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K. R. Gabriel

**Institutions:** University of North Carolina at Chapel Hill

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SIMULTANEOUS TEST PROCEDURES  
IN MULTIVARIATE ANALYSIS OF VARIANCE

by

K. R. Gabriel

University of North Carolina  
and Hebrew University, Jerusalem

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DEPARTMENT OF BIostatISTICS

UNIVERSITY OF NORTH CAROLINA

Chapel Hill, N. C.

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1. INTRODUCTION AND SUMMARY

Multivariate analysis of variance (MANOVA) provides significance tests for a null hypothesis on a number of linear parametric functions (LPFs) of the expectations of several variables. A significant result is taken to mean that the hypothesis does not hold as a whole. However, such a result does not indicate which of the LPFs are non-null and on what variables or linear combinations of variables (LCVs). Simultaneous test procedures (STPs) allow some resolution of overall significant results into rejection of detailed hypotheses on subsets of LPFs and of LCVs, down to single LPFs and LCVs. Associated with these procedures are simultaneous confidence bounds (SCBs) on single LPFs of single LCVs.

The MANOVA STPs presented here use statistics based on the characteristic roots of the product of the hypothesis and inverse error matrices. Such statistics include Roy's maximum root, Hotelling-Lawley's trace, Pillai's trace and Wilks's likelihood ratio. The maximum root MANOVA STP was implicit in the work of S. N. Roy, and the corresponding SCBs are due to Roy and Bose (1953). The formal presentation and extension of these results to a wider class of statistics appears to be novel. It is based on the application of general results obtained for STPs and SCBs in any statistical set-up in an earlier paper by Gabriel (1967b).

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Comparison of the properties of all MANOVA STPs considered shows the maximum root MANOVA STP to be preferable to all others in that, for a given probability of any type I error, it achieves the most resolution into significant detail, that is, it provides greatest power for hypotheses on single LPFs of single LCVs. Correspondingly, the maximum root SCBs are narrower than any other of the SCBs considered.

Earlier attempts at providing simultaneous inferences on subhypotheses in MANOVA are due to Roy and Gnanadesikan (1957) and to Bhapkar (1965) who set SCBs on certain non-centrality parameters. As far as inference on null hypotheses is concerned, Bhapkar's SCBs are equivalent to the maximum root STP whilst Roy-Gnanadesikan's SCBs are inferior since they provide less resolution for the same stated type I error probability. Neither of these SCBs is likely to be of much practical usefulness in providing actual bounds, since the functions of non-centrality which they bound are too complicated to have much practical meaning. Moreover, the use of these SCBs is encumbered by certain incoherences which may occur between the bounds on different functions.

Mudholkar (1966) has recently given a general derivation of these and other SCBs. He has also shown that the SCBs using Roy's maximum characteristic root statistic are the narrowest within a certain class, which includes those based on Hotelling-Lawley's trace, but not those based on Wilks's likelihood ratio.

## 2. ONE-WAY MANOVA STP — AN EXAMPLE

Data on skull measurements of anteaters due to Reeve (1941) (quoted by Seal (1965, p. 134)) have been chosen for an illustration. 48 specimens of Tamandua tetradactyla were obtained in collections from six localities (belonging to four subspecies) and three skull measurements were recorded for each specimen. The means of these measurements (after logarithmic transformation) are reproduced in Table 1 along with the within, or error, matrix of sums of squares and products, as

computed by Seal (1965).

TABLE 1. MEANS AND ERROR SUMS OF SQUARES AND PRODUCTS

Locality	Number of Skulls	Means			Subspecies
		X	Y	Z	
1. Sta. Marta, Colombia	21	2.054	2.066	1.621	instabilis
2. Minas Geraes, Brazil	6	2.097	2.100	1.625	chapadensis
3. Matto Grosso, Brazil	9	2.091	2.095	1.624	chapadensis
4. Sta. Cruz, Bolivia	3	2.099	2.102	1.643	chapadensis
5. Panama	4	2.092	2.110	1.703	chiriquensis
6. Mexico	5	2.099	2.107	1.671	mexicana
TOTAL	48				

42 d.f. Error Sum of Squares and Products

	X	Y	Z
X	0.013631	0.012769	0.016438
Y	0.012769	0.012923	0.017135
Z	0.016438	0.017135	0.036152

Note: X, Y, Z are common logarithms of, respectively, basal length excluding the premaxilla, occipito nasal length and greatest length of the nasals.

MANOVA of these data tests the hypothesis that the expectation of each one of the three measurement variables, X, Y and Z, are the same at all six localities. If such an overall hypothesis is rejected, as it is in this case (Seal (1965), p. 135) further questions arise. Are all six localities different from one another, or are there some homogeneous groupings? Where differences do exist, on what variables or LCVs do these differences occur? What localities differ on which variables? Are there any variables which have the same expectations at all localities? etc.. Some answers to questions of this kind may be obtained from MANOVA STPs.

The overall hypothesis of equality of all six localities on each of the three variables can be resolved into component hypotheses of equality in subgroups of two or more localities on one or more variables. There are altogether  $2^6 - 6 - 1 = 57$  groups of localities and  $2^3 - 1 = 7$  sets of variables, so  $57 \times 7 = 399$  hypotheses may be considered, including the overall hypothesis. Further hypotheses on LCVs other than X, Y and Z themselves, may also be considered, as well as on contrasts in three or more locality expectations. All the hypotheses considered must be implied by the overall hypothesis, but those are not the only implication relations among this family of hypotheses. Other instances are that if a larger group is homogeneous this implies every one of its subgroups also is, and if homogeneity exists on some set of variables that implies it also exists on any linear transformation of that set and on every subset of these variables. Clearly, the decisions of a test procedure will be coherent only if they preserve these implication relations.

An STP consists of rejecting all those hypotheses for which the statistic exceeds the common critical value, and of "accepting" all others. In the present MANOVA example Roy's maximum characteristic root statistic is used and the critical value, for a 5% level STP, is the upper 5% point of the null distribution of that statistic for the overall hypothesis. Since there are  $p_o = 3$  variables,  $r_o = 6 - 1 = 5$  independent comparisons of localities and  $n_e = 42$  d.f. for error, the parameters of that distribution are  $\underline{s} = \min(p_o, r_o) = 3$ ,  $\underline{m} = (|r_o - p_o| - 1) / 2 = 1/2$  and  $\underline{n} = (n_e - p_o - 1) / 2 = 19$ , for which the upper 5% point is  $\theta_{.95} = 0.3323$ , as interpolated in Pillai's Tables (1960) or Heck's Charts (1960).

The 5% level of an STP means that the probability of rejecting one or more true hypotheses out of all those tested is no more than 5%, whether the overall hypothesis, and hence each of its components, is true, or whether only some of them are true. Thus the probability of not wrongly rejecting any hypothesis of the family

tested is at least 95%. Also, the 95% probability includes that of having all parametric values covered in the appropriate SCBs.

To obtain Roy's statistic for a hypothesis of equality of a group G of localities on the expectations of all variables of a set V, the following computations are required. Let  $\bar{y}_j$  denote the (column) vector of means at locality j based on a sample of size  $n_j$  and let S denote the matrix of sums of squares and products for error. The hypothesis, or "between", matrix is

$$H^G = \sum_{j \in G} n_j \bar{y}_j \bar{y}_j' - \bar{\bar{y}}_G \bar{\bar{y}}_G' \sum_{j \in G} n_j$$

where

$$\bar{\bar{y}}_G = \sum_{j \in G} n_j \bar{y}_j / \sum_{j \in G} n_j.$$

The principal minors of the rows and columns corresponding to the variables of set V will be denoted  $S_V$  and  $H_V^G$ , for matrices S and  $H^G$ , respectively. Writing  $c_1 [ \ ]$  for the maximum characteristic root, Roy's statistic is  $c_1 [S_V^{-1} H_V^G]$  or, equivalently,

$$\theta = c_1 [(S_V + H_V^G)^{-1} H_V^G] = c_1 [S_V^{-1} H_V^G] / (1 + c_1 [S_V^{-1} H_V^G]).$$

The tabulated percentage values are those of  $\theta$ , but  $\theta_{1-\alpha} / (1 - \theta_{1-\alpha})$  could equivalently be used as  $1-\alpha$  percentage point for  $c_1 [S_V^{-1} H_V^G]$ .

