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SIMULTANEOUS SIMILARITY, BOUNDED GENERATION AND AMENABILITY

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Abstract. We prove that a discrete group G is amenable if and only if it is strongly unitarizable in the following sense: every unitarizable representation π on G can be unitarized by an invertible chosen in the von Neumann algebra generated by the range of π. Analogously, a C^*-algebra A is nuclear if and only if any bounded homomorphism $u : A \to B(H)$ is strongly similar to a $*$-homomorphism in the sense that there is an invertible operator ξ in the von Neumann algebra generated by the range of u such that $a \to \xi u(a)\xi^{-1}$ is a $*$-homomorphism. An analogous characterization holds in terms of derivations. We apply this to answer several questions left open in our previous work concerning the length $L(A \otimes_{\max} B)$ of the maximal tensor product $A \otimes_{\max} B$ of two unital C^*-algebras, when we consider its generation by the subalgebras $A \otimes 1$ and $1 \otimes B$. We show that if $L(A \otimes_{\max} B) < \infty$ either for $B = B(\ell_2)$ or when B is the C^*-algebra (either full or reduced) of a non-Abelian free group, then A must be nuclear. We also show that $L(A \otimes_{\max} B) \le d$ if and only if the canonical quotient map from the unital free product $A * B$ onto $A \otimes_{\max} B$ remains a complete quotient map when restricted to the closed span of the words of length at most d.

1. Introduction. In 1950, Dixmier and Day (see [24]) proved that any amenable group G is unitarizable, i.e. any uniformly bounded representation $\pi : G \to B(H)$ is similar to a unitary representation. More precisely, there is an invertible operator $\xi : H \to H$ such that $\xi\pi(\cdot)\xi^{-1}$ is a unitary representation of G. The proof uses a simple averaging argument, from which it can be seen that ξ can be chosen with the additional property that ξ commutes with any unitary U commuting with the range of π. Equivalently (see Remark 5 below), ξ can be chosen in the von Neumann algebra generated by $\pi(G)$. (See [15] for more on this.) For convenience, let us say that π (resp. G) is strongly unitarizable if it has this additional property (resp. if every uniformly bounded π on G is strongly unitarizable).

It is still an open problem whether 'unitarizable' implies 'amenable' (see [24]). However, we will show that G is amenable if and only if it is strongly unitarizable. Moreover, we will show an analogous result for C^*-algebras, as follows.

THEOREM 1. *The following properties of a C^*-algebra A are equivalent.*

(i) *A is nuclear.*

(ii) *For any completely bounded (c.b.) homomorphism $u : A \to B(H)$ there is an invertible operator ξ on H belonging to the von Neumann algebra generated by $u(A)$ such that $a \to \xi u(a)\xi^{-1}$ is a $*$-homomorphism.*

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(iii) *For any C^*-algebra B, the pair (A, B) has the following simultaneous similarity property*: *for any pair $u\colon A \to B(H)$, $v\colon B \to B(H)$ of c.b. homomorphisms with commuting ranges there is an invertible ξ on H such that both $\xi u(\cdot)\xi^{-1}$ and $\xi v(\cdot)\xi^{-1}$ are $*$-homomorphisms.*

(iv) *Same as* (iii) *but with v assumed to be itself a $*$-homomorphism.*

REMARK 2. It is possible that (iii) or (iv) for a fixed given B implies that $A \otimes_{\min} B = A \otimes_{\max} B$, but this is not clear (at the time of writing).

COROLLARY 3. *If a discrete group G is strongly unitarizable then G is amenable.*

PROOF. Let $A = C^*(G)$. Any bounded homomorphism $u\colon A \to B(H)$ restricts to a uniformly bounded representation π on G. Note that $\pi(G)$ and $u(A)$ generate the same von Neumann algebra M. Thus, if G is strongly amenable, A satisfies (ii) in Theorem 1; hence, it is nuclear and, as is well known, this implies that G is amenable in the discrete case (see [13]). □

Actually, we obtain a stronger statement.

COROLLARY 4. *If every* unitarizable *representation π on a discrete group G is strongly unitarizable then G is amenable.*

PROOF. Indeed, in Theorem 1, u is assumed c.b. on $A = C^*(G)$, so the corresponding π is unitarizable. □

REMARK 5. Assume that G is amenable with invariant mean ϕ. Consider a uniformly bounded representation π on G; then, the proof of the Day-Dixmier theorem is as follows: essentially we can define ξ by the (non-rigorous) formula

$$\xi = \left(\int \pi(g)^* \pi(g) \phi(dg) \right)^{1/2}.$$

However, it is obvious how to make this rigorous: for any h in H we define $x_h(g) = \langle \pi(g)h, \pi(g)h \rangle$ (note that $x_h \in L_\infty(G)$) and then define ξ by setting $\langle \xi^2 h, h \rangle = \phi(x_h)$. Clearly (by the invariance of ϕ) ξ unitarizes π, and the above formula makes it clear that ξ is in the von Neumann algebra generated by the range of π.

REMARK 6. Note that, by [8], $C^*(G)$ is nuclear for any separable, connected locally compact group G; hence, every continuous unitarizable representation on G is strongly unitarizable; therefore we definitely must restrict the preceding Corollary 4 to the discrete case.

REMARK 7. The following elementary fact will be used repeatedly: let U be a unitary operator on H and let $\xi \geq 0$ be an invertible on H such that $\xi U \xi^{-1}$ is still unitary. Then $\xi U \xi^{-1} = U$. Indeed, we have $(\xi U \xi^{-1})^*(\xi U \xi^{-1}) = I$; hence, $U^* \xi^2 U = \xi^2$. Equivalently, ξ^2 commutes with U and hence $\xi = \sqrt{\xi^2}$ also commutes with U.

The above results are proved in the first part of the paper. The second part is devoted to the length of a pair of (unital) C^*-algebras A, B, introduced in [21] and denoted below by $L(A \otimes_{\max} B)$. Let $W_{\leq d}$ be the closed span of the words of length at most d in the unital free product $A * B$. We will prove that $L(A \otimes_{\max} B) \leq d$ if and only if the restriction to $W_{\leq d}$ of the canonical quotient map from $A * B$ onto $A \otimes_{\max} B$ is a complete quotient map (i.e. it yields a complete isomorphism after passing to the quotient by the kernel). This gives a more satisfactory reformulation of the definition in [21]. To establish this, we need to prove that $W_{\leq d}$ decomposes naturally (completely isomorphically) into a direct sum of Haagerup tensor products of copies of A and B of order $0 \leq j \leq d$ (see Lemma 20). The latter result seems to be of independent interest.

2. Notation and background. While the first part uses mostly basic C^*-algebra theory and c.b. maps (for which we refer the reader to [26, 17]), the second part requires more background from operator space theory, e.g. the Haagerup tensor product, and its connection with free products of operator algebras, for which we refer the reader to [18] (see also [2, 9]).

Recall that a linear map $v\colon Y \to X$ between operator spaces is called c.b. if the maps $v_n = \mathrm{Id} \otimes v\colon M_n(Y) \to M_n(X)$ are uniformly bounded, and we set

$$\|v\|_{\mathrm{cb}} = \sup_n \|v_n\| .$$

Equivalently, if we denote $K(Y) = K \otimes_{\min} Y$, we have $\|v\|_{\mathrm{cb}} = \|\mathrm{Id} \otimes v\colon K(Y) \to K(X)\|$.

A c.b. map v is called a 'complete surjection' (or a 'complete quotient map') if there is a constant c such that, for any $n \geq 1$, and any x in $M_n(X)$ with $\|x\| < 1$, there is a y in $M_n(Y)$ such that $[x_{ij}] = [q(y_{ij})]$ with $\|y\| < c$.

When this holds with $c = 1$ we say that v is a complete metric surjection. Note that, when $n = 1$, any surjection satisfies this for some $c > 0$. When this holds for $c = 1$ (and for $n = 1$), we say that v is a metric surjection. Equivalently, v is a complete (resp. metric) surjection if and only if $\mathrm{Id} \otimes v\colon K(Y) \to K(X)$ is a (resp. metric) surjection.

Let A, B be unital C^*-algebras and let $u\colon A \to B(H)$ and $v\colon B \to B(H)$ be linear maps. We denote by $u \cdot v\colon A \otimes B \to B(H)$ the linear map defined on the *algebraic* tensor product $A \otimes B$ by $u \cdot v(a \otimes b) = u(a)v(b)$.

We will say (for short) that $u \cdot v$ is c.b. on $A \otimes_{\min} B$ (resp. $A \otimes_{\max} B$) if $u \cdot v$ extends to a c.b. map on $A \otimes_{\min} B$ (resp. $A \otimes_{\max} B$). We will use a similarly shortened terminology for ordinary boundedness instead of the complete one.

