mathbb{1}_{\{\theta=\theta^{(1)}\}}\big]$$

$$= \int_{\mathcal{S}^1}\int_0^{s_1} \mathbb{E}\big[f\big((\sqrt{x} + \pi_1(z)\psi_{\alpha,\lfloor\delta\rfloor}(t))^2 + \psi_{\alpha,\lfloor\delta\rfloor}^2(t)(1 - \pi_1^2(z)) + X_t^2\big)\mathbb{1}_{\{\theta^{(2)}>t\}}\big]$$

$$\tag{5.11}$$

$$\times U_1(t)\,dt\,\sigma(dz)$$

$$= \int_{\mathcal{S}^1}\int_0^{s_1} \mathbb{E}\big[f\big(x + 2\sqrt{x}\pi_1(z)\psi_{\alpha,\lfloor\delta\rfloor}(t) + \psi_{\alpha,\lfloor\delta\rfloor}^2(t) + X_t^2\big)\mathbb{1}_{\{\theta^{(2)}>t\}}\big]U_1(t)\,dt\,\sigma(dz).$$

We denote here by $\mathcal{S}^1$ the unit sphere in $\mathbb{R}^{\lfloor\delta\rfloor}$, $\sigma(dz)$ the uniform surface measure on this sphere and $\pi_1(z)$ the projection on the first coordinate and α is chosen so that (5.7) is satisfied. Let us notice that the variable $x + 2\sqrt{x}\pi_1(z)\psi_{\alpha,\lfloor\delta\rfloor}(t) + \psi_{\alpha,\lfloor\delta\rfloor}^2(t) + X_t^2$ always stays in the interval $[0, L^2]$ on the event $\theta^{(2)} > t$. Consider the Taylor expansion of f in a neighborhood of x. By using (H) we have that $f^{(4)}(t) \leq 0$ on the whole interval $[0, L^2]$. Hence,

$$P_1 f(x) \leq f(x)G_1 + f'(x)G_2 + \frac{1}{2}f''(x)G_3 + \frac{1}{6}f^{(3)}(x)G_4. \tag{5.12}$$

It suffices now to compute G_k for $k = 1, 2, 3, 4$ where G_k is defined by

$$G_k := \int_{\mathcal{S}^1}\int_0^{s_1} \mathbb{E}\big[(\psi_{\alpha,\lfloor\delta\rfloor}^2(t) + 2\sqrt{x}\pi_1(z)\psi_{\alpha,\lfloor\delta\rfloor}(t) + X_t^2)^{k-1}\mathbb{1}_{\{\theta^{(2)}>t\}}\big]U_1(t)\,dt\,\sigma(dz). \tag{5.13}$$

In particular $G_1 = \mathbb{P}[\theta^{(2)} > \theta^{(1)}]$. Using symmetry arguments, the term associated to the projection vanishes, and we can split G_2 into two parts, $G_2 = G_{2,1} + G_{2,2}$:

$$G_{2,1} := \int_0^{s_1} \psi_{\alpha,\lfloor\delta\rfloor}^2(t)\mathbb{P}(\theta^{(2)} > t)U_1(t)\,dt \quad \text{and} \quad G_{2,2} := \int_0^{s_1} \mathbb{E}\big[X_t^2\mathbb{1}_{\{\theta^{(2)}>t\}}\big]U_1(t)\,dt.$$

By changing the variable $s_1 u = t$ and the scaling properties developed in Step 2, we obtain:

$$G_{2,1} = \int_0^1 \psi_{\alpha,\lfloor\delta\rfloor}^2(s_1 u)\mathbb{P}(\theta^{(2)} > s_1 u)U_1(s_1 u)s_1\,du$$
$$= \int_0^1 s_1\lfloor\delta\rfloor\Phi^2(u)\left(\int_u^1 U_2(s_1 w)\,dw\right)\frac{c_{\lfloor\delta\rfloor}}{s_1 u}\Phi^{\lfloor\delta\rfloor}(u)s_1\,du.$$

Using now the equality $s_1 = s_2$ and a suitable change of variable, we get

$$G_{2,1} = s_1\lfloor\delta\rfloor c_{\lfloor\delta\rfloor}\int_0^1\int_u^1 \frac{1}{u}\Phi^{\lfloor\delta\rfloor+2}(u)s_1 U_2(s_1 v)\,dv\,du = s_1\kappa_{2,1}, \qquad (5.14)$$

where

$$\kappa_{2,1} := \lfloor\delta\rfloor c_{\lfloor\delta\rfloor}c_{\delta'}\int_0^1\int_u^1 \frac{1}{uv}\Phi^{\lfloor\delta\rfloor+2}(u)\Phi^{\delta'}(v)\,dv\,du > 0. \qquad (5.15)$$

