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Simultaneous Partitions of Measures by k-Fans*

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Abstract. A k-fan is a point in the plane and k semilines emanating from it. Motivated by a neat question of Kaneko and Kano, we study equipartitions by k-fans of two or more probability measures in the plane, as well as partitions in other prescribed ratios. One of our results is: for any two measures there is a 4-fan such that one of its sectors contains two-fifths of both measures, and each of the the remaining three sectors contains one-fifth of both measures.

1. Introduction

For an integer $k \geq 2$, we define a *k-fan* as a point x (the *center*) in the plane and k semilines emanating from x (the *rays*); Fig. 1(a) shows an example of a 3-fan. A k-tuple of parallel lines, as in Fig. 1(b), is also considered to be a k-fan (this is a limit case for x receding to infinity). In this case, it is even possible that some of the parallel lines are also at infinity. The k rays emanating from x are numbered as $\ell_1, \ell_2, \ldots, \ell_k$ in a cyclic order (clockwise or counterclockwise) around x. Each k-fan has an *orientation* (clockwise or counterclockwise) associated to it; for $k \geq 3$, the orientation is given by the labeling of the lines, and for $k = 2$, it is extra information attached to the 2-fan.

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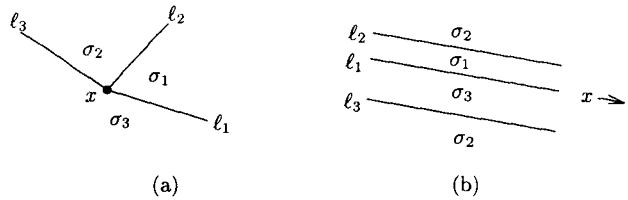

Fig. 1. Examples of 3-fans.

The open angular sector between ℓ_i and ℓ_{i+1} is denoted by σ_i (for $k = 2$, σ_1 is the sector following ℓ_1 in the given orientation of the 2-fan). For x at infinity, the two unbounded regions together form one sector; see Fig. 1(b) (unless, of course, one of the rays is at infinity). We also allow for $\sigma_i = \emptyset$; in this case, ℓ_i and ℓ_{i+1} coincide. A k-fan is called *convex* if all of its sectors are convex.

Let $\mu_1, \mu_2, \ldots, \mu_m$ be Borel probability measures in the plane, $m \geq 2$, and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a vector of nonnegative real numbers whose components sum up to 1. We say that a k-fan with sectors $\sigma_1, \sigma_2, \ldots, \sigma_k$ *α-partitions* the measure μ_j if the following holds: For any $i_1, i_2 \in \{1, 2, \ldots, k\}$, the open angular sector between ℓ_{i_1} and ℓ_{i_2} (in the sense given by the orientation of the k-fan) has μ_j-measure at most $\alpha_{i_1} + \alpha_{i_1+1} + \cdots + \alpha_{i_2}$ (where the indices are taken in the cyclic order, with 1 following k). If μ_j is such that any line has μ_j-measure 0, then this definition can of course be simplified to $\mu_j(\sigma_i) = \alpha_i$, $i = 1, 2, \ldots, k$. For measures partially concentrated on some of the ℓ_i, the part of μ_j on ℓ_i can be arbitrarily divided between the adjacent sectors; this is captured by the rather complicated general definition above (which also covers the case of several ℓ_i's coinciding). If $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1/k$ we speak of an *equipartition* of μ_j.

In this paper we investigate the following problem: for what combinations of m, k, and α can any $\mu_1, \ldots, \mu_m$ be simultaneously α-partitioned by some k-fan?

This problem takes its origin from a very nice question of Kaneko and Kano [KK]. Given an integer $n \geq 2$ and two measures μ_1 and μ_2 in the plane (finite point sets, in fact) with $\mu_1(\mathbf{R}^2) = \mu_2(\mathbf{R}^2) = n$, does there exist a convex partition $C_1, \ldots, C_n$ of $\mathbf{R}^2$ such that $\mu_1(C_i) = \mu_2(C_i) = 1$ for all i. (As expected, $C_1, \ldots, C_n$ form a convex partition of $\mathbf{R}^2$ by definition if the C_i are convex sets that are pairwise internally disjoint and their union is $\mathbf{R}^2$.) The case $n = 2$ is easy: it is the planar ham-sandwich theorem. The case $n = 3$ leads immediately to the problem of whether a convex 3-fan equipartition exists for any two measures.

As we learned during the preparation of this paper, results answering Kaneko and Kano's original question, in various levels of generality, were proved, independent of our work, in several very recent papers. Akiyama et al. [ARNU] prove the special case when the two measures are the surface area and perimeter of a plane convex body. Ito et al. [IUY] show the case $n = 3$ of the conjecture. Finally Bespamyatnikh et al. [BKS] and Sakai [Sa], independently of each other, prove the conjecture in full generality. The argument of Bespamyatnikh et al. [BKS] and Sakai [Sa] is similar to a proof of the two-dimensional Brouwer fixed-point theorem via Sperner's lemma, but it is considerably more complicated. The method does not seem to extend to α-partitions with $\alpha \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

The following theorem summarizes our results.

Theorem 1.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ have no zero component.*

(i) (a) *For any $k \geq 2$ and any α, there are four measures that cannot be simultaneously α-partitioned by a k-fan.*
(b) *For any $k \geq 3$ and any α, there are three measures that cannot be simultaneously α-partitioned by a k-fan.*
(c) *For any $k \geq 5$ and any α, there are two measures that cannot be simultaneously α-partitioned by a k-fan.*
(d) *For $k = 4$ and any α, there are two measures that cannot be simultaneously α-partitioned by a convex 4-fan.*

(ii) (a) *Any two measures can be simultaneously α-partitioned by a 2-fan, for all α. The center of the 2-fan can be prescribed arbitrarily.*
(b) *Any three measures can be simultaneously α-partitioned by a 2-fan for $\alpha = (\frac{1}{2}, \frac{1}{2})$ and for $\alpha = (\frac{2}{3}, \frac{1}{3})$.*
(c) *Any two measures can be simultaneously equipartitioned by a 3-fan, or $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$-partitioned by a 3-fan. They can also be $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$-partitioned by a 4-fan.*

Note that the last statement in (ii)(c) implies that any two measures can be simultaneously $(\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$-partitioned and $(\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$-partitioned by a 3-fan. The results are summarized in Fig. 2.

The negative results in (i) are proved, using simple counterexamples, in Section 2. In parts (b) and (c), the reason for the nonexistence of partitions is that k-fans do not have enough "degrees of freedom," while in (a) and (d), there is a geometric constraint (although the degrees of freedom of 2-fans appear sufficient to partition four measures).

Part (ii)(a) follows by a rather simple averaging argument. Also the $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$-partition is very simple. The other positive results in (ii) are proved using equivariant topology.

