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Simultaneous Measurement, Phase-Space Distributions, and Quantum State Determination

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Abstract. Noting that a classical phase-space probability distribution $w(q, p)$ may be calculated from moment expectation values $\{\langle q^m p^n \rangle\}$, we inquire as to whether similar data in quantum mechanics would be adequate to determine the statistical operator ϱ. For the family of simultaneous (q, p) measurement schemes investigated, it turns out that such moments do not suffice to fix ϱ. Comparison of the empirical information that is adequate to determine ϱ with that required to find $w(q, p)$ reveals that in a sense more data are needed for state determination in quantum statistics than are needed in the classical case.

Simultan-Messung, Phasenraum-Verteilungen und Bestimmtheit der Quantenzustände

Inhaltsübersicht. Da die klassische Wahrscheinlichkeitsverteilung $w(q, p)$ im Phasenraum aus den Erwartungswerten der Momente $\{\langle q^m p^n \rangle\}$ berechenbar ist, erhebt sich die Frage, ob sich nicht in ähnlicher Weise der statistische Operator der Quantenmechanik bestimmen läßt. Für die Gruppe der untersuchten simultanen (q, p)-Messungen stellt es sich heraus, daß die o.g. Momente nicht zur Bestimmung von ϱ ausreichen. Der Vergleich der zur Festlegung von ϱ benötigten empirischen Informationen mit denen zur Bestimmung von w zeigt, daß in gewissem Sinne mehr Daten zur Festlegung der Zustände in der Quantenmechanik erforderlich sind als im klassischen Falle.

1. Introduction

The statistical operator ϱ in quantum mechanics is sometimes regarded as a theoretical analog to the Gibbsian phase-space probability density $w(q, p)$. Pursu ing this analogy, one may reasonably ask whether the same kinds of statistical data that suffice to determine $w(q, p)$ in the classical case might also provide an empirical determination of the quantal ϱ. To be specific, it is well known that classically the characteristic function method permits the joint probability distribution $w(q, p)$ to be expressed in terms of the moments $\{\langle q^m p^n \rangle \mid n, m = 0, 1, 2, \ldots\}$, where $\langle q^m p^n \rangle \equiv \int dq \int dp\, q^m p^n w(q, p)$. The question then is whether this same type of data—mean values of products of powers of simultaneous (q, p) measurement results—would be adequate in a quantum-mechanical context to determine ϱ.

We are of course well aware that there are staunch adherents of a rather extreme form of complementarity who would declare the question just posed to be meaningless because in their philosophy simultaneous (q, p) data cannot exist. However, having rejected this viewpoint for many reasons that have been elaborated elsewhere [1], and being therefore unencumbered by "neo Bohrian" dogma, we feel quite free intellectually to proceed with the proposed investigation.

In the analysis which follows, we focus attention on one particular family of simultaneous (q, p) measurements, defined by the special feature that a direct measurement of q alone is in fact tantamount to a concurrent measurement of p. The idea here is that a p datum may be inferred from each q datum by substitution in a prescribed function; i.e., a relation of the form $p = s(q)$ holds for the (q, p) data even though of course no such functional dependence is valid for the corresponding noncommuting operators for position and momentum. We find that for simultaneous measurement schemes of this type, the (q, p) data analysis which is adequate to determine the joint distribution $w(q, p)$ is in general insufficient for empirical determination of the statistical operator ϱ. Motivated by the breakdown in this sense of the analogy between ϱ and $w(q, p)$, we are led next to explore the relationship of this discovery to the established theory of quantum-mechanical phase-space distribution functions and the concomitant problem of finding rules of association between quantal operators and classical phase functions. The discussion concludes by comparing the inadequate data set $\{\langle q^m p^n \rangle\}$ with what we call the quorum [2] data—the information actually needed in quantum mechanics for the empirical determination of ϱ. As expected, the quorum for ϱ turns out to embody more empirical information than that required to determine the classical $w(q, p)$.

As we have argued in other publications [3] concerned more generally with the quantum theory of measurement, there is one structural feature paradigmatic of the quantum description of any operational measurement scheme: a physical arrangement is quantum-theoretically certified to be a bona-fide measurement procedure for A, the observable of interest, if it can be proved that the probability distribution for A-data matches the probability distribution for D-Data, where, measurement of D is the directly performable act (e.g., reading a meter). This point of view, solidly grounded in the statistical interpretation of quantum mechanics, has been attacked by a few authors—JAUCH [4] for example—who apparently believe that more than such probability matching is required to validate a quantum measurement scheme. However, we remain convinced that quantum theory cannot possibly transcend its own intrinsically probabilistic mode of reasoning; and hence the foregoing characterization of measurement is not only proper but is in fact quantally complete.