Table 2 presents the  $\theta$  statistics for each of the 399 hypotheses of equality of groups of localities on sets of variable expectations. These statistics were computed on the University of North Carolina's IBM 360 computer by means of a special FORTRAN program. The MANOVA  $c_1$ -STP of level 5% consists of rejecting all those hypotheses whose  $\theta$  statistic exceeds the common critical value  $\theta_{.95} = 0.3323$ . In Table 2 all statistics which are thus significant are underlined, so that the STP decision can be seen at a glance.

TABLE 2. MAXIMUM CHARACTERISTIC ROOT STATISTICS FOR ALL GROUPS OF LOCALITIES AND ALL SETS OF VARIABLES.

LOCATIONS	VARIABLES						
	XYZ	XY	XZ	YZ	X	Y	Z
1 2 3 4 5 6	<u>.7133</u>	<u>.6260</u>	<u>.7074</u>	<u>.6382</u>	<u>.5952</u>	<u>.5466</u>	<u>.4613</u>
1 2 3 4 5	<u>.7118</u>	<u>.6055</u>	<u>.7057</u>	<u>.6356</u>	<u>.5599</u>	<u>.5032</u>	<u>.3982</u>
1 2 3 4 6	<u>.6960</u>	<u>.6257</u>	<u>.6936</u>	<u>.6340</u>	<u>.5830</u>	<u>.5023</u>	<u>.2344</u>
1 2 3 5 6	<u>.6968</u>	<u>.6001</u>	<u>.6917</u>	<u>.6238</u>	<u>.5757</u>	<u>.5330</u>	<u>.4599</u>
1 2 4 5 6	<u>.6429</u>	<u>.5815</u>	<u>.6368</u>	<u>.5691</u>	<u>.5672</u>	<u>.5315</u>	<u>.4494</u>
1 3 4 5 6	<u>.6415</u>	<u>.5750</u>	<u>.6337</u>	<u>.5602</u>	<u>.5593</u>	<u>.5238</u>	<u>.4552</u>
2 3 4 5 6	<u>.6037</u>	<u>.4200</u>	<u>.5831</u>	<u>.5177</u>	<u>.0237</u>	<u>.0606</u>	<u>.3902</u>
1 2 3 4	<u>.6929</u>	<u>.6047</u>	<u>.6905</u>	<u>.6298</u>	<u>.5396</u>	<u>.4356</u>	<u>.0341</u>
1 2 3 5	<u>.6945</u>	<u>.5701</u>	<u>.6891</u>	<u>.6206</u>	<u>.5303</u>	<u>.4810</u>	<u>.3938</u>
1 2 4 5	<u>.6303</u>	<u>.5334</u>	<u>.6228</u>	<u>.5470</u>	<u>.5104</u>	<u>.4723</u>	<u>.3920</u>
1 3 4 5	<u>.6328</u>	<u>.5305</u>	<u>.6234</u>	<u>.5423</u>	<u>.5028</u>	<u>.4637</u>	<u>.3954</u>
2 3 4 5	<u>.5839</u>	<u>.4135</u>	<u>.5576</u>	<u>.4824</u>	<u>.0163</u>	<u>.0470</u>	<u>.3507</u>
1 2 3 6	<u>.6792</u>	<u>.6001</u>	<u>.6775</u>	<u>.6197</u>	<u>.5593</u>	<u>.4807</u>	<u>.2245</u>
1 2 4 6	<u>.6347</u>	<u>.5798</u>	<u>.6318</u>	<u>.5685</u>	<u>.5458</u>	<u>.4729</u>	<u>.2291</u>
1 3 4 6	<u>.6296</u>	<u>.5741</u>	<u>.6262</u>	<u>.5601</u>	<u>.5382</u>	<u>.4641</u>	<u>.2323</u>
2 3 4 6	<u>.2760</u>	<u>.0910</u>	<u>.2755</u>	<u>.2576</u>	<u>.0211</u>	<u>.0359</u>	<u>.1862</u>
1 2 5 6	<u>.5994</u>	<u>.5415</u>	<u>.5952</u>	<u>.5360</u>	<u>.5368</u>	<u>.5124</u>	<u>.4487</u>
1 3 5 6	<u>.6019</u>	<u>.5348</u>	<u>.5954</u>	<u>.5276</u>	<u>.5288</u>	<u>.5044</u>	<u>.4543</u>
2 3 5 6	<u>.5994</u>	<u>.4073</u>	<u>.5803</u>	<u>.5166</u>	<u>.0196</u>	<u>.0605</u>	<u>.3897</u>
1 4 5 6	<u>.5074</u>	<u>.5073</u>	<u>.5074</u>	<u>.4981</u>	<u>.5073</u>	<u>.4974</u>	<u>.4389</u>
2 4 5 6	<u>.5554</u>	<u>.3847</u>	<u>.5310</u>	<u>.4558</u>	<u>.0095</u>	<u>.0221</u>	<u>.3088</u>
3 4 5 6	<u>.5548</u>	<u>.3879</u>	<u>.5282</u>	<u>.4595</u>	<u>.0212</u>	<u>.0597</u>	<u>.3503</u>
1 2 3	<u>.6747</u>	<u>.5700</u>	<u>.6729</u>	<u>.6143</u>	<u>.5015</u>	<u>.3967</u>	<u>.0029</u>
1 2 4	<u>.6175</u>	<u>.5320</u>	<u>.6143</u>	<u>.5468</u>	<u>.4686</u>	<u>.3697</u>	<u>.0341</u>
1 2 5	<u>.5794</u>	<u>.4623</u>	<u>.5731</u>	<u>.5012</u>	<u>.4565</u>	<u>.4368</u>	<u>.3888</u>
1 2 6	<u>.5934</u>	<u>.5332</u>	<u>.5919</u>	<u>.5331</u>	<u>.5054</u>	<u>.4390</u>	<u>.2209</u>
1 3 4	<u>.6153</u>	<u>.5301</u>	<u>.6115</u>	<u>.5412</u>	<u>.4631</u>	<u>.3620</u>	<u>.0340</u>
1 3 5	<u>.5895</u>	<u>.4614</u>	<u>.5808</u>	<u>.5019</u>	<u>.4506</u>	<u>.4287</u>	<u>.3916</u>
1 3 6	<u>.5908</u>	<u>.5290</u>	<u>.5886</u>	<u>.5265</u>	<u>.4985</u>	<u>.4305</u>	<u>.2232</u>
1 4 5	<u>.4173</u>	<u>.4038</u>	<u>.3996</u>	<u>.4114</u>	<u>.3953</u>	<u>.4038</u>	<u>.3868</u>
1 4 6	<u>.4910</u>	<u>.4789</u>	<u>.4862</u>	<u>.4294</u>	<u>.4625</u>	<u>.4082</u>	<u>.2252</u>
1 5 6	<u>.4786</u>	<u>.4668</u>	<u>.4700</u>	<u>.4784</u>	<u>.4502</u>	<u>.4667</u>	<u>.4387</u>
2 3 4	<u>.0355</u>	<u>.0164</u>	<u>.0295</u>	<u>.0345</u>	<u>.0150</u>	<u>.0117</u>	<u>.0235</u>
2 3 5	<u>.5817</u>	<u>.4031</u>	<u>.5568</u>	<u>.4824</u>	<u>.0099</u>	<u>.0460</u>	<u>.3505</u>
2 3 6	<u>.2757</u>	<u>.0883</u>	<u>.2750</u>	<u>.2557</u>	<u>.0178</u>	<u>.0347</u>	<u>.1839</u>
2 4 5	<u>.5492</u>	<u>.3844</u>	<u>.5214</u>	<u>.4405</u>	<u>.0071</u>	<u>.0189</u>	<u>.2933</u>
2 4 6	<u>.2436</u>	<u>.0800</u>	<u>.2428</u>	<u>.2215</u>	<u>.0010</u>	<u>.0105</u>	<u>.1381</u>
2 5 6	<u>.5357</u>	<u>.3529</u>	<u>.5149</u>	<u>.4440</u>	<u>.0082</u>	<u>.0205</u>	<u>.2978</u>
3 4 5	<u>.5451</u>	<u>.3862</u>	<u>.5141</u>	<u>.4379</u>	<u>.0107</u>	<u>.0470</u>	<u>.3243</u>
3 4 6	<u>.2263</u>	<u>.0810</u>	<u>.2253</u>	<u>.2108</u>	<u>.0195</u>	<u>.0359</u>	<u>.1644</u>
3 5 6	<u>.5399</u>	<u>.3632</u>	<u>.5164</u>	<u>.4519</u>	<u>.0156</u>	<u>.0597</u>	<u>.3462</u>
4 5 6	<u>.3952</u>	<u>.2958</u>	<u>.3429</u>	<u>.2479</u>	<u>.0095</u>	<u>.0085</u>	<u>.1481</u>
1 2	<u>.5647</u>	<u>.4519</u>	<u>.5628</u>	<u>.5010</u>	<u>.3876</u>	<u>.2945</u>	<u>.0021</u>
1 3	<u>.5677</u>	<u>.4574</u>	<u>.5654</u>	<u>.4999</u>	<u>.3875</u>	<u>.2908</u>	<u>.0016</u>
2 3	<u>.0178</u>	<u>.0108</u>	<u>.0178</u>	<u>.0152</u>	<u>.0094</u>	<u>.0069</u>	<u>.0001</u>
1 4	<u>.3681</u>	<u>.3304</u>	<u>.3579</u>	<u>.2820</u>	<u>.2806</u>	<u>.2084</u>	<u>.0340</u>
2 4	<u>.0351</u>	<u>.0006</u>	<u>.0295</u>	<u>.0345</u>	<u>.0006</u>	<u>.0006</u>	<u>.0176</u>
3 4	<u>.0282</u>	<u>.0110</u>	<u>.0221</u>	<u>.0238</u>	<u>.0105</u>	<u>.0085</u>	<u>.0220</u>
1 5	<u>.4169</u>	<u>.3761</u>	<u>.3847</u>	<u>.3915</u>	<u>.2625</u>	<u>.3348</u>	<u>.3846</u>
2 5	<u>.5336</u>	<u>.3519</u>	<u>.5103</u>	<u>.4342</u>	<u>.0044</u>	<u>.0182</u>	<u>.2877</u>
3 5	<u>.5340</u>	<u>.3631</u>	<u>.5066</u>	<u>.4346</u>	<u>.0002</u>	<u>.0460</u>	<u>.3234</u>
4 5	<u>.3808</u>	<u>.2776</u>	<u>.3324</u>	<u>.2424</u>	<u>.0061</u>	<u>.0084</u>	<u>.1458</u>
1 6	<u>.3777</u>	<u>.3772</u>	<u>.3767</u>	<u>.3469</u>	<u>.3750</u>	<u>.3444</u>	<u>.2183</u>
2 6	<u>.2418</u>	<u>.0701</u>	<u>.2413</u>	<u>.2213</u>	<u>.0008</u>	<u>.0102</u>	<u>.1377</u>
3 6	<u>.2254</u>	<u>.0766</u>	<u>.2246</u>	<u>.2108</u>	<u>.0149</u>	<u>.0346</u>	<u>.1642</u>
4 6	<u>.0908</u>	<u>.0465</u>	<u>.0826</u>	<u>.0631</u>	<u>.0000</u>	<u>.0036</u>	<u>.0391</u>
5 6	<u>.2364</u>	<u>.1798</u>	<u>.1874</u>	<u>.1162</u>	<u>.0079</u>	<u>.0015</u>	<u>.0592</u>