Notice that $\kappa_{2,1}$ is a constant which depends only on the dimension δ. We can also prove that there exists a constant $\kappa_{2,2}$ independent of x such that $G_{2,2} = s_1\kappa_{2,2}$. Indeed, we have

$$G_{2,2} = \int_0^{s_1}\mathbb{E}\big[X_t^2\mathbb{1}_{\{\theta^{(2)}>t\}}\big]U_1(t)\,dt = s_1\int_0^1\mathbb{E}\big[X_{s_1 u}^2\mathbb{1}_{\{\theta^{(2)}>s_1 u\}}\big]U_1(s_1 u)\,du$$
$$= \int_0^1\mathbb{E}\big[X_{s_1 u}^2\mathbb{1}_{\{\forall r\leq s_1 u: X_r\leq\psi_{\beta,\delta'}(r)\}}\big]\frac{c_{\lfloor\delta\rfloor}}{u}\Phi^{\lfloor\delta\rfloor}(u)\,du.$$

Using the scaling property of the Bessel process, we get

$$G_{2,2} = \int_0^1\mathbb{E}\big[s_1 X_u^2\mathbb{1}_{\{\forall r\leq u: X_r\leq\psi_{\beta,\delta'}(s_1 r)/\sqrt{s_1}\}}\big]\frac{c_{\lfloor\delta\rfloor}}{u}\Phi^{\lfloor\delta\rfloor}(u)\,du = s_1\kappa_{2,2}$$

since $s_1 = s_2$ and thus $\psi_{\beta,\delta'}(s_1 r)/\sqrt{s_1} = \sqrt{\delta'}\Phi(r)$ does not depend on x but only on δ. To sum up, we have proved the existence of two constants $\kappa_{2,i}$, $i = 1, 2$, independent of x satisfying

$$G_2 = \kappa_2 s_1, \qquad \text{where } \kappa_2 = \kappa_{2,1} + \kappa_{2,2}. \qquad (5.16)$$

Let us now focus our attention on G_3 defined in (5.13).

While developing the square of $\psi_{\alpha,\lfloor\delta\rfloor}^2(t) + 2\sqrt{x}\pi_1(z)\psi_{\alpha,\lfloor\delta\rfloor}(t) + X_t^2$, we obtain 6 terms:

$$H_1 = \psi_{\alpha,\lfloor\delta\rfloor}^4(t), \qquad H_2 = 4x\pi_1^2(z)\psi_{\alpha,\lfloor\delta\rfloor}^2(t), \qquad H_3 = X_t^4,$$
$$H_4 = 4\psi_{\alpha,\lfloor\delta\rfloor}^3(t)\sqrt{x}\pi_1(z), \qquad H_5 = 4\sqrt{x}\pi_1(z)\psi_{\alpha,\lfloor\delta\rfloor}(t)X_t^2, \qquad H_6 = 2\psi_{\alpha,\lfloor\delta\rfloor}^2(t)X_t^2.$$

Therefore G_3 can be split into 6 terms: $G_3 = \sum_{j=1}^6 G_{3,j}$ with

$$G_{3,j} := \int_{\mathcal{S}^1}\int_0^{s_1}\mathbb{E}[H_j\mathbb{1}_{\{\theta^{(2)}>t\}}]U_1(t)\,dt\sigma\,(dz).$$

Now, let us compute $G_{3,j}$ for $j = 1, \ldots, 6$. First, we note that, due to symmetry properties of the variable z, $G_{3,4} = G_{3,5} = 0$. By similar arguments as those included in the computation of G_2, we get:

$$G_{3,1} = s_1^2 \kappa_{3,1} \qquad \text{with } \kappa_{3,1} := \lfloor \delta \rfloor^2 c_{\lfloor \delta \rfloor} c_{\delta'} \int_0^1 \int_u^1 \frac{1}{uv} \Phi^{\lfloor \delta \rfloor + 4}(u) \Phi^{\delta'}(v) \, \mathrm{d}v \, \mathrm{d}u > 0, \qquad (5.17)$$

$$G_{3,2} = x s_1 \kappa_{3,2} \qquad \text{with } \kappa_{3,2} := 4\kappa_{2,1} \int_{\mathcal{S}^1} \pi_1^2(z) \sigma(\mathrm{d}z) > 0, \qquad (5.18)$$

$$G_{3,3} = s_1^2 \kappa_{3,3} \qquad \text{with } \kappa_{3,3} := \int_0^1 \mathbb{E}\left[X_u^4 \mathbb{1}_{\{\forall r \leq u : X_r \leq \sqrt{\delta'} \Phi(r)\}} \right] \frac{c_{\lfloor \delta \rfloor}}{u} \Phi^{\lfloor \delta \rfloor}(u) \, \mathrm{d}u > 0, \qquad (5.19)$$

$$G_{3,6} = s_1^2 \kappa_{3,6}$$

$$\text{with } \kappa_{3,6} := 2 \int_0^1 \mathbb{E}\left[X_u^2 \mathbb{1}_{\{\forall r \leq u : X_r \leq \sqrt{\delta'} \Phi(r)\}} \right] \frac{c_{\lfloor \delta \rfloor}}{u} \lfloor \delta \rfloor \Phi^{\lfloor \delta \rfloor + 2}(u) \, \mathrm{d}u > 0.$$