The concept of probability matching has many routine practical applications in explaining the operation of measurement devices, but our need for it here is of a more esoteric nature. We seek state preparations (statistical operators ϱ) for which a probability matching occurs between the distributions for momentum p and some function s of position q, a rather exotic property to be sure, but not an unrealizable one. For a particle prepared in such a state it becomes possible to measure p by directly measuring q and then computing $p = s(q)$, a procedure justified by the probability match; in other words, for such states we have a scheme for simultaneous measurement of q and p.

In quantum simultaneous measurement theory, it is of utmost importance to avoid misinterpretation of any expression such as $p = s(q)$ which is valid only within the context of its derivation (i.e., for a certain family of quantum states). In particular, $p = s(q)$ must not be regarded as a general relation between the quantal observables (dynamical variables) Q and P, for it is in fact merely a connection between measurement results (data) q and p, which is applicable only under certain circumstances which must be specified in the formulation of the simultaneous measurement scheme. Thus $p = s(q)$, a datal relationship derived from a probability match valid for particular states, does not imply $P = s(Q)$. Indeed the latter expression would generate dynamical absurdities. The time-of-flight illustration given below provides a familiar example of a particular $p = s(q)$ which is empirically useful in both classical and quantum mechanics but which obviously does not lead in either theory to $P = s(Q)$ as a general relation between dynamical variables. This technicality has been discussed more thoroughly in a general article [1] on simultaneous measurement, and the point is also elucidated in a recent paper [11] on the determination of quantum states from data.

Before turning to the mathematical analysis of this situation, we should like to remark that nothing that either has been or will be said here in any way violates the uncertainty principle, for no statistical operator can possibly do so. Moreover, we do not claim that the particular type of simultaneous (q, p) measurement being considered here is the only kind, or even the most interesting kind; it is, however, adequate for the present theoretical discussion. For a recent attempt to formulate a comprehensive quantum theory of simultaneous measurement and joint probability distributions, the reader is referred to the work [5] of DE MUYNCK et al.

2. A Family of Simultaneous Measurement Schemes

To derive a condition on ϱ such that the associated probability distributions for p and $s(q)$ match, let Q, P denote the position and momentum operators, $|\rangle_Q$, $|\rangle_P$ their respective eigenkets, and $W_A(a\varepsilon[a_1, a_2]; \varrho)$ the probability that an A-measurement on a system prepared in the state ϱ will yield a value in the interval $[a_1, a_2]$. From the ordinary rules of quantum theory we have that

$$W_P(p\varepsilon[p_1, p_2]; \varrho) = \int_{p_1}^{p_2} dp \, {}_P\langle p|\varrho|p\rangle_P , \tag{1}$$

and

$$W_{s(Q)}(s\varepsilon[p_1, p_2]; \varrho) = \int_{s^{-1}(p_1)}^{s^{-1}(p_2)} dq \, {}_Q\langle q|\varrho|q\rangle_Q \tag{2}$$

where s^{-1} denotes the inverse function to s, i.e.,

$$s^{-1}[s(q)] = q . \tag{3}$$

The probability match condition to be imposed on ϱ is

$$W_P(p\varepsilon[p_1, p_2]; \varrho) = W_{s(Q)}(s\varepsilon[p_1, p_2]; \varrho), \tag{4}$$

for every p_1, p_2. If we change the dummy variable in (2) by letting $q = s^{-1}(p)$, the integral takes the form

$$\int_{s^{-1}(p_1)}^{s^{-1}(p_2)} dq \, {}_Q\langle q|\varrho|q\rangle_Q = \int_{p_1}^{p_2} dp \frac{ds^{-1}(p)}{dp} \, {}_Q\langle s^{-1}(p)|\varrho|s^{-1}(p)\rangle_Q , \tag{5}$$

which permits (4) to be written, using (1) and (2), as

$$\int_{p_1}^{p_2} dp \, {}_P\langle p|\varrho|p\rangle_P = \int_{p_1}^{p_2} dp \frac{ds^{-1}(p)}{dp} \, {}_Q\langle s^{-1}(p)|\varrho|s^{-1}(p)\rangle_Q , \tag{6}$$

for every p_1, p_2. The arbitrariness of the limits of integration then implies that

$${}_P\langle p|\varrho|p\rangle_P = \frac{ds^{-1}(p)}{dp} \, {}_Q\langle s^{-1}(p)|\varrho|s^{-1}(p)\rangle_Q . \tag{7}$$