Now assume that u, v are unital homomorphisms with commuting ranges. Then $u \cdot v$ is a homomorphism on the incomplete algebra $A \otimes B$. By [10], $u \cdot v$ is c.b. on $A \otimes_{\max} B$ if and only if there is an invertible ξ in $B(H)$ such that $\xi u(\cdot)\xi^{-1}$ and $\xi v(\cdot)\xi^{-1}$ are both $*$-homomorphisms. More precisely, we have

$$\|u \cdot v\colon A \otimes_{\max} B \to B(H)\|_{\mathrm{cb}} = \inf\{\|\xi\|\|\xi^{-1}\|\}, \tag{1}$$

where the infimum runs over all ξ satisfying this.

Now assume that v is a unital $*$-homomorphism. In that case, we have

$$\|u \cdot v\colon A \otimes_{\max} B \to B(H)\|_{\mathrm{cb}} = \inf\{\|\xi\|\|\xi^{-1}\|\}, \tag{2}$$

where the infimum runs over all ξ in $v(B)'$ such that $a \to \xi u(a)\xi^{-1}$ is a $*$-homomorphism. Indeed, this is an immediate consequence of Remark 7 (since $b \to \xi v(b)\xi^{-1}$ is a $*$-homomorphism if and only if it maps unitaries to unitaries).

Let $r: A \to B(H)$ and $\sigma: B \to B(\mathcal{H})$ be unital $*$-homomorphisms, and let $\pi: A \otimes_{\min} B \to B(H \otimes \mathcal{H})$ be their tensor product, i.e. $\pi(a \otimes b) = r(a) \otimes \sigma(b)$.

By an r-derivation $d: A \to B(H)$ we will mean a derivation with respect to r (i.e. $d(a_1a_2) = r(a_1)d(a_2) + d(a_1)r(a_2)$). Let $r_1: A \to B(H \otimes \mathcal{H})$ and $\sigma_1: B \to B(H \otimes \mathcal{H})$ be the representations defined by $r_1(a) = a \otimes I$ and $\sigma_1(b) = I \otimes \sigma(b)$. Let $\delta: A \to B(H \otimes \mathcal{H})$ be an r_1-derivation such that $\delta(A) \subset (I \otimes \sigma(B))'$. It is easy to check that $\delta \cdot \sigma_1$ is a π-derivation on the (incomplete) algebra $A \otimes B$. For any T in $B(H \otimes \mathcal{H})$, we denote

$$\delta_T(a) = (r(a) \otimes 1)T - T(r(a) \otimes 1)\,.$$

By a result due to Christensen [6], we then have

$$\|\delta \cdot \sigma_1: A \otimes_{\min} B \to B(H \otimes \mathcal{H})\|_{\text{cb}} = 2\inf\{\|T\| \mid T \in \sigma_1(B)',\ \delta = \delta_T\}\,. \tag{3}$$

Actually, in the present special situation this is also equal to the c.b. norm of $\delta \cdot \sigma_1$ on $A \otimes_{\max} B$. Indeed, let $C_{\max}$ be the latter c.b. norm. Then, Christensen's result implies that the above $2\inf\|T\|$ is $\leq C_{\max}$, but since π is *continuous* on $A \otimes_{\min} B$ it follows that the c.b. norm of δ_T on $A \otimes_{\min} B$ is $\leq 2\|T\|$.

3. Proof of the main result. Actually, we will prove a more general result than the above Theorem 1. Indeed, we will show that it suffices for A to be nuclear that (iii) or (iv) holds for a 'large enough' C^*-algebra B. It may be that any non-nuclear B can be used but we cannot prove this. Instead, we introduce the notion of a 'liberal' C^*-algebra, which is close to being the 'opposite' of nuclearity.

To describe this, we need to introduce the following notion.

We denote by E_λ^n the operator space that is the linear span of $\lambda(g_1), \dots, \lambda(g_n)$ in the von Neumann algebra generated by the left regular representation of the free group $\boldsymbol{F}_n$ (recall that $g_1, \dots, g_n$ denote the free generators of $\boldsymbol{F}_n$).

NOTE. We could use $R_n \cap C_n$ instead of E_λ^n (see [18, p. 184]), but it is easier to see the connection with the preceding argument using E_λ^n.

DEFINITION 8. We say that $\{E_\lambda^n\}$ factors uniformly through an operator space B if, for any $n \geq 1$, there are mappings

$$v_n: E_\lambda^n \to B\,, \quad w_n: B \to E_\lambda^n$$

such that $w_n v_n = \text{Id}$ and $\sup_n \|v_n\|_{\text{cb}}\|w_n\|_{\text{cb}} < \infty$.

DEFINITION 9. We say that a C^*-algebra B is 'liberal' if it admits a representation $\sigma: B \to B(H)$ such that $\{E_\lambda^n\}$ factors uniformly through the commutant $\sigma(B)'$.

REMARK. Examples of liberal C^*-algebras are $C^*(\boldsymbol{F}_\infty)$, $C_\lambda^*(\boldsymbol{F}_\infty)$ or the von Neumann algebra generated by $\lambda(\boldsymbol{F}_\infty)$. This follows from [12, Theorem 4.1]. *A fortiori* (since $\boldsymbol{F}_\infty$ embeds in $\boldsymbol{F}_2$), the same is true for $\boldsymbol{F}_n$ for any $n \geq 2$. By [1, p. 205] and [28], $B(\ell_2)$ or the Calkin algebra $B(\ell_2)/K(\ell_2)$ are liberal. Clearly, since nuclear passes to quotients, any liberal C^*-algebra is non-nuclear by [12, Theorem 4.1]. Apparently, there is no known counterexample to the converse.

THEOREM 10. *Assume that $A \subset B(H)$ and let B be a liberal (unital) C^*-algebra. The properties in Theorem 1 are equivalent to the following.*

(v) *For any $*$-homomorphism $\sigma\colon B \to B(\mathcal{H})$ there is a constant C such that for any c.b. derivation $\delta\colon A \to B(H) \otimes \sigma(B)'$ (relative to the embedding $A \simeq A \otimes I$) for which the associated $\delta \cdot \sigma_1$ is also c.b. on $A \otimes_{\min} B$, there is an operator T in the von Neumann algebra generated by $A \otimes I$ and $\delta(A)$ such that*

$$\|T\| \le C\|\delta\|_{\text{cb}} \quad \text{and} \quad \delta(a) = aT - Ta \quad (a \in A)\,.$$

(vi) *There is a non-decreasing function $F\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ satisfying the following: for any c.b. homomorphism $u\colon A \to B(H)$ and any $*$-homomorphism $v\colon B \to B(H)$, with commuting ranges such that $u \cdot v\colon a \otimes b \to u(a)v(b)$ extends to a c.b. homomorphism on $A \otimes_{\min} B$, there is an invertible ξ in $v(B)'$ with $\|\xi\|\|\xi^{-1}\| \le F(\|u\|_{\text{cb}})$ such that $\xi u(\cdot)\xi^{-1}$ is a $*$-homomorphism.*

PROOF OF THEOREMS 1 AND 10. We will assume A and u unital for simplicity. (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (ii) is easy. Indeed, let M denote the von Neumann algebra generated by $u(A)$. Let $v\colon M' \to B(H)$ be the inclusion mapping. If (iv) holds, we can find a ξ such that, for any unitary pair a, b in A, $\xi u(a)\xi^{-1}$ and $\xi v(b)\xi^{-1}$ are both unitary. By polar decomposition of ξ we may assume that $\xi > 0$. Then, by the preceding remark, ξ must commute with $v(M') = M'$, and hence $\xi \in M'' = M$. The implication (i) $\Rightarrow$ (iii) follows from the basic properties of the so-called δ-norm, as presented in [18, Theorem 12.1] (see also [14]). Indeed, let u, v be as in (iii). Clearly, the mapping $v \cdot u\colon b \otimes a \to v(b)u(a)$ satisfies $\|v \cdot u\colon B \otimes_\delta A \to B(H)\|_{\text{cb}} \le \|v\|_{\text{cb}}\|u\|^2_{\text{cb}}$. Now, if A is nuclear, $B \otimes_\delta A = B \otimes_{\min} A = B \otimes_{\max} A$, and hence $v \cdot u$ defines a c.b. homomorphism ρ on $B \otimes_{\max} A$. By [10], there is an invertible ξ such that, $\xi\rho(\cdot)\xi^{-1}$ is a $*$-homomorphism, from which we conclude that (iii) holds.

For Theorem 1, it remains only to prove that (ii) implies (i). We will show (ii) $\Rightarrow$ (vi) $\Rightarrow$ (v) $\Rightarrow$ (i).

Let u be as in (ii). Let M_u be the von Neumann algebra generated by $u(A)$. We first claim that the mapping $\hat{u}\colon x \otimes y \to u(x)y$ extends to a c.b. homomorphism from $A \otimes_{\max} M_u'$ to $B(H)$. Indeed, since $\xi \in M_u$, $\xi\hat{u}(\cdot)\xi^{-1}$ is a $*$-homomorphism on $A \otimes M_u'$. By a routine direct sum argument, (ii) implies that there is a non-decreasing function $F\colon \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that, for all u as in (ii), we have

$$\|\hat{u}\|_{\text{cb}} \le F(\|u\|_{\text{cb}})\,. \tag{4}$$

By (2), this clearly shows that (ii) implies (vi).