To sum up, there exist two positive constants κ_3 and $\tilde{\kappa}_3$ independent of x such that

$$G_3 = \kappa_3 x s_1 + \tilde{\kappa}_3 s_1^2. \qquad (5.21)$$

Finally, we consider the expression G_4 defined by (5.13). We are not going to compute it explicitly as we have just performed it already for the first terms. Due to the symmetry property of the variable z, the expansion of G_4 coupled to the computation of $(\psi_{\alpha, \lfloor \delta \rfloor}^2(t) + 2\sqrt{x} \pi_1(z) \psi_{\alpha, \lfloor \delta \rfloor}(t) + X_t^2)^3$ leads to terms which are either positive or equal to 0. Hence, we can conclude that

$$G_4 \geq 0. \qquad (5.22)$$

Step 4. *Computation of* $P_2 f(x)$. Using the definition of $P_2 f$, we obtain:

$$P_2 f(x) = \mathbb{E}\left[f\left(\|\underline{\sqrt{x}} + B_\theta\|^2 + (X_\theta)^2 \right) \mathbb{1}_{\{\theta = \theta^{(2)}\}} \right]$$

$$= \int_0^{s_2} \mathbb{E}\left[f\left(\|\underline{\sqrt{x}} + B_t\|^2 + \psi_{\beta, \delta'}^2(t) \right) \mathbb{1}_{\{\theta^{(1)} > t\}} \right] U_2(t) \, \mathrm{d}t$$

$$= \int_0^{s_2} \mathbb{E}\left[f\left(x + 2\langle \underline{\sqrt{x}}, B_t \rangle + \|B_t\|^2 + \psi_{\beta, \delta'}^2(t) \right) \mathbb{1}_{\{\theta^{(1)} > t\}} \right] U_2(t) \, \mathrm{d}t.$$

By hypothesis (H), f'' is non positive on the support of the Markov chain. Then, by using a Taylor expansion, the following bound holds: for any $x \in [0, L^2 - \varepsilon]$

$$P_2 f(x) \leq f(x) \mathbb{P}\left(\theta^{(1)} > \theta^{(2)} \right)$$

$$+ f'(x) \int_0^{s_2} \mathbb{E}\left[\left(2\langle \underline{\sqrt{x}}, B_t \rangle + \|B_t\|^2 + \psi_{\beta, \delta'}^2(t) \right) \mathbb{1}_{\{\theta^{(1)} > t\}} \right] U_2(t) \, \mathrm{d}t.$$

The integral expression contains three distinct terms. The first one associated to the scalar product is equal to zero since the distribution of B_t given $\{\theta^{(1)} > t\}$ is rotationally invariant. The second

and third terms are positive. We therefore deduce that for any function f satisfying (H), the following bound holds:

$$P_2 f(x) \leq f(x)\mathbb{P}\big(\theta^{(1)} > \theta^{(2)}\big). \tag{5.23}$$

Step 5. *Application to a particular function f.* Let us introduce the function $f_\varepsilon : [0, L^2) \mapsto \mathbb{R}$ defined by

$$f_\varepsilon(x) = \log\left(\frac{L^2 - x}{(1 - \gamma)\varepsilon}\right), \tag{5.24}$$

where $\gamma < 1$ is the constant close to 1 already introduced in (5.10). Let us assume that the Markov chain M starts with the initial value $x \in [0, L^2 - \varepsilon]$ that is $R(0) = x$. We will prove that $f_\varepsilon(R(1))$ is a non-negative random variable. Indeed for any y in the support of $R(1)$ we have:

$$0 \leq y \leq (\sqrt{x} + W_{\alpha, \lfloor \delta \rfloor})^2 + W_{\beta, \delta'}^2,$$

where $W_{\alpha, \lfloor \delta \rfloor}$ is defined by (5.8) and $W_{\beta, \delta'}$ by (5.9). Using the identity (5.10), we obtain

$$f_\varepsilon(y) \geq \log\left(\frac{(L^2 - x)}{\varepsilon}\right) \geq 0$$

since $L^2 - x \geq \varepsilon$. We deduce that f_ε is a non-negative function on the support of the Markov chain stopped at the first exit time of the interval $[0, L^2 - \varepsilon]$.

Let us now apply the operator P defined by (5.2) to the function f_ε. Since

$$f_\varepsilon'(x) = -\frac{1}{L^2 - x} < 0, \qquad f_\varepsilon^{(k)}(x) = -\frac{(k-1)!}{(L^2 - x)^k} < 0, \qquad \text{for } k = 2, 3, 4$$

the condition (H) is satisfied and we obtain by (5.12) and the computation of $G_1, \ldots, G_4$:

$$P_1 f_\varepsilon(x) \leq f_\varepsilon(x)\mathbb{P}\big(\theta^{(2)} > \theta^{(1)}\big) - \frac{s_1}{(L^2 - x)^2}\big(\kappa_2(L^2 - x) + \kappa_3 x\big), \tag{5.25}$$

by using (5.16) and (5.21).