The l.h.s. of (7) may be expressed in the Q-representation by inserting identities, so that with $\hbar = 1$ we obtain

$$\begin{aligned} {}_P\langle p|\varrho|p\rangle_P &= \int dq' \int dq'' \, {}_P\langle p|q'\rangle_Q \, {}_Q\langle q'|\varrho|q''\rangle_Q \, {}_Q\langle q''|p\rangle_P \\ &= \frac{1}{2\pi} \int dq' \int dq'' \, e^{-ip(q'-q'')} \, {}_Q\langle q'|q|q''\rangle_Q . \end{aligned} \tag{8}$$

Combining (7) and (8) and substituting $p = s(q)$ finally yields

$$\langle q\,|\varrho|\,q\rangle = \frac{1}{2\pi}\frac{ds}{dq}\int dq' \int dq''\, e^{-i(q'-q'')s(q)} \langle q'\,|\varrho|\,q''\rangle, \tag{9}$$

where the subscript Q has been suppressed since all density-matrix elements are now in the position representation. Equation (9) is the desired condition which ϱ must satisfy if it embodies a probability match between P and $s(Q)$.

If, say, the r.h.s. is multiplied by λ, then (9) assumes the standard form of a homogeneous Fredholm integral equation of the second kind in two independent variables; the density matrices of interest to us are then the eigensolutions of this equation belonging to eigenvalue $\lambda = 1$. Although we are not concerned in the present analysis with a general mathematical study of (9), it is perhaps of interest to note that there does exist at least one family of solutions having a familiar physical interpretation.

To see this, consider the particular function

$$s(q) = \alpha q, \quad \alpha \geq 0 \tag{10}$$

and the statistical operator

$$\varrho = e^{-\frac{iP^2}{2\alpha}} \varrho_0 e^{\frac{iP^2}{2\alpha}} \tag{11}$$

where ϱ_0 is any statistical operator having the property

$$\langle q\,|\varrho_0|\,q\rangle = 0 \quad \text{if} \quad q \notin [-q_0, q_0]. \tag{12}$$

Since the nonnegative definiteness of ϱ_0 implies that

$$\langle \hat{q}\,|\varrho_0|\,\tilde{q}\rangle \leq \sqrt{\langle \hat{q}\,|\varrho_0|\,\hat{q}\rangle \langle \tilde{q}\,|\varrho_0|\,\tilde{q}\rangle}, \tag{13}$$

it follows by combining (12) and (13) that

$$\langle \hat{q}\,|\varrho_0|\,\tilde{q}\rangle = 0 \text{ if either } \hat{q} \notin [-q_0, q_0] \text{ or } \tilde{q} \notin [-q_0, q_0]. \tag{14}$$

For the ϱ given by (11), the Q-representation density matrix has the form

$$\begin{aligned}\langle q'\,|\varrho|\,q''\rangle = {} & \frac{1}{(2\pi)^2}\int d\hat{q} \int d\tilde{q} \int dp' \int dp'' \\ & \times e^{ip'q'}\, e^{-\frac{ip'^2}{2\alpha}}\, e^{-ip'\hat{q}} \,\langle \hat{q}\,|\varrho_0|\,\tilde{q}\rangle\, e^{ip''\tilde{q}}\, e^{\frac{ip''^2}{2\alpha}}\, e^{-ip''q''},\end{aligned} \tag{15}$$

which becomes, after elementary integration,

$$\begin{aligned}\langle q'\,|\varrho|\,q''\rangle = {} & \frac{\alpha}{2\pi} e^{\frac{i\alpha}{2}(q'^2-q''^2)} \\ & \times \int d\hat{q} \int d\tilde{q}\, e^{\frac{i\alpha}{2}(\hat{q}^2-2\hat{q}q')} \langle \hat{q}\,|\varrho_0|\,\tilde{q}\rangle\, e^{-\frac{i\alpha}{2}(\tilde{q}^2-2\tilde{q}q'')}.\end{aligned} \tag{16}$$