Previous work. Equipartitions of measures in $\mathbf{R}^d$ by simple geometric configuration have been studied in many papers. The primary example is the well-known ham-sandwich theorem, stating that any d measures in $\mathbf{R}^d$ can be simultaneously bisected by a hyper-

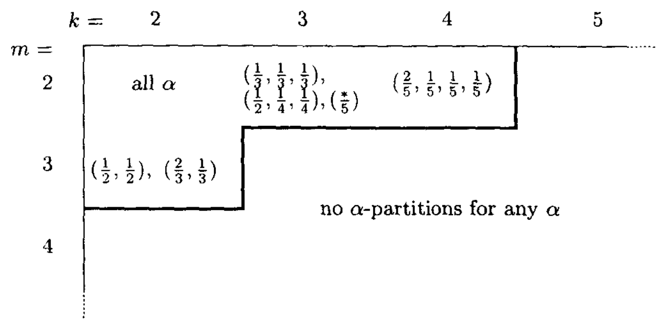

Fig. 2. Results on α-partitioning m measures by k-fans ($*/5$ means all combinations with denominators 5).

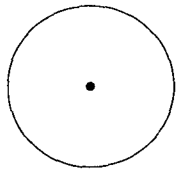

Fig. 3. Two measures that cannot be α-partitioned by a line for $\alpha \neq (\frac{1}{2}, \frac{1}{2})$.

plane. In the plane, any two measures can be simultaneously bisected by a line. Note that no α-partition by a line with $\alpha \neq (\frac{1}{2}, \frac{1}{2})$ is possible in general. This is shown by the example in Fig. 3, where one measure is uniformly distributed on the circle and the other one is concentrated in its center.

An easy consequence of the two-dimensional ham-sandwich theorem is the possibility of partitioning a measure in the plane into four equal parts by two lines. Various generalizations of this fact, most notably partitions of m measures in $\mathbf{R}^d$ into 2^k equal parts by k hyperplanes, have been considered by several researchers; see [Ra] for recent results and references, and the survey by Živaljević [Ži3] for a description of still newer results of Petrović et al. The most challenging problem in this area is probably partitioning a measure in $\mathbf{R}^4$ into 16 equal parts by four hyperplanes, which still remains unsolved.

Another interesting equipartition result, namely equipartitioning a measure into eight parts by a "cobweb" (two lines and a convex quadrilateral with vertices on the lines and surrounding the intersection of the lines), is due to Schulman [Sc]. Makeev [Ma] established the existence of 6-partitions by suitable cones in $\mathbf{R}^3$; for example, for any measure, there is a cube C such that the six cones with apex in the center of C and with the facets of C as bases form an equipartition.

In the literature we are aware of, all measure partition results of this type only concern equipartitions. For k-fan partitions, partitions other than equipartitions are sometimes possible, and this, in our opinion, makes the problem of partitioning by k-fans quite interesting.

2. Counterexamples

Four Measures. To prove part (a) of Theorem 1.1(i), it is sufficient to consider 2-fans and arbitrary α, since α-partitions by k-fans imply α'-partitions by $(k-1)$-fans by "omitting a ray."

We consider four points $p_1, \dots, p_4$ such that one of them is in the convex hull of the other three (Fig. 4). Let μ_j be concentrated in p_j, $j = 1, \dots, 4$. Let $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 > 0$ and $\alpha_2 > 0$. If a 2-fan α-partitions μ_j, then p_j must lie on a ray of the 2-fan. However, it is not possible that all of $p_1, \dots, p_4$ simultaneously lie on the rays of a 2-fan.

Three Measures. In part (b), it is enough to consider 3-fans. We choose a set $P = \{p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}\}$ of six points in *strongly general position*, meaning that no

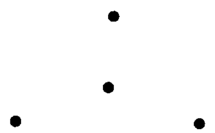

Fig. 4. Four measures that cannot be partitioned by a 2-fan.

three of them are collinear and no three lines determined by disjoint pairs of points of P have a common intersection (or are parallel). Note that such P cannot be simultaneously covered by the rays of a 3-fan (for at most two rays can cover two points each).

Next, given $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we choose weights $w_1, w_2 > 0$ with $w_1 + w_2 = 1$ and $w_1, w_2 \notin \{\alpha_1, \alpha_2, \alpha_3\}$. The measure μ_j is concentrated on p_{j1} and p_{j2}, and we put $\mu_j(p_{ji}) = w_i$, $i = 1, 2$. We claim that if a 3-fan α-partitions μ_j, then *either one of* p_{j1}, p_{j2} *is the center of the* 3*-fan, or both* p_{j1} *and* p_{j2} *lie on rays of the* 3*-fan*. Indeed, supposing that the center is not one of p_{j1}, p_{j2}, it is clear that at least one of these points, say p_{j1}, must lie on a ray (see Fig. 5). If p_{j2} is inside a sector, then this sector must be adjacent to the ray containing p_{j1}, otherwise the sector would have the wrong measure w_2. However, then there is a sector of measure 0. Thus, the claim holds, and from the strongly general position of P, it follows that a simultaneous α-partition of μ_1, μ_2, μ_3 by a 3-fan is impossible.

Two Measures by 5*-Fans.* In part (c) of Theorem 1.1(i), it is enough to consider the case $k = 5$ and $\alpha = (\alpha_1, \ldots, \alpha_5)$ with all $\alpha_j > 0$.

There is a very simple construction showing the impossibility of equipartitioning. Namely, choose a set P of eight points in strongly general position. Let μ_1 be uniformly distributed on some four of the points of P, and let μ_2 be uniformly distributed on the remaining four points. Since the rays of a 5-fan can cover at most seven points of P, there is a point of P lying inside some sector, and so the measure of that sector is at least $\frac{1}{4}$. Hence a 5-fan cannot equipartition these measures.

For α-partitioning by a 5-fan with arbitrary α, we use the following construction. Let μ_1 be the uniform measure whose support S_1 is the segment $[(-1, 1), (1, 1)]$, and let μ_2 be the measure whose support S_2 is the x-axis and whose distribution function $F(t)$ is continuous with $0 < F(t) < 1$ and, most importantly, no line intersects the graph of F in more than three points.

We claim that there is no α-partition by a 5-fan for these two measures. Assume there is one. Then four consecutive rays must intersect each support as otherwise there are no five pieces. So the center is not between S_2 and the line of S_1: it is either below S_2 or above S_1. In both cases the four consecutive rays intersecting S_1 intersect S_2 as well.

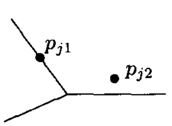

Fig. 5. A 2-point measure and a 3-fan.

Denote the intersection points on S_2 by x_1, x_2, x_3, x_4 in this order. Computing the μ_1 measures of the sectors we get, with some positive h and suitable j,

$$x_2 - x_1 = h\alpha_j, \qquad x_3 - x_2 = h\alpha_{j+1}, \qquad x_4 - x_3 = h\alpha_{j+2},$$

where $j+1$, $j+2$ are taken mod 5. The μ_2 measures of the sectors give

$$F(x_2) - F(x_1) = \alpha_j, \qquad F(x_3) - F(x_2) = \alpha_{j+1}, \qquad F(x_4) - F(x_3) = \alpha_{j+2}.$$

This shows four points on the graph of $F(t)$ contained in a line, namely, the ones corresponding to x_1, x_2, x_3, x_4.