Due to the explicit expression of s_1 in (5.4), there exists a constant $\kappa > 0$ independent of x and ε such that

$$P_1 f_\varepsilon(x) \leq f_\varepsilon(x)\mathbb{P}\big(\theta^{(2)} > \theta^{(1)}\big) - \kappa, \qquad \forall x \in \big[0, L^2 - \varepsilon\big). \tag{5.26}$$

Combining this with (5.23), we get

$$P f_\varepsilon(x) - f_\varepsilon(x) \leq -\kappa, \qquad \forall x \in \big[0, L^2 - \varepsilon\big].$$

By using a comparison theorem of the classical potential theory, see Norris [28] (Theorem 4.2.3, page 139), we deduce that

$$\mathbb{E}_x\big[N^\varepsilon\big] \leq \frac{f_\varepsilon(x)}{\kappa}. \tag{5.27}$$

In particular, for $x = 0$, the desired result in the statement of the theorem is proved.

Step 6. The time Θ_{N^ε} given by the algorithm is close to the first hitting time τ_L.

Let us denote by F (resp. F^ε) the cumulative distribution function of the random variable τ_L (resp. Θ_{N^ε}). We construct these two random variables on the same paths; the law of the squared Bessel process of dimension δ is obtained as a succession of sums of two independent squared Bessel processes of dimension $\lfloor\delta\rfloor$ denoted by $Y^{\lfloor\delta\rfloor,k}$ and δ' denoted by $\hat{Y}^{\delta',k}$ on random time intervals $[\Theta_{k-1}, \Theta_k)$ until the hitting time Θ_{N^ε} and afterwards, the paths are generated just by a squared Bessel process of dimension δ starting in $M(N^\varepsilon)$ and independent from the past. Since $\tau_L \geq \Theta_{N^\varepsilon}$ a.s. we immediately obtain the first bound

$$F(t) \leq F^\varepsilon(t), \qquad t \geq 0.$$

Furthermore, for any small $\alpha > 0$,

$$1 - F(t) = \mathbb{P}(\tau_L > t) = \mathbb{P}(\tau_L > t, \Theta_{N^\varepsilon} \leq t - \alpha) + \mathbb{P}(\tau_L > t, \Theta_{N^\varepsilon} > t - \alpha)$$

$$\leq \mathbb{P}(\tau_L > t, \Theta_{N^\varepsilon} \leq t - \alpha) + \mathbb{P}(\Theta_{N^\varepsilon} > t - \alpha) \tag{5.28}$$

$$\leq \mathbb{P}(\tau_L > t, \Theta_{N^\varepsilon} \leq t - \alpha) + 1 - F^\varepsilon(t - \alpha).$$

At time Θ_{N^ε} the sum of the two squared Bessel processes is in the small neighborhood of the threshold L^2, namely $L^2 - \|Y^{\lfloor\delta\rfloor,N^\varepsilon}_{\Theta_{N^\varepsilon}}\|^2 - \|\hat{Y}^{\delta',N^\varepsilon}_{\Theta_{N^\varepsilon}}\|^2 \leq \varepsilon$. Using the strong Markov property, we obtain

$$\mathbb{P}(\tau_L > t, \Theta_{N^\varepsilon} \leq t - \alpha) \leq F^\varepsilon(t - \alpha) \sup_{y \in [\sqrt{L^2-\varepsilon}, L]} \mathbb{P}_y(\tau_L > \alpha). \tag{5.29}$$

Applying Shiga and Watanabe's result (2.5), the Bessel process of dimension $\delta > 1$ is stochastically larger than the Bessel process of dimension 1 which has the same law as $|B_t|$. Here B stands for a 1-dimensional Brownian motion. By these observations, the following upper bound holds, for any $y \in [\sqrt{L^2 - \varepsilon}, L]$:

$$\mathbb{P}_y(\tau_L > \alpha) \leq \mathbb{P}_{L^\varepsilon}(\tau_L > \alpha) \leq \mathbb{P}_{L^\varepsilon}\left(\sup_{0 \leq t \leq \alpha} |B_t| < L\right), \qquad \text{with } L^\varepsilon = \sqrt{L^2 - \varepsilon}$$

$$\leq \mathbb{P}_{L^\varepsilon}\left(\sup_{0 \leq t \leq \alpha} B_t < L\right) \leq \mathbb{P}_0\left(\sup_{0 \leq t \leq \alpha} B_t < L - L^\varepsilon\right) \tag{5.30}$$

$$\leq \mathbb{P}_0\left(\sup_{0 \leq t \leq \alpha} B_t < \frac{\varepsilon}{L}\right) \leq \frac{\varepsilon}{L\sqrt{2\alpha\pi}}.$$

By combining (5.28), (5.29) and (5.30), we have

$$F^\varepsilon(t - \alpha)\left(1 - \frac{\varepsilon}{L\sqrt{2\alpha\pi}}\right) \leq F(t) \leq F^\varepsilon(t), \qquad t \geq 0. \tag{5.31}$$

