SIMULTANEOUS NUMERICAL APPROXIMATION OF MICROSTRUCTURES AND RELAXED MINIMIZERS

ZHIPING LI

SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, P.R.CHINA

Abstract. The problem of minimizing multiple integral functionals with nonquasiconvex integrands is considered. A numerical method, which is based on an alternative minimizing problem to the relaxed problem and thus uses no quasiconvex envelope of the integrands nor its numerical approximation in the computation, is introduced to approximate simultaneously the highly oscillating minimizing sequences, or in other words microstructures, and the minimizers of the corresponding relaxed problem. Existence and convergence of the discrete solutions are proved and an error estimate is obtained. A numerical example is given.

1. Introduction

In many physical problems, for example in material sciences and nonlinear elasticity, one is lead to consider problems of minimizing nonquasiconvex energies [1, 2], or in other words, the problem of minimizing an integral functional

$$F(u) = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx \tag{1.1}$$

with nonquasiconvex integrand $f : \Omega \times R^m \times R^{n\times m} \to R$ in a set of admissible functions

$$\mathbb{A} = \{u \in W^{1,p}(\Omega; R^m) : u = u_0, \text{ on } \partial\Omega\}. \tag{1.2}$$

It is well known that in general such problems fail to have a solution [3, 4, 5]. However, the minimizing sequences of $F(\cdot)$ in $\mathbb{A}$, which consist of finer and finer oscillations, can converge in the sense of Young measures and lead to microstructures [1, 6, 7]. The minimizing sequences of $F(\cdot)$ in $\mathbb{A}$ are also

1991 *Mathematics Subject Classification.* 65N12,65N30,49J45,49M20.

Key words and phrases. microstructure, relaxed minimizer, minimizing sequence, simultaneous numerical approximation, finite element solutions, convergence.

The research was supported by the National Natural Science Foundation of China.

found to be closely related to the minimizers of the relaxed problem, which is the minimizing problem obtained by replacing the original integrand f by its quasiconvex envelope, that is the greatest quasiconvex function less than or equal to f [5, 7, 8]. In fact, under certain general hypotheses, it is proved that the limit of every weakly convergent minimizing sequence of $F(\cdot)$ is a minimizer of the relaxed problem and vice versa [5, 7, 8, 11].

It is extremely difficult in practice to compute such highly oscillating minimizing sequences by a numerical method based on a straight forward discretization of (1.1) and (1.2). An alternative is to make use of the relationship between the microstructure and the relaxed minimizer, that is the minimizer of the relaxed problem. Many efforts have been made on the numerical methods for approximating microstructures for multiwell problems under proper affine boundary conditions (see for example [12, 13, 14, 15, 16] among many others). Efforts have also been made on the numerical methods for the computation of microstructures for general problems under nonlinear boundary conditions by making use of the numerical solutions to the relaxed problem [17, 18].

The main purpose of the present paper is to develop, for the problem of minimizing an integral functional $F(\dot{})$ with nonquasiconvex integrand in $\mathbb{A}$ with general boundary conditions, a numerical method, which uses no quasiconvex envelope of the energy density nor its numerical approximation during the computation, to approximate the highly oscillating minimizing sequences and the minimizers of the corresponding relaxed problem, or in other words the microstructures and relaxed minimizers, simultaneously. The idea is to approximate the relaxed minimizer by a relatively coarse mesh and the microstructure by a refined mesh, and to improve the approximations successively by using only the current information of the two approximating discrete solutions, more precisely, the two approximations are taking turns to be improved and each uses only the information of the current position of the other. To achieve the aim, we first introduce, in Sec. 2, an alternative minimizing problem which is equivalent to the relaxed problem in the sense that they have exactly the same set of minimizers. The properties of the new problem and the corresponding functionals are studied in Sec. 2 to provide us a theoretical base for the discrete problems and the numerical method to be introduced in Sec. 3 and Sec. 4. In Sec. 3, the existence and convergence of solutions to the discrete problems are proved and an error estimate is obtained. A numerical method to compute the discrete solutions is investigated in Sec. 4, and a numerical example with nonlinear boundary conditions is given to show the effectiveness of the method.

The idea of approximating the relaxed minimizer with a coarse mesh and the microstructures with a fine mesh can be traced back to Kohn and Vogelius [18]. In [18], the relaxed minimizer on the coarse mesh is calculated by first finding the quasiconvex envelope of the integrand constructively and analytically and then to solve the relaxed problem numerically. Since this works only when the quasiconvex envelope can be found analytically, its application is restricted. In contrast, the method given in this paper needs no information on the quasiconvex envelope and solves the problem completely numerically, and thus can be applied to solve general problems.

In cases when the integrands are of the form $f(\nabla u)$ and the boundary conditions are affine, the relaxed minimizer is known to be simply the same affine function as the boundary data. Thus, in such cases, there is no need to calculate the coarse mesh approximation, only the fine mesh approximated microstructures need to be calculated and this can be done by applying the existing numerical methods such as those given in [12, 13, 14, 17]. For general problems, these numerical techniques can also be applied to calculate on the fine mesh the function $w^i_{h_1} = u^i_{h_1} - v^i_{h_0}$ which is the oscillating component of the current approximated microstructure, where $v^i_{h_0}$ is the current approximated relaxed minimizer (see Sec.4).

2. Alternative minimizing problems

Let $\Omega \subset R^n$ be a bounded connected open set with Lipschitz continuous boundary $\partial\Omega$. Let $W^{1,p}(\Omega)$ be the usual Sobolev space [19], and let $W^{1,p}(\Omega; R^m) = (W^{1,p}(\Omega))^m$. Let the integrand $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy the hypotheses

(H1): f is continuous and there exist constants $C, \hat{C} \in R$, $0 < C_1 < C_2$ and $1 < p < +\infty$ such that for all $(x, u, P) \in \Omega \times R^m \times R^{n\times m}$

$$C + C_1(|u|^p + |P|^p) \leq f(x, u, P) \leq \hat{C} + C_2(|u|^p + |P|^p).$$

(H2): Either $f(x, u, P) = f(x, P)$ or

$$|f(x, u_1, P) - f(x, u_2, P)| \leq \omega(x, |u_1 - u_2|)\beta(|P|),$$

where $\omega : \Omega \times R \to R^+$ is a Carathéodory function, $\omega(x, 0) = 0$, and $\beta(\cdot)$ is increasing and nonnegative.

Denote the quasiconvex envelope of $f(x, u, \cdot)$ by $\hat{f}(x, u, \cdot)$ and define $\hat{F} : \mathbb{A} \to R$ by

$$\hat{F}(v) = \int_\Omega \hat{f}(x, v(x), \nabla v(x))\, dx. \tag{2.1}$$

It is well known that, under the hypotheses (H1) and (H2), $\hat{F}(\cdot)$ is the greatest sequentially weakly lower semicontinuous functional less than or equal to $F(\cdot)$ (see [5], [7]-[11] for more general results in this direction). It is worth noticing that this is why the hypothesis (H2), which does not explicitly appear in the arguments in the paper, is required for our purpose. As a consequence, we also have the following property.