Two Measures by Convex 4-Fan. For part (d), we use the previous setting but the second measure μ_2 will be different: let its support, S_2, be the segment $[(-2, 0), (0, 2)]$ and let it be uniform on S_2 for the time being. We modify it soon. Assume there is convex α-partition by a 4-fan. Then three consecutive rays intersect each support and the center cannot be between the two supports. It cannot be below S_2 either because then one sector would meet S_2 in an interval too short to have the prescribed μ_2-measure. So the center is above the lines of S_1 and S_2. Then there are three downward rays and the fourth goes upwards to make the partition convex. It is evident that the three downward rays, together with the center, are uniquely determined by α. Now we modify μ_2 near the intersection of the middle downward ray with S_2, by pushing a little mass from the left to the right. This changes only the middle downward ray, and this ray will not pass through the intersection of the other two downward rays.

3. Preliminary Reductions

In this section we make some preliminary steps for the proof of the positive results in Theorem 1.1.

A Reformulation on the Sphere. First, we transfer the problem to the two-dimensional sphere S^2. Let S^2 be the unit sphere in $\mathbf{R}^3$ centered at the origin, and let $\mathbf{R}^2$ be embedded in $\mathbf{R}^3$ as the horizontal plane ρ tangent to S^2 from below (i.e., with equation $z = -1$). Let π denote the central projection from the origin. This π gives a homeomorphism of the upper open hemisphere of S^2 onto ρ, and similarly for the lower open hemisphere. A given Borel measure in $\mathbf{R}^2$ is transferred by π^{-1} to a Borel measure in the upper open hemisphere.

A *k-fan* in S^2 is a point $x \in S^2$ and a collection $(\ell_1, \ell_2, \dots, \ell_k)$ of great semicircles emanating from x, in such a way that the semicircles $\ell_1, \dots, \ell_k$ are ordered clockwise around x when viewed from the center of the sphere. We write this k-fan as $(x; \ell_1, \dots, \ell_k)$. The sectors of a k-fan and α-partition of a measure by a k-fan are defined in an obvious analogy to the planar case.

To any k-fan $(x; \ell_1, \dots, \ell_k)$, we assign a k-fan in the plane as follows. The center of the corresponding planar k-fan is $\pi(x)$ (for x on the equator, $\pi(x)$ is formally at the infinity). The image of the ray ℓ_i is obtained as the intersection of ρ with the halfplane that intersects S^2 in the great semicircle ℓ_i. Also the orientation of the k-fan is transferred by the projection (hence k-fans with centers at the upper hemisphere induce counterclockwise planar k-fans, and k-fans with centers on the lower hemisphere induce clockwise planar k-fans).

If any m measures on S^2 can be α-partitioned by k-fans, then any m planar measures can be α-partitioned by k-fans. In fact, the above correspondence of planar and spherical k-fans is bijective, but the spherical problem is more general, since the measures obtained from planar situations only live in the upper open hemisphere.

Nice Measures. Next, we note that by standard arguments, we can restrict ourselves to special measures. We formulate a somewhat stronger result than we actually need.

Lemma 3.1. *Let k, m, and α be given. Any m Borel probability measures in S^2 can be α-partitioned by a k-fan if (and only if) all m-tuples of measures from the following special classes can be so partitioned*:

(i) *Measures concentrated on finitely many points in S^2.*
(ii) *Measures that are absolutely continuous with respect to the Lebesgue measure and such that any nonempty open set has a strictly positive measure.*

Sketch of Proof. It follows easily from the results of Vapnik and Chervonenkis [VC] that given a Borel probability measure μ on S^2, for any $\varepsilon > 0$ there is a measure μ_ε concentrated on finitely many points such that for any sector σ (the open region delimited by two great semicircles), we have $|\mu(\sigma) - \mu_\varepsilon(\sigma)| \leq \varepsilon$. The argument is finished by letting $\varepsilon \to 0$ and noting that the space of all spherical k-fans with the natural topology is compact.

The transformation to measures as in (ii) is used in almost all of the equipartition results in the literature. For a measure concentrated on finitely many points, we can replace each point by a spherical cap of radius ε with the appropriately scaled Lebesgue measure on it, and then add ε-times the Lebesgue measure on S^2. Letting $\varepsilon \to 0$ and using compactness works again. We remark that for an arbitrary Borel probability measure, we can take convolution with a suitable measure ν_ε (whose density function is a narrow peak) and obtain a measure as in (ii) directly. □

4. Easy Positive Results

First we prove part (a) of Theorem 1.1(ii), α-partitions of two measures by 2-fans. This result can easily be proved by the methods below involving equivariant topology, but here we show a simple averaging argument suggested by Attila Pór. Fix the center x of the considered 2-fans, and let γ be the circle of unit length centered at x. We may assume that μ_1 and μ_2 are measures on γ, and by considerations analogous to those in the proof of Lemma 3.1, we may suppose that they are sufficiently nice. Namely, we suppose that after a suitable reparameterization of γ, μ_2 is the one-dimensional Lebesgue measure on γ, and that μ_1 is given by a density function g on γ (with $\int_\gamma g(t)\,dt = 1$).

We want to show that there is an arc $a = (s, s+\alpha_1) \subset \gamma$ of length α_1 with $\mu_1(a) = \alpha_1$. Define the function $f(s) = \mu_1((s, s+\alpha_1))$. This is a continuous function, and if it attains no value α_1, then it is always strictly below α_1 or always strictly above α_1. However, we have

$$\int_\gamma f(s)\,\mathrm{d}s = \int_\gamma \int_s^{s+\alpha_1} g(t)\,\mathrm{d}t\,\mathrm{d}s = \int_\gamma \int_{t-\alpha_1}^{t} g(t)\,\mathrm{d}s\,\mathrm{d}t = \alpha_1.$$

This proves part (a) of Theorem 1.1(ii).

Another easy case is the $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$-partition of two measures by 3-fans in part (c) of Theorem 1.1(ii). Two measures can be simultaneously bisected by a line by the ham-sandwich theorem. We bisect the halves in one of the halfplanes by another line, and we obtain a (convex) 3-fan providing the desired partition.

5. Tools from Equivariant Topology

We now interrupt our discussion of k-fans and discuss some general results from equivariant topology.

Let G be a group and let X be a topological space. We recall that an *action* ω of G on X is a homomorphism from G into the group of homeomorphisms of X. Explicitly, for each $g \in G$, we have a homeomorphism $\omega_g\colon X \to X$, and $\omega_g \circ \omega_h = \omega_{gh}$ holds for all $g, h \in G$. The action ω is called *free* if for $g \in G$ distinct from the unit element, the homeomorphism ω_g has no fixed points.

If $G = \mathbf{Z}_q$ is a cyclic group with generator g, it is sufficient to specify the single homeomorphism ω_g. In this case, as is usual in the literature, we write ω instead of ω_g and we speak of "the $\mathbf{Z}_q$-action ω."