We now show that (vi) implies (v). Assume (vi). Let δ, σ be as in (v). Let $u\colon\ A \to M_2(B(H \otimes \mathcal{H}))$ be the homomorphism

$$u\colon\ a \to \begin{pmatrix} a\otimes 1 & \delta(a) \\ 0 & a \otimes 1 \end{pmatrix} \in M_2(B(H\otimes\mathcal{H}))\,,$$

and define $v\colon\ B \to M_2(B(H\otimes\mathcal{H}))$ by

$$u\colon\ b \to \begin{pmatrix} \sigma_1(b) & 0 \\ 0 & \sigma_1(b) \end{pmatrix} \in M_2(B(H\otimes\mathcal{H}))\,.$$

Note that u, v have commuting ranges and $\|u\|_{\mathrm{cb}} \le 1+\|\delta\|_{\mathrm{cb}}$; also $u\cdot v$ is c.b. on $A\otimes_{\min} B$ because we assume in (v) that it is so for $\delta\cdot\sigma_1$ (and the representation $\pi = r_1\cdot\sigma_1$ is continuous, hence c.b. on $A\otimes_{\min} B$). Then, since we assume (vi), we obtain that there is an invertible ξ in $VN(u(A))$ with $\|\xi\|\|\xi^{-1}\| \le F(\|u\|_{\mathrm{cb}})^2$ such that $\xi u(\cdot)\xi^{-1}$ is a $*$-homomorphism. Reviewing an argument of Paulsen (see either [17] or [20, p. 80]), we find that there is an operator T with $\|T\| \le 2(\|\xi\|\|\xi^{-1}\|)^2 \le 2F(\|u\|_{\mathrm{cb}})$ in the von Neumann algebra generated by $A\otimes 1$ and $\delta(A)$ such that $\delta(a) = [a\otimes 1, T]$ for all a in A. To deduce (v), by homogeneity, we may assume $\|\delta\|_{\mathrm{cb}} = 1$ and then $\|u\|_{\mathrm{cb}} \le 2$ and we find (v) with $C = 2F(2)^2$. To complete the proof, it remains to show that (v) implies (i), i.e. that (v) implies that A is nuclear.

Let $\mathcal{H} = \ell_2(\boldsymbol{F}_\infty)$. Let $W \subset B(\mathcal{H})$ be the von Neumann algebra generated by the left regular representation λ on the free group $\boldsymbol{F}_\infty$ with n generators, denoted by $g_1,\dots,g_n,\dots$. We will first show that (v) implies (i) in the particular case $B = W'$. Let $r\colon A \to B(H)$ be a $*$-homomorphism. By the well-known Connes-Choi-Effros results (see [18]), it suffices to show that $r(A)''$ is always injective. For simplicity, we replace A by $r(A)$. Thus, it suffices to show that $N = A''$ is injective or equivalently that N' is injective. By [20, Theorem 2.9], N' is injective if and only if there is a constant β such that for all n, for all $y_i \in N'$ there are elements $a_i, b_i \in N'$ with $y_i = a_i + b_i$ such that

$$\left\|\sum a_i a_i^*\right\|^{1/2} + \left\|\sum b_i^* b_i\right\|^{1/2} \le \beta\|(y_i)\|_{R+C}\,, \tag{5}$$

where

$$\|(y_i)\|_{R+C} = \inf\left\{\left\|\sum \alpha_i\alpha_i^*\right\|^{1/2} + \left\|\sum \beta_i^*\beta_i\right\|^{1/2}\right\}$$

and where the infimum runs over all the possible decompositions $y_i = \alpha_i + \beta_i$ with α_i, β_i in $B(H)$.

Note: when there is a c.b. projection $P\colon B(H) \to N'$, then we can take $a_i = P\alpha_i$, $b_i = P\beta_i$ and $\beta = \|P\|_{\mathrm{cb}}$.

To prove (5), we first consider an n-tuple (y_i) in N' with $\|(y_i)\|_{R+C} < 1$, so that $y_i = \alpha_i + \beta_i$ with

$$\left\|\sum \alpha_i\alpha_i^*\right\| < 1 \quad \text{and} \quad \left\|\sum \beta_i^*\beta_i\right\| < 1\,.$$

We introduce the derivation $\delta\colon A \to B(H) \otimes W$ defined as follows:

$$\delta(a) = \sum\nolimits_1^n [a, \alpha_i] \otimes \lambda(g_i) \quad \text{for all } a \in A .$$

It is well known (see [12] or [19, p. 185]) that there is a decomposition $\lambda(g_i) = s_i + t_i$ with $s_i, t_i \in B(\mathcal{H})$ satisfying

$$\Big\| \sum s_i^* s_i \Big\|^{1/2} \leq 1 \quad \text{and} \quad \Big\| \sum t_i t_i^* \Big\|^{1/2} \leq 1 .$$

Thus, since $y_i \in A'$ (and hence $[a, \alpha_i] = -[a, \beta_i]$), we have

$$\delta(a) = [a \otimes 1, \theta] ,$$

where $\theta = \sum_1^n \alpha_i \otimes s_i - \beta_i \otimes t_i$. Therefore,

$$\|\delta\|_{\mathrm{cb}} \leq \|\theta\| \leq \Big\| \sum \alpha_i \alpha_i^* \Big\|^{1/2} \Big\| \sum s_i^* s_i \Big\|^{1/2} + \Big\| \sum \beta_i^* \beta_i \Big\|^{1/2} \Big\| \sum t_i t_i^* \Big\|^{1/2} \leq 2 .$$

Note that $\delta \cdot \sigma_1$ is a finite sum of maps that are obviously c.b. on $A \otimes_{\min} B$. Since we assume (v), there is an operator T with $\|T\| \leq C\|\delta\|_{\mathrm{cb}} \leq 2C$ in the von Neumann algebra generated by $A \otimes 1$ and $\delta(A)$ such that $\delta(a) = [a \otimes 1, T]$ for all a in A. Note that T belongs to $B(H) \overline{\otimes} W$. Let $Q\colon W \to W$ be the orthogonal projection onto the span of $\lambda(g_1), \ldots, \lambda(g_n)$. It is known (see, e.g., [18, p. 184]) that $\|Q\|_{\mathrm{cb}} \leq 2$; hence, if we set

$$T_1 = (1 \otimes Q)(T) , \quad \text{we have } \|T_1\| \leq 4C$$

and, moreover, since $\delta(a) = (1 \otimes Q)(\delta(a)) = (1 \otimes Q)[a \otimes 1, T] = [a \otimes 1, T_1]$, we have

$$[a \otimes 1, \theta] = [a \otimes 1, T_1] .$$

We can write $T_1 = \sum z_i \otimes \lambda(g_i)$. We have

$$\max\left\{ \Big\| \sum z_i z_i^* \Big\|^{1/2}, \Big\| \sum z_i^* z_i \Big\|^{1/2} \right\} \leq \|T_1\| \leq 4C \tag{6}$$

and, since $\sum [a, \alpha_i] \otimes \lambda(g_i) = \sum [a, z_i] \otimes \lambda(g_i)$, we find that $\alpha_i - z_i \in A'$. To conclude, we set

$$a_i = \alpha_i - z_i , \quad b_i = \beta_i + z_i .$$

We have $a_i \in A'$, $b_i = y_i - a_i \in A'$ and moreover by (6)

$$\Big\| \sum a_i a_i^* \Big\|^{1/2} \leq \Big\| \sum \alpha_i \alpha_i^* \Big\|^{1/2} + \Big\| \sum z_i z_i^* \Big\|^{1/2} \leq 1 + 4C$$

and similarly

$$\Big\| \sum b_i^* b_i \Big\|^{1/2} \leq 1 + 4C .$$

Thus, we obtain (5) with $\beta = 2(1 + 4C)$, which proves that A is nuclear. This completes the proof that (v) implies (i) in the case $B = W'$. However, if B is liberal, the preceding argument extends: we replace W by $\sigma(B)'$ and $\lambda(g_i)$ by the elements in $\sigma(B)'$ corresponding to the basis of E_λ^n. We skip the easy details. □

REMARK. The preceding proof obviously shows that A is nuclear if and only if

(vii) For any $*$-homomorphism $\sigma : A \to B(H)$ and any c.b. σ-derivation $\delta : A \to B(H)$ (i.e. $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$), there is a T in the von Neumann algebra generated by the ranges of σ and δ such that $\delta(a) = \sigma(a)T - T\sigma(a)$ for all a in A.

Note that any nuclear A is amenable [11] and hence has a virtual diagonal, i.e. there is a net $t_i = \sum_k a_k(i) \otimes b_k(i)$ bounded in $A \hat{\otimes} A$ such that, for any a in A, $a \cdot t_i - t_i \cdot a$ tends to zero in $A \hat{\otimes} A$. Then, it is easy to see that if δ is as above, and if T is a weak$*$-cluster point of $\sum_k \delta(a_k(i))\sigma(b_k(i))$, we have $\delta(\cdot) = \sigma(\cdot)T - T\sigma(\cdot)$, and T lies manifestly in the von Neumann algebra (or even in the weak closure of the algebra) generated by $\sigma(A) \cup \delta(A)$. This remark should be compared with what is known on 'strongly amenable' C^*-algebras (a smaller class than the nuclear ones), for which we refer the reader to, e.g., [16, Section 1.31].