Note -- 5% significant statistics underlined. Critical value 0.3323.



The rejections and acceptances of the STP are coherent in the sense that if one hypothesis implies another, the former cannot be accepted if the latter is rejected. Thus, for instance, the hypothesis of equality of all six localities on all three variables cannot be accepted since, among others, it implies the hypothesis of equality of localities (1,2,3) on variable X, and the latter hypothesis is rejected, its statistic being 0.5015 ( $>0.3323$ , the critical value). Another instance is that since localities (2,3,4) are not found to differ significantly on variables X, Y, Z ( $\theta=0.0355 < 0.3323$ ), and since that implies that these localities do not differ on any pair XY, XZ or YZ of variables or any single variable X, Y or Z and it also implies that none of the pairs (2,3), (2,4), (3,4) so differs, coherence is maintained in that none of these implied hypotheses have a significant statistic. These and other instances of coherence are readily checked in Table 2.

However, a certain type of dissonance will be noted to occur among the decisions of Table 2. Thus, for example, localities (4,5) differ significantly on XZ, but neither on X separately nor on Z separately. Yet if there were no differences in the expectations of either of these variables, there could be no differences in both together. Another example is that whereas localities (1,2,3) differ significantly on Y neither (1,2) nor (1,3) nor (2,3) differ significantly on Y. In these dissonances the STP fails to provide full resolution of the rejection of the more general hypotheses into rejection of some of their components. Note, however, that if the family of hypotheses included all possible locality contrasts and all possible LCVs, the  $c_1$ -STP would not show any dissonances at all -- see property III in section 4, below.

The existence of coherence allows a brief but complete summarization of the STP's decisions by means of either of the following two classes of hypotheses.

Minimal rejected hypotheses are those which are rejected but do not imply any other rejected hypothesis. In other words, omitting one or more localities and/or one or more variables from any such rejected hypothesis would lead to an accepted hypothesis. Maximal accepted hypotheses, on the other hand, are those which are not rejected though any other hypothesis implying them is rejected. In other words, adding one or more localities and/or one or more variables to those covered by any such accepted hypothesis would lead to a rejected hypothesis. By virtue of coherence, any hypothesis is known to be rejected if, and only if, it implies a minimal rejected hypothesis, and accepted if, and only if, it is implied by a maximal accepted hypothesis. Hence, a listing of either of these classes of hypotheses, as in Table 3, suffices to identify all STP decisions.

TABLE 3. MINIMAL REJECTED AND MAXIMAL ACCEPTABLE COMBINATIONS OF LOCALITIES AND VARIABLES — MAXIMUM CHARACTERISTIC ROOT STP AT 5% LEVEL.

<u>Minimal Rejected</u>				<u>Maximal Accepted</u>			
<u>Localities</u>				<u>Localities</u>			
1 2		X	YZ	2 3 4 6		XYZ	
1 3		X	YZ		5 6	XYZ	
1 4			XZ	1 4		XY YZ	
1 5		Y Z			4 5 6	XY YZ	
1 6	X Y			2 3 4 5 6			X Y
2 5			XY XZ YZ	1 5			X
3 5			XY XZ YZ	1 2			Y
4 5			XZ	1 3			Y
1 2 3		Y		1 2 3 4 6			Z
1 2 4		Y		2 4 5 6			Z
1 3 4		Y		3 4 5			Z
1 4 5	X						
2 3 5			Z				
3 5 6			Z				

Considering, then, the summary of the STP decisions in Table 3 one notes that locality 1 differs from all others; localities 2, 3, 4 do not differ among themselves but differ from locality 5; locality 6 does not differ significantly from either group (2,3,4) or from 5. As far as the STP's decisions go, the correct groupings could be either 1, (2,3,4), (5,6) or 1, (2,3,4,6), 5. It is gratifying to note that the three localities where anteaters belong to the same subspecies chapadensis, namely 2, 3 and 4 have been correctly grouped together. The grouping separates out subspecies instabilis (locality 1) and subspecies chiriquensis, (locality 5), and is indefinite as to whether subspecies mexicana (locality 6), belongs with chapadensis or chiriquensis.

Consideration of the variables to which the differences between groups are due shows that at least two variables enter each one of these differences. No simple pattern emerges from Table 3. However, since the MANOVA STP allows exploration of LCVs as well as the original variables, it was decided to investigate the group differences on a total  $T=X+Y+Z$  and on contrasts  $C = (C_1, C_2) = (Y-X, Y-Z)$ . (Any choice of independent contrasts would have yielded the same results). Table 4 shows the  $\theta$  statistics for the two possible groupings on T and C as well as on TC, the latter being the course equivalent to XYZ. (The small difference between the XYZ and TC statistics is due to the omission of within group variability from the latter).

From Table 4 it appears that the locality groups (2,3,4) and (5,6) (or (2,3,4,6) and 5) differ on contrasts only and not on the total of all three variables. On the other hand, locality 1 differs from (5,6) (or 5) only on the total and from (2,3,4) (or (2,3,4,6)) on the total and probably on contrasts as well. In other words, subspecies chapadensis, chiriquensis and mexicana are not found to differ in overall size, but only in the relative sizes of the different measures. Subspecies instabilis differs from chiriquensis, and possibly also from

mexicana only on overall size but not in the relative sizes of the different measures.