If X is a topological space with an action ω of G and Y is another topological space with an action ν of G, a *G-equivariant map* (with respect to ω and ν), or simply a *G-map*, from X to Y is a continuous mapping $f\colon X \to Y$ that commutes with the actions; that is, $f \circ \omega_g = \nu_g \circ f$ for all $g \in G$.

There are numerous results in combinatorics and in geometry that are derived by showing the nonexistence of an equivariant map between suitable topological spaces. This method was elaborated in a number of papers by Lovász, Alon, Bárány, Shlosman, Szűcs, Sarkaria, Živaljević, Vrećica, Ramos, and others; a recent survey is [Ži2] (also see [Ži1] for an expanded version), and fairly advanced tools are discussed in its continuation [Ži3].

The following theorem of Dold [Do] can be used to exclude the existence of equivariant mappings $X \to Y$, provided that the actions of G on both X and Y are free. It is a far-reaching generalization of the famous Borsuk–Ulam theorem (the latter claims that there is no $\mathbf{Z}_2$-equivariant mapping $S^n \to S^{n-1}$, where both spheres are considered with the antipodal actions $x \mapsto -x$).

We recall that a topological space X is *n-connected* if for each $j \le n$, any continuous mapping $f\colon S^j \to X$ can be extended continuously to $\bar{f}\colon B^{j+1} \to X$, where B^{j+1} is the $(j+1)$-dimensional ball bounded by the S^j.

Theorem 5.1. *Let G be a finite group, $|G| > 1$, let X be an n-connected space with a free action of G, and let Y be a (paracompact) topological space of dimension at most n with a free action of G. Then there is no G-equivariant map $f\colon X \to Y$.*

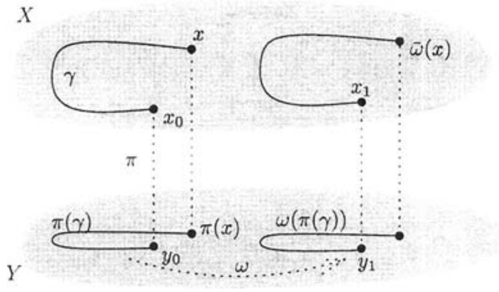

Fig. 6. Lifting a $\mathbf{Z}_q$-action.

In our applications of the theorem, we deal with a space X which is homeomorphic to the real projective space $P\mathbf{R}^3$, and thus it is not even 1-connected (the fundamental group is $\mathbf{Z}_2$). However, there is a double-covering map $\pi\colon S^3 \to P\mathbf{R}^3$, and S^3 is 2-connected. The following lemma shows that for odd q, a $\mathbf{Z}_q$-action can be lifted from our X to S^3, and so X is as good as a 2-connected space in this case.

Lemma 5.2. *Let $\pi\colon X \to Y$ be a p-fold covering map, with X arcwise connected and simply connected, and let ω be a $\mathbf{Z}_q$-action on Y, where $(p, q) = 1$. Then there is a $\mathbf{Z}_q$-action $\tilde{\omega}$ on X such that π is a $\mathbf{Z}_q$-map. If ω is free, then $\tilde{\omega}$ is free too.*

This simple result is most likely known, but we have not found a reference.

Proof. Fix $x_0 \in X$. Let $y_0 = \pi(x_0)$, $y_1 = \omega(y_0)$, and fix some $x_1 \in \pi^{-1}(y_1)$. Define a map $\bar{\omega}\colon X \to X$, as follows. For $x \in X$, choose a path γ in X connecting x_0 to x. The path $\omega(\pi(\gamma))$ in Y has endpoints y_1 and $\omega(\pi(x))$. Lift the endpoint y_1 to x_1; the other endpoint of this lifted path defines $\bar{\omega}(x)$; see Fig. 6.

This is a well-defined mapping. If γ' is another path from x_0 to x, then it is homotopic to γ (rel$\{x_0, x\}$). Applying π and then ω to the system of paths witnessing the homotopy of γ and γ' yields a system of homotopic paths from y_1 to $\omega(\pi(x))$, and the liftings thus all have the same endpoints.

Clearly, $\bar{\omega}$ is continuous. By the construction, we have $\omega \circ \pi = \pi \circ \bar{\omega}$. (From this, we also have $\omega^k \circ \pi = \pi \circ \bar{\omega}^k$, for all $k \geq 1$.)

One can define a continuous inverse to $\bar{\omega}$: to find the inverse image of $z \in X$, consider a path γ from x_1 to z, apply π, then ω^{-1}, then lift to X with y_0 lifted to x_0. Thus, $\bar{\omega}$ is a homeomorphism.

Consider $\nu = \bar{\omega}^q$. This is a homeomorphism lifting the identity. It acts on $F = \pi^{-1}(y_0)$ as a permutation. Moreover, if some ν^k has a fixed point, then it is the identity map, as is easy to check. Hence all cycles of the permutation $\nu|_F$ have the same length, and consequently ν^k is the identity for any k divisible by p. Choosing k as a multiple of p with $k \equiv 1 \pmod{q}$, we get that $\tilde{\omega} = \bar{\omega}^k$ satisfies $\tilde{\omega}^q = 1_X$ and $\pi \circ \tilde{\omega} = \omega \circ \pi$. □

We remark that, for example, a $\mathbf{Z}_2$-action generally cannot be lifted from $P\mathbf{R}^3$ to S^3.

6. Fan Partitions by Equivariant Topology

Here we resume our discussion of (spherical) k-fans. From now on, we assume that the measures $\mu_1, \dots, \mu_m$ on S^2 are as in Lemma 3.1(ii). Actually, we use the following two properties: any nonempty sector has a strictly positive measure, and if the angle of a sector goes to zero, then its measure goes to zero.

The Candidate Space and Test Maps. Let $q \geq 2$ be a given integer and let the probability measures $\mu_1, \dots, \mu_m$ on S^2 be fixed. To each pair $x, y \in S^2$ of orthonormal unit vectors we assign a q-fan in S^2, as follows. The center is x, and ℓ_1 is the intersection of S^2 with the halfplane having 0 and x on its boundary and containing y. The rays $\ell_2, \dots, \ell_q$ are uniquely determined by the condition that the resulting q-fan equipartitions the measure μ_m. In this way, the space X_q of all q-fans equipartitioning μ_m is identified with the space $V_2(\mathbf{R}^3)$ of all pairs (x, y) of orthonormal vectors in $\mathbf{R}^3$, with the natural topology ($V_2(\mathbf{R}^3)$ is called the *Stiefel manifold* of orthonormal 2-frames in $\mathbf{R}^3$). Write $\iota_q\colon V_2(\mathbf{R}^3) \to X_q$ for this identification, that is,

$$\iota_q(x, y) = (x; \ell_1, \dots, \ell_q).$$

It is easy to see that $V_2(\mathbf{R}^3)$ is homeomorphic with $SO(3)$, the group of rotations around the origin in $\mathbf{R}^3$, and as is well known, this is homeomorphic with the projective space $P\mathbf{R}^3$ (see, e.g., p. 164 of [Br]).