REMARK. In the group case the above argument should be compared with [5].

REMARK 11. An alternate argument (more direct but somewhat less 'constructive') for (v) $\Rightarrow$ (i) can be obtained as in the following sketch. We use the same notation as in the preceding proof. We argue that δ has range into $B(H) \overline{\otimes} W$ and note that the latter commutes with $1 \otimes W'$. Assume that $\|\delta\|_{\mathrm{cb}} = 1$. Then our assumption (v) implies that the map $\delta \cdot \sigma_1$ extends to a bounded map on $A \otimes_{\min} W'$ with norm $\leq 2\|T\| \leq 2C$. Then, for any $x = \sum_{t \in \boldsymbol{F}_\infty} x(t) \otimes \rho(t)$ in $A \otimes W'$, we have

$$\left\| \sum_{t \in \boldsymbol{F}_\infty} \delta(x(t))(1 \otimes \rho(t)) \right\| \leq 2C\|x\|_{\min}. \tag{7}$$

Composing the operator on the left of (7) with $\mathrm{Id} \otimes_\varphi$, where $\varphi(T) = \langle T\delta_e, \delta_e \rangle$, we find that

$$\left\| \sum_i [x(g_i), \alpha_i] \right\| \leq C\|x\|_{\min}. \tag{8}$$

Note that

$$\max\left\{ \left\| \sum x(g_i)x(g_i)^* \right\|^{1/2}, \left\| \sum x(g_i)^* x(g_i) \right\|^{1/2} \right\} \leq \|x\|_{\min}. \tag{9}$$

Hence, by the Cauchy-Schwarz inequality

$$\left\| \sum \alpha_i x(g_i) \right\| \leq \|x\|_{\min} \tag{10}$$

and

$$\left\| \sum x(g_i)\beta_i \right\| \leq \|x\|_{\min}. \tag{11}$$

Thus, (8) implies that

$$\left\| \sum_i x(g_i) y_i \right\| \leq (C + 2)\|x\|_{\min}. \tag{12}$$

Letting $x = \sum x_i \otimes \rho(g_i)$, we find that for all (x_i) in A we have

$$\Big\| \sum_i x_i y_i \Big\| \le 2(C+2) \max\Big\{ \Big\| \sum x_i^* x_i \Big\|^{1/2}, \Big\| \sum x_i x_i^* \Big\|^{1/2} \Big\}.$$

Clearly, this remains valid for all (x_i) in $A'' = N$ and, hence, by [23, Corollary 5], N' is injective.

REMARK 12. Actually, the preceding argument shows that A is nuclear if there is a constant C such that the ordinary norm of $\delta \cdot \sigma_1$ on $A \otimes_{\min} W'$ is $\le C\|\delta\|_{\mathrm{cb}}$.

4. Length for a pair of C^*-subalgebras. Let G_1, G_2 be two subgroups generating a group G. One says that G_1, G_2 generate G with bounded length (more precisely with length at most d) if every element in G can be written as a product of a bounded number of elements either in G_1 or in G_2 (resp. a product of at most d such elements). Equivalently, let $\psi\colon\ G_1 * G_2 \to G$ be the canonical homomorphism from the free product onto G; then generation with length $\le d$ is the same as saying that the restriction of ψ to the subset formed by all the words of length $\le d$ is surjective.

It turns out there is a natural analogue of this in the C^*-algebra (or operator algebra) context, already considered in [21], as follows.

Let A, B be C^*-subalgebras of a C^*-algebra Z. By convention, we will view $M_n(A)$ and $M_n(B)$ as subalgebras of $M_n(Z)$, so that if $x_1 \in M_n(A)$ and $x_2 \in M_n(B)$, then the product x_1x_2 belongs to $M_n(Z)$, and similarly for a product of rectangular matrices.

Now let $d \ge 1$ be an integer. We will say that $L(Z; A, B) \le d$ or more simply (when there is no ambiguity) that $L(Z) \le d$ if there is a constant C such that for any n and any x in $M_n(Z)$ with $\|x\|_{M_n(Z)} < 1$ and for any $\varepsilon > 0$, there is an integer N for which we can find matrices $x_1, x_2, \ldots, x_d$ and $y_1, y_2, \ldots, y_d$, with entries either all in A or all in B, where $x_1, x_2, \ldots, x_d$ are of sizes resp. $n \times N$, $N \times N, \ldots, N \times N$ and $N \times n$, and similarly for $y_1, y_2, \ldots, y_d$, satisfying

$$\prod_1^d \|x_j\| + \prod_1^d \|y_j\| < C$$

and finally such that

$$\Big\| x - \prod_1^d x_j - \prod_1^d y_j \Big\|_{M_n(Z)} < \varepsilon\,. \tag{13}$$

If this holds but only for $n = 1$, then we say that $L_1(Z; A, B) \le d$ or simply that $L_1(Z) \le d$. Note that the two products are needed because one of them 'starts' in B and the other 'starts' in A. So, we will make the convention that x_1 is a matrix with entries in B while y_1 is one with entries in A. To eliminate the ε-error term, we need to use infinite matrices, as follows. Let us denote $K(A) = K \otimes_{\min} A$. We may identify $K(A)$ and $K(B)$ with subalgebras of $K(Z)$. Then $L(Z) \le d$ (resp. $L_1(Z) \le d$) if and only if any x in $K(Z)$ can be written as the sum of two products

$$x = x_1 x_2 \cdots x_d + y_1 y_2 \cdots y_d \tag{14}$$

with each x_j, y_j either in $K(A)$ or in $K(B)$, with the first terms x_1 in $K(B)$ and y_1 in $K(A)$, and satisfying

$$\prod_1^d \|x_j\| + \prod_1^d \|y_j\| \le C\|x\|_{K(Z)}\,.$$

Let $D \ge 1$ be another integer and let $d = 2D + 1$. We will say that $L^A(Z) \le D$ if there is a constant C such that the same as before holds but with $x = x_1 x_2 \cdots x_d$ or equivalently with the y_j all vanishing. More precisely, we have $x_{2j+1} \in K(B)$ for $j = 0, 1, 2, \dots, D$ and $x_{2j} \in K(A)$ for $j = 1, 2, \dots, D$. For convenience, we say that $L^B(Z) \le D$ if there is a constant C such that the same as before holds but with $x = y_1 x_2 \cdots y_d$ or equivalently with the x_j all vanishing.

Finally, if $d = 2D + 1$ and if the property used above to define $L(Z) \le d$ holds for $n = 1$ but with the y_j all vanishing, we say that $L_1^A(Z) \le D$. Again, we say that $L_1^B(Z) \le D$ if this holds for $n = 1$ but with the x_j all vanishing. Needless to say $L(Z) = L(Z; A, B)$ is defined as the smallest integer $d > 0$ such that $L(Z) \le d$, and similarly for $L^A(Z)$, $L^B(Z)$, $L_1^A(Z)$, and so on.

Roughly, $L(Z) \le d$ corresponds to factorizations of length at most d jointly in $K(A)$ and $K(B)$, while $L^A(Z) \le D$ corresponds to factorizations of length D in $K(A)$ (and *a fortiori* of length at most $2D + 1$ jointly in $K(A)$ and $K(B)$).

REMARK 13. By an elementary counting argument, we find that

$$2L^A(Z) - 1 \le L(Z) \le 2L^A(Z) + 1\,,$$
$$2L_1^A(Z) - 1 \le L_1(Z) \le 2L_1^A(Z) + 1\,.$$

Moreover (this is obvious if A is unital, otherwise we use an approximate unit),

$$L^A(Z) \le L^B(Z) + 1 \quad \text{and} \quad L_1^A(Z) \le L_1^B(Z) + 1\,.$$

REMARK. Note that length at most d obviously passes to quotients: for any ideal $I \subset Z$, we have

$$L(Z/I) \le L(Z)\,. \tag{15}$$

Let A, B be two C^*-algebras. Let $A \mathbin{\dot{*}} B$ be their (non-unital) C^*-algebraic free product. This is obtained by completion of the algebraic free product with respect to the maximal C^*-norm on it. Let $V_d \subset A \mathbin{\dot{*}} B$ be the subspace generated by elements of the form $x_1 x_2 \cdots x_d$ with $x_k \in A$ or $x_k \in B$ in such a way that x_k and x_{k+1} do not belong to the same subalgebra (A or B).

Similarly, let V_d^A (resp. V_d^B) be the closed span of elements of the form $x_1 x_2 \cdots x_d$ as above but such that $x_1 \in A$ (resp. $x_1 \in B$). Note that V_d is obviously the closure of $V_d^A + V_d^B$.