TABLE 4. MAXIMUM CHARACTERISTIC ROOT STATISTICS FOR GROUPINGS OF LOCALITIES AND TOTAL AND CONTRASTS OF VARIABLES

LOCATIONS	VARIABLES		
	TC=XYZ	T=X+Y+Z	Contrasts C
1 (2,3,4) (5,6)	<u>.6930</u>	<u>.4943</u>	<u>.6008</u>
1 (2,3,4)	<u>.6910</u>	<u>.2787</u>	<u>.4942</u>
1 (5,6)	<u>.4748</u>	<u>.4717</u>	<u>.1702</u>
(2,3,4) (5,6)	<u>.5489</u>	<u>.1580</u>	<u>.5480</u>
1 (2,3,4,6) 5	<u>.6979</u>	<u>.4753</u>	<u>.5997</u>
1 (2,3,4,6)	<u>.6848</u>	<u>.3696</u>	<u>.4298</u>
1 5	<u>.4169</u>	<u>.3677</u>	<u>.2854</u>
(2,3,4,6) 5	<u>.5407</u>	<u>.1044</u>	<u>.5387</u>

To study the last comparison more closely, SCBs have been set on the differences of the expectations of each variable between locality group (5,6) and locality 1. The method of setting these bounds is due to Roy and Bose (1953) (see also Morrison (1967, section 5.4) or Seal (1965, pp. 94-97)) and is presented in detail in Table 5. Note, again, that the probability that all true hypotheses of the STP be accepted and all SCBs include the true parametric functions is 95%.

It appears from Table 5 that chiriquensis and mexicana have all (logarithmic) length variables larger than instabilis. Along with the finding, noted above, that these subspecies do not differ significantly on contrasts in the variables, this confirms Reeve's (1941) statement that "Instabilis appears to be a small edition of mexicana and chiriquensis". This last finding was not brought out in the canonical analysis of these data by Seal (1965, chapter 7) -- this suggests the present STP analysis may be a more sensitive tool than canonical analysis (see section 6, below).

TABLE 5. SIMULTANEOUS CONFIDENCE BOUNDS ON SOME DIFFERENCES IN EXPECTATIONS

Locality	Number of anteaters	Mean		
		X	Y	Z
(5,6)	9	2.0959	2.1083	1.6852
1	21	2.0540	2.0660	1.6210
Difference = d		0.0419	0.0423	0.0642
Error Squares = S <sup>2</sup>		(0.1168) <sup>2</sup>	(0.1137) <sup>2</sup>	(0.1901) <sup>2</sup>
$\zeta_{.95} S \sqrt{1/9+1/21} = \delta$		0.0328	0.0320	0.0534
Lower bound = d- $\delta$		+0.0082	+0.0103	+0.0108
Upper bound = d+ $\delta$		+0.0738	+0.0743	+0.1176

$$\zeta_{.95} = \sqrt{\theta_{.95}/(1-\theta_{.95})} = 0.7055$$

The MANOVA  $c_1$ -STP has given some resolution of the rejection of the overall hypothesis into significant detail. However, this resolution is incomplete in, for example, not indicating with which other localities one should group locality 6. Such incomplete resolution, or dissonances, will generally be found with STPs and unique conclusions about population differences will not be obtained. Even so, the extent of resolution that STPs do provide is often of considerable value, as in the present example.

MANOVA STPs may also be obtained by means of other statistics such as Hotelling-Lawley's trace  $T_o^2 = n_e \text{tr}[S_V^{-1} H_V^G]$ , Wilks's likelihood ratio  $U = |S_V|/|S_V+H_V^G|$  or Pillai's trace  $V = \text{tr}[(S_V+H_V^G)^{-1} H_V^G]$ . (see any text on multivariate analysis such as Anderson (1958), Morrison (1967), Rao (1952, 1965) or Seal (1965)). In each case a table of statistics corresponding to those in Table 2 would be computed and tested against the critical value from the distribution of the statistic for the overall hypothesis.

MANOVA STPs by means of the Hotelling-Lawley  $T_o^2$  and the Wilks U have been carried out on the anteatater data at level 5%. In the former case the common critical value for  $T_o^2$  is  $\underline{n}_e = 42$  times the value of 0.6745 obtained by interpolation for  $\underline{s}=3$ ,  $\underline{m}=1/2$  and  $\underline{n}=19$  in Pillai's (1960) tables. In the latter the critical value for  $-\ln U$  is  $24.9958 \times 1.0015 / 42.5 = 0.5890$ , the values in this expression being, respectively, the upper 5% point of the 15 d.f. chi-square distribution, Schatzoff's (1966a) exact correction term and Bartlett's factor  $n_e + (q_o - p_o - 1)/2$ .

TABLE 6. MINIMAL REJECTED COMBINATIONS OF LOCALITIES AND VARIABLES:  
TRACE AND LIKELIHOOD RATIO STPs AT 5% LEVEL

Location	<u>Hotelling-Lawley Trace</u> (Crit. Value 0.6745 for $T_o^2/n_e$ )	<u>Wilks's Likelihood Ratio</u> (Crit. Value 0.5890 for $-\ln U$ )
12	XY XZ YZ	XY XZ YZ
13	XY XZ YZ	XY XZ YZ
15	XYZ	
25	XZ YZ	XZ
35	XZ YZ	XZ
123	X	X
124	X	X
125	X Y	X
126	X Y	X
134	X	X
135	X Y	X
136	X Y	X
145	Y XZ	XY XZ YZ
146	X Y	X YZ
156	X Y Z	X Y
235	XY	YZ
356		XZ YZ
1234	Y	
1235		Y
1236		Y
1245		Y
1246		Y
1256		Y Z
1345		Y
1346		Y
1356		Y Z
3456		YZ

The results of these two STPs which are summarized in Table 6 do not differ greatly from those of the  $c_1$ -STP. However, the  $T_0^2$ -STP is less resolute than the  $c_1$ -STP in that every hypothesis rejected by the  $T_0^2$ -STP is also rejected by the  $c_1$ -STP, but the latter rejects some which are accepted by the former -- compare Tables 3 and 6. For example, localities 1,5 are found by the  $T_0^2$ -STP to differ on some variable(s) but not on any particular variable. The  $c_1$ -STP, on the other hand, specifically rejects the equality of these localities on variable Y as well as on variable Z. Finally, the U-STP does not reject the equality of these localities at all. This illustrates the fact that the U-STP is less resolute than the  $T_0^2$ -STP and, a fortiori, than the  $c_1$ -STP. Again, this may be checked in comparing the two columns of Table 6.

This order of resolution of the different MANOVA STPs holds generally. Property V of STPs (section 4, below) states that the  $c_1$ -STP is the most resolute and by similar reasoning<sup>(\*)</sup> the  $T_0^2$ -STP can be shown to be more resolute than the U-STP. One may therefore prefer the  $c_1$ -STP to any other MANOVA STP since, for any given probability of type I error, it provides most resolution into significant detail.

### 3. A LEMMA ON THE ROOTS OF MATRICES

A matrix result is presented in this section in general mathematical terms, to be used in the proofs of the properties of MANOVA STPs in section 5, below.

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(\*) The proof is similar to that of property V in section 5, below, using the inequality  $T_0^2/n_e = \sum c_v \geq \prod_v (1+c_v) - 1 = U^{-1} - 1$  which is strict unless all characteristic roots  $c_2, c_3, \dots$  of  $S^{-1}H$ , except  $c_1$ , are zero.

LEMMA: Let A and B be symmetric (p×p) matrices, non-negative definite and positive definite, respectively. Then for any (p×q) matrix D of rank q

$$c_{\nu}[B^{-1}A] \geq c_{\nu}[(D'BD)^{-1}D'AD]$$

$\nu=1,2,\dots,q$ , where  $c_{\nu}[M]$  denotes the  $\nu$ -th largest characteristic root of M.

PROOF: Let P denote the (px(ν-1)) matrix whose columns are the characteristic vectors of  $B^{-1}A$  corresponding to the ν-1 largest characteristic roots,  $\nu=1,2,\dots,p$  (for  $\nu=1$  P is a (p×1) null vector). It is known that, for  $\nu=1,2,\dots,p$ ,

$$c_{\nu}[B^{-1}A] = \sup_{\underline{X}'BP=\underline{0}} \frac{\underline{X}'AX}{\underline{X}'BX}$$

(as follows by an extension of lf.2.4 of Rao (1965)) and that, for  $i=1,2,\dots,q$ ,

$$c_{\nu}[(D'BD)^{-1}D'AD] = \inf_Q \sup_{\underline{Y}'Q=\underline{0}} \frac{\underline{Y}'D'ADY}{\underline{Y}'D'BDY}$$

where Q denotes any (qx(ν-1)) matrix (for  $\nu=1$  Q is taken as a null vector) (See Rao (1965, 22.2 of p. 59)).