Furthermore, by construction, Θ_{N^ε} is an increasing sequence of positive random variables as ε goes to 0. We deduce that, for any fixed $t \geq 0$, $F^\varepsilon(t)$ decreases as ε goes to 0. Moreover, the right-hand side of (5.30) ensures a lower bound for the sequence which leads to the convergence

of $F^\varepsilon(t)$. The limit is denoted by $F^0(t)$ and (5.30) implies that $F^0(t) = F(t)$ for any $t \geq 0$. Consequently, Θ_{N^ε} converges to τ_L in distribution as ε goes to 0. This ends the proof. $\square$

Remark 5.2. The first step of the algorithm could be simplified when the starting position x_0 is equal to 0. The procedure then consists in first choosing α_1 such that $\psi_{\alpha_1, \delta}(t) < L$ for all $t \geq 0$. Then we simulate the first hitting time of this moving boundary which is given by a transformation of the Gamma distribution. We denote this random variable by θ_1 and we compute the value of the process at this time:

$$M(1) = \psi_{\alpha_1, \delta}(\theta_1).$$

Finally, we set $\Theta_1 = \Theta_0 + \theta_1$. Even if this modified version of the Algorithm (NI) seems simpler (just the first step is different), in the following we prefer to illustrate the convergence results associated to the Algorithm (NI).

6. Numerical results

In this section, we will discuss some numerical results based on the algorithm for approaching the hitting time (developed in Section 5). A particularly important task in such an iterative method is to estimate the number of steps or even the number of times the uniform random generator is used. The Algorithm (NI) presented in Section 5 allows to simulate hitting times for Bessel processes of non-integer dimensions $\delta > 1$. We will therefore only present results in that context and refer to the previous work [9] for Bessel processes of integer dimensions.

6.1. Number of steps versus ε

The number of steps of the algorithm is of prime interest. Classical time splitting method, in order to simulate particular paths of stochastic processes, can be used for classical diffusion processes with regular diffusion and drift coefficients if the study is restricted to some fixed time intervals. Here the diffusion is singular, the classical methods could not be applied, nevertheless the approximation procedure (NI) developed in Section 5 is different from classical splitting methods and holds for any given time. In particular, we are not able to compare our algorithm with other methods and we will just describe the relevance of (NI) by the estimation of the average number of steps. The algorithm used in order to simulate the hitting time of the level L by the Bessel process, gives an approximated hitting time Θ_{N^ε} and the corresponding position $M(N^\varepsilon)$ which satisfies:

$$L^2 - \left(M\left(N^\varepsilon\right)\right)^2 \leq \varepsilon.$$

The number of iterations will decrease with respect to the parameter ε. The average number of steps $\mathbb{E}[N^\varepsilon]$ is bounded from above by the logarithm of ε up to a multiplicative constant (Theorem 5.1). Let us therefore choose different values of ε and approximate through a law
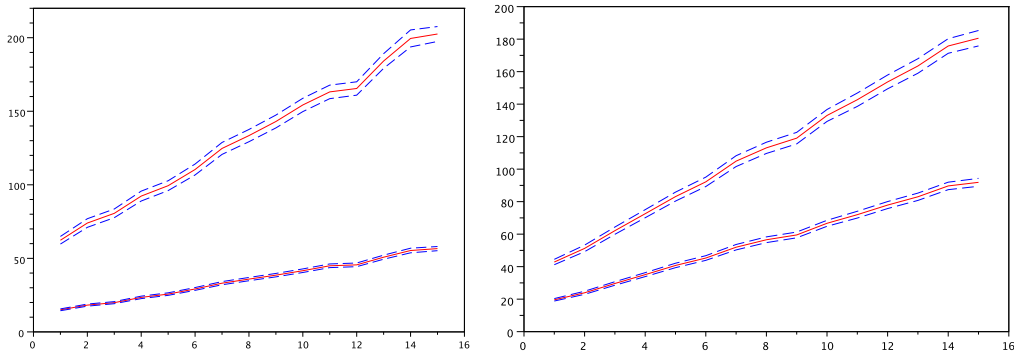
Figure 3. The average number of steps N^ε for different precisions $\varepsilon_k = 0.5^k$ together with its 95%-confidence interval (dashed lines) based on $\aleph = 1000$ simulations of the hitting time, for dimension $\delta = 2.2$ (left) and $\delta = 4.7$ (right).

of large number this average (we denote by $\aleph$ the number of independent simulations; here $\aleph = 1000$).

$$\varepsilon_k = 0.5^k, \qquad k = 1, \dots, 15.$$

The results concern two different dimensions of the Bessel process $\delta = 2.2$ and $\delta = 4.7$ and we fix the parameter $\gamma = 0.95$ in the Algorithm (NI) (this parameter is fixed for the whole numerical section). Note that each step of the (NI) algorithm is associated to a comparison between particular hitting times $\theta_n^{(1)}$ and $\theta_n^{(2)}$, the first one is associated with a Bessel process of dimension $\lfloor \delta \rfloor$ and the second one is associated with a Bessel process of dimension $\delta - \lfloor \delta \rfloor$. In the following we are also interested in the average number of steps $N_{\text{integer}}^\varepsilon$ satisfying

$$N_{\text{integer}}^\varepsilon := \#\big\{1 \le n \le N^\varepsilon : \theta_n^{(1)} < \theta_n^{(2)}\big\}.$$

The Figures 3 represent both the estimated average number of steps $\mathbb{E}[N^\varepsilon]$ and its confidence interval (three upper curves) and the estimated average number $\mathbb{E}[N_{\text{integer}}^\varepsilon]$ and its 95%-confidence interval (three lower curves).