Given C^*-subalgebras A, B as above, we denote by

$$\dot{Q}_Z\colon A \dot{*} B \to Z$$

the (surjective) $*$-homomorphism canonically extending the inclusions $A \subset Z$ and $B \subset Z$.

THEOREM 14. *Let $A \subset Z$ and $B \subset Z$ be C^*-subalgebras as above. Let $d \geq 1$ be an integer. The following assertions are equivalent.*

(i) $L(Z; A, B) \leq d$.

(ii) *The restriction of the canonical quotient map $\dot{Q}_Z\colon A \dot{*} B \to Z$ to $V_{\leq d}$ is a complete surjection.*

Moreover, (i) *or* (ii) *implies the following.*

(iii) *Every x in $K(Z)$ can be written as a product*

$$x = x_1 \cdots x_{d+1}$$

with $x_1, \dots, x_{d+1}$ either in $K(A)$ or in $K(B)$.

For completeness, we also state the analogue for L_1.

THEOREM 15. *Let $A \subset Z$, $B \subset Z$ and $d \geq 1$ be as above. The following assertions are equivalent.*

(i) $L_1(Z; A, B) \leq d$.

(ii) *The restriction of the canonical quotient map $\dot{Q}_Z\colon A \dot{*} B \to Z$ to $V_{\leq d}$ is a surjection.*

Moreover, (i) *or* (ii) *implies the following.*

(iii) *Every x in Z can be written as a product*

$$x = x_1 \cdots x_{d+1}$$

with $x_1, \dots, x_{d+1}$ either in $K(A)$ or in $K(B)$, with the understanding that x_1 is a $1 \times \infty$ matrix and x_{d+1} is a $\infty \times 1$ matrix.

To prove these statements, we will use the 'Haagerup tensor product' of operator spaces, for which we refer the reader to [2, 9, 18]. The main relevant fact for our purpose is the following.

LEMMA 16. *The space V_d^A (resp. V_d^B) is completely isomorphic to the Haagerup tensor product*

$$A \otimes_h B \otimes_h A \cdots \ (\textit{resp. } B \otimes_h A \otimes_h B \cdots)$$

with a total of d factors.

PROOF. For this last fact (apparently due to the present author), we refer the reader to [18, Exercise 5.8, p. 108 and pp. 433–434]. This is a refinement of results originally presented in [7]. □

REMARK 17. It will be convenient to use the universal C^*-algebra generated by two projections p, q, denoted by C_2. We define C_2 as follows: let x be a formal linear combination of the set

$$J = \{1, p, q, (pq)^j, (qp)^j, (pq)^j p, (qp)^j q \mid j \geq 1\}.$$

We set $\|x\|$ equal to the supremum of the norm of x in $B(H)$ when we replace p, q by an arbitrary pair of orthogonal projections in $B(H)$, H being an arbitrary Hilbert space. Then C_2 can be defined as the completion of the space of these x equipped with this norm. Let $\varepsilon_1 = p - (1-p) = 2p - 1$ and $\varepsilon_2 = 2q - 1$. Note that $\varepsilon_1, \varepsilon_2$ are unitaries with $\varepsilon_1^2 = \varepsilon_2^2 = 1$, which generate C_2 as a C^*-algebra. Consequently (see [25] for more on this), C_2 can be identified with $C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$ the C^*-algebra of the (amenable) dihedral group, with ε_1 and ε_2 corresponding to the free generators of the two (free) copies of $\mathbf{Z}_2$. For convenience, we introduce the following notation:

$$p_{2j} = (pq)^j, \quad p_{2j+1} = (pq)^j p,$$
$$q_{2j} = (qp)^j, \quad q_{2j+1} = (qp)^j q.$$

We will use the observation that the family J is linearly independent in C_2. This is easy to check by observing that

$$(pq)^j = (\varepsilon_1\varepsilon_2)^j + \text{lower-order terms},$$
$$(qp)^j = (\varepsilon_2\varepsilon_1)^j + \text{lower-order terms},$$

and similarly for $(pq)^j p$ and $(qp)^j q$.

This observation implies that, if we fix $d \geq 1$, there is a constant $K(d)$ such that for any finitely supported families of scalars $(\lambda_j)_{j\geq 1}$ and $(\mu_j)_{j\geq 1}$ we have

$$\tag{16} \begin{aligned} K(d)^{-1} \max\left\{\sup_{j\leq d} |\lambda_j|, \sup_{j\leq d} |\mu_j|\right\} &\leq \left\|\sum_{j\leq d} \lambda_j p_j + \sum_{j\leq d} \mu_j q_j\right\| \\ &\leq K(d) \max\left\{\sup_{j\leq d} |\lambda_j|, \sup_{j\leq d} |\mu_j|\right\}. \end{aligned}$$

Actually, the sum $V_d^A + V_d^B$ is a direct sum of operator spaces. More precisely, we have the following.

LEMMA 18. *Let $V_{\leq d}$ be the closed span of V_j for $1 \leq j \leq d$. Then we have for each fixed $d \geq 1$ the following complete isomorphisms*:

$$\tag{17} V_{\leq d} \simeq V_1 \oplus \cdots \oplus V_d,$$

$$\tag{18} V_d \simeq V_d^A \oplus V_d^B$$

and consequently

$$\tag{19} V_{\leq d} \simeq V_1^A \oplus V_1^B \oplus \cdots \oplus V_d^A \oplus V_d^B.$$

PROOF. The proof is elementary. Consider an element x in $V_1^A + V_1^B + \cdots + V_d^A + V_d^B$; say we have $x = \alpha_1 + \beta_1 + \cdots + \alpha_d + \beta_d$ with $\alpha_j \in V_j^A$, $\beta_j \in V_j^B$. Let p, q and C_2 be as in the above Remark 17. We may then consider the pair of (non-unital) representations $\pi_p \colon A \to (A * B) \otimes_{\min} C_2$ and $\pi_q \colon B \to (A * B) \otimes_{\min} C_2$ defined by $\pi_p(a) = a \otimes p$ and

$\pi_q(b) = b \otimes q$. Let $\pi : A \dot{*} B \to (A \dot{*} B) \otimes_{\min} C_2$ be the representation canonically extending (jointly) π_p and π_q.

Note that

$$\pi(x) = \sum_{j=1}^{d} \alpha_j \otimes p_j + \beta_j \otimes q_j \,. \tag{20}$$

By (16), the span of $\{p_j, q_j \mid 1 \le j \le d\}$ is (completely) isomorphic to $\boldsymbol{C}^{2d}$; therefore, (19) follows immediately from (20). Then (18) follows by restricting to $x = \alpha_d + \beta_d$ and (17) is but a combination of (18) and (19). □

PROOF OF THEOREMS 14 AND 15. To simplify the notation, let us denote by $X_d = A \otimes_h B \otimes_h A \otimes_h \cdots$ (resp. $Y_d = B \otimes_h A \otimes_h B \cdots$), where each X_d and Y_d have exactly d factors. By definition of the Haagerup tensor product the assumption that $L(Z) \le d$ (resp. $L^A(Z) \le d$) equivalently means that the product map $X_d + Y_d \to Z$ (resp. the product map $Y_{2d+1} \to Z$) is a complete surjection (see e.g. [18, Corollary 5.3, p. 91]). Here $X_d + Y_d$ is defined as the operator quotient space of the direct sum (say in the ℓ_1-sense) $X_d \oplus Y_d$ by the kernel of the map $(x, y) \to x + y$ (see [18, p. 55] for more information). Since we just saw (by (18) and Lemma 16) that V_d (resp. V^B_{2d+1}) can be identified with $X_d \oplus Y_d$ (resp. with Y_{2d+1}), this holds if and only if the restriction of $\dot{Q}_Z$ to V_d (resp. V^B_{2d+1}) is a complete surjection. The same argument yields the analogous statement concerning $L_1(Z) \le d$ (resp. $L^A_1(Z) \le d$). Finally, the assertions (iii) are proved using a unit if it exists, or an approximate unit otherwise. □

REMARK. Similarly, $L^B(Z) \le d$ if and only if $\dot{Q}_Z$ restricted to V^A_{2d+1} is a complete surjection.

The analogous notation and statements for the *unital* free product are as follows. Let $A * B$ be the unital free product of A and B (both assumed unital). Clearly, there is a canonical surjective $*$-homomorphism $\kappa\colon A \dot{*} B \to A * B$. Let $Q_Z\colon A * B \to Z$ be the natural (quotient) unital $*$-homomorphism. We have obviously $Q_Z\, \kappa = \dot{Q}_Z$.

Let $W_{\le d}$ be the subspace generated by elements of the form $x_1 x_2 \cdots x_d$ with either x_k in A or x_k in B for each $k = 1, \dots, d$. Let φ (resp. ψ) be a state on A (resp. B). Let $\mathring{A}$ (resp. $\mathring{B}$) denote the subspace of A (resp. B) formed of all elements on which φ (resp. ψ) vanishes.

We will keep this choice of states φ (resp. ψ) fixed throughout the rest of the paper. Note that $\mathring{A}$ (resp. $\mathring{B}$) implicitly depends on this initial choice, even though the notation does not indicate it.