Now, for  $\nu=1,2,\dots,q$ ,

$$\begin{aligned} c_{\nu}[B^{-1}A] &= \sup_{\underline{X}'BP=\underline{0}} \frac{\underline{X}'AX}{\underline{X}'BX} \geq \sup_{\substack{\underline{X}'BP=\underline{0} \\ \underline{X}'=\underline{Y}'D'}} \frac{\underline{X}'AX}{\underline{X}'BX} \\ &= \sup_{\underline{Y}'D'BP=\underline{0}} \frac{\underline{Y}'D'ADY}{\underline{Y}'D'BDY} \\ &\geq \inf_Q \sup_{\underline{Y}'Q=\underline{0}} \frac{\underline{Y}'D'ADY}{\underline{Y}'D'BDY} \\ &= c_{\nu}[(D'BD)^{-1}D'AD], \end{aligned}$$

as was to be proved.



A function  $\Psi[M]$  of a matrix argument  $M$  will be said to be an Increasing Root Function if it depends on that matrix through its characteristic roots so that

(1)  $\Psi[M]$  is continuous monotone increasing in the largest root of  $M$  and at least non-decreasing in all other roots of  $M$ , and if

(2)  $\Psi[M_1] = \Psi[M_2]$  whenever  $M_1$  and  $M_2$  have all non-zero roots equal, irrespective of how many additional zero roots  $M_1$  and/or  $M_2$  may have.

To avoid functions which are one-to-one transformations of one another one may impose the further requirement that

(3)  $\Psi[M] = c_1[M]$  whenever  $M$  has a single non-zero root.

With this definition the following corollary is obtained from the above Lemma.

COROLLARY: Let  $\Psi$  be an increasing root function,  $A$  and  $B$  symmetric ( $p \times p$ ) matrices, non-negative and positive definite, respectively, and  $D$  any ( $p \times q$ ) matrix of rank  $q$ , then

$$\underline{\Psi[B^{-1}A] \geq \Psi[(D'BD)^{-1}D'AD]}.$$

Some examples of increasing root functions are:  $c_1[M]$ ;  $\text{tr}[M] = \sum_{\nu} c_{\nu}[M]$ ;  $V[M]$ , where  $V[M]/(1+V[M]) = \text{tr}[(I+M^{-1})^{-1}] = \sum_{\nu} \{c_{\nu}[M]/(1+c_{\nu}[M])\}$ ;  $1/U[M]-1$ , where  $U[M] = 1/|I+M| = \prod_{\nu} (1+c_{\nu}[M])^{-1}$ . The relation of these functions to the MANOVA statistics mentioned in section 2 is evident on introducing  $S^{-1}H$  for  $M$ . Note that the increasing root function definition cannot be satisfied by any other single root, except the first.

#### 4. SIMULTANEOUS TEST PROCEDURES FOR THE GENERAL MULTIVARIATE LINEAR HYPOTHESIS

STPs will be formulated for the general multivariate normal linear hypothesis and certain properties stated in this section. Proofs will follow in

section 5. This general treatment includes the special case of one-way MANOVA illustrated in section 2, above, using any increasing root function of the inverse error times hypothesis matrix, including the  $c_1$ ,  $T_0^2$  and U statistics illustrated.

Consider a matrix Y ( $n \times p$ ) whose rows are n independent p-variate observations with common variance  $\Sigma(p \times p)$  and expectations such that  $EY = X'\theta$ , where X ( $m \times n$ ) is a known design matrix and  $\theta(m \times p)$  is a matrix of parameters. Under this model any linear null hypothesis can be written in the form

$$\omega_i: \phi_i = C_i'\theta D_i = 0 \quad (4.1)$$

for some ( $m \times r_i$ ) matrix  $C_i$  of rank  $r_i$  and ( $p \times p_i$ ) matrix  $D_i$  of rank  $p_i$ . Least squares estimates of the parametric functions  $\phi_i$  may be written

$$\hat{\phi}_i = C_i'\hat{\theta}D_i, \quad (4.2)$$

where

$$\hat{\theta} = (XX')*XY, \quad (4.3)$$

the asterisk indicating a generalized inverse (See Rao (1965, p. 24)). Denoting

$$W_i = C_i'(XX')*C_i, \quad (4.4)$$

the  $\omega_i$  hypothesis sum of squares and products matrix is

$$H_i = \hat{\phi}_i'W_i^{-1}\hat{\phi}_i \quad (4.5)$$

and the corresponding matrix of sums of squares and products for error is

$$S_i = D_i'(Y'Y - Y'X'\hat{\theta})D_i, \quad (4.6)$$

both these matrices being ( $p_i \times p_i$ ).

For the model at hand consider a family of hypotheses  $\Omega = \{\omega_i | i \in I\}$ , I being an index set (not necessarily denumerable), such that

$$\omega_0: \phi_0 = C_0' \theta D_0 = 0 \quad (4.7)$$

belongs to  $\Omega$ , and all other  $\omega_i$  of  $\Omega$  are implied by  $\omega_0$ . In other words, for every  $i \in I$  there exist matrices  $B_i (r_0 \times r_i)$  and  $E_i (p_0 \times p_i)$  such that

$$C_i = C_0 B_i \quad \text{and} \quad D_i = D_0 E_i. \quad (4.8)$$

The family  $\Omega$  may consist of all linear hypotheses implied by  $\omega_0$  or only of part of these hypotheses.

Of particular importance are one-root hypotheses  $\omega_i$ , such that  $s_i = \min(r_i, p_i) = 1$  and  $S_i^{-1} H_i$  has only one non-zero root. On such hypotheses  $\Psi[S_i^{-1} H_i] = c_1 [S_i^{-1} H_i]$  for any increasing root function  $\Psi$ , by virtue of requirement (3) of the definition of these functions. The condition  $s_i = 1$  means  $\omega_i$  is either on a single LPF,  $r_i = 1$ , or univariate,  $p_i = 1$ , or both. This last type of univariate single LPF hypothesis  $\omega_j$  with  $s_j = r_j = p_j = 1$  is referred to as minimal in that such hypotheses imply no other hypotheses of  $\Omega$ . Their importance is in their being the ultimate detail tested in the MANOVA, e.g., pairwise population comparisons on single variables in a one-way set-up.

An  $\alpha$ -level  $\Psi$ -STP for family  $\Omega$ , where  $\Psi$  is some increasing root function, is defined as the family of tests rejecting any  $\omega_i \in \Omega$  if and only if

$$\Psi[S_i^{-1} H_i] > \Psi_{1-\alpha}, \quad (4.9)$$

where  $\Psi_{1-\alpha}$  is a constant satisfying

$$P_{\omega_0} (\Psi[S_0^{-1} H_0] > \Psi_{1-\alpha}) = \alpha. \quad (4.10)$$

The adjective simultaneous in STP indicates that all hypotheses  $\omega_i \in \Omega$  are tested simultaneously against the same critical value  $\Psi_{1-\alpha}$  and without the need of referring from the test of one hypothesis to that of another.

In addition to the STP it is possible, without increasing the probability of type I error, to set SCBs for all scalar parametric functions  $\phi_j = \underline{c}'_j \underline{\theta}_j$  (these correspond to minimal hypotheses). The bounds are

$$\hat{\phi}_j - \sqrt{\Psi_{1-\alpha} W_j S_j} \leq \phi_j \leq \hat{\phi}_j + \sqrt{\Psi_{1-\alpha} W_j S_j} \quad , \quad (4.11)$$

$W_j$  and  $S_j$  being defined as in (4.4) and (4.6), respectively. These bounds for  $\Psi=c_1$  are due to Roy and Bose (1953) and have been extended by Mudholkar (1966) to a class of statistics including  $\Psi=T_0^2$ . The extension to all increasing root functions follows from the correspondences established by Gabriel (1967b, section 5) between STPs and SCBs in general. Indeed, from the form of the bounds (4.11) it appears that the choice of function  $\Psi$  affects only the width of the SCBs.