Now, when the expression (10) for $s(q)$ is inserted, the integral equation (9) becomes

$$\langle q\,|\varrho|\,q\rangle = \frac{\alpha}{2\pi}\int dq' \int dq''\, e^{-i(q'-q'')\alpha q} \langle q'\,|\varrho|\,q''\rangle, \tag{17}$$

and it may be readily demonstrated as follows that (16) satisfies (17) in the limit $\alpha \to 0$. First note that restriction (14) permits taking this limit inside the integral in (16) and thereby eliminating factors not containing q' or q''. Thus, for $\alpha \to 0$,

$$\begin{aligned}\langle q'\,|\varrho|\,q''\rangle \cong {} & \frac{\alpha}{2\pi} e^{\frac{i\alpha}{2}(q'^2-q''^2)} \\ & \times \int d\hat{q} \int d\tilde{q}\, e^{-i\alpha(\hat{q}q'-\tilde{q}q'')} \langle \hat{q}\,|\varrho_0|\,\tilde{q}\rangle.\end{aligned} \tag{18}$$

Next substitute (18) into the r.h.s. of (17); when the resulting elementary integrals are worked out we have, for $\alpha \to 0$,

$$\frac{\alpha}{2\pi} \int dq' \int dq'' \, e^{-i(q'-q'')\alpha q} \langle q' | \varrho | q'' \rangle$$

$$\cong \left(\frac{\alpha}{2\pi}\right)^2 \int d\hat{q} \int d\tilde{q} \left(\frac{2\pi}{\alpha}\right) e^{\frac{-i\alpha(q-\hat{q})^2}{2}} e^{\frac{-i\alpha(q+\tilde{q})^2}{2}} \langle \hat{q} | \varrho_0 | \tilde{q} \rangle \tag{19}$$

$$\cong \frac{\alpha}{2\pi} e^{-i\alpha q^2} \int d\hat{q} \int d\tilde{q} \, e^{-i\alpha q(\hat{q}-\tilde{q})} \langle \hat{q} | \varrho_0 | \tilde{q} \rangle, \tag{20}$$

where step (20) invoked again restriction (14). Finally use (18) to evaluate the l.h.s. of (17), obtaining, as $\alpha \to 0$,

$$\langle q | \varrho | q \rangle \cong \frac{\alpha}{2\pi} \int d\hat{q} \int d\tilde{q} \, e^{-i\alpha q(\hat{q}-\tilde{q})} \langle \hat{q} | \varrho_0 | \tilde{q} \rangle \tag{21}$$

which for any q obviously shares the common limit zero with (20) as $\alpha \to 0$.

The physical intrepretation of this family of solutions is a time-of-flight experiment that has in the past occasionally been a subject of philosophical controversy. If we interpret ϱ_0 as a state preparation at $t = 0$ then (12) and hence (14) mean only that the initial preparation localizes the particle to within a finite interval $[-q_0, q_0]$, a very mild stricture from an empirical standpoint. If we further take $\alpha = \frac{m}{t}$, where m is the mass of the particle, then (11) describes the free-flight evolution of the statistical operator to time t. The integral equation therefore expresses the intuitively reasonable theorem that as $t \to \infty$, the probability distribution for $\frac{mq}{t}$ matches that for p, provided the particle was finitely localized about $q = 0$ at $t = 0$.

Returning now to the general case, let us suppose that for some given $s(q)$, a set of density matrices satisfying integral equation (9) has been found. (The foregoing time-of-flight illustration exemplifies this possibility.) For state preparations belonging to that set, it will be true that probability distributions for observables $s(Q)$ and P match and hence that each Q-measurement amounts to a simultaneous measurement of Q and P.

3. Phase-space Distributions

The joint probability density $w(q, p)$ describing the distribution of (q, p) results that would be obtained from simultaneous measurements performed in the particular manner just discussed is obviously given by

$$w(q, p) = \langle q | \varrho | q \rangle \, \delta[p - s(q)], \tag{22}$$

an expression valid only when ϱ satisfies the integral equation (9). From (22) it is immediately apparent that $w(q, p)$ depends only upon the diagonal elements of ϱ in the Q-representation. It is therefore conceivable that many distinct statistical operators, all satisfying (9) might share one single distribution $w(q, p)$. (The free-flight case in the preceding section is an example of this.) We are thus led to conclude that even in a quantal situation where it is in fact possible to gather (q, p) data and hence compute $w(q, p)$, such information is not necessarily adequate for a complete empirical determination of ϱ; for the family of simultaneous measurement schemes we have explicitly considered, knowledge of $w(q, p)$ is definitely insufficient to determine ϱ. This means in particular that, despite the seeming analogy between the classical version of $w(q, p)$ and ϱ, the moments $\{\langle q^m p^n \rangle\}$ generally do not determine the quantal ϱ.