Note that $A \simeq \boldsymbol{C}1_A \oplus \mathring{A}$ and $B \simeq \boldsymbol{C}1_B \oplus \mathring{B}$. We denote by W^A_d (resp. W^B_d) the closed span in $A * B$ of all elements y of the form

$$y = y_1 y_2 \cdots y_d \tag{21}$$

with each y_k either in $\mathring{A}$ or in $\mathring{B}$ in such a way that no two consecutive elements belong to the same set $\mathring{A}$ or $\mathring{B}$ (so if $y_k \in \mathring{A}$ then $y_{k+1} \in \mathring{B}$) and finally such that $y_1 \in \mathring{A}$ (resp. $y_1 \in \mathring{B}$).

Roughly speaking, W_d^A (resp. W_d^B) is spanned by the elements of length exactly equal to d, that start in A (resp. B). We denote by W_d the closure of $W_d^A + W_d^B$.

Let us denote by $\mathcal{W}_d^A$ (resp. $\mathcal{W}_d^B$) the linear span in $A * B$ of all elements y of the form (21) with y_1 in A (resp. y_1 in B). Let

$$\mathcal{W}_d = \mathcal{W}_d^A + \mathcal{W}_d^B .$$

With this notation, W_d^A (resp. W_d^B) appears as the closure of $\mathcal{W}_d^A$ (resp. $\mathcal{W}_d^B$) in $A * B$, and W_d is the closure of $\mathcal{W}_d$.

Note that $\mathcal{W}_d^A$ (resp. $\mathcal{W}_d^B$) is clearly linearly isomorphic to the algebraic tensor product $\overset{\circ}{A} \otimes \overset{\circ}{B} \otimes \cdots$ (resp. $\overset{\circ}{B} \otimes \overset{\circ}{A} \otimes \cdots$). Therefore, using the canonical embeddings $A \subset A \dot{*} B$ and $B \subset A \dot{*} B$ we can unambiguously define linear embeddings of $\mathcal{W}_d^A$ and $\mathcal{W}_d^B$ into $A \dot{*} B$. This gives us a linear embedding of $\mathcal{W}_d$ into $A \dot{*} B$. Let us denote by Λ the linear map extending the preceding one to the linear span of $\{\mathcal{W}_j \mid j \geq 1\}$. Then we have the following.

LEMMA 19. *For any $d \geq 1$, the mapping Λ extends to a complete isometry from $W_1 + \cdots + W_d$ into $A \dot{*} B$, which lifts the canonical surjective $*$-homomorphism $\kappa\colon A \dot{*} B \to A * B$.*

PROOF. Let $x = \omega_1 + \cdots + \omega_d$ with $\omega_j \in \mathcal{W}_j$ for any $j \geq 1$. We will show that $\|\Lambda(x)\|_{A \dot{*} B} = \|x\|_{A * B}$. Note that Λ is trivially a lifting of κ, so that $q(\Lambda(x)) = x$ and hence $\|x\| \leq \|\Lambda(x)\|$ is immediate. To prove the converse, consider a pair of representations

$$\pi_1\colon A \to B(H) \quad \text{and} \quad \pi_2\colon B \to B(H)$$

such that the associated representation π on $A \dot{*} B$ is isometric, so that $\|\pi(\Lambda(x))\| = \|\Lambda(x)\|$. Let $\pi_1(1_A) = p$ and $\pi_2(1_B) = q$. We may introduce the maps $\hat{\pi}_1$ and $\hat{\pi}_2$ on A and B, respectively, by setting

$$\hat{\pi}_1(a) = \pi_2(a) + \varphi(a)(1-p) ,$$
$$\hat{\pi}_2(b) = \pi_2(b) + \psi(b)(1-q) .$$

Note that since $\pi_1(a) = p\pi_1(a)p$ and $\pi_2(b) = q\pi_2(b)q$, $\hat{\pi}_1$ and $\hat{\pi}_2$ are unital completely positive (c.p. in short) maps. By [3] (see also [4]), there is a unital completely positive map $\hat{\pi}\colon A * B \to B(H)$ such that for any y as in (21) we have

$$\hat{\pi}(y) = \hat{\pi}_1(y_1)\hat{\pi}_2(y_2)\cdots \quad \text{if } y_1 \in \overset{\circ}{A}$$

and

$$\hat{\pi}(y) = \hat{\pi}_2(y_1)\hat{\pi}_1(y_2)\cdots \quad \text{if } y_1 \in \overset{\circ}{B} .$$

But, for any y_1 in $\overset{\circ}{A}$ (resp. y_2 in $\overset{\circ}{B}$), we have $\hat{\pi}_1(y_1) = \pi_1(y_1)$ (resp. $\hat{\pi}_2(y_2) = \pi_2(y_2)$).

Hence, this shows that $\hat{\pi}(\omega_j) = \pi(\Lambda(\omega_j))$ for any j and hence

$$\hat{\pi}(x) = \pi(\Lambda(x)) .$$

Thus, we conclude that

$$\|\pi(\Lambda(x))\| \leq \|\hat{\pi}\|\|x\| \leq \|x\| ,$$

and since we choose π so that $\|\pi(\Lambda(x))\| = \|\Lambda(x)\|$, we obtain as announced $\|\Lambda(x)\| \leq \|x\|$. This shows that Λ is isometric. The proof that it is completely isometric is entirely similar; we leave the routine details to the reader. $\square$

We then have the following.

LEMMA 20. *For each fixed $d \geq 1$, we have the following complete isomorphisms*:

$$W_{\leq d} \simeq \boldsymbol{C} \oplus W_1 \oplus \cdots \oplus W_d\,, \tag{22}$$

$$W_d \simeq W_d^A \oplus W_d^B\,, \tag{23}$$

$$W_{\leq d} \simeq \boldsymbol{C} \oplus W_1^A \oplus W_1^B \oplus \cdots \oplus W_d^A \oplus W_d^B\,. \tag{24}$$

Moreover, we have complete isomorphisms

$$W_d^A \simeq \overset{\circ}{A} \otimes_h \overset{\circ}{B} \otimes_h \cdots \quad \textit{and} \quad W_d^B \simeq \overset{\circ}{B} \otimes_h \overset{\circ}{A} \otimes_h \cdots\,.$$

PROOF. Let us first check that the sums $W_1 \oplus \cdots \oplus W_d$ and $W_d^A \oplus W_d^B$ are direct ones. Using the lifting in Lemma 19, this is an immediate consequence of (17) and (18), since we have, of course, $\Lambda(W_d) \subset V_d$, $\Lambda(W_d^A) \subset V_d^A$ and $\Lambda(W_d^B) \subset V_d^B$. In particular, this proves (23). Let $\phi * \psi$ denote the free product state on $A * B$ (see [27, p. 4]). Note that $\phi * \psi$ vanishes on $W_1 + \cdots + W_d$ and hence (22) follows. Then (24) is but a recapitulation. Finally, the last assertion follows from Lemmas 16 and 19 using again $\Lambda(W_d^A) \subset V_d^A$ and $\Lambda(W_d^B) \subset V_d^B$, and the injectivity of the Haagerup tensor product (cf., e.g., [18, p. 93]). $\square$

In the next statement, we denote by $W_{\leq 2d+1}^B$ the sum $\sum_{j=1}^{2d+1} W_j^B$. Note that the latter sum is closed since, by Lemma 20, it is a direct sum. Equivalently, this is the closed span of all alternated products in $\overset{\circ}{A}$ and $\overset{\circ}{B}$ but with at most d factors in $\overset{\circ}{A}$.

THEOREM 21. *Let A, B, Z be unital C^*-algebras as above. Then $L(Z) \leq d$ (resp. $L^A(Z) \leq d$) if and only if the restriction of $Q_Z\colon A * B \to Z$ to $W_{\leq d}$ (resp. $W_{\leq 2d+1}^B$) is a complete surjection. Moreover, $L_1(Z) \leq d$ (resp. $L_1^A(Z) \leq d$) if and only if the restriction of Q_Z to $W_{\leq d}$ (resp. $W_{\leq 2d+1}^B$) is a surjection.*

PROOF. Assume that $L(Z) \leq d$. Then, since $Q_Z\,\kappa = \dot{Q}_Z$, Theorem 14 implies that Q_Z restricted to $\kappa(V_d)$ is a complete surjection. Since $\kappa(V_d) \subset W_{\leq d}$, it follows that Q_Z restricted to $W_{\leq d}$ is a complete surjection. Conversely, assume that Q_Z restricted to $W_{\leq d}$ is a complete surjection. Since $W_{\leq d}$ is spanned by the unit of $A * B$ and $W_1 + \cdots + W_d$, Lemma 19, recalling (22), implies that there is a completely bounded map $\hat{\Lambda}\colon\ W_{\leq d} \to A \dot{*} B$ lifting κ, defined e.g. by $\hat{\Lambda}(\lambda 1 + x) = \lambda 1_A + \Lambda(x)$. Note that $\hat{\Lambda}(W_{\leq d}) \subset V_{\leq d}$ and hence $\dot{Q}_Z$ restricted to $V_{\leq d}$ is a complete surjection.