For assessing the "quality" of a given q-fan from X_q with respect to the other measures, we introduce *test maps* $f_j\colon X_q \to \mathbf{R}^q$, $j = 1, 2, \dots, m-1$, by letting

$$f_j(F) = \left(\mu_j(\sigma_1) - \tfrac{1}{q}, \mu_j(\sigma_2) - \tfrac{1}{q}, \dots, \mu_j(\sigma_q) - \tfrac{1}{q}\right),$$

where $\sigma_1, \dots, \sigma_q$ are the sectors of the k-fan F. As is easy to check, the conditions on the measures μ_j imply that each f_j is continuous. We also observe that the image of each f_j is actually contained in the hyperplane $Z = \{y \in \mathbf{R}^q\colon y_1 + y_2 + \cdots + y_q = 0\}$.

If we want to prove the existence of a q-fan equipartitioning μ_1 through μ_m, we must show that $f_1, \dots, f_{m-1}$ have a common zero. More generally, for proving the existence of simultaneous α-partitions by k-fans, where $\alpha = (a_1/q, a_2/q, \dots, a_k/q)$ with $a_1, a_2, \dots, a_q$ being natural numbers summing up to q, it is sufficient to show that there is a q-fan $F \in X_q$ such that $f_j(F) \in L$ for all $j = 1, 2, \dots, m-1$, where L is the linear subspace

$$L = L(\alpha) = \{x \in \mathbf{R}^q\colon x_1 + x_2 + \cdots + x_{a_1} = 0, x_{a_1+1} + \cdots + x_{a_1+a_2} = 0, \dots, \\ x_{a_1+\cdots+a_{k-1}+1} + \cdots + x_q = 0\}. \tag{1}$$

The results in Theorem 1.1 are obtained by considering suitable group actions on the candidate space X_q such that the test maps are equivariant, and then using the results in Section 5 for showing that no equivariant map missing L can exist.

The Group Action. On our space X_q, we have a natural free action ω of the group $\mathbf{Z}_q$ (integers with addition modulo q). It is given by "turning by one sector"; formally, $\omega(x; \ell_1, \dots, \ell_q) = (x; \ell_2, \ell_3, \dots, \ell_q, \ell_1)$. Let $F \in X_q$ be a q-fan, and let $f_j(F) =$

$(y_1, \dots, y_q) \in Z$. We have $f_j(\omega(F)) = (y_2, y_3, \dots, y_q, y_1)$. Thus, if we define the $\mathbf{Z}_q$-action ν on the hyperplane Z by $\nu(y_1, y_2, \dots, y_q) = (y_2, y_3, \dots, y_q, y_1)$, then f_j is equivariant.

We mention that there is a natural $\mathbf{Z}_q$ action ρ_q on $V_2(\mathbf{R}^3)$: $\rho_q(x, y) = (x, y')$ where y' is obtained from y by turning clockwise by angle $2\pi/q$ around x when viewed from the center of the sphere. With this map the identification ι_q becomes a $\mathbf{Z}_q$ map:

$$\iota_q \circ \rho_q = \omega \circ \iota_q.$$

The advantage is that the $\mathbf{Z}_q$ action does not depend on the measure μ_m any more.

We should remark that the $\mathbf{Z}_q$-action ν is free on $Z \setminus \{0\}$ if and only if q is a prime, and so we can expect that Dold's Theorem 5.1 will only be applicable for prime q's.

Next, we discuss the few specific cases where we can prove the existence of k-fan partitions by the equivariant topology methods.

Equipartition of Two Measures by 3-Fans. The possibility of equipartitioning two measures on $\mathbf{R}^2$ by a 3-fan was proved in [BKS], [IUY], [Sa]; they even get a convex 3-fan. Here we give a topological proof, of the slightly more general case of S^2, as it is the simplest case in our discussion. We consider the space X_3 of 3-fans that equipartition μ_2 as introduced above. It suffices to show that there is no $\mathbf{Z}_3$-map f_1: $X_3 \to Z \setminus \{0\}$, where Z is the plane $\{y_1 + y_2 + y_3 = 0\}$. Supposing that f_1 exists and using Lemma 5.2, we can lift the $\mathbf{Z}_3$-action from X_3 to S^3 (obtaining a free action), and define a $\mathbf{Z}_3$-map $\tilde{f}_1$: $S^3 \to Z \setminus \{0\}$. Finally, it is convenient (although not strictly necessary here) to reduce the dimension of the target space. Namely, if $S(Z) = \{y \in Z\colon \|y\| = 1\}$ denotes the unit sphere in Z (an S^1 in this case), we define h: $Z \setminus \{0\} \to S(Z)$ by $h(y) = y/\|y\|$. Then $h \circ \tilde{f}_1$: $S^3 \to S(Z)$ is a $\mathbf{Z}_3$-map. Since the $\mathbf{Z}_3$-action on $S(Z)$ is free, Theorem 5.1 applies and excludes the existence of such a map (as S^3 is 2-connected and $S(Z)$ is one-dimensional; this is even more than we need).

There is another, almost elementary, argument showing the nonexistence of a $\mathbf{Z}_3$-map $h \circ f_1$: $X_3 \to S^1$, which we now describe. Assume there is such a map and combine it with the identification ι_3 to obtain a $\mathbf{Z}_3$-map g: $V_2(\mathbf{R}^3) \to S^1$. Consider the set $S(a) = \{(a, y) \in V_2(\mathbf{R}^3)\}$ with $a \in S^2$ fixed; this is an S^1, and so we have a $\mathbf{Z}_3$-map g_1: $S^1 \to S(a)$, say the isometric embedding. Then $g \circ g_1$: $S^1 \to S^1$ is also a $\mathbf{Z}_3$-map, where the action on both S^1's is the rotation by $2\pi/3$. Then the degree of $g \circ g_1$ is 1 mod 3, as it is well known and easy to check (see [KZ], or [BSS]). Define now S^* as the union of two copies of S^1 glued together at a single point. Define g_2: $S^* \to V_2(\mathbf{R}^3)$ as winding twice around S^1 while mapping to $S(a) \subset V_2(\mathbf{R}^3)$ the same way as g_1 does. The degree of $g \circ g_2$ is clearly twice that of $g \circ g_1$, and so it is 2 mod 3. However, as the fundamental group of $V_2(\mathbf{R}^3)$ is $\mathbf{Z}_2$, the cycle that goes around $S(a)$ twice is homotopic to zero. Thus the degree of $g \circ g_2$ is zero. Contradiction.

Using this proof one can place extra restrictions on the 3-fan realizing the equipartition: any subset of $V_2(R^3)$ in which twice $S(a)$ can be deformed to a point ought to contain such a 3-fan. For example, it follows that for two measures in the plane, there exists a 3-fan equipartition with one of the halflines being vertical.

Other Partitions by Dold's Theorem. Another case where Dold's theorem can be applied are α-partitions of two measures by 4-fans with $\alpha = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ (consequently, we also get all α-partitions by 3-fans, where all the components of α are fractions with denominator 5).