The quantity $w(q, p)$, defined here in terms of relative frequencies of (q, p) data from simultaneous measurements, is mathematically a joint probability distribution whose marginal distributions are the usual quantum mechanical ones associated with the single observables Q and P; thus if $w(q, p)$ has the form (22), it is easy to verify that

$$\int dp\, w(q, p) = {}_Q\langle q \,|\varrho|\, q\rangle_Q \tag{23}$$

and

$$\int dq\, w(q, p) = {}_P\langle p \,|\varrho|\, p\rangle_P \,. \tag{24}$$

Eq. (23) follows immediately upon inspection of (22); to derive (24), note that the change of variable $q = s^{-1}(\hat{p})$ yields

$$\int dq\, w(q, p) = \frac{ds^{-1}(\hat{p})}{d\hat{p}}\, d\hat{p}\; {}_Q\langle s^{-1}(p) \,|\varrho|\, s^{-1}(\hat{p})\rangle_Q\; \delta(p - p)$$

$$= \frac{ds^{-1}(p)}{dp}\, {}_Q\langle s^{-1}(p) \,|\varrho|\, s^{-1}(p)\rangle_Q \tag{25}$$

$$= {}_P\langle p \,|\varrho|\, p\rangle_P \,, \tag{26}$$

where step (26) is based on the relation (7).

There is in the literature of quantum physics an extensive theory of phase-space distribution functions $w(q, p)$ satisfying marginal conditions (23) and (24). However, the original context of that theory was not the problem of simultaneous measurement; indeed the $w(q, p)$ functions usually considered are only quasiprobability distributions which, lacking nonnegative-definiteness, cannot be interpreted physically as relative frequencies of (q, p) measurement results. Nevertheless, one formulation of quantal phase-space distributions, that due to COHEN [6] is rich enough to accommodate nonnegative forms of $w(q, p)$ such as (22); in this theory, the following canonical form associates with any pure state ψ a function $w(q, p)$ whose marginal distributions will match the Q and P distributions inherent in ψ:

$$\begin{aligned} w(q, p) = \frac{1}{(2\pi)^2} \int d\theta \int d\tau \int du\; e^{-i\theta q - i\tau p + i\theta u} f(\theta, \tau) \\ \times {}_Q\langle u + \frac{\tau}{2} \,|\psi\rangle \langle\psi|\, u - \frac{\tau}{2}\rangle_Q \end{aligned} \tag{27}$$

where

$$f(0, \tau) = f(\theta, 0) = 1. \tag{28}$$

Unlike others (cf. Refs. [7, 8] and other work cited therein) who have considered similar mappings from ψ to $w(q, p)$, COHEN does not insist that f be independent of ψ; and it is essentially for this reason that only his theory is broad enough to encompass the distributions arising from simultaneous measurement schemes.

Indeed if f is not permitted to be state-dependent, then a theorem due to WIGNER [9] establishes that $w(q, p)$ cannot be nonnegative definite. However, WIGNER [10] has discovered an alternative distribution function $Z(q, p)$ which is always nonnegative, even when f is state-independent and $w(q, p)$ is consequently indefinite as to sign. Wigner's function $Z(q, p)$ is related to $w(q, p)$ by the defining equation

$$Z(q, p) \equiv \int dq' \int dp'\, w(q', p') \exp\left\{-\frac{1}{2}\left[a^2(p - p')^2 + \frac{1}{a^2}(q - q'^2)\right]\right\}.$$

While the marginal distributions derivable from $Z(q, p)$ do not equal the quantal probability densities associated with the state $|\psi\rangle$, there are nevertheless connections between $Z(q, p)$ and such densities; for example, according to WIGNER [10].

$$\int dq\, Z(q, p) = \int dp'\, |{}_P\langle p' \mid \psi\rangle|^2 \exp\left\{-\frac{1}{2}\, a^2(p - p')^2\right\}.$$

Whether there exists a useful interpretation of the nonnegative $Z(q, p)$ in terms of simultaneous (q, p) measurement results is at present unknown to us. Therefore we shall continue to limit the present discussion to nonnegative definite $w(q, p)$ and hence to the case where Cohen's function f is state dependent.