Now observe that the completely isometric mapping

$$x \to 1_A x \quad (\text{resp. } x \to 1_B x)$$

takes V_{d-1}^B (resp. V_{d-1}^A) to V_d^A (resp. V_d^B) and we have $Q_Z(x) = Q_Z(1_A x) = Q_Z(1_B x)$, and similarly for V_{d-2}^B (resp. V_{d-2}^A), V_{d-3}^B (resp. V_{d-3}^A), and so on. Since, by Lemma 18, $V_{\leq d}$

decomposes as a direct sum of $V_j^A + V_j^B$ $(j \le d)$, it is easy to use the preceding observation to replace the elements of $V_{\le d}$ by suitably chosen ones in V_d in order to show that $\dot{Q}_Z$ restricted to the smaller subspace $V_d \subset V_{\le d}$ is a complete surjection. By Theorem 14 again, we conclude that $L(Z) \le d$. This proves the part of the statement concerning $L(Z)$, but actually the same proof also establishes the part concerning $L_1(Z)$ by removing 'complete' from 'complete surjection'. The other part is proved similarly. We leave the details to the reader. □

Finally, we give the basic result that connects the length with the first part of the paper.

LEMMA 22. *Assume that $L(Z) \le d$. Then there is a constant C such that for any bounded homomorphism $\Phi\colon Z \to B(H)$ we have*

$$\|\Phi\|_{\mathrm{cb}} \le C(\max\{\|\Phi_{|A}\|_{\mathrm{cb}}, \|\Phi_{|B}\|_{\mathrm{cb}}\})^d .$$

Moreover, for any bounded derivation $\Delta\colon Z \to B(H)$ (relative to a representation of Z on $B(H)$),

$$\|\Delta\|_{\mathrm{cb}} \le C(d \max\{\|\Delta_{|A}\|_{\mathrm{cb}}, \|\Delta_{|B}\|_{\mathrm{cb}}\}) .$$

PROOF. Let $t = \max\{\|\Phi_{|A}\|_{\mathrm{cb}}, \|\Phi_{|B}\|_{\mathrm{cb}}\}$. Let $\Phi_n = \mathrm{Id} \otimes \Phi\colon M_n(Z) \to M_n(B(H))$. Let $x \in M_n(Z)$ with $\|x\|_{M_n(Z)} < 1$. With the notation in (13) we have

$$\|\Phi_n(x - x_1 x_2 \cdots x_d - y_1 y_2 \cdots y_d)\| \le \|\Phi_n\|\varepsilon .$$

But we have clearly

$$\|\Phi_n(x_1 x_2 \cdots x_d + y_1 y_2 \cdots y_d)\| \le t^d \left(\prod_1^d \|x_j\| + \prod_1^d \|y_j\| \right) \le C t^d .$$

Hence $\|\Phi_n(x)\| \le Ct^d + \|\Phi_n\|\varepsilon$ and hence letting $\varepsilon \to 0$ (here we crucially use the fact that Φ is continuous!) we find that $\|\Phi_n\| \le Ct^d$ and $\|\Phi\|_{\mathrm{cb}} \le Ct^d$. A similar argument gives the other inequality. □

The main result of [21] says that essentially we have a converse (note however that this is not exactly the converse; see Remark 25 below).

THEOREM 23. *Let $A \subset Z$, $B \subset Z$ as before. Assume A, B, Z unital with unital embeddings. Let $\mathrm{Alg}(A, B)$ denote the (dense) subalgebra generated by A and B. If there is a constant C such that any homomorphism $\Phi\colon \mathrm{Alg}(A, B) \to B(H)$ satisfies $\|\Phi\|_{\mathrm{cb}} \le C(\max\{\|\Phi_{|A}\|_{\mathrm{cb}}, \|\Phi_{|B}\|_{\mathrm{cb}}\})^d$, then*

$$L(Z) \le d .$$

PROOF. This is a particular case of Theorem 6 in [21]. □

REMARK. For simplicity, in the definition of length and in Theorems 14 and 21, we have restricted our description to pairs A, B of subalgebras, but it is easy to extend these statements (by a simple iteration) to triples A, B, C of subalgebras, or to any finite given number N of them. Of course, the resulting (complete) isomorphism constants in Lemmas 18 and 20 will now depend both on d and N.

REMARK. Throughout this section, we have restricted attention to C^*-algebras, but it is easy to verify that the same results remain valid when A, B, Z are non-self-adjoint operator algebras with minor changes in the proofs.

5. Length for the maximal tensor product. In this section, we will specialize the preceding to the situation when $Z = A \otimes_{\max} B$, where A, B are two unital C^*-algebras embedded into Z via the mappings $a \to a \otimes 1$ and $b \to 1 \otimes b$. We will identify A with $A \otimes 1$ and B with $1 \otimes B$, and view them as subalgebras of $Z = A \otimes_{\max} B$.

We will denote for simplicity

$$L(A \otimes_{\max} B) = L(A \otimes_{\max} B;\, A \otimes 1, 1 \otimes B)$$

and similarly for $L_1(A \otimes_{\max} B)$, $L_1^A(A \otimes_{\max} B)$ and $L_1^B(A \otimes_{\max} B)$.

We will similarly denote $L(A \otimes_{\min} B) = L(A \otimes_{\min} B;\, A \otimes 1, 1 \otimes B)$, and similarly for L_1, L_1^A, L_1^B.

In [21], the present author introduced the 'similarity degree' of a pair of unital C^*-algebras A, B as follows: Changing the notation from [21] slightly, we will say here that $d(A, B) \leq d$ (resp. $d_1(A, B) \leq d$) if there is a constant C such that for any pair $u\colon A \to B(H)$, $v\colon B \to B(H)$ of c.b. unital homomorphisms with commuting ranges, the homomorphism $u \cdot v\colon\ A \otimes B \to B(H)$ (taking $a \otimes b$ to $u(a)v(b)$) is c.b. (resp. is bounded) on $A \otimes_{\max} B$ with c.b. norm (resp. with norm) $\leq C \max(\|u\|_{\mathrm{cb}}, \|v\|_{\mathrm{cb}})^d$.

Of course, the number $d(A, B)$ (resp. $d_1(A, B)$) is defined as the infimum of the numbers $d \geq 1$ such that this property holds.

If (A, B) is such that for any (u, v) as above, the map $u \cdot v$ is c.b. (resp. bounded) on $A \otimes_{\max} B$, then $d(A, B) < \infty$ (resp. $d_1(A, B) < \infty$) (see [22]).

Let us denote by $d^A(A, B)$ (resp. $d_1^A(A, B)$) the smallest d with the following property: there is a constant C such that for any c.b. unital homomorphism $u : A \to B(H)$ and any unital $*$-homomorphism $\sigma : B \to B(H)$ with commuting ranges (i.e. we have $\sigma(B) \subset u(A)'$), the product mapping $u.\sigma$ is c.b. (resp. bounded) on $A \otimes_{\max} B$ with c.b. norm (resp. with norm) $\leq C\|u\|_{\mathrm{cb}}^d$. Moreover, we set by convention

$$d^B(A, B) = d^B(B, A)\,, \quad d_1^B(A, B) = d_1^B(B, A)\,.$$

When A is nuclear, we claim that this holds with $C = 1$ and $d = 2$ and then it even holds for all *complete contractions* $\sigma : B \to u(A)'$. Indeed, if A is nuclear, for any operator space B, the mapping $q : A \otimes_h B \otimes_h A \to A \otimes_{\min} B$ defined by $q(a \otimes b \otimes a') = aa' \otimes b$ is a complete metric surjection (see [19, pp. 240–241]). Let $P_3 : B(H) \otimes_h B(H) \otimes_h B(H) \to B(H)$ be the product map, which is clearly a complete contraction. We have obviously

$$u \cdot \sigma(aa' \otimes b) = (u \cdot \sigma)q(a \otimes b \otimes a') = P_3(u \otimes \sigma \otimes u)(a \otimes b \otimes a')\,.$$

Hence, since q is a complete metric surjection, we have

$$\|u \cdot \sigma\|_{CB(A \otimes_{\min} B, B(H))} = \|(u.\sigma)q\|_{\mathrm{cb}} = \|P_3(u \otimes \sigma \otimes u)\|_{\mathrm{cb}} \leq \|u\|_{\mathrm{cb}}^2 \|\sigma\|_{\mathrm{cb}}\,.$$

This holds for any operator space B. *A fortiori*, when B is a C^*-algebra, we can replace the min-norm by the max-one and this proves the above claim.

REMARK. In [22], we introduced the similarity degree $d(A)$ of a C^*-algebra A. This is defined as the smallest $d \geq 1$ such that there is a constant C so that any bounded homomorphism $u\colon A \to B(H)$ satisfies

$$\|u\|_{\mathrm{cb}} \leq C\|u\|^d.$$

This is related to the number $d(A, B)$ via the following obvious estimate:

$$d(A \otimes_{\max} B) \leq d(A, B) \max\{d(A), d(B)\}.$$

As explained in [21], the number $d(A, B)$ and its other variants are closely related to the length $L(A \otimes_{\max} B)$. Let us briefly recall this here.