6.2. Number of steps versus the dimension of the Bessel process

In [9], the authors pointed out that the number of steps of the WoMS algorithm increases as the dimension of the integer Bessel process becomes larger. This means in particular that each step of the Algorithm (NI) will take more time as the dimension of the Bessel process becomes large. This feature can be illustrated by a simple experiment: let us sample a sequence of independent random variables $(U_n)_{n \ge 1}$ uniformly distributed on the sphere in $\mathbb{R}^\delta$, with $\delta \in \mathbb{N}$, centered in $(0.5, 0, 0, \dots, 0)$ and with radius 0.5. We stop the trials as soon as one random variable U_n belongs to the ε-neighborhood of the unit sphere. We observe that the averaged number of trials increases with the dimension since the proportion of the small sphere surface which is close to
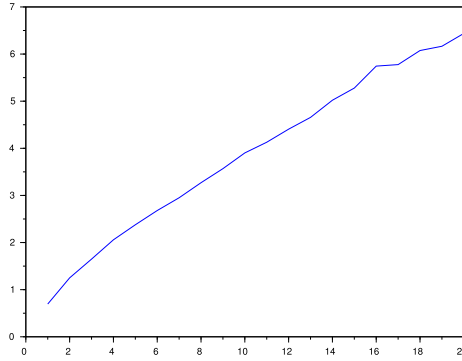
Figure 4. Logarithm of the average number of trials before hitting a ε-neighborhood ($\varepsilon = 0.1$) of the unit sphere in dimension δ ($\delta = 1, \ldots, 20$) based on $10\,000$ simulations.

the unit sphere decreases with the dimension. Figure 4 represents the logarithm of the average number of trials when the dimension increases.

This simple experiment heuristically explains why in the integer dimension case, the walk on moving spheres needs more steps as soon as the dimension increases. We can illustrate this feature by the cpu time needed by the Algorithm (NI).

Let us focus our attention on the non-integer case. We observe unexpected effects with respect to the dimension: on one hand if $\lfloor \delta \rfloor$ is fixed and the dimension increases then the average number of steps decreases, on the other hand if $\delta - \lfloor \delta \rfloor$ is fixed and the dimension increases, so does the number of steps. For the simulation, we set $\aleph = 1000$, $\varepsilon = 0.01$ and the level height $L = 5$ (Figure 5).

Let us explain what happens in a simplified way. Each step of the Algorithm (NI) is based on the simulation of two independent random variables: A the value of the squared Bessel process of integer dimension $\lfloor \delta \rfloor$, B the value of the squared Bessel process of dimension δ', both stopped
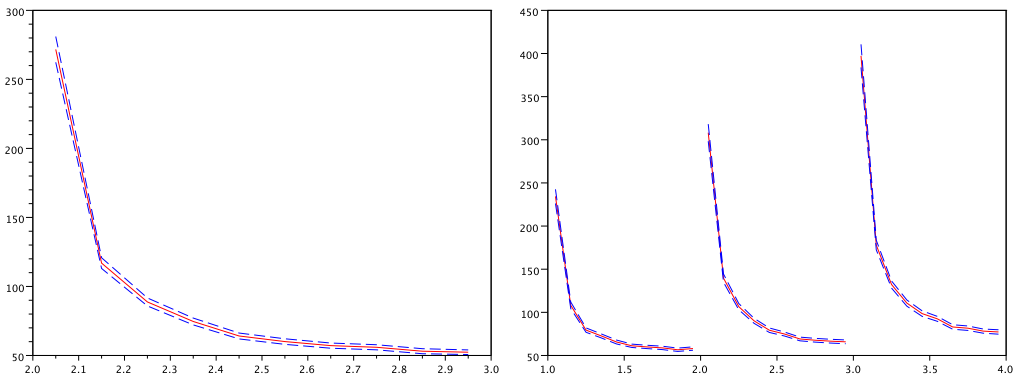
Figure 5. The average number of steps N^ε for varying δ. We set $\aleph = 1000$, $\varepsilon = 0.01$ and the level height $L = 5$. On the right $\lfloor \delta \rfloor$ is fixed and on the left $\delta - \lfloor \delta \rfloor$ is fixed.
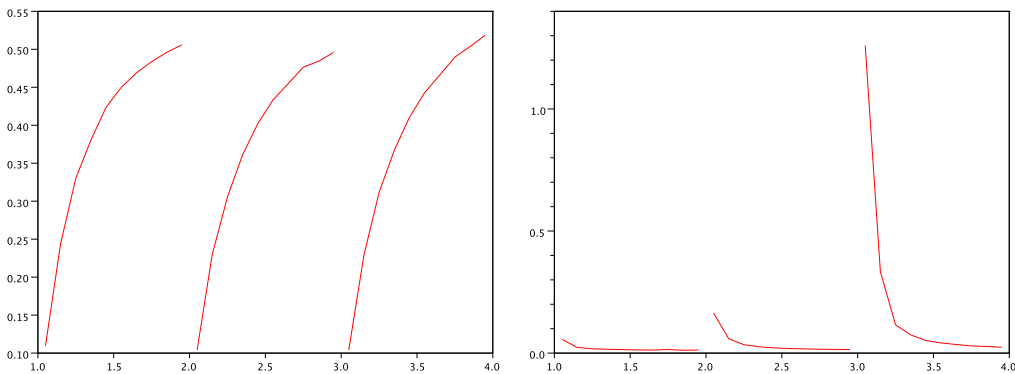
Figure 6. Averaged proportion $\mathbb{E}[N^{\varepsilon}_{\text{integer}}/N^{\varepsilon}]$ versus δ with $\aleph = 1000$ (left) and average cpu time (seconds) for Algorithm (NI) on a 2.3 GHz Intel Core i5 processor with the Scilab software versus δ with $\aleph = 100$ on a (right). For both figures, we set $\varepsilon = 0.01$ and the level height $L = 5$.