Each choice of an increasing root function  $\Psi$  will determine a particular STP and corresponding SCBs. For  $\Psi[S^{-1}H] = c_1[S^{-1}H]$ , Roy's statistic, percentage points of  $\theta=c_1/(1+c_1)$  may be obtained from Heck's (1960) charts or Pillai's (1960, 1964, 1965a, 1965b) tables, or Foster's (1957a, 1957b, 1957c) tables. For  $\Psi[S^{-1}H] = T_0^2[S^{-1}H]/n_e = \text{tr}[S^{-1}H]$ , the Hotelling-Lawley trace, and  $\Psi[S^{-1}H] = V[S^{-1}H] = \text{tr}[(S+H)^{-1}H]/\{1-\text{tr}[(S+H)^{-1}H]\}$ , the Pillai trace, small sample percentage points have been tabulated by Pillai (1960) whereas for large numbers of degrees of freedom  $n_e$  each of  $T_0^2[S_0^{-1}H_0]/(1-(r_0-p_0-1)/2n_e)$  and  $n_e V[S_0^{-1}H_0]/(1+r_0/n_e)$  has approximately a chi-square distribution with  $r_0 p_0$  d.f.. Finally, if  $\Psi[S^{-1}H] = U^{-1}[S^{-1}H]-1 = |S+H|/|S|-1$ , the Wilks likelihood ratio criterion, then  $-(n_e+(r_0-p_0-1)/)\ln U[S^{-1}H]$  is approximately a chi-square variable with  $r_0 p_0$  d.f. when  $n_e$  is large. For small  $n_e$  exact percentage points can be obtained by means of correction factors due to Schatzoff (1966a).

A number of properties of MANOVA  $\Psi$ -STPs, where  $\Psi$  is an increasing root function of  $S^{-1}H$ , will now be stated and discussed, their proofs following in section 5, below.

- I. If  $\omega_i, \omega_j \in \Omega$  and  $\omega_i$  implies  $\omega_j$ , then the STP rejects  $\omega_i$  whenever it rejects  $\omega_j$ . Conversely, if  $\omega_i$  is accepted, so is  $\omega_j$  which is implied by  $\omega_i$ . This is referred to as coherence.
- II. The probability of making one or more type I errors in all the tests of an  $\alpha$ -level STP is exactly  $\alpha$  if  $\omega_0$  holds. Otherwise it is no greater than  $\alpha$ .
- III. If a hypothesis  $\omega_i$  is rejected, this does not always entail rejection of some other hypothesis  $\omega_j$  implied by  $\omega_i$  (provided such implied hypotheses are among those tested). In allowing such dissonances, the STP is not, in general, consonant. The only consonant MANOVA STP for a family including  $\omega_0$  and all minimal hypotheses implied by  $\omega_0$ , is the  $c_1$ -STP.
- IV. Among all STPs giving identical decisions on all one-root hypotheses (i.e., having equal critical values, since on such hypotheses all the statistics are equal), the  $c_1$ -STP is most parsimonious in that it has the lowest level. In other words, it provides the same single root tests at least probability of any type I error.
- V. Among all STPs of level  $\alpha$  the  $c_1$ -STP is most resolvent in the sense that in testing every one-root hypothesis it will reject every such hypothesis rejected by any other STP, and possibly some more. Thus, the  $c_1$ -STP provides more resolution into significant detail than any other STP.

Corresponding properties also hold for the corresponding SCBs:

- VI. The SCBs corresponding to an  $\alpha$ -level STP have joint confidence coefficient  $1-\alpha$ . Moreover, with probability  $1-\alpha$  simultaneously all SCB statements are true and no type I error is made by the STP.
- VII. Among all SCBs of joint confidence  $1-\alpha$ , the  $c_1$ -SCBs give the narrowest confidence intervals for each parametric function, the  $c_1$  intervals being contained

in the corresponding intervals of any other SCBs.

Power considerations must enter in the choice of which MANOVA STP to use. However, it is not only the power of the overall test that is relevant, but even more so, the power of the tests of component hypotheses and particularly of minimal hypotheses. It would be difficult to formalize these considerations and assign appropriate weights to the different hypotheses tested. Furthermore, it would not generally be possible to evaluate these powers, and average weighted powers, since not enough is known about the requisite non-central distributions. One is therefore led back to intuitive considerations in the light of what is known about the different STPs and their statistics.

TABLE 7: POWERS OF VARIOUS 5% LEVEL MANOVA STPs  
FOR HYPOTHESES ON SINGLE LPFs OF SINGLE LCVs.  
ONE WAY 6 GROUP 3 VARIABLE MANOVA WITH 42 ERROR DF.

STP based on	Non-centrality parameter $\delta = (\mu - \mu_0) / \sigma$				
	1.6	3.2	4.8	6.4	8.0
Roy's $c_1$	0.004	0.116	0.592	0.951	0.999
Hotelling-Lawley's trace	0.001	0.036	0.339	0.830	0.990
Wilks's likelihood ratio	0.0002	0.015	0.209	0.704	0.969
Pillai's trace	0.0000	0.002	0.060	0.393	0.841

Computations based on the following critical values:

$$\theta_{.95} / (1 - \theta_{.95}) = 0.4977$$

$$\sqrt{42c_{.95}} = 4.572$$

$$T_{o,.95}^2 / n_e = 0.6745$$

$$\sqrt{T_{o,.95}^2} = 5.323$$

$$U_{.95}^{-1} - 1 = e^{0.5890} - 1 = 0.8022 \quad \sqrt{42(U_{.95}^{-1} - 1)} = 5.805$$

$$V_{.95} = 1.0345$$

$$\sqrt{42V_{.95}} = 6.592.$$

The probabilities were read off from the New Tables of the Non-central t distribution due to Locks, Alexander and Byars (1963) using the 40 d.f. distribution as an approximation to the requisite 42 d.f. t distribution.

In choosing, for instance, between the  $c_1$ -STP and the U-STP one would have to weigh the former's greater resolution (property V) against the knowledge that the U test is, under most circumstances, more powerful than the  $c_1$  test<sup>(\*)</sup>. These differences may balance out in a way that would allow the power on minimal hypotheses to be greater in the  $c_1$ -STP than in the U-STP.

To illustrate such considerations Table 7 is presented, showing the powers of the MANOVA STPs of the example in section 2 on a test of a minimal hypothesis. Since on a minimal hypothesis  $\Psi[S_j^{-1}H_j] = c_1[S_j^{-1}H_j] = H_j/S_j$  and  $n_e H_j/S_j$  is an F statistic with 1 and  $n_e$  d.f., powers could be read off from tables of the non-central t distribution using  $\sqrt{n_e \Psi_{1-\alpha}}$  as critical value, as explained underneath Table 7.

The table indeed shows the power of the  $c_1$ -STP for minimal hypotheses to be considerably greater than that of the other STPs, including the U-STP. So, if one is willing to forego some overall power in order to gain greater power for the important minimal hypotheses, one is led to prefer the  $c_1$ -STP.

##### 5. PROOFS OF THE PROPERTIES OF MANOVA STPs

A general discussion of STPs and SCBs has been given in an earlier paper by Gabriel (1967b). The MANOVA STPs are special cases of those discussed there and their properties are properties established for STPs in general in certain theorems of that paper. It will therefore suffice to show that the conditions for these theorems hold in the MANOVA set-up. The theorems will then be quoted to show how the properties stated in section 4 follow from them.

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(\*) In separate Monte Carlo studies of the power of MANOVA tests, both Schatzoff (1966) and Genizi (1967) have found the likelihood ratio test to be rather more powerful than the maximum root test, except when all parametric characteristic roots beyond the first were close to zero.

For any increasing root function  $\Psi$ , the statistic  $\Psi[S_i^{-1}H_i]$  can be used to test hypothesis  $\omega_i$  of (4.1),  $H_i$  and  $S_i$  being defined as in (4.5) and (4.6), respectively.  $S_i$  has a central Wishart distribution and is independent of  $H_i$  which has a Wishart distribution with non-centrality parameter  $\phi_i'W_i^{-1}\phi_i$ ,  $W_i$  defined as in (4.4). Thus, the distribution of  $S_i$  and  $H_i$ , and therefore also of  $\Psi[S_i^{-1}H_i]$ , is completely specified under  $\omega_i$ . This establishes  $\{(\omega_i, \Psi[S_i^{-1}H_i]) | i \in I\}$  as a testing family as defined by Gabriel (1967b, section 2).

For any subfamily  $\tilde{\Omega} = \{\omega_i | i \in \tilde{I}\}$  of hypotheses of  $\Omega$ , i.e.,  $\tilde{I} \subseteq I$ , the intersection hypothesis  $\omega_o = \bigcap_{i \in \tilde{I}} \omega_i: \tilde{\phi}_o = 0$  is obviously also a linear null hypothesis in  $\theta$ . The joint distribution of all  $(S_i, H_i)$ ,  $i \in \tilde{I}$ , depends only on the common variance and on parameters affecting the values of  $\phi_i$ ,  $i \in \tilde{I}$ . Thus, under  $\tilde{\omega}_o: \tilde{\phi}_o = 0$  one would have  $\phi_i = 0$  for all  $i \in \tilde{I}$  and this would completely specify the above joint distribution. This establishes  $\{(\omega_i, \Psi[S_i^{-1}H_i]) | i \in \tilde{I}\}$  as a joint testing family in the terms of Gabriel (1967b).