Lemma 2.1. *([5, 7, 8]) Let $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy (H1) and (H2). Then, for any $v \in \mathbb{A}$, there is a sequence $\{u_i\}_{i=1}^\infty \subset \mathbb{A}$ such that*

$$u_i \rightharpoonup v \quad in\ W^{1,p}(\Omega; R^m),$$

where "$\rightharpoonup$" means "converges weakly to", and

$$\hat{F}(v) = \lim_{i\to\infty} F(u_i).$$

For $\alpha > 0$, define functionals $F_\alpha : \mathbb{A} \times \mathbb{A} \to R$ and $\hat{F}_\alpha : \mathbb{A} \to R$ by

$$F_\alpha(u, v) = F(u) + \alpha \int_\Omega |u(x) - v(x)|^p dx, \tag{2.2}$$

and

$$\hat{F}_\alpha(v) = \inf_{u\in\mathbb{A}} F_\alpha(u, v) \tag{2.3}$$

respectively. Instead of considering the problem of minimizing $F(\cdot)$ in $\mathbb{A}$ and the problem of minimizing $\hat{F}(\cdot)$ in $\mathbb{A}$, we consider the problem of minimizing $F_\alpha(\cdot)$ in $\mathbb{A} \times \mathbb{A}$ and the problem of minimizing $\hat{F}_\alpha$ in $\mathbb{A}$. The following results reveal some important properties about the relationship of these functionals and of the corresponding minimizing problems.

Lemma 2.2. *For any $\alpha > 0$,*

$$\inf_{(u,v)\in\mathbb{A}\times\mathbb{A}} F_\alpha(u, v) = \inf_{u\in\mathbb{A}} F(u). \tag{2.4}$$

Proof. It follows from

$$F_\alpha(u,u) = F(u) \quad \forall u \in \mathbb{A}$$

that

$$\inf_{(u,v)\in\mathbb{A}\times\mathbb{A}} F_\alpha(u,v) \le \inf_{u\in\mathbb{A}} F(u).$$

On the other hand, since

$$F_\alpha(u,v) \ge F(u) \quad \forall (u,v) \in \mathbb{A}\times\mathbb{A},$$

we have

$$\inf_{(u,v)\in\mathbb{A}\times\mathbb{A}} F_\alpha(u,v) \ge \inf_{u\in\mathbb{A}} F(u).$$

Hence the conclusion of the lemma follows. □

Lemma 2.3. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy (H1) and (H2). Then, we have*

$$\hat{F}_\alpha(v) \le \hat{F}(v) \quad \forall v \in \mathbb{A} \text{ and } \alpha > 0, \tag{2.5}$$

$$\lim_{\alpha\to+\infty} \hat{F}_\alpha(v) = \hat{F}(v) \qquad \forall v \in \mathbb{A}. \tag{2.6}$$

Proof. For any $v \in \mathbb{A}$, by lemma 2.1, there exists a sequence $\{u_i\}_{i=1}^\infty \subset \mathbb{A}$ such that

$$u_i \rightharpoonup v \quad \text{in } W^{1,p}(\Omega; R^m),$$

$$\int_\Omega \hat{f}(x, v(x), \nabla v(x))\,dx = \lim_{i\to\infty} \int_\Omega f(x, u_i(x), \nabla u_i(x))\,dx.$$

Since, by (2.3),

$$\hat{F}_\alpha(v) \le F_\alpha(u_i, v) \quad \forall i,$$

we have

$$\hat{F}_\alpha(v) \le \hat{F}(v).$$

This proves (2.5).

For $\alpha > 0$, by (2.3) there exists a $u_\alpha \in \mathbb{A}$ such that

$$F_\alpha(u_\alpha, v) \le \hat{F}_\alpha(v) + \frac{1}{\alpha}.$$

By (H1), this implies that

$$u_\alpha \rightharpoonup v \quad \text{in } W^{1,p}(\Omega; R^m) \text{ as } \alpha \to \infty.$$

Thus, by the sequentially weakly lower semicontinuity of $\hat{F}(\cdot)$, we have

$$\begin{aligned}\hat{F}(v) &\le \liminf_{\alpha\to\infty} \hat{F}(u_\alpha) \le \liminf_{\alpha\to\infty} F(u_\alpha)\\ &\le \liminf_{\alpha\to\infty} F_\alpha(u_\alpha, v) \le \liminf_{\alpha\to\infty} \hat{F}_\alpha(v).\end{aligned}$$

This and (2.5) give (2.6). □

Theorem 2.1. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy (H1) and (H2). Then, v is a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$ if and only if*

$$\hat{F}_\alpha(v) = \hat{F}(v) \quad \forall \alpha > 0. \tag{2.7}$$

Proof. Let v be a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$. Let $\alpha > 0$ be given. By the definition of $\hat{F}_\alpha(v)$, there is a sequence $\{u_i\}_{i=1}^\infty \subset \mathbb{A}$ such that

$$\lim_{i\to\infty} F_\alpha(u_i, v) = \hat{F}_\alpha(v). \tag{2.8}$$

It follows from the hypothesis (H1) that u_i are bounded in $W^{1,p}(\Omega; R^m)$ and thus there exists a subsequence of $\{u_i\}_{i=1}^\infty$, again denoted by $\{u_i\}_{i=1}^\infty$, and a function $\hat{v} \in \mathbb{A}$ such that

$$u_i \rightharpoonup \hat{v} \quad \text{in } W^{1,p}(\Omega; R^m).$$

By the sequentially weakly lower semicontinuity of $\hat{F}(\cdot)$ in $W^{1,p}(\Omega; R^m)$ and (2.8), we have

$$\begin{aligned}\hat{F}(\hat{v}) &\le \liminf_{i\to\infty} \hat{F}(u_i) \le \liminf_{i\to\infty} F(u_i)\\ &\le \lim_{i\to\infty} F_\alpha(u_i, v) = \hat{F}_\alpha(v).\end{aligned}$$

Since v is a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$, this and lemma 2.3 give (2.7).

Now, suppose v is not a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$. Let $\hat{v} \in \mathbb{A}$ be such that

$$\hat{F}(\hat{v}) < \hat{F}(v). \tag{2.9}$$

Let $\{u_i\}_{i=1}^\infty \subset \mathbb{A}$ be such that

$$\lim_{i\to\infty} F_\alpha(u_i, \hat{v}) = \hat{F}_\alpha(\hat{v}). \tag{2.10}$$

By the hypothesis (H1), without loss of generality, we may assume that

$$u_i \rightharpoonup \tilde{v} \quad \text{in } W^{1,p}(\Omega; R^m), \tag{2.11}$$

for some $\tilde{v} \in \mathbb{A}$. By the Sobolev's imbedding theorem [19], (2.11) implies

$$u_i \to \tilde{v} \quad \text{in } L^p(\Omega; R^m). \tag{2.12}$$

Thus, by the definition of $\hat{F}_\alpha(\cdot)$, (2.10), (2.12) and lemma 2.3, we have

$$\begin{aligned}\hat{F}_\alpha(v) &\leq \liminf_{i\to\infty} F_\alpha(u_i, v)\\ &\leq \lim_{i\to\infty} F_\alpha(u_i, \hat{v}) + \lim_{i\to\infty} \alpha \int_\Omega |u_i - v|^p \, dx\\ &= \hat{F}_\alpha(\hat{v}) + \alpha \int_\Omega |\tilde{v} - v|^p \, dx\\ &\leq \hat{F}(\hat{v}) + \alpha \int_\Omega |\tilde{v} - v|^p \, dx.\end{aligned}$$