as soon as one of them is sufficiently large (let us say: larger than λ_A and λ_B). Then the algorithm stops if the sum $A + B$ is sufficiently large, say larger than $\lambda_A + \lambda_B$. What should be observed in such a context?

- If B/λ_B is stochastically much larger than A/λ_A, then it is particularly difficult for the sum $A + B$ to reach the level $\lambda_A + \lambda_B$. Therefore, the algorithm needs a lot of steps for the condition to be satisfied. This situation appears in fact when δ' is small: the proportion $N^{\varepsilon}_{\text{integer}}/N^{\varepsilon}$ is much smaller than 50% (see Figure 6).
- If B/λ_B is stochastically of the same order as A/λ_A, then the sum $A + B$ is more likely to overcome the threshold $\lambda_A + \lambda_B$; therefore the algorithm needs less steps. This situation appears when δ' is close to 1: the proportion $N^{\varepsilon}_{\text{integer}}/N^{\varepsilon}$ is close to 50% (see Figure 6).

We observe the averaged proportion $\mathbb{E}[N^{\varepsilon}_{\text{integer}}/N^{\varepsilon}]$ as the dimension of the Bessel process increases, see Figure 6. This proportion seems to depend mainly on the fractional part of the dimension. One way to increase the efficiency of Algorithm (NI) would be to choose other values for λ_A and λ_B in such a way that the proportion always stays close to 50%. The thresholds λ_A and λ_B in this informal discussion corresponds in fact in the Algorithm (NI) to the coefficients $I(\delta, x)$ and $N(\delta, x)$ defined by (3.3) and (3.4). The only condition required for the convergence of the Algorithm is:

$$\gamma\left(L^2 - x\right) = 2\sqrt{x}\frac{\sqrt{\lfloor\delta\rfloor}}{\sqrt{e}}\left(\frac{I(\delta, x)}{\Gamma(\frac{\lfloor\delta\rfloor}{2})2^{\lfloor\delta\rfloor/2-1}}\right)^{1/\lfloor\delta\rfloor} + \frac{\lfloor\delta\rfloor}{e}\left(\frac{I(\delta, x)}{\Gamma(\frac{\lfloor\delta\rfloor}{2})2^{\lfloor\delta\rfloor/2-1}}\right)^{2/\lfloor\delta\rfloor}$$
$$+ \frac{\delta'}{e}\left(\frac{N(\delta, x)}{\Gamma(\frac{\delta'}{2})2^{\delta'/2-1}}\right)^{2/\delta'},$$

where $\delta' = \delta - \lfloor\delta\rfloor$. We observe that there is one degree of freedom in the choice of the couple $(I(\delta, x), N(\delta, x))$.
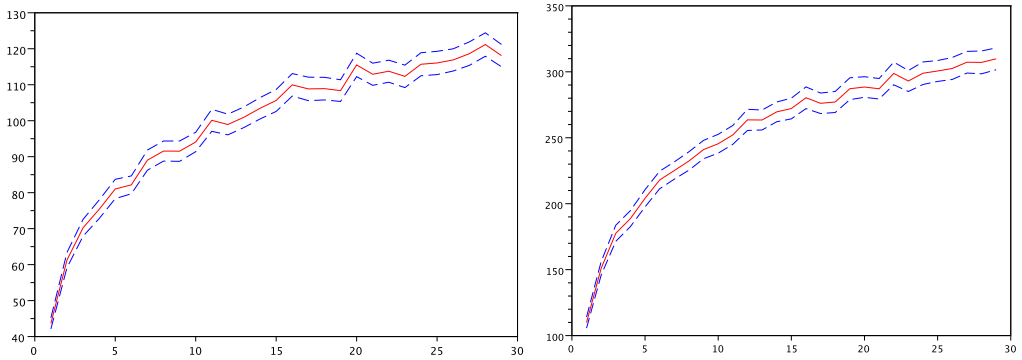
Figure 7. Averaged number of step versus the level L. Numerical results obtained for $\aleph = 1000$, $\varepsilon = 0.01$ with dimension $\delta = 3.8$ (on the left) and $\delta = 5.2$ (on the right).