In this case, we choose $q = 5$, and we let $L = L((\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})) = \{y \in \mathbf{R}^5\colon y_1 + y_2 = 0, y_3 = y_4 = y_5 = 0\}$ and $\mathcal{L} = \{L, \nu(L), \nu^2(L), \ldots, \nu^{q-1}(L)\}$, where ν is the coordinate shift action on $\mathbf{R}^q$. If the desired 4-fan α-partition did not exist, we would get a $\mathbf{Z}_5$-map $f_1\colon X_5 \to Z \setminus \bigcup \mathcal{L}$. By lifting the $\mathbf{Z}_5$-action from X_5 to S^3, we obtain a $\mathbf{Z}_5$-map $\tilde{f}_1\colon S^3 \to Z \setminus \bigcup \mathcal{L}$, with free $\mathbf{Z}_5$-actions on both sides.

This time, even if we compose $\tilde{f}_1$ with the mapping $h\colon Z \setminus \{0\} \to S(Z)$ as in the previous case, the target space is still three-dimensional; it is a S^3 with 10 points (i.e. five copies of S^0) deleted. The dimension is too large to apply Dold's theorem directly, but it can be easily reduced by 1, by equivariantly contracting $Z \setminus \bigcup \mathcal{L}$ to a two-dimensional subspace $Y \subset S(Z)$. In the following lemma, we give a simple geometric construction of a suitable Y.

Lemma 6.1. *Let $\mathcal{L}$ be a finite collection of linear subspaces of $\mathbf{R}^n$, each of dimension between 1 and $n - 1$, and let $\mathcal{L}$ be closed under a $\mathbf{Z}_q$-action ν on $\mathbf{R}^n$, whose homeomorphisms are isometries of $\mathbf{R}^n$ fixing the origin. Suppose that the linear span of $\bigcup \mathcal{L}$ is the whole $\mathbf{R}^n$. Then there is a subset $Y \subset S^{n-1}$ of dimension at most $n - 2$, closed under the action ν, and a $\mathbf{Z}_q$-map $g\colon \mathbf{R}^n \setminus \bigcup \mathcal{L} \to Y$.*

We postpone the proof to the end of this section. In the problem of $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$-partitioning of two measures by 4-fans, we thus obtain a $\mathbf{Z}_5$-mapping $g \circ \tilde{f}_1\colon S^3 \to Y$, where Y is a two-dimensional space. This is ruled out by Dold's Theorem 5.1.

A similar situation arises for $(\frac{2}{3}, \frac{1}{3})$-partitioning of three measures by a 2-fan. Here we have two $\mathbf{Z}_3$-mappings $f_1, f_2\colon X_3 \to Z$, which can be regarded as a single $\mathbf{Z}_3$-map to the product $Z \times Z = \{(y_1, y_2, \ldots, y_6) \in \mathbf{R}^6\colon y_1 + y_2 + y_3 = 0, y_4 + y_5 + y_6 = 0\}$. This time, the excluded subspace is $L(\frac{2}{3}, \frac{1}{3}) \times L(\frac{2}{3}, \frac{1}{3}) = \{y \in \mathbf{R}^6\colon y_1 + y_2 = y_3 = y_4 + y_5 = y_6 = 0\}$, plus its two images under the $\mathbf{Z}_3$-action $\nu \times \nu$. The sphere $S(Z \times Z)$ is again an S^3, and three copies of S^1 are deleted this time. By Lemma 6.1, the target space can be $\mathbf{Z}_3$-mapped to a two-dimensional subspace, and an application of Lemma 5.2 and of Theorem 5.1 finishes the proof.

Equipartitions of Three Measures by 2-Fans. Here we have the mappings $f_1, f_2\colon X_2 \to Z$, where the target space is one-dimensional. By considering this as a single mapping into the product $Z \times Z$ and by composing with the mapping $h\colon Z \times Z \setminus \{0\} \to S(Z \times Z)$, we obtain a mapping : $X_2 \to S^1$. Combining it with the identification ι_2 we get an $f\colon V_2(\mathbf{R}^3) \to S^1$ map which is a $\mathbf{Z}_2$-map. In fact, it is straightforward to check that f is antipodal in the second variable, that is $f(x, -y) = -f(x, y)$.

We cannot use Lemma 5.2 since p and q are not coprime (both are equal to 2). However, the second, almost elementary, proof of the 3-fan equipartition works. Namely, f, when restricted to $S(a)$, is an $S^1 \to S^1$ antipodal map so it has odd degree. Nevertheless, when extending it to S^* by winding twice around S^1 we get that this map is homotopic to zero, and its degree (which is twice that of f) is zero. A contradiction again.

We remark that another proof can be obtained using the ideal-valued cohomological index of Fadell and Husseini (discussed in [Ži3]).

Proof of Lemma 6.1. Define $Y_0 \subset \mathbf{R}^n \setminus \bigcup \mathcal{L}$ as the set of all points y that have at least two nearest neighbors in $\bigcup \mathcal{L}$ (thus, Y_0 consists of the boundaries of the cells of the Voronoi diagram of $\mathcal{L}$). This Y_0 is a union of finitely many pieces of quadratic surfaces (the surfaces equidistant to some $L_i, L_j \in \mathcal{L}$) and so $\dim Y_0 \leq n - 1$. We define a mapping g_0: $\mathbf{R}^n \setminus \bigcup \mathcal{L} \to Y_0$; by composing g_0 with the mapping h: $\mathbf{R}^n \setminus \{0\} \to S^{n-1}$, we then obtain the desired g (with $Y = S^{n-1} \cap Y_0$).

For $y \in Y_0$, we let $g_0(y) = y$. For $y \notin Y_0$, let $\bar{y}$ be the (unique) point of $\bigcup \mathcal{L}$ nearest to y, and define the semiline s as the part of the line $y\bar{y}$ starting at y and not containing $\bar{y}$.

We claim that s intersects Y_0. If $L_i \in \mathcal{L}$ is the linear subspace containing $\bar{y}$, then s is perpendicular to L_i. If we travel distance t from y along the semiline s, the distance to L_i increases by t. On the other hand, since $\mathcal{L}$ spans $\mathbf{R}^n$, not all of its spaces can be perpendicular to s, and thus there is a $L_j \in \mathcal{L}$ such that by traveling distance t from y along s, the distance to L_j increases by no more than βt for a fixed $\beta < 1$. Therefore, by going far enough along s, we reach a point that is closer to L_j than to L_i, and we must have passed a point of Y_0 on the way. This proves the claim, and we define the first intersection of s with Y_0 as $g_0(y)$.