Let

$$\alpha_1 = \begin{cases} (\hat{F}(v) - \hat{F}(\hat{v}))/\int_\Omega |\tilde{v} - v|^p \, dx & \text{if } \tilde{v} \neq v,\\ +\infty & \text{if } \tilde{v} = v.\end{cases}$$

Then for all $\alpha \in (0, \alpha_1)$, which by (2.9) is not an empty set, we have

$$\hat{F}_\alpha(v) \leq \hat{F}(\hat{v}) + \alpha \int_\Omega |\tilde{v} - v|^p \, dx < \hat{F}(v).$$

This completes the proof of the theorem. □

Corollary 2.1. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy (H1) and (H2). If v is not a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$, then there exists $\alpha_0 \in (0, \infty]$ such that*

$$\hat{F}_\alpha(v) < \hat{F}(v) \quad \forall \alpha \in (0, \alpha_0).$$

Proof. Let $\alpha_0 = \sup\{\alpha > 0 : \hat{F}_\alpha(v) < \hat{F}(v)\}$, which by theorem 2.1 is well defined. It is easily seen that $\hat{F}_\alpha(v)$ is nondecreasing as a function of α, thus the result follows. □

Definition 2.1. Let $v, \hat{v} \in \mathbb{A}$, define

$$\beta(v, \hat{v}) = \liminf_{t\to 0^+} t^{-1}(\hat{F}(v + t\,(\hat{v} - v)) - \hat{F}(v)).$$

If $v \in \mathbb{A}$ is such that

$$\beta(v, \hat{v}) \geq 0 \quad \forall \hat{v} \in \mathbb{A},$$

then v is said to be a lower stationary point of $\hat{F}(\cdot)$ in $\mathbb{A}$.

Remark 2.1. If $\hat{F}$ is Fréchet differentiable at v, then v is a lower stationary point of $\hat{F}(\cdot)$ in $\mathbb{A}$ if and only if $D\hat{F}(v) = 0$.

Theorem 2.2. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy (H1) and (H2). If $v \in \mathbb{A}$ is not a lower stationary point of $\hat{F}(\cdot)$ in $\mathbb{A}$, then*

$$\hat{F}_\alpha(v) < \hat{F}(v) \quad \forall \alpha > 0. \tag{2.13}$$

Proof. Since $v \in \mathbb{A}$ is not a lower stationary point of $\hat{F}(\cdot)$ in $\mathbb{A}$, there exists a $\hat{v} \in \mathbb{A}$, a sequence of positive numbers t_μ with $\lim_{\mu\to\infty} t_\mu = 0$ and $\beta(v, \hat{v}) < 0$ such that

$$\hat{F}(v + t_\mu (\hat{v} - v)) \leq \hat{F}(v) + t_\mu \beta(v, \hat{v}) + o(t_\mu). \tag{2.14}$$

For a fixed t_μ, by lemma 2.1, there exists a sequence $\{u_i\}_{i=1}^\infty \subset \mathbb{A}$ such that

$$u_i \rightharpoonup v + t_\mu (\hat{v} - v) \quad \text{in } W^{1,p}(\Omega; R^m), \tag{2.15}$$

$$\lim_{i\to\infty} F(u_i) = \hat{F}(v + t_\mu (\hat{v} - v)). \tag{2.16}$$

It follows from (2.15) and Sobolev's imbedding theorem [19] that

$$u_i \to v + t_\mu (\hat{v} - v) \quad \text{in } L^p(\Omega; R^m). \tag{2.17}$$

Thus, by (2.14), (2.16) and (2.17), for any $\alpha > 0$ and t_μ

$$\begin{aligned} \hat{F}_\alpha(v) &\leq \lim_{i\to\infty} F_\alpha(u_i, v) \\ &= \hat{F}(v + t_\mu (\hat{v} - v)) + \alpha t_\mu^p \int_\Omega |\hat{v} - v|^p dx \\ &\leq \hat{F}(v) + t_\mu \beta(v, \hat{v}) + \alpha t_\mu^p \int_\Omega |\hat{v} - v|^p dx + o(t_\mu). \end{aligned} \tag{2.18}$$

Since $\beta(v, \hat{v}) < 0$ and $t_\mu > 0$ can be arbitrarily small, (2.18) implies (2.13). □

Corollary 2.2. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy (H1) and (H2). If $v \in \mathbb{A}$ is not a lower stationary point of $\hat{F}(\cdot)$ in $\mathbb{A}$, then for any $\alpha > 0$ and any $W^{1,p}-$weak neighborhood $D(v)$ of v in $\mathbb{A}$*

$$\inf_{u\in D(v)} F_\alpha(u, v) < \hat{F}(v).$$

Proof. The conclusion follows directly from the proof of theorem 2.2. □

Theorem 2.3. *Let $f : \Omega \times R^m \times R^{n\times m} \to R$ satisfy (H1) and (H2). Then, a necessary and sufficient condition for $v \in \mathbb{A}$ being a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$ is that v is a minimizer of $\hat{F}_\alpha(\cdot)$ in $\mathbb{A}$.*

Proof. First, assume that v is a minimizer of $\hat{F}_\alpha(\cdot)$ in $\mathbb{A}$. Let $\{u_i\}_{i=1}^\infty \subset \mathbb{A}$ be such that

$$\lim_{i\to\infty} F_\alpha(u_i, v) = \hat{F}_\alpha(v). \tag{2.19}$$

By (2.19) and the hypothesis (H1), there is a subsequence of $\{u_i\}_{i=1}^\infty$, again denoted by $\{u_i\}_{i=1}^\infty$, and a function $\tilde{v} \in \mathbb{A}$ such that

$$u_i \rightharpoonup \tilde{v} \quad \text{in } W^{1,p}(\Omega; R^m).$$

Thus we have, by the sequentially weakly lower semicontinuity of $\hat{F}(\cdot)$ in $W^{1,p}(\Omega; R^m)$

$$\hat{F}(\tilde{v}) \le \lim_{i\to\infty} F(u_i), \tag{2.20}$$

and, by the Sobolev's imbedding theorem [19]

$$u_i \to \tilde{v} \quad \text{in } L^p(\Omega; R^m). \tag{2.21}$$

We claim that $\tilde{v} = v$, since otherwise, by (2.19)-(2.21), one would lead to

$$\begin{aligned}\hat{F}_\alpha(\tilde{v}) \le \hat{F}(\tilde{v}) &\le \liminf_{i\to\infty} F(u_i) \\ &< \liminf_{i\to\infty} F(u_i) + \alpha \int_\Omega |\tilde{v} - v|^p dx \\ &= \liminf_{i\to\infty} F_\alpha(u_i, v) = \hat{F}_\alpha(v),\end{aligned}$$

which is a contradiction to the assumption that v is a minimizer of $\hat{F}_\alpha(\cdot)$ in $\mathbb{A}$. Hence, by (2.19) and (2.20),

$$\hat{F}(v) \le \liminf_{i\to\infty} F(u_i) \le \lim_{i\to\infty} F_\alpha(u_i, v) = \hat{F}_\alpha(v).$$

This and lemma 2.3 give

$$\hat{F}(v) = \hat{F}_\alpha(v),$$

which, by theorem 2.2, implies that v is a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$.