Let us finally note that, for small δ', several steps concern extremely small stopping times. We recommend in this situation to remove these small stopping times (for instance smaller than 10^{-30}), it is just a slight modification of the algorithm.

6.3. Number of steps versus the level height

In the previous simulations, the threshold to reach was fixed at 5. Let us now study the dependence of the number of steps with respect to the threshold. The numerical results are obtained for $\aleph = 1000$, $\varepsilon = 0.01$ and two different Bessel processes (of dimension $\delta = 3.8$ and 5.2). Let us note that this dependence is sublinear and rather weak, the dimension of the Bessel process seems to play a more important role as illustrated in Figure 7. Observe also that ε is an upper-bound of $L^2 - (M(N^\varepsilon))^2$. We deduce that

$$L - M(N^\varepsilon) \leq \frac{\varepsilon}{2L},$$

and therefore the error of the approximation diminishes as L increases. This particular remark is also emphasized by the dependence of L in the bounds (5.31). We present an illustration of this phenomenon in Figure 8 for which ε/L is fixed and equal to 0.01.

6.4. Number of generated random variables

Finally, let us study the number of random variables used in the simulation of the Bessel hitting times. Each step of the Algorithm (NI) requires many calls of the uniform random generator in order to simulate the first coordinate of the uniform variable on the sphere of dimension $\lfloor \delta \rfloor$, the Gamma distributed variable which appears in the simulation of the hitting times of curved boundaries (here we use Johnk's algorithm, see, for instance, [10], page 418), and finally the rejection method for the conditional law described in the *Realization of the algorithm*. The Figures 9 represent simulations for the parameters $L = 5$ and $\varepsilon = 0.01$.
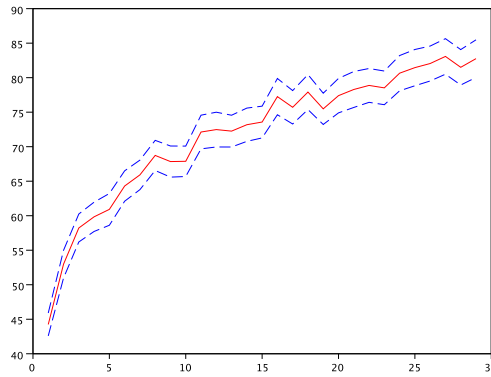
Figure 8. Averaged number of steps versus L for $\aleph = 1000$, $\varepsilon/L = 0.01$ and $\delta = 3.8$.

We observe also that the Algorithm (NI) is rather difficult to use when the fractional part of the dimension δ i.e. $\delta - \lfloor\delta\rfloor$ is small, the number of steps becomes huge. Furthermore, intuitively the parameter γ should be equal to 1. However, for technical reasons in the proof of the main result we have to consider $\gamma < 1$. Thus, for numerical purposes we suggest to take the parameter γ as close as possible to 1 as it has a weak impact on the convergence rate of the algorithm.

Acknowledgment
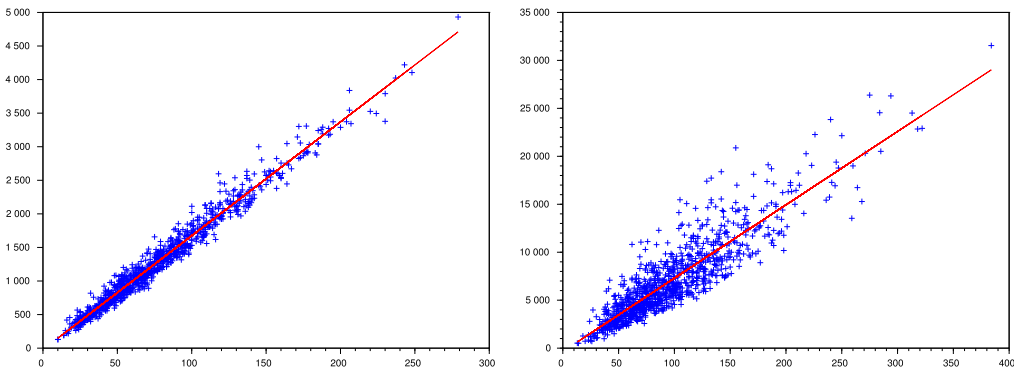
Figure 9. Number of random variables required versus number of steps for $L = 5$, $\varepsilon = 0.01$ $\delta = 2.5$ (on the left) and $\delta = 4.8$ (on the right).

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