Since g_0 is defined using metric properties and the action ν is an isometry, g_0 commutes with ν. It remains to check the continuity of g_0. First, let $y \in Y_0$; we have $g_0(y) = y$ and so want to show that points close to y are mapped close to y. Let $\mathcal{L}' \subseteq \mathcal{L}$ be the collection of the linear subspaces containing points nearest to y. For $L_i \in \mathcal{L}'$, let $\bar{y}_i \in L_i$ be the point of L_i nearest to y. First, we note that $y\bar{y}_i$ cannot be perpendicular to all $L_j \in \mathcal{L}'$. Indeed, if it were the case, $\mathcal{L}'$ would be contained in the hyperplane h perpendicular to $y\bar{y}_i$ and containing $\bar{y}_i$, and $\bar{y}_i$ is the unique point of h nearest to y, which contradicts y having at least two nearest neighbors in $\bigcup \mathcal{L}$. Now if we choose a point y' in a sufficiently small δ-neighborhood of y, the nearest point $\bar{y}' \in \bigcup \mathcal{L}$ used in the definition of $g_0(y')$ lies in some $L_i \in \mathcal{L}'$. Moreover, there is an $L_j \in \mathcal{L}'$ such that the line $y'\bar{y}'$ is not perpendicular to it, and so by traversing distance t along the line $y'\bar{y}'$, the distance to L_j changes by at most βt for some $\beta < 1$ (independent of δ). Since the distances of y' to L_i and to L_j differ by no more than 2δ, we see that $g_0(y')$ lies at a distance at most $2\delta/(1 - \beta)$ from y'. This proves the continuity of g_0 at y.

Finally, we consider $y \notin Y_0$, and let y' lie very close to y. We let $\bar{y}'$ be the point of $\bigcup \mathcal{L}$ nearest to y', and we let y'' be the point of the line $y'\bar{y}'$ closest to $g_0(y)$. By choosing y' sufficiently near to y, we can guarantee that y'' and $g_0(y)$ are arbitrarily close. Then we can apply the continuity of g_0 at $g_0(y)$. □

7. Discussion and Open Problems

The most natural open question is probably whether, for two measures, a 4-fan equipartition exists. The reasons why our topological approach cannot provide a positive answer will be briefly discussed below.

Our results show that α-partitions of two measures by 3-fans and by 4-fans are possible for *some* values of α. A very intriguing question is whether they exist for *all* values of α. It seems hard to imagine why some α's should not work when others do. On the other hand, our proof method cannot provide many more values of α; the reasons are indicated below. Maybe equivariant topology is not the right tool here.

Equipartitions by 4-Fans. The space X_4 of 4-fans is naturally equipped by the $\mathbf{Z}_4$-action ω, but the corresponding coordinate shift action ν of the target sphere $S(Z)$ is not free. There are some theorems excluding the existence of an equivariant map even for nonfree action, discovered by Özaydin, by Sarkaria (unpublished manuscripts), and by Volovikov [Vo] (see [Ži3] for discussion). These theorems involve actions of noncyclic groups like $\mathbf{Z}_p \times \mathbf{Z}_p$. In our case, there are natural $(\mathbf{Z}_2 \times \mathbf{Z}_2)$-actions on the spaces involved.

On X_q, we have the free $\mathbf{Z}_2$-action ω' corresponding to "changing the orientation" of a q-fan. Namely, $\omega'(x; \ell_1, \dots, \ell_q) = (-x; \ell_1, \ell_q, \ell_{q-1}, \dots, \ell_2)$. We have $f_j(\omega'(F)) = \nu'(y_1, \dots, y_q) = (y_q, y_{q-1}, \dots, y_2, y_1)$, and so the mappings f_j are also equivariant with respect to ω' and the action ν' given by reversing the order of coordinates.

For $q = 4$, we can take the $\mathbf{Z}_2$-action ω^2 of X_4 (turning by two sectors) and combine it with the ω' defined above. As is easy to check, we obtain a free action of the group $G = \mathbf{Z}_2 \times \mathbf{Z}_2$ (direct product) on X_4, and f_1 becomes a G-map. Unfortunately, we found that there is a G-map $X_4 \to S(Z)$. While this provides no counterexample to the 4-fan equipartitioning problem, it shows that this particular proof method fails.

Here comes an explicit description of such a G-map. Recall that X_4 is just $V_2(R^3) = \{(x, y) \in S^2 \times S^2\colon \langle x, y\rangle = 0\}$. The actions of the generators of G are

$$\omega^2(x, y) = (x, -y), \qquad \omega'(x, y) = (-x, y).$$

In this case $S(Z)$ is a S^2 and the actions corresponding to ω^2, resp. ω', on the target space are g and h with

$$g(u, v, w) = (-u, -v, w), \qquad h(u, v, w) = (-u, v, -w).$$

With $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ we define $f\colon V_2(\mathbf{R}^3) \to S^2$ as

$$f(x, y) = (|x_1|a - x_2 b, |x_1|b + x_2 a, x_3),$$

where $a = (x_1 y_2 - x_2 y_1)/\sqrt{x_1^2 + x_2^2}$ and $b = \operatorname{sgn}(y_3)\sqrt{1 - a^2}$. The expression for $f(x, y)$ is undefined if $x_3 = \pm 1$ but one can check that for all y, $f(x, y) \to (0, 0, \pm 1)$ as $x_3 \to \pm 1$, respectively, and so f extends continuously to these points. An easy calculation verifies that $\|f(x, y)\| = 1$ for all unit x, y. Finally, f is continuous at $y_3 = 0$ too since, as further simple calculation reveals, $(x_1^2 + x_2^2)(1 - a^2) = 0$ whenever $y_3 = 0$. Clearly, $f(x, y)_1$ is odd in both x and y and $f(x, y)_2$ is even in x and odd in y.

Equivariant Maps for Larger q. In the above proofs, the existence of equivariant maps could be excluded using Dold's theorem or other tools. Unfortunately, it turns out that for sufficiently large q, equivariant maps between the considered $\mathbf{Z}_q$-spaces do exist, although it is not clear if such maps can actually arise in the k-fan partition problem. We describe two concrete examples of such equivariant maps. The first one concerns the α-partitioning of three measures by 2-fans, where the presentation of the mapping is particularly simple.

Let $\alpha = (1/q, 1 - 1/q)$, $q \geq 3$ odd. Recall that for α-partitioning of three measures by a 2-fan, the test mappings provide a $\mathbf{Z}_q$-map $S^3 \to Z \times Z$, where Z is the hyperplane $\{x_1 + x_2 + \cdots + x_q = 0\}$ in $\mathbf{R}^q$ and where the $\mathbf{Z}_q$-action ν on $\mathbf{R}^q$ is the coordinate shift. The nonexistence of the α-partition implies that this mapping avoids the subspace $L(1/q, 1-1/q) \times L(1/q, 1-1/q) = \{(x, y)\colon x \in Z, y \in Z, x_1 = 0, y_1 = 0\}$. The following example shows that for $q \geq 6$, an equivariant map exists in this situation.

Proposition 7.1. *For any integer $q \geq 6$, there exists a $\mathbf{Z}_q$-map $f\colon S^3 \to \mathbf{R}^q \times \mathbf{R}^q$ (where S^3 has a free $\mathbf{Z}_q$-action and both the $\mathbf{R}^q$'s in the target space are equipped with the coordinate shift action), such that for all $x \in S^3$, $\sum_{i=1}^{q} f(x)_{1i} = \sum_{i=1}^{q} f(x)_{2i} = 0$ and $f(x)_{11}$ and $f(x)_{21}$ are never simultaneously zero (the points of the image are indexed as $2 \times q$ matrices).*

Proof. For simplicity of notation, let $q = 6$ (the construction for larger q is entirely analogous; we stress that the primality or nonprimality of q plays no role here). It is sufficient to construct a continuous map $g\colon S^3 \to \mathbf{R}^2 \setminus \{0\}$ satisfying

$$\sum_{i=0}^{5} g(\omega^i x) = 0 \tag{2}$$

for all $x \in S^3$, where ω is a free $\mathbf{Z}_q$-action on S^3. Indeed, with such a g at our disposal, the mapping f given by $f(x)_{ji} = g(\omega^i x)_j$ is as required.