Next, assume that v is a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$. For any given $\hat{v} \in \mathbb{A}$, let $\{u_i\}_{i=1}^{\infty} \subset \mathbb{A}$ be such that

$$\lim_{i\to\infty} F_\alpha(u_i, \hat{v}) = \hat{F}_\alpha(\hat{v}). \tag{2.22}$$

Then, by theorem 2.2 and (2.22),

$$\begin{aligned}\hat{F}_\alpha(v) = \hat{F}(v) = \inf_{u\in\mathbb{A}} \hat{F}(u) &\le \liminf_{i\to\infty} \hat{F}(u_i)\\ &\le \liminf_{i\to\infty} F(u_i) \le \lim_{i\to\infty} F_\alpha(u_i, \hat{v}) = \hat{F}_\alpha(\hat{v}).\end{aligned}$$

This proves that v is a minimizer of $\hat{F}_\alpha(\cdot)$ in $\mathbb{A}$. □

3. Existence and Convergence of Discrete Solutions

Throughout this section, for simplicity, (H1) and (H2) are assumed to be satisfied by $f(\cdot,\cdot,\cdot)$ in $\Omega_1 \times R^m \times R^{n\times m}$, where $\Omega_1 \supset \bar{\Omega}$ is a bounded open set in R^n. Let $\{\mathfrak{T}_{h_i}\}_{i=1}^{\infty}$ be regular triangulations of Ω [20] with $\Omega_{h_i} \subset \Omega_1$, where Ω_{h_i} is the interior of the set $\cup_{K\in\mathfrak{T}_{h_i}} K$, and mesh sizes $h_i > 0$ satisfying $\lim_{i\to\infty} h_i = 0$. Denote

$$\begin{aligned}\mathbb{A}_{h_i} = \{u \in C(\bar{\Omega}_{h_i}; R^m) :\ & u|_K \text{ is affine}, \forall K \in \mathfrak{T}_{h_i},\\ &\text{and } u(x) = u_0(x) \text{ if } x \text{ is a node on } \partial\Omega_{h_i}\ \}.\end{aligned} \tag{3.1}$$

Our method is to approximate the microstructures and relaxed minimizers by solving the finite problem of minimizing the integral functional $F_\alpha(u, v)$ in the set of admissible functions $\mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$ with properly chosen $\alpha > 0$ and $h_i \gg h_j > 0$.

Theorem 3.1. *For any given $\alpha > 0$, $j \ge 1$ and $i \ge 1$, there exists a solution to the problem of minimizing $F_\alpha(\cdot,\cdot)$ in $\mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$.*

Proof. The conclusion follows directly from the coerciveness and continuity of the functional $F_\alpha(\cdot,\cdot)$ in $W^{1,p}(\Omega; R^m)$ (see (H1)) and that $\mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$ is of finite dimension. □

Lemma 3.1. *For any $\epsilon > 0$, there exist $I(\epsilon) \ge 1$ and $J(\epsilon) \ge 1$ such that for all $\alpha > 0$*

$$\inf_{(u,v)\in\mathbb{A}_{h_j}\times\mathbb{A}_{h_i}} F_\alpha(u, v) \le \inf_{v\in\mathbb{A}} \hat{F}(v) + (\alpha + 1)\epsilon \quad \forall i \ge I(\epsilon) \text{ and } j \ge J(\epsilon).$$

Proof. By (H1) and lemma 2.1, there exists a minimizer $\hat{v}$ of $\hat{F}(\cdot)$ in $\mathbb{A}$ and a sequence $\{u_k\} \subset \mathbb{A}$ such that

$$u_k \rightharpoonup \hat{v} \quad \text{in } W^{1,p}(\Omega; R^m), \tag{3.2}$$

$$F(u_k) \longrightarrow \hat{F}(\hat{v}). \tag{3.3}$$

For given $\epsilon > 0$, by (3.2), (3.3) and Sobolev's imbedding theorem [19], there exists $k(\epsilon) \geq 1$ such that

$$\|u_k - \hat{v}\|_{0,p} < \frac{1}{3}\epsilon^{\frac{1}{p}} \qquad \forall k \geq k(\epsilon), \tag{3.4}$$

$$F(u_k) \leq \hat{F}(\hat{v}) + \frac{\epsilon}{2} \qquad \forall k \geq k(\epsilon), \tag{3.5}$$

where $\|\cdot\|_{0,p}$ is the usual norm of $L^p(\Omega; R^m)$. For $u_{k(\epsilon)}$ and $\hat{v}$, by the interpolation properties of the finite element function spaces [20], there exists $(u_{k(\epsilon)})_{h_j} \in \mathbb{A}_{h_j}$ and $\hat{v}_{h_i} \in \mathbb{A}_{h_i}$ such that

$$\|(u_{k(\epsilon)})_{h_j} - u_{k(\epsilon)}\|_{1,p} \longrightarrow 0 \quad \text{as } j \to \infty, \tag{3.6}$$

$$\|\hat{v}_{h_i} - \hat{v}\|_{1,p} \longrightarrow 0 \quad \text{as } i \to \infty, \tag{3.7}$$

where $\|\cdot\|_{1,p}$ is the usual norm of $W^{1,p}(\Omega; R^m)$. By (3.6) and the continuity of $F(\cdot)$ in $W^{1,p}(\Omega; R^m)$ (see (H1)), there exists $J_1(\epsilon) \geq 1$ such that

$$F((u_{k(\epsilon)})_{h_j}) \leq \hat{F}(\hat{v}) + \epsilon \quad \forall j \geq J_1(\epsilon). \tag{3.8}$$

By (3.6) and (3.7), there exist $J_2(\epsilon) \geq 1$ and $I(\epsilon) \geq 1$ such that

$$\|(u_{k(\epsilon)})_{h_j} - u_{k(\epsilon)}\|_{1,p} \leq \frac{1}{3}\epsilon^{\frac{1}{p}} \quad \forall j \geq J_2(\epsilon), \tag{3.9}$$

$$\|\hat{v}_{h_i} - \hat{v}\|_{1,p} \leq \frac{1}{3}\epsilon^{\frac{1}{p}} \qquad \forall i \geq I(\epsilon). \tag{3.10}$$

Let $J(\epsilon) = \min\{J_1(\epsilon), J_2(\epsilon)\}$. Then, by (3.8)-(3.10),

$$\begin{aligned} F_\alpha((u_{k(\epsilon)})_{h_j}, \hat{v}_{h_i}) &= F((u_{k(\epsilon)})_{h_j}) + \alpha \, \|(u_{k(\epsilon)})_{h_j} - \hat{v}_{h_i}\|_{0,p}^p \\ &\leq \hat{F}(\hat{v}) + (\alpha + 1)\epsilon \quad \forall i \geq I(\epsilon) \text{ and } j \geq J(\epsilon). \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $h_{j(k)} > 0$ and $h_{i(k)} > 0$ satisfy $\lim_{k\to\infty} h_{j(k)} = 0$ and $\lim_{k\to\infty} h_{i(k)} = 0$. Let $(u_k, v_k) \in \mathbb{A}_{h_{j(k)}} \times \mathbb{A}_{h_{i(k)}}$ be minimizers of $F_\alpha(\cdot, \cdot)$ in*