Equation (27) may be generalized to embrace all quantum states, pure and mixed, by replacing $|\psi\rangle\langle\psi|$ with a statistical operator ϱ. Then, noting that

$$\left\{\begin{array}{l} \left| u - \dfrac{\tau}{2}\right\rangle_Q = e^{\frac{iP\tau}{2}}\, |u\rangle_Q \\ {}_Q\left\langle u + \dfrac{\tau}{2}\right| = {}_Q\langle u|\, e^{\frac{iP\tau}{2}} \end{array}\right\} \tag{29}$$

and

$$\int du\, e^{i\theta u}\, {}_Q\langle u|\, e^{\frac{iP\tau}{2}}\, \varrho e^{\frac{iP\tau}{2}}\, |u\rangle_Q$$
$$= Tr\left(\varrho e^{\frac{iP\tau}{2}}\, e^{i\theta Q}\, e^{\frac{iP\tau}{2}}\right) \tag{30}$$
$$= Tr\{\varrho e^{i(\theta Q + \tau P)}\}, \tag{31}$$

we obtain the general transformation

$$w(q, p) = \frac{1}{(2\pi)^2} \int d\theta \int d\tau\, f(\theta, \tau)\, e^{-i(\theta q + \tau p)}\, Tr\{\varrho e^{i(\theta Q + \tau P)}\}, \tag{32}$$

which has the inverse

$$f(\theta, \tau) = [Tr\{\varrho e^{i(\theta Q + \tau P)}\}]^{-1} \int dq \int dp\, w(q, p)\, e^{i(\theta q + \tau p)}, \tag{33}$$

as may be verified directly by substitution back into (32). The algebraic steps connecting (30) and (31) are straightforward manipulations involving repeated use of the formula

$$e^{i\alpha Q + i\beta P} = e^{i\alpha Q} e^{i\beta P} e^{\frac{i\alpha\beta}{2}}. \tag{34}$$

Later we shall need an expression for the $f(\theta, \tau)$ associated with joint distributions of the form (22). Thus, using (33), we have

$$\begin{aligned} f(\theta, \tau) &= [Tr\{\varrho e^{i(\theta Q + \tau P)}\}]^{-1} \int dq \int dp\, \langle q\, |\varrho|\, q\rangle\, \delta[p - s(q)]\, e^{i(\theta q + \tau p)} \\ &= [Tr\{\varrho e^{i(\theta Q + \tau P)}\}]^{-1} \int dq\, \langle q\, |\varrho|\, q\rangle\, e^{i[\theta q + \tau s(q)]} \\ &= [Tr\{\varrho e^{i(\theta Q + \tau P)}\}]^{-1}\, Tr\{\varrho e^{i[\theta Q + \tau s(Q)]}\}, \end{aligned} \tag{35}$$

an expression which, as expected, is not independent of ϱ.

4. Rules of Association

Intimately linked with the quasiprobability distributions are so-called rules of association; such a rule defines an operator G corresponding to each phase function $g(q, p)$ to that the following equality will hold:

$$\int dq \int dp\, w(q, p)\, g(q, p) = Tr(\varrho G). \tag{36}$$

Given $g(q, p)$, $w(q, p)$, and hence via (33), $f(\theta, \tau)$, a solution to (36) that is customarily chosen is the general rule of association

$$G = \int d\theta \int d\tau\, \gamma(\theta, \tau)\, f(\theta, \tau)\, e^{i(\theta Q + \tau P)}, \tag{37}$$

where

$$\gamma(\theta, \tau) = \frac{1}{(2\pi)^2} \int dq \int dp\, g(q, p)\, e^{-i(\theta q + \tau p)}. \tag{38}$$

It is of particular interest in the present investigation to ask what quantal operator G_{mn} corresponds to the moment phase function $q^m p^n$ when ϱ satisfies (9); i.e., when the state preparation is such that there is a known method (Sec. 2) for measuring $q^m p^n$. On the basis of conclusions drawn in the work of PARK and MARGENAU [1] dealing in a different way with the problem of simultaneous measurement, we would anticipate that no ordinary (state-independent) operator G_{mn} corresponding to $q^m p^n$ can exist, because $q^m p^n$ is a member of the class of compound observables for which there apparently cannot be operator counterparts unless the fundamental quantum axioms are somehow modified. Nevertheless, it is instructive to approach this matter along the alternative path afforded by the general rule of association (37).