The following theorem establishes that this testing family is monotone as defined in Gabriel (1967b).

THEOREM A: Let  $\omega_i$  and  $\omega_j$  be linear hypotheses under a multivariate normal linear model, and let  $S_i, H_i$  and  $S_j, H_j$  be the corresponding error and hypothesis matrices, respectively. Let  $\Psi$  be any increasing root function, then

$$\underline{\Psi[S_i^{-1}H_i] \geq \Psi[S_j^{-1}H_j]}$$

if  $\omega_i$  implies  $\omega_j$ .

Proof: Writing  $\omega_i: C_i' \theta D_i = 0$  and  $\omega_j: C_j' \theta D_j = 0$  as in (4.1) the implication relation means that there exist non-degenerate  $B$  and  $E$  such that  $C_j' = B' C_i'$  and  $D_j = D_i E$ .

Using (4.6) one obtains

$$S_j = E' S_i E,$$



and from (4.2) and (4.4)

$$\hat{\phi}_j = B' \hat{\phi}_i E$$

and

$$W_j = B' W_i B,$$

so that, by (4.5),

$$H_j = E' \hat{\phi}_i' B (B' W_i B)^{-1} B' \hat{\phi}_i E .$$

The Lemma of section 3 may now be applied to the following matrices. For  $\nu=1,2,\dots,p_j$  (the order of  $S_j$  and  $H_j$ )

$$\begin{aligned} c_\nu [S_i^{-1} H_i] &= c_\nu [S_i^{-1} \hat{\phi}_i' W_i^{-1} \hat{\phi}_i] \\ &\geq c_\nu [(E' S_i E)^{-1} E' \hat{\phi}_i' W_i^{-1} \hat{\phi}_i E] \\ &= c_\nu [S_j^{-1} E' \hat{\phi}_i' W_i^{-1} \hat{\phi}_i E]. \end{aligned}$$

Let  $p_j^+$  be the number of positive roots of  $S_j^{-1} E' \hat{\phi}_i' W_i^{-1} \hat{\phi}_i E$ , then for  $\nu=1,2,\dots,p_j^+$

$$\begin{aligned} c_\nu [S_j^{-1} E' \hat{\phi}_i' W_i^{-1} \hat{\phi}_i E] &= c_\nu [W_i^{-1} \hat{\phi}_i E S_j^{-1} E' \hat{\phi}_i'] \\ &\geq c_\nu [(B' W_i B)^{-1} B' \hat{\phi}_i E S_j^{-1} E' \hat{\phi}_i' B] \\ &= c_\nu [W_j^{-1} \hat{\phi}_j S_j^{-1} \hat{\phi}_j'] \\ &= c_\nu [S_j^{-1} H_j]. \end{aligned}$$

For  $\nu=p_j^++1, p_j^++2, \dots, p_j$  the same inequality obviously holds since both sides are zero.

Hence, for  $\nu=1,2,\dots,p_j$

$$c_\nu [S_i^{-1} H_i] \geq c_\nu [S_j^{-1} H_j].$$

Finally,  $\Psi[S_i^{-1} H_i]$  depends on  $p_j$  roots, each of which has been shown to be

no less than the corresponding root on which  $\Psi[S_j^{-1}H_j]$  depends, as well as on  $p_i - p_j$  additional non-negative roots. Since  $\Psi$  is non-decreasing in all these arguments the inequality of the theorem follows.

Next, consider  $\Omega = \{\omega_i | i \in I\}$  such that it contains  $\omega_0$  and all the minimal hypotheses implied by  $\omega_0$ . Whether  $\Omega$  contains any other hypotheses or not, the testing family  $\{(\omega_i, \Psi[S_i^{-1}H_i]) | i \in I\}$  is strictly monotone, as defined by Gabriel (1967b, section 2) if, and only if,  $\Psi = c_1$ . This follows from the next theorem.

**THEOREM B:** If  $\omega_i$  is a linear hypothesis under the normal multivariate linear model and  $\{\omega_j | j \in I_i\}$  are all the single LPF single LCV linear hypotheses implied by  $\omega_i$ , and if  $\Psi$  is an increasing root function, then

$$\underline{\Psi[S_i^{-1}H_i] = \max_{j \in I_i} \Psi[S_j^{-1}H_j] \quad \text{a.e.},}$$

if and only if  $\Psi = c_1$ .

**Proof:** Any minimal (single LPF single LCV) hypothesis  $\omega_j: \phi_j = 0$  is implied by  $\omega_i: \phi_i = 0$  if, and only if, there exist vectors  $\underline{b}_j$  and  $\underline{e}_j$  such that  $\phi_j = \underline{b}'_j \phi_i \underline{e}_j$ . In other words,  $j \in I_i$  whenever  $\underline{b}_j$  and  $\underline{e}_j$  are any non-null vectors of suitable orders.

As in the proof of Theorem A, it follows that  $S_j = \underline{e}'_j S_i \underline{e}_j$  and  $H_j = (\underline{b}'_j \hat{\phi}_i \underline{e}_j)^2 / \underline{b}'_j W_i \underline{b}_j$ , so that

$$c_1[S_j^{-1}H_j] = H_j/S_j = (\underline{b}'_j \hat{\phi}_i \underline{e}_j)^2 / \underline{b}'_j W_i \underline{b}_j \underline{e}'_j S_i \underline{e}_j.$$

Next,

$$\begin{aligned} \max_{j \in I_i} c_1[S_j^{-1}H_j] &= \max_{\underline{b}_j} \max_{\underline{e}_j} (\underline{b}'_j \hat{\phi}_i \underline{e}_j)^2 / \underline{b}'_j W_i \underline{b}_j \underline{e}'_j S_i \underline{e}_j \\ &= \max_{\underline{b}_j} \underline{b}'_j \hat{\phi}_i S_i^{-1} \hat{\phi}_i' \underline{b}_j / \underline{b}'_j W_i \underline{b}_j \end{aligned}$$

by virtue of the Cauchy-Schwarz inequality. (As in Rao (1965) 1f.1.1). Finally,

applying A.2.2 of Roy (1957),

$$\begin{aligned} \max_{\underline{b}_j} \underline{b}'_j \hat{\phi}_i S_i^{-1} \hat{\phi}'_i \underline{b}_j / \underline{b}'_j W_i \underline{b}_j &= c_1[\hat{\phi}_i S_i^{-1} \hat{\phi}'_i W_i^{-1}] \\ &= c_1[S_i^{-1} \hat{\phi}'_i W_i^{-1} \hat{\phi}_i] \end{aligned}$$

by virtue of A.1.18 of Roy (1957). Thus

$$\max_{j \in I_i} c_1[S_j^{-1} H_j] = c_1[S_i^{-1} H_i]$$

proving the if statement in the theorem.

Next, write  $\Psi_1, \Psi_2, \dots$  for the functions such that

$$\Psi[M] = \Psi_p(c_1[M], c_2[M], \dots, c_p[M])$$

if M has p non-zero roots. In particular, if M has only one root,  $\Psi[M] = \Psi_1(c_1[M]) = c_1[M]$ , by definition. For any minimal  $\omega_j$

$$\Psi[S_j^{-1} H_j] = \Psi_1(c_1[S_j^{-1} H_j]) = c_1[S_j^{-1} H_j].$$

For any  $\omega_i$ , in view of the definition of increasing root functions,

$$\begin{aligned} \Psi[S_i^{-1} H_i] &= \Psi_{p_i}(c_1[S_i^{-1} H_i], c_2[S_i^{-1} H_i], \dots, c_{p_i}[S_i^{-1} H_i]) \\ &\geq \Psi_{p_i}(c_1[S_i^{-1} H_i], 0, \dots, 0) \\ &= \Psi_1(c_1[S_i^{-1} H_i]) \\ &= c_1[S_i^{-1} H_i] \end{aligned}$$

and equality holds a.e. only if  $\Psi = c_1$ .

Thus,

$$\Psi[S_i^{-1} H_i] \geq \max_{j \in I_i} \Psi[S_j^{-1} H_j],$$

with equality holding a.e. only if  $\Psi=c_1$ .

In view of Gabriel's (1957b) LEMMA B: A testing family is strictly monotone if and only if its statistics are related by Roy's Union-Intersection principle, the above theorem confirms that the testing family with maximum root statistics is so related.