$\mathbb{A}_{h_{j(k)}} \times \mathbb{A}_{h_{i(k)}}$. *Then, there exists a subsequence of* $\{(u_k, v_k)\}$*, again denoted by* $\{(u_k, v_k)\}$*, and a function* $\hat{v} \in \mathbb{A}$ *such that*

$$u_k \rightharpoonup \hat{v} \qquad \text{in } W^{1,p}(\Omega; R^m)$$

and

$$\hat{F}(\hat{v}) = \inf_{v \in \mathbb{A}} \hat{F}(v).$$

Proof. In view of lemma 3.1, we have

$$\lim_{k\to\infty} F_\alpha(u_k, v_k) = \inf_{v\in\mathbb{A}} \hat{F}(v).$$

This and (H1) imply that $\{u_k\}$ is bounded in $W^{1,p}(\Omega; R^m)$ and thus contains a subsequence which converges weakly to a function $\hat{v} \in W^{1,p}(\Omega; R^m)$. Since $\mathbb{A}$ is weakly closed in $W^{1,p}(\Omega; R^m)$, $\hat{v}$ belongs to $\mathbb{A}$. Therefore, it follows from the sequentially weakly lower semicontinuity of $\hat{F}(\cdot)$ and $\hat{F}(u_k) \le F(u_k) \le F_\alpha(u_k, v_k)$ that

$$\inf_{v\in\mathbb{A}} \hat{F}(v) \le \hat{F}(\hat{v}) \le \lim_{k\to\infty} F_\alpha(u_k, v_k) = \inf_{v\in\mathbb{A}} \hat{F}(v).$$

This completes the proof. □

Remark 3.1. It is easy to verify that in theorem 3.2, we also have $v_k \to \hat{v}$ in $L^p(\Omega; R^m)$.

In practice, h_j should be taken to be much smaller than h_i so that the solution to the problem of minimizing $\|u_{h_j} - \cdot\|_{0,p}$ in $\mathbb{A}_{h_i}$ contains only low frequency components of u_{h_j} and thus could be a better approximation of a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$ than u_{h_j}. More precisely, we may expect $v_k \to \hat{v}$ in $W^{1,p}(\Omega; R^m)$. In fact, we have the following result.

Theorem 3.3. *Suppose other than (H1) and (H2) the following two hypotheses are also satisfied.*

- **(H3):** *There is a minimizer* $\hat{v}_1$ *of* $\hat{F}(\cdot)$ *in* $\mathbb{A}$ *and a constant* $r > \max\{\frac{np}{n+p}, \frac{n}{2}\}$ *such that* $\hat{v}_1 \in W^{2,r}(\Omega; R^m)$*;*
- **(H4):** *There are constants* $q \in (0, (2 + n(\frac{1}{p} - \frac{1}{r}))p)$*,* $\delta > 0$ *and* $C_1 > 0$ *such that*

$$F(u) - \inf_{v\in\mathbb{A}} F(v) \ge C_1 \min\{\delta, \inf_{v\in\hat{\mathbb{A}}} \|u - v\|_{0,p}^q\},$$

where $\hat{\mathbb{A}} = \{\hat{v} \in \mathbb{A} : \hat{F}(\hat{v}) = \inf_{v\in\mathbb{A}} \hat{F}(v)\}$.

Suppose the regular triangulations $\{\mathfrak{T}_{h_i}\}_{i=1}^{\infty}$ *of* Ω *satisfy the inverse assumptions* [20]

$$\frac{h_i}{h_K} \le \nu \quad \forall i \text{ and } K \in \mathfrak{T}_{h_i} \tag{3.11}$$

for a constant $\nu > 0$*. Then, there exist nondecreasing functions* $I(\alpha) \ge 1$ *and* $J(i) \ge 1$*, constants* $\hat{C}_1 > 0$ *and* $\gamma > 0$*, and a function* $\hat{C}_2(\alpha) > 0$*, which is bounded on every compact subset of* $(0, +\infty)$*, such that*

$$F(u_{h_j}) \le \inf_{v\in\mathbb{A}} \hat{F}(v) + (\alpha+1)\hat{C}_1 h_i^{(2+n(\frac{1}{p}-\frac{1}{r}))p}, \tag{3.12}$$

$$\inf_{\hat{v}\in\hat{\mathbb{A}}} \|v_{h_i} - \hat{v}\|_{1,p} \le \hat{C}_2(\alpha) h_i^{\gamma}, \tag{3.13}$$

provided $i \ge I(\alpha)$*,* $j \ge J(i)$ *and*

$$F_\alpha(u_{h_j}, v_{h_i}) = \inf_{(u,v)\in\mathbb{A}_{h_j}\times\mathbb{A}_{h_i}} F_\alpha(u, v).$$

Proof. Let $\hat{v}_1 \in W^{2,r}(\Omega; R^m) \cap \mathbb{A}$ be a minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$ (see (H3)). Since $r > n/2$, $\hat{v}_1 \in C(\bar{\Omega})$. Let $\hat{v}_{h_i} \in \mathbb{A}_{h_i}$ be the interpolation of $\hat{v}_1$ in $\mathbb{A}_{h_i}$, by the interpolation properties of the finite element function spaces [20], there exists a constant $C_2 > 0$ such that

$$\|\hat{v}_{h_i} - \hat{v}_1\|_{1,p} \le C_2 h_i^{1+n(\frac{1}{p}-\frac{1}{r})} |\hat{v}_1|_{2,r}, \tag{3.14}$$

$$\|\hat{v}_{h_i} - \hat{v}_1\|_{0,p} \le C_2 h_i^{2+n(\frac{1}{p}-\frac{1}{r})} |\hat{v}_1|_{2,r}. \tag{3.15}$$

On the other hand, with the same argument as that in the proof of lemma 3.1, there exists a sequence of functions $\hat{u}_{h_j} \in \mathbb{A}_{h_j}$ such that

$$\hat{u}_{h_j} \rightharpoonup \hat{v}_1 \quad \text{in } W^{1,p}(\Omega; R^m), \tag{3.16}$$

$$F(\hat{u}_{h_j}) \to \hat{F}(\hat{v}_1) = \inf_{v\in\mathbb{A}} \hat{F}(v). \tag{3.17}$$

It follows from (3.15)-(3.17) that there exists $J(i) \ge 1$ such that

$$F_\alpha(\hat{u}_{h_j}, \hat{v}_{h_i}) \le \hat{F}(\hat{v}_1) + (\alpha+1)(2C_2)^p h_i^{(2+n(\frac{1}{p}-\frac{1}{r}))p} (1 + |\hat{v}_1|_{2,r})^p \quad \forall i \ge 1 \text{ and } j \ge J(i). \tag{3.18}$$

Let $\hat{C}_1 = (2C_2(1 + |\hat{v}_1|_{2,r}))^p$, then (3.18) implies

$$\inf_{(u,v)\in \mathbb{A}_{h_j}\times \mathbb{A}_{h_i}} F_\alpha(u,v) \le \inf_{v\in\mathbb{A}} \hat{F}(v) + (\alpha+1)\hat{C}_1 h_i^{(2+n(\frac{1}{p}-\frac{1}{r}))p}$$

$$\forall i \ge 1 \text{ and } j \ge J(i). \quad (3.19)$$

Let $I(\alpha) \ge 1$ be such that (see (H4))

$$(\alpha+1)\hat{C}_1 h_i^{(2+n(\frac{1}{p}-\frac{1}{r}))p} \le C_1\delta \quad \forall i \ge I(\alpha). \qquad (3.20)$$