Substituting $g(q, p) = q^m p^n$ into (38), we obtain

$$\gamma_{mn} = \frac{1}{(2\pi)^2} \int dq \int dp\, q^m p^n e^{-i(\theta q + \tau p)}, \tag{39}$$

$$= i^{m+n}\, \delta^{(m)}(\theta)\, \delta^{(n)}(\tau), \tag{40}$$

since the mth derivative of the Dirac delta has integral representation

$$\delta^{(m)}(\theta) = \frac{1}{2\pi i^m} \int dq\, q^m e^{-i\theta q}. \tag{41}$$

Combining (40) and (37) then yields

$$\begin{aligned} G_{mn} &= i^{m+n} \int d\theta \int d\tau\, \delta^{(m)}(\theta)\, \delta^{(n)}(\tau)\, f(\theta, \tau)\, e^{i(\theta Q + \tau P)} \\ &= (-i)^{m+n} \frac{\partial^{m+n}}{\partial\theta^m\, \partial\tau^n} [f(\theta, \tau)\, e^{i(\theta Q + \tau P)}] \big|_{\theta = \tau = 0}. \end{aligned} \tag{42}$$

To exhibit the strong dependence G_{mn} has on ϱ in the simultaneous measurement schemes discussed in Sec. 2, it suffices to consider just the relatively simple case $m = n = 1$. Since $f(0, \tau) = f(\theta, 0) = 1$, (42) reduces to

$$G_{11} = -\left\{\frac{\partial^2 e^{i(\theta Q + \tau P)}}{\partial\theta\, \partial\tau} + \frac{\partial^2 f}{\partial\theta\, \partial\tau} 1\right\}_{\theta = \tau = 0} \tag{43}$$

It is a routine exercise to demonstrate that

$$\left.\frac{\partial^2 e^{i(\theta Q + \tau P)}}{\partial\theta\, \partial\tau}\right|_{\theta = \tau = 0} = -\frac{1}{2}(QP + PQ) \tag{44}$$

Hence we have

$$G_{11} = \frac{1}{2}(QP + PQ) - \left.\frac{\partial^2 f}{\partial\theta\, \partial\tau}\right|_{\theta = \tau = 0} 1 \tag{45}$$

as the operator corresponding to the product qp. To complete the evaluation of G_{11}, we must study the mixed derivative of the particular $f(\theta, \tau)$ associated with the simultaneous measurement schemes of Sec. 2. Differentiation of (35) with the aid of (44) and the fact that ϱ has unit trace leads, after algebraic simplification, to the expression

$$\begin{aligned} \left.\frac{\partial^2 f}{\partial\theta\, \partial\tau}\right|_{\theta = \tau = 0} &= Tr\left\{\varrho \frac{1}{2}(QP + PQ)\right\} - Tr\{\varrho Q s(Q)\} \\ &\quad + Tr(\varrho Q)\, [Tr\{\varrho s(Q)\} - Tr\, (\varrho P)]. \end{aligned} \tag{46}$$

Examination of (45) and (46) bears out our expectation that no normal quantum-mechanical operator can correspond to the compound observable qp; indeed G_{11} is highly dependent on ϱ. Moreover, if we recall that the theory of rules of association is based entirely upon seeking operator solutions to (36), it is evident that if G solves (36), then so does

$$K = G + J, \tag{47}$$

provided

$$Tr(\varrho J) = 0. \tag{48}$$

Hence we would be equally justified in regarding either G or K as the operator associated with $g(q, p)$. Applied to G_{11}, this observation implies that the simplest operator counterpart to qp is just the state-dependent c-number

$$K_{11} = Tr\{\varrho Q s(Q)\}\, 1 - Tr(\varrho Q)\, [Tr\{\varrho s(Q)\} - Tr(\varrho P)]. \tag{49}$$