Let A be the perimeter of a regular hexagon $a_0 a_1 \cdots a_5$; that is, A is the one-dimensional simplicial complex with the six vertices $a_0, a_1, \ldots, a_5$ and with 1-simplices $a_i a_{i+1}$, $i = 0, 1, \ldots, 5$ (indices taken modulo 6). Similarly, B is the perimeter of a hexagon $b_0 b_1 \cdots b_5$. The sphere S^3 is homeomorphic to the join $A * B$. The three-dimensional simplices of this join have the form $a_i a_{i+1} b_j b_{j+1}$. A $\mathbf{Z}_q$-action ω is defined on $A * B$ by letting $a_i \mapsto a_{i+1}$, $b_i \mapsto b_{i+1}$, and extending linearly on each simplex.

First we describe a simpler mapping $g_0\colon A*B \to \mathbf{R}^2$. We choose three nonzero vectors $u, v, w \in \mathbf{R}^2$ with $4u + v + w = 0$. Put $g_0(a_0) = g_0(b_0) = v$, $g_0(a_2) = g(b_2) = w$, and $g_0(a_i) = g_0(b_i) = u$ for $i = 1, 3, 4, 5$. This defines g_0 on the vertices of $A * B$, and we extend it linearly on each simplex of $A * B$. Clearly, (2) is satisfied for g_0, but there are simplices in $A * B$ whose image contains 0. We note that all such simplices contain a_0 or b_0, and also a_2 or b_2.

Next, we form a simplicial complex K by subdividing some simplices in $A * B$. We let c_{02} be the midpoint of the edge $a_0 b_2$, and we let $O_1 = \{c_{02}, c_{13}, \ldots, c_{51}\}$ be the orbit of c_{02} under ω. Similarly, c_{20} is the midpoint of the edge $a_2 b_0$ and $O_2 = \{c_{20}, c_{31}, \ldots, c_{15}\}$ is its orbit; see the schematic Fig. 7. Each edge of $A * B$ containing a point of $O_1 \cup O_2$ is subdivided, and the higher-dimensional simplices containing such

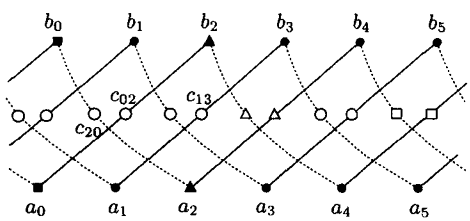

Fig. 7. The subdivided edges in $A * B$.

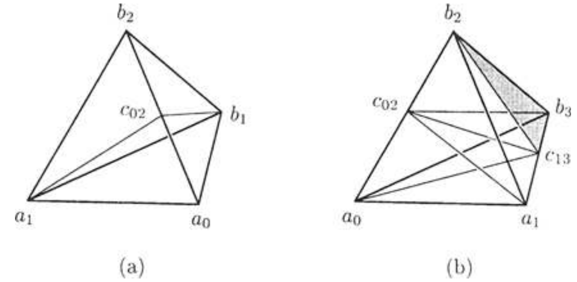

Fig. 8. Subdividing the three-dimensional simplices.

edges are appropriately subdivided too. In particular, three-dimensional simplices may have one subdivided edge (such as the simplex $a_0a_1b_1b_2$) or two subdivided edges (such as the simplex $a_0a_1b_2b_3$). The simplices with a subdivided edge are subdivided into two simplices as in Fig. 8(a) and those with two subdivided edges into four simplices as in Fig. 8(b) (note that the subdivided edges are never adjacent). This defines the simplicial complex K. Now we define a new mapping g on the vertex set of K; the values at the a_i and b_j are as those for g_0, and we put $g(c_{24}) = g(c_{42}) = w$, $g(c_{40}) = g(c_{04}) = v$, and $g(c_{ij}) = u$ for all other $c_{ij} \in O_1 \cup O_2$. In Fig. 7 the vertices with values v are indicated by squares, those with values w by triangles, and those with values u by circles. Finally, g is extended linearly on the whole polyhedron of K. It is easy to check that for each simplex of K, g attains at most two distinct values at the vertices. Thus, no point of the polyhedron of K is mapped to 0 by g. □

Remark. Another, somewhat more complicated example of this type was constructed independently by Attila Pór.

The second example is only sketched, since it is rather similar to the previous one. It is relevant for α-partitioning by 3-fans for $\alpha = (1/q, 1/q, 1-2/q)$. The equivariant topology argument fails if there is a $\mathbf{Z}_q$-map $f\colon S^3 \to Y$, where $Y = \{x \in \mathbf{R}^q\colon x_1 + \cdots + x_q = 0 \text{ and } (x_1 \neq 0 \text{ or } x_2 \neq 0)\}$. It suffices to construct the mapping g specifying the first coordinate ($g(x) = f(x)_1$). The conditions are $\sum_{i=0}^{q-1} g(\omega^i x) = 0$ and for all $x \in S^3$, $g(x) \neq 0$ or $g(\omega x) \neq 0$.

As in the other example, let A and B be the circumferences of the regular q-gon. This time, we subdivide each edge a_ia_{i+1} by a new point $a_{i+1/2}$, obtaining a complex A', and similarly for B'. The $\mathbf{Z}_q$-action ω sends a_i to a_{i-1} and b_i to b_{i-1} (indices taken modulo q).

Let $u, v \in \mathbf{R}$ be nonzero with $(q-1)u + v = 0$. Put $g_0(a_0) = g_0(b_0) = g_0(a_{5/2}) = g_0(b_{5/2}) = v$ and $g_0(a_i) = g_0(b_i) = u$ for the other i.

If we extend g_0 linearly on all simplices of $A' * B'$, what are the bad simplices (possibly containing x with $g_0(x) = g_0(\omega x) = 0$)? They must contain a vertex mapped to v *and* a vertex whose ω-image is mapped to v. Thus, we subdivide the edges a_0b_1, a_1b_0, $a_{5/2}b_{7/2}$, $a_{7/2}b_{5/2}$, $a_0b_{7/2}$, $a_{7/2}b_0$, $a_1b_{5/2}$, and $a_{5/2}b_1$, as well as the edges in their orbits. In each of these eight orbits, we choose one edge whose subdividing vertex is mapped to v, while all others go to u. What we want to achieve is that they do not interfere with each other or with the a_i and b_i vertices mapped to v, where x interferes with y if x and ωy occur in the same simplex of the (subdivided) complex. It is easy to see that if q is sufficiently large, interference can be avoided. We have not tried to find the smallest possible value of q for which such a construction works; however, note that $q \geq 6$ follows from the last statement of Theorem 1.1.

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