Let $i \ge I(\alpha)$ and $j \ge J(i)$, and let $(u_{h_j}, v_{h_i}) \in \mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$ be a minimizer of $F_\alpha(\cdot,\cdot)$ in $\mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$. Then, as a consequence of (3.19), we have (3.12) and

$$\|u_{h_j} - v_{h_i}\|_{0,p} \le (\frac{\alpha+1}{\alpha}\hat{C}_1)^{1/p} h_i^{(2+n(\frac{1}{p}-\frac{1}{r}))}. \qquad (3.21)$$

By (3.14), (3.20), (H4) and the fact that the set $\hat{\mathbb{A}}$ is weakly compact in $W^{1,p}(\Omega; R^m)$, there exists a $\hat{v} \in \hat{\mathbb{A}}$ such that

$$\|u_{h_j} - \hat{v}\|_{0,p} \le ((\alpha+1)C_1^{-1}\hat{C}_1)^{1/q} h_i^{(2+n(\frac{1}{p}-\frac{1}{r}))\frac{p}{q}}$$

$$\forall i \ge I(\alpha) \text{ and } j \ge J(i). \qquad (3.22)$$

Thus, it follows from (3.15), (3.21) and (3.22) that

$$\|v_{h_i} - \hat{v}_{h_i}\|_{0,p} \le \tilde{C}_2(\alpha) h_i^{1+\gamma}, \qquad (3.23)$$

where

$$\gamma = \min\{1 + n(\frac{1}{p} - \frac{1}{r}),\ (2 + n(\frac{1}{p} - \frac{1}{r}))\frac{p}{q} - 1\} > 0$$

$$\tilde{C}_2(\alpha) = (\frac{\alpha+1}{\alpha}\hat{C}_1)^{1/p} + ((\alpha+1)C_1^{-1}\hat{C}_1)^{1/q} + C_2|\hat{v}_1|_{2,r}.$$

It follows from (3.23) and the inverse inequalities of the finite element function spaces [20] that there exists $C(n,\nu) > 0$ such that

$$\|v_{h_i} - \hat{v}_{h_i}\|_{1,p} \le C(n,\nu)\tilde{C}_2(\alpha) h_i^{\gamma}.$$

This and (3.14) give (3.13) with $\hat{C}_2(\alpha) = (1 + C(n,\nu))\tilde{C}_2(\alpha)$. □

4. Numerical Method for the Discrete Solutions

To solve the problem of minimizing $F_\alpha(\cdot,\cdot)$ in $\mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$ with $h_j \ll h_i$, we begin with the following theorem.

Theorem 4.1. *Define*

$$\hat{F}_{\alpha,h_j}(v) = \inf_{u \in \mathbb{A}_{h_j}} F_\alpha(u,v). \tag{4.1}$$

Then, we have

$$\inf_{v \in \mathbb{A}_{h_i}} \hat{F}_{\alpha,h_j}(v) = \inf_{(u,v) \in \mathbb{A}_{h_j} \times \mathbb{A}_{h_i}} F(u,v). \tag{4.2}$$

Proof. Since the function spaces $\mathbb{A}_{h_j}$ and $\mathbb{A}_{h_i}$ are of finite dimension and the functionals are continuous, there exist $\hat{v} \in \mathbb{A}_{h_i}$ and $u_j(\hat{v}_i) \in \mathbb{A}_{h_j}$ such that

$$\hat{F}_{\alpha,h_j}(\hat{v}) = \inf_{v \in \mathbb{A}_{h_i}} \hat{F}_{\alpha,h_j}(v) \tag{4.3}$$

and

$$\hat{F}_\alpha(u_j(\hat{v}_i), \hat{v}_i) = \inf_{u \in \mathbb{A}_{h_j}} F_\alpha(u, \hat{v}_i). \tag{4.4}$$

It is easy to verify that

$$\hat{F}_\alpha(u_j(\hat{v}_i), \hat{v}_i) = \inf_{(u,v) \in \mathbb{A}_{h_j} \times \mathbb{A}_{h_i}} F(u,v). \tag{4.5}$$

By combining the definition (4.1) with (4.3), (4.4) and (4.5), we get (4.2). □

In view of theorem 4.1, to find a minimizer of $F_\alpha(\cdot,\cdot)$ in $\mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$ with $h_j \ll h_i$, we start from a initial function $v^0 \in \mathbb{A}_{h_i}$ and produce a sequence of functions $(u^k, v^k) \in \mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$ by solving the following problems recursively

$$F_\alpha(u^k, v^k) = \inf_{u \in \mathbb{A}_{h_j}} F_\alpha(u, v^k), \tag{4.6}$$

$$\|u^k - v^{k+1}\|_{0,p} = \inf_{v \in \mathbb{A}_{h_i}} \|u^k - v\|_{0,p}. \tag{4.7}$$

It is obvious that the sequence thus produced satisfies

$$F_\alpha(u^{k+1}, v^{k+1}) \le F_\alpha(u^k, v^k) \quad \forall k, \tag{4.8}$$

where the equality holds if and only if $v^{k+1} = v^k$. For α sufficiently large, the solutions to (4.6) are in a L^p-neighborhood of v^k, and can be located by a gradient type method starting from a small random perturbation of v^k. In fact, we only need to solve (4.6) locally to achieve (4.8). Generally, the procedure leads to local minimizers of $F_\alpha(\cdot,\cdot)$ in $\mathbb{A}_{h_j} \times \mathbb{A}_{h_i}$. However, as suggested by corollary 2.2, for $h_j \ll h_i$ sufficiently small, a fixed point of the above procedure should be a good approximation to a stationary point of $\hat{F}(\cdot)$ in $\mathbb{A}$. Therefore we have reason to expect that the limit $v^\infty \in \mathbb{A}_{h_i}$ of the sequence $\{v^k\}$ is a good approximation to a local minimizer of $\hat{F}(\cdot)$ in $\mathbb{A}$ and the corresponding $u^\infty \in \mathbb{A}_{h_j}$ is a good approximation to a microstructure.

To avoid to be stuck in a local minimizer of $F(\cdot)$ in $\mathbb{A}_{h_i}$, particularly to eliminate oscillations in v^k and increase the stability of the method, the following problem may be used to replace (4.7), that is to find a function $v^{k+1} \in \mathbb{A}_{h_i}$ such that

$$\|u^k - v^{k+1}\|_{0,p}^p + \gamma\|\overline{\nabla v^{k+1}} - \nabla v^{k+1}\|_{0,p}^p = \inf_{v\in\mathbb{A}_{h_i}} (\|u^k - v\|_{0,p}^p + \gamma\|\overline{\nabla v} - \nabla v\|_{0,p}^p). \quad (4.9)$$

where γ is a parameter and for each $K \in \mathfrak{T}_h$ $\overline{\nabla v}|_K$ is a weighted average of ∇v in a neighborhood of K. For example, $\overline{\nabla v}|_K$ can be defined by

$$\overline{\nabla v}|_K = \sigma(K)^{-1} \sum_{K'\in\mathfrak{T}_h} \sigma(K,K')\nabla v|_{K'},$$

where $\sigma(K,K') = \{$number of common nodes shared by K and $K'\}$ and $\sigma(K) = \sum_{K'\in\mathfrak{T}_h} \sigma(K,K')$. The reason is established upon the observation that if $v_i^\infty \in \mathbb{A}_{h_i}$ converges strongly in $W^{1,p}(\Omega; R^m)$ to a function $v \in \mathbb{A}$ then

$$\lim_{i\to\infty} \|\overline{\nabla v_i^\infty} - \nabla v_i^\infty\|_{0,p} = 0.$$

Since the speed of the convergence depends on the regularity of the function $v \in \mathbb{A}$ to be approximated, which is not known in advance, the parameter γ, which is used somehow to balance the two terms in (4.9), usually need to be decided by experiments in computation.