Since in our simultaneous measurement scheme ϱ is restricted by the probability-matching condition (4), it follows that

$$Tr\{\varrho s(Q)\} = Tr(\varrho P), \tag{50}$$

which makes the last term of (49) vanish. Thus in effect, for the simultaneous measurements here envisaged, the operator associated with qp reduces to the trivial form

$$K_{11} = Tr\{\varrho Q s(Q)\}\, 1. \tag{51}$$

The state dependence of G_{11} and K_{11} is of course only a manifestation of the state dependence of $f(\theta, \tau)$, which, as we noted earlier, is a consequence of the nonnegative definiteness of the functions $w(q, p)$ that are derived from simultaneous measurement schemes. We are therefore led to conclude that for the family of simultaenous measurements described in Sec. 2, the theory of rules of association provides no state-independent quantal operator corresponding to qp; and it is quite apparent that an extension of the foregoing argument establishes in general that the theory generates no acceptable operators corresponding to such compound observables as $\{q^m p^n\}$ or functions thereof. This conclusion, however, is not a total surprise, as the literature on joint distributions [5] and on simultaneous measurements [1] already contains at least two independent arguments against the existence of such correspondences.

5. The Quorum of Observables

We have seen that even in quantum-mechanical situations where there is a known method for measuring $\{\langle q^m p^n \rangle\}$ and hence determining $w(q, p)$, such information proves to be insufficient to determine the statistical operator ϱ. In these same situations, standard rules of association yield no ascceptable operators corresponding to joint functions of q and p.

Therefore the question remains open as to what data would in principle be sufficient to determine ϱ for an ordinary nonrelativistic quantum system characterized by observables Q, P and functions of these operators.

Interestingly, there is a *purely formal* operator generalization of the classical method of characteristic functions that does indeed yield a list of theoretical observables whose mean values would suffice to fix the statistical operator. We call such a list a *mathematical quorum*.

We have demonstrated in another article [11] that the set $\left\{\frac{1}{2}(Q^m P^n + P^n Q^m)\right\}$ constitutes a quorum for a spinless particle moving in one dimension. It is noteworthy that these quorum operators have a *mathematical form* reminiscent of the classical functions $\{q^m p^n\}$ but they definitely have a quite different physical meaning; for, as we have shown, there are no useful operators corresponding to $\{q^m p^n\}$. Moreover, even if there were operator counterparts for $\{q^m p^n\}$, their mean values would determine only $w(q, p)$ but not ϱ, whereas the quorum means $\left\{\left\langle \frac{1}{2}(Q^n P^m + P^m Q^n)\right\rangle\right\}$ do in fact suffice to determine ϱ.

An interesting parallel to this situation may be found in quantum optics, where the "optical equivalence theorem" due to SUDARSHAN [12, 13] establishes that in a certain representation the quantum theory of coherence bears a formal resemblance to the classical theory of analytic signals despite the physical inequivalence of the two descriptions.

We have also reported elsewhere [11] a procedure for obtaining the mathematical quorum information $\left\{\left\langle \frac{1}{2}(Q^n P^m + P^m Q^n)\right\rangle\right\}$ in terms of the quantities $\left\{\left\langle \frac{d^n Q^m}{dt^n}\right\rangle\right\}$ whenever the system Hamiltonian has the common form $H = \frac{P^2}{2M} + V(Q)$. The latter, being mean nth-order rates of change of powers of Q, admit of a simple empirical interpretation in terms of position measurements at various instants. The rates $\left\{\frac{d^n Q^m}{dt^n}\right\}$ constitute therefore a physically-interpreted quorum of observables for the determination of ϱ.

The classical and quantal prescription for empirical determination of statistical states may now be summarized as follows:

(a) To determine the classical phase-space distribution $w(q, p)$, it suffices to measure the expectation values of $\{q^m p^n\}$.

(b) In quantal situations where the observables $\{q^m p^n\}$ are measurable, such information is inadequate to determine the statistical operator ϱ. Furthermore, these compound observables do not correspond to normal state-independent operators.

(c) To determine the statistical operator for a spinless, nonrelativistic particle with Hamiltonian of the form $H = \frac{P^2}{2M} + V(Q)$, it suffices to measure the expectation values of $\left\{\frac{d^n Q^m}{dt^n}\right\}$.

Finally we note that in the classical case the data required to compute rates analogous to those described in (c) would be more than adequate to determine $\{\langle q^m p^n\rangle\}$ and hence $w(q, p)$. Therefore, in view of point (b), it may be concluded that state determination in quantum mechanics requires in a sense more data than the corresponding problem in classical statistics.

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