For a pair (A, B) of C^*-algebras, we will consider the following properties (see also [14]).

(SP) For any pair $u\colon A \to B(H)$, $v\colon B \to B(H)$ of c.b. homomorphisms with commuting ranges, the product map $u \cdot v$ is c.b. on $A \otimes_{\max} B$.

$(\mathrm{SP})_1$ For any pair (u, v) as in (SP), the product map $u \cdot v$ is bounded on $A \otimes_{\max} B$. Then the main result concerning $d(A, B)$ in [21] can be stated as follows.

THEOREM 24 ([21]). *Assume* (SP) (*resp.* $(\mathrm{SP})_1$). *Then necessarily* $d(A, B) < \infty$ (*resp.* $d_1(A, B) < \infty$) *and, moreover,*

$$d(A, B) = L(A \otimes_{\max} B) \quad (\textit{resp. } d_1(A, B) = L_1(A \otimes_{\max} B)),$$
$$d^A(A, B) = L^A(A \otimes_{\max} B) \quad (\textit{resp. } d_1^A(A, B) = L_1^A(A \otimes_{\max} B)).$$

REMARK 25. Assuming $(\mathrm{SP})_1$, we get by Lemma 22 that $d(A, B) \leq L(A \otimes_{\max} B)$ and also $d_1(A, B) \leq L_1(A \otimes_{\max} B)$ or $d^A(A, B) \leq L^A(A \otimes_{\max} B)$ and $d_1^A(A, B) \leq L_1^A(A \otimes_{\max} B)$. However, it may be worthwhile to insist on an unpleasant feature of this particular setting involving $A \otimes_{\max} B$: if we only assume that $L(A \otimes_{\max} B) < \infty$ (resp. $L_1(A \otimes_{\max} B) < \infty$), we cannot verify in full generality that

$$d(A, B) \leq L(A \otimes_{\max} B) \quad (\text{resp. } d_1(A, B) \leq L_1(A \otimes_{\max} B))$$

because we do not see how to check that (SP) (resp. $(\mathrm{SP})_1$) holds. The difficulty lies in the fact that we have an approximate factorization in (13) relative to a norm (the max-norm) for which we do not know yet that $u \cdot v$ is continuous! See the proof of Lemma 22 for clarification. An equivalent difficulty arises with (14). Fortunately, in the situations of interest to us, $(\mathrm{SP})_1$ holds (or $u \cdot v$ is continuous), so there is no problem.

REMARK. Note that (iii) in Theorem 1 means that (A, B) satisfies (SP) for all B. Note also that this is formally equivalent to saying that (A, B) satisfies $(\mathrm{SP})_1$ for all B. Indeed, the latter implies the existence of a function F such that $\|u \cdot v\colon A \otimes_{\max} B \to B(H)\| \leq F(\|u\|_{\mathrm{cb}}, \|v\|_{\mathrm{cb}})$ holds for all B, u, v. We can then show that $\|u \cdot v\colon A \otimes_{\max} B \to B(H)\|_{\mathrm{cb}} \leq F(\|u\|_{\mathrm{cb}}, \|v\|_{\mathrm{cb}})$ by replacing B by $M_n(B)$ to estimate the cb-norm of $u \cdot v$. Thus, A is nuclear if and only if for any B the pair (A, B) satisfies $(\mathrm{SP})_1$.

Given unital C^*-algebras A and B we denote by

$$\dot{Q}_{A,B}\colon A \dot{*} B \to A \otimes_{\max} B$$

the (surjective) $*$-homomorphism canonically extending the inclusions $A \subset A \otimes_{\max} B$ and $B \subset A \otimes_{\max} B$.

The next two statements recapitulate what we know from Theorems 14 and 21.

PROPOSITION 26. *Let A, B be unital C^*-algebras satisfying* (SP). *Then $L(A \otimes_{\max} B) \le d$ (resp. $L^A(A \otimes_{\max} B) \le d$) if and only if the restriction of $\dot{Q}_{A,B}\colon A \dot{*} B \to A \otimes_{\max} B$ to V_d (resp. V^B_{2d+1}) is a complete surjection. Moreover, $L_1(A \otimes_{\max} B) \le d$ (resp. $L^A_1(A, B) \le d$) if and only if the restriction of $\dot{Q}_{A,B}$ to V_d (resp. V^B_{2d+1}) is a surjection.*

PROPOSITION 27. *Let A, B be unital C^*-algebras satisfying* (SP). *Then $L(A \otimes_{\max} B) \le d$ (resp. $L^A(A \otimes_{\max} B) \le d$) if and only if the restriction of $Q_{A,B}\colon A * B \to A \otimes_{\max} B$ to $W_{\le d}$ (resp. $W^B_{\le 2d+1}$) is a complete surjection. Moreover, $L_1(A \otimes_{\max} B) \le d$ (resp. $L^A_1(A, B) \le d$) if and only if the restriction of $Q_{A,B}$ to $W_{\le d}$ (resp. $W^B_{\le 2d+1}$) is a surjection.*

The preceding Theorem 1 shows that (SP) holds for all B if and only if A is nuclear (and in that case $d(A, B)$ as defined above is ≤ 3, and also $d^A(A, B) \le 2$ and $d^B(A, B) \le 1$). Now we have conversely the following.

THEOREM 28. *Let A, B be C^*-algebras. If B is liberal and if $L(A \otimes_{\max} B) < \infty$ or if more generally $L(A \otimes_{\min} B) < \infty$, then A is nuclear.*

PROOF. Note $L(A \otimes_{\min} B) \le L(A \otimes_{\max} B)$. Using Lemma 22 with $Z = A \otimes_{\min} B$ and recalling (1), we see that $L(A \otimes_{\min} B) < \infty$ implies property (vi) in Theorem 10. □

As a corollary, we can answer several questions raised in [21].

COROLLARY 29. *Assume that* $\dim H = \infty$. *Let G be a discrete group. Then $L(C^*(G) \otimes_{\max} B(H)) < \infty$ if and only if G is amenable. Therefore, $L(C^*(\boldsymbol{F}_n) \otimes_{\max} B(H)) = \infty$ for any $n \ge 2$. Moreover, $L(B(H) \otimes_{\max} B(H)) = \infty$.*

PROOF. Recall that $C^*(G)$ is nuclear if and only if G is amenable (see [13]). Moreover, by [28], $B(H)$ is not nuclear. Obviously $C^*(\boldsymbol{F}_\infty)$ is liberal. A combination of [1, p. 205] and [28] shows that $B(\ell_2)$ (or even the Calkin algebra $B(\ell_2)/K(\ell_2)$) is liberal. Therefore, this corollary follows from the preceding theorem. □

As mentioned already in [21], if A is nuclear and B an arbitrary C^*-algebra, then $L^A(A \otimes_{\max} B) \le 2$ and $L(A \otimes_{\max} B) \le 3$. Note the obvious inequality $L(A \otimes_{\min} B) \le L(A \otimes_{\max} B)$. Then here is a final recapitulation.

THEOREM 30. *Let A be a C^*-algebra. The following are equivalent.*

(i) *A is nuclear.*
(ii) *For any C^*-algebra B, we have $L(A \otimes_{\max} B) < \infty$.*
(ii)$_1$ *For any C^*-algebra B, we have $L_1(A \otimes_{\max} B) < \infty$.*
(iii) *For any C^*-algebra B, we have $L^A(A \otimes_{\max} B) \le 2$.*

(iv) *For any C^*-algebra B, we have $L^B(A \otimes_{\max} B) \leq 1$.*
Moreover, these are all equivalent to the same properties with respect to $A \otimes_{\min} B$.

PROOF. The fact that nuclear implies (iii) or (iv) (and *a fortiori* any of the other properties) follows from properties of the so-called δ-norm in [18, p. 240]. The converses all follow from the preceding theorem, recalling Remarks 12 and 13. $\square$

REMARK. As already mentioned in Remark 2, it may be true that $L(A \otimes_{\max} B) < \infty$, or even merely $L_1(A \otimes_{\min} B) < \infty$, for a fixed pair A, B implies that $A \otimes_{\max} B = A \otimes_{\min} B$. One simple-minded approach to prove this would be as follows: let $Z = A \otimes_{\min} B$ and $n = 1$ in (13). Consider $x \in A \otimes B$ (*algebraic* tensor product) with $\|x\|_{\min} < 1$. The problem is simply to prove that the property expressed by (13) (i.e. the fact that $L_1(A \otimes_{\min} B) \leq d$) automatically implies another representation as in (13) but for $\varepsilon = 0$. Indeed, it is clear (here recall that $n = 1$) that $\|\prod_1^d x_j + \prod_1^d y_j\|_{\max} < C$, so we would conclude that $\|\cdot\|_{\max} \leq C\|\cdot\|_{\min}$. Note that if such a simple-minded proof (in particular not using [8]) is found, it would give a more direct way to show that nuclear passes to quotients.

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