For a better performance of the method, the parameter α should be related to the refined mesh size h_j. In the numerical example given below, we took $\alpha = O(h_j^{-3/2})$. An analysis on such a relationship can be found in [17], where microstructures are approximated by solving (4.6) with $h_i = h_j$ and v^k being a given approximate relaxed minimizer.

Example. Let $\Omega = (0,\ 1) \times (0,\ 1) \subset R^2$. Let

$$f(\nabla u) = ((u'_x)^2 - 1)^2 + (u'_y)^2, \tag{4.10}$$

$$\mathbb{A} = \{u \in W^{1,4}(\Omega) : u(x,y) = 24(x-0.5)^2(y-0.5)^3, \quad \forall (x,y) \in \partial\Omega\}. \tag{4.11}$$

Let $N_x > 1$, $N_y > 1$ be integers. Let $h = N_x^{-1}$ and $t = N_y^{-1}$. Introduce on Ω a triangulation with $2N_xN_y$ triangles by the following lines

$$\begin{cases} x = ih, & i = 0, 1, \ldots, N_x; \\ y = jt, & j = 0, 1, \ldots, N_y; \\ y = ht^{-1}(x - kh), & k = -N_y, -N_y + 1, \ldots, N_x - 1. \end{cases}$$

Notice that there are two potential wells in $f(\cdot)$ and the boundary condition is nonlinear. To simplify the computation, p in (4.9) is taken to be 2 instead of 4.

First, a 20×20 coarse mesh ($N_x = N_y = 20$) was used to compute the relaxed minimizer and, with a refinement parameter 3, a 60×60 refined mesh was used to compute the microstructure. $v^0(x,y) = 24(x-0.5)^2(y-0.5)^3$ was taken to be the initial function, and α in (4.6) and γ in (4.9) was set to 100 and 10^{-2} respectively in the computation. Then, with a refinement parameter 7 and with $\alpha = 1400$ and $\gamma = 10^{-3}$, a 140×140 refined mesh was used to couple with the same coarse mesh to compute the relaxed minimizer and the microstructure. The numerical results are shown in Fig 1, Fig 2 and Table 1, where in Table 1 S-method represents the method developed in this paper.

Finally, a 50×50 coarse mesh ($N_x = N_y = 50$) was used to compute the relaxed minimizer and, with a refinement parameter 3, a 150×150 refined mesh was used to compute the microstructure. $v^0(x,y) = 24(x-0.5)^2(y-0.5)^3$ was again taken to be the initial function, while α and γ were set to 1500 and 10^{-3} respectively in the computation. The numerical results are shown in Fig 3 and Table 1.

The numerical results suggest that, for reasonably chosen parameters, the relaxed energy $\hat{F}(v_h^\infty)$ depends mainly on the coarse mesh size h_i and the energy $F(u_h^\infty)$ mainly depends on the refined mesh size h_j.

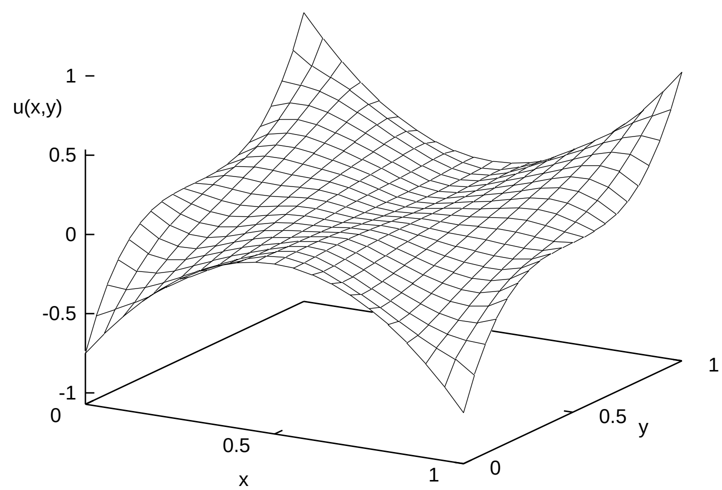

FIGURE 1. 20×20 mesh solution for relaxed minimizer

Where in Table 1

$$A^+(a) = \text{meas}\{x \in \Omega : \sqrt{(u'_x(x) - 1)^2 + (u'_y(x))^2} < a\},$$

$$A^-(a) = \text{meas}\{x \in \Omega : \sqrt{(u'_x(x) + 1)^2 + (u'_y(x))^2} < a\}$$

are the measures of the regions in which the gradient of u_h^∞ falls into a neighborhood of the potential wells.

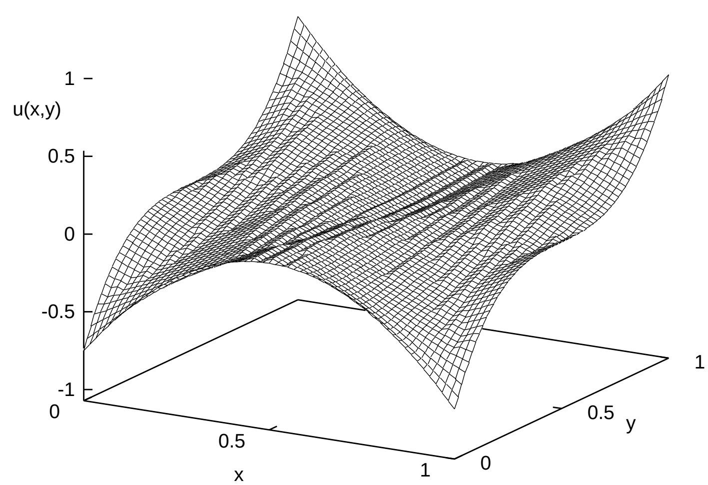

FIGURE 2. 60×60 mesh solution for microstructure

For comparison, we provide also the numerical results obtained by a numerical method given by Li in [17], which is mentioned as Q-method in Fig 4, Fig 5 and Table 1. Notice that it is because that the Q-method made use of the quasiconvex envelope $\hat{f}$ of f, which can be given explicitly in this case by

$$\hat{f}(\nabla u(x)) = \begin{cases} f(\nabla u(x)), & \text{if } |u'_x| > 1; \\ (u'_y(x))^2, & \text{if } |u'_x| \le 1, \end{cases}$$

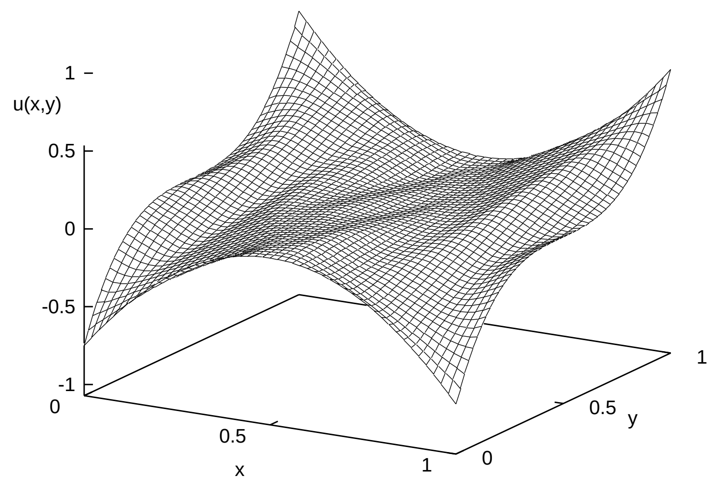

FIGURE 3. 50×50 mesh solution for relaxed minimizer

that the better numerical results were obtained. By comparing the numerical results, we see that the method given in this paper produced satisfactory approximation to both the relaxed minimizer and the microstructure.

References

[1] J. M. Ball and R. D. James, *Fine phase mixtures as minimizers of energy.* **Arch. Rat. Mech. Anal., 100**(1987)1, 13-52.

Method	Mesh		Energy		Area in Wells	
	Coarse	Refined	$\hat{F}(v_h^\infty)$	$F(u_h^\infty)$	$A^+(0.1)$	$A^-(0.1)$
S-	20×20	60×60	1.2293	1.0605	0.3842	0.3817
	20×20	140×140	1.2265	1.0155	0.3894	0.3942
	50×50	150×150	1.0240	1.0081	0.3878	0.3999
Q-	20×20		1.2063	1.3768	0.2875	0.3225
	50×50		1.0013	1.0782	0.3576	0.3568
	150×150		0.9841	1.0298	0.3722	0.3727

TABLE 1. Energies and areas in potential wells

[2] Z.-P. Li, *Existence of minimizers and microstructures in nonlinear elasticity.* **Nonlinear Anal.: Theo., Meth. Appl., 27**(1996), 297-308.

[3] C. B. Morrey, **Multiple Integrals in the Calculus of Variations**. Springer, New York, 1966.

[4] J. M. Ball and F. Murat $W^{1,p}$*-quasiconvexity and variational problems for multiple integrals.* **J. Funct. Anal., 58**, (1984)225-253.

[5] E. Acerbi and N. Fusco, *Semicontinuity problems in the calculus of variations.* **Arch. Rat. Mech. Anal. 86**(1984), 125-145.

[6] J.M. Ball, *A version of the fundamental theorem for Young measures.* In **Partial Differential Equations and Continuum models of Phase Transitions**, Lecture Notes in Physics, No. 344, M. Rascle, D. Serre and M. Slemrod, eds., Springer-Verlag, (1989), 207-215.

[7] B. Dacorogna, **Direct Methods in the Calculus of Variations**, Applied Mathematical Sciences, 78, Springer-Verlag, Berlin, 1989.

[8] G. Buttazzo, **Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations**. Pitman Research Notes in Mathematics Series 207. Longman, 1989.

[9] Z.-P. Li, *A theorem on lower semicontinuity of integral functionals.* **Proc. R. Soc. Edinburgh 126A**(1996), 363-374.

[10] Z.-P. Li, *Lower semicontinuity of multiple integrals and convergent integrands.* **ESAIM: Control, Optimisation and Calculus of Variations 1**(1996), 169-189.

[11] Z.-P. Li, *An integral representation theorem on lower semicontinuous envelope of integral functionals.* Research Report No.12, Institute of Mathematics and School of Mathematical Sciences, Peking University, 1996.

[12] M. Chipot, *Numerical analysis of oscillations in nonconvex problems.* **Numer. Math., 59** (1991), 747-767.

[13] C. Collins and M. Luskin, *Optimal-order error estimates for the finite element approximation of the solution of a nonconvex variational problem.* **Math. Comput., 57**(1991), 621-637.

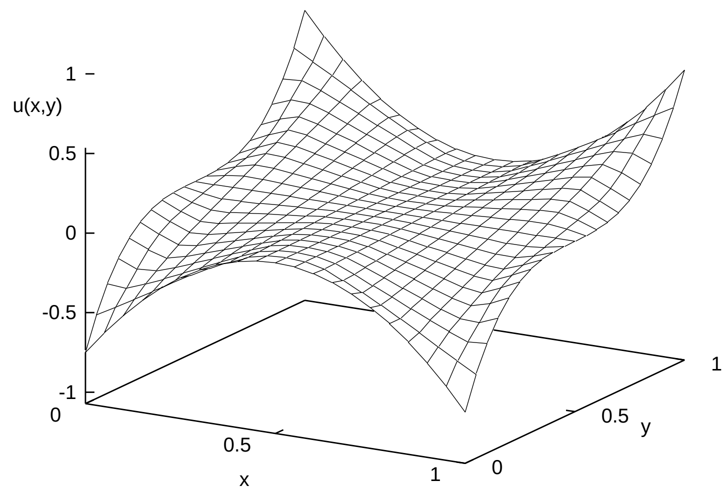

Figure 4. 20×20 mesh relaxed minimizer obtained by Q-method

[14] C. Collins, D. Kinderlehrer and M. Luskin, *Numerical approximation of the solution of a variational problem with a double well potential*. **SIAM J. Numer. Anal., 28**(1991), 321-332.

[15] P.-A. Gremaud, *Numerical analysis of a nonconvex variational problem related to solid-solid phase transitions*. **SIAM J. Numer. Anal., 31**(1994), 111-127.

[16] M. Luskin, *On the computation of crystalline microstructure*. To appear in **Acta Numerica**, (1996).

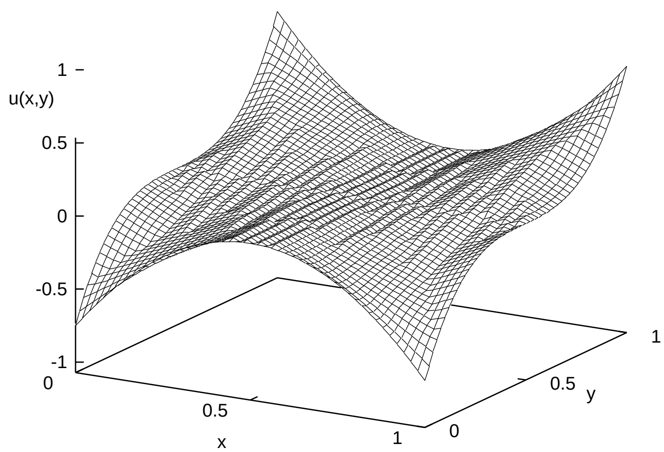

FIGURE 5. 50×50 mesh microstructure obtained by Q-method

[17] Z.-P. Li, *Computation of microstructures by using quasiconvex envelope*. Research Report No.61, Institute of Mathematics and Department of Mathematics, Peking University, 1995.

[18] R.T. Kohn and M. Vogelius, *Relaxation of a variational method for impedance computed tomography*. **Commu. Pure Appl. Math., 40**(1987), 745-777.

[19] R.A. Adams, **Sobolev Spaces**. Academic Press, New York, 1975.

[20] P.G. Ciarlet, **The Finite Element Method for Elliptic Problems**. North-Holland, Amsterdam, 1978.

E-mail address: zpli@math.pku.edu.cn