Coherence, property I, of MANOVA  $\Psi$ -STPs follows as Theorem A establishes monotonicity of the testing family  $\{(\omega_i, \Psi[S_i^{-1}H_i]) | i \in I\}$  and Gabriel (1967b) states THEOREM 1: STPs are coherent if and only if they are based on monotone testing families. The probability statement, property II, follows since  $\{(\omega_i, \Psi[S_i^{-1}H_i]) | i \in I\}$  was shown to be a joint testing family and Gabriel (1967b) states THEOREM 2: The probability that a coherent STP of level  $\alpha$  for testing family  $\{\Omega, Z\}$  rejects at least one true hypothesis  $\omega_i$  of  $\Omega$  is  $\alpha$  if  $\omega_0 = \bigcap_I \omega_i$  is true; it is at most  $\alpha$  irrespective of the truth of  $\omega_0$ , provided  $\{\Omega, Z\}$  is either closed or joint. The probability of rejecting any particular true  $\omega_i$  of  $\Omega$  is no more than the above probability.

Property III, that the  $c_1$ -STP is the only consonant STP for families including all minimal hypotheses implied by  $\omega_0$ , follows from Theorem B as this shows the  $c_1$  statistic to provide the only strictly monotone testing family, in view of Gabriel's (1967b) THEOREM 3: STPs are coherent and consonant if and only if they are based on strictly monotone testing families.

Next, the greater parsimony and resolution of the  $c_1$ -STP — properties IV and V — follow from Gabriel's (1967b) COROLLARY 3(I): Let  $\{\Omega, Z\}$  and  $\{\Omega^*, Z^*\}$  be monotone testing families such that  $\Omega \subseteq \Omega^*$ ,  $\Omega_{\min} \subseteq \Omega^*_{\min}$  (these being the subfamilies of minimal hypotheses) and  $\omega_0 = \omega_0^*$ , and let there exist a function  $g$  which is

- (1) continuous monotone increasing, and such that
- (2)  $Z_j^* = g(Z_j)$  a.e. for all minimal  $\omega_j$ .

Then  $\{\Omega, Z\}$  provides more parsimonious and more resolvent STPs than  $\{\Omega^*, Z^*\}$  if

- (3) for  $\omega_0$   $Z_0^* \neq g(Z_0)$  a.e.,  
and  $\{\Omega, Z\}$  is UI related.

In comparing a  $c_1$ -STP to any other increasing root function  $\Psi$ -STP for the same MANOVA testing families the containment relations of the Corollary obviously hold. Letting  $Z_i = c_1[S_i^{-1}H_i]$  and  $Z_i^* = \Psi[S_i^{-1}H_i]$ , (1) and (2) hold by using identity function  $g$  since  $\Psi[S_j^{-1}H_j] = c_1[S_j^{-1}H_j]$  for minimal hypotheses. Condition (3) is essentially equivalent to the requirement that  $\Psi \neq c_1$ . Finally, it was noted that the maximum root testing family is UI related (See also Gabriel (1967a)) so that Corollary 3(I) establishes the greater parsimony and resolution of  $c_1$ -STPs.

Finally, the statement on  $1-\alpha$  joint confidence of the corresponding SCBs -- property VI -- and the statement that  $c_1$ -SCBs are contained within the corresponding SCBs for any other increasing root function, -- property VII -- follow from the analogous results for simultaneous confidence statements. (Theorem 2' and Corollary 3'(I), respectively, in Gabriel (1967b)).

## 6. COMPARISON WITH OTHER TECHNIQUES

A simple graphical method of studying the detail of a MANOVA is to plot the projection of the mean vectors on the plane of the first two discriminants (or canonical variables as Seal (1965) calls them), adding circular confidence sets for the expectation vectors. Examples are given by Rao (1952, p. 370 and p. 374) and by Seal (1965, p. 136). This plot usually shows the pattern of expectations pretty clearly but, unlike an STP, it does not provide exact significance tests. Moreover, the graphical representation is in terms of two LCVs chosen not for their physical meaning but because they maximize certain statistics. The pattern graphed in those co-ordinates cannot always be interpreted meaningfully in terms of the actual variables used. Here STPs have the advantage of providing decisions on the original variables or LCVs chosen by the experimenter for their relevance. A comparison of the two

types of analysis on the same data in section 2, above, illustrates the more detailed information given by the STP.

Simultaneous confidence statements on positive definite functions of non-centrality parameters have been used by Roy and his associates to infer on the details of a significant MANOVA. For hypothesis  $\omega_i: \phi_i=0$ , bounds were set on the maximum root of  $\phi_i'W_i^{-1}\phi_i$ , a non-centrality parameter of the distribution of  $H_i$ . Roy and Gnanadesikan (1957) proposed, for each  $i \in I$ , the bounds

$$c_1^{\frac{1}{2}}[H_i] - \zeta_{1-\alpha}^{\frac{1}{2}} c_1^{\frac{1}{2}}[S_i] \leq c_1^{\frac{1}{2}}[\phi_i'W_i^{-1}\phi_i] \leq c_1^{\frac{1}{2}}[H_i] + \zeta_{1-\alpha}^{\frac{1}{2}} c_1^{\frac{1}{2}}[S_i],$$

where  $\zeta_{1-\alpha} = \theta_{1-\alpha}/(1-\theta_{1-\alpha})$ , and Bhapkar (1965) improved the lower bound and proposed

$$c_{\min}^{\frac{1}{2}}[S_i] \{c_1^{\frac{1}{2}}[S_i^{-1}H_i] - \zeta_{1-\alpha}^{\frac{1}{2}}\} \leq c_1^{\frac{1}{2}}[\phi_i'W_i^{-1}\phi_i] \leq c_1^{\frac{1}{2}}[H_i] + \zeta_{1-\alpha}^{\frac{1}{2}} c_1^{\frac{1}{2}}[S_i].$$

These SCBs are arrived at by means of inequalities so their joint confidence coefficient is not known exactly, except that it exceeds  $1-\alpha$ .

For scalar  $\phi_j$ s, i.e., minimal  $\omega_j$ s, either of these bounds reduces to the Roy-Bose bounds (4.11). For vector or matrix  $\phi_i$ s, these are bounds on a function  $c_1^{\frac{1}{2}}[\phi_i'W_i^{-1}\phi_i]$  which is unlikely to have a physically meaningful interpretation in many practical applications. (No useful example has been published to date). The utility of this function is mainly, if not entirely, in being positive definite in  $\phi_i$ , that is, being zero if and only if  $\phi_i=0$ , i.e., if  $\omega_i$  is true. The main use of Roy-Gnanadesikan and Bhapkar bounds would therefore be in determining whether or not hypotheses  $\omega_i$ ,  $i \in I$ , are tenable, that is, essentially in the same role as STPs.

The Bhapkar SCBs contain zero for  $\phi_i$  if  $c_1[S_i^{-1}H_i] \leq \zeta_{1-\alpha}$ , so that their decisions on null hypotheses coincide exactly with those of the  $c_1$ -STP of level  $\alpha$ .

The Roy-Gnanadesikan bounds for  $\phi_i$ , on the other hand, contain zero if  $c_1[H_i]/c_1[S_i] \leq \zeta_{1-\alpha}$ , but in view of Roy's (1954) theorem  $c_1[H_i]/c_1[S_i] < c_1[S_i^{-1}H_i]$  a.e., so that these bounds contain the Bhapkar bounds. Hence they will never allow rejection of any  $\omega_i$  not also rejected by Bhapkar's bounds, that is, by the  $c_1$ -STP, but will sometimes accept when the latter reject. As these SCBs are wider, and

the corresponding tests less powerful, than those of the  $c_1$ -STP, there is no need to consider them further.

Coming back to the Bhapkar SCBs and supposing some use can be found for the actual bounds, a further point needs to be made. (This also applies, mutatis mutandis, to Roy-Gnanadesikan SCBs). Let  $\phi_i$  and  $\phi_j$  be parametric functions such that for some B and E,  $\phi_j = B' \phi_i E$ . It then follows from the Lemma of section 3 and A1.22.5 of Roy (1957) that

$$c_1[\phi_i' W_i^{-1} \phi_i] \geq c_1[\phi_j' W_j^{-1} \phi_j]$$

provided E is chosen so that  $c_1(EE')=1$ . However, the Bhapkar lower bound for  $c_1^{1/2}[\phi_i' W_i^{-1} \phi_i]$  may be less than that for  $c_1^{1/2}[\phi_j' W_j^{-1} \phi_j]$ , leading to incoherent results.

(For a discussion of coherence of confidence statements see Gabriel (1967b, section 5)). As a simple illustration let  $\hat{\phi}_i = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $W_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $S_i = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ ;  $E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  so that  $\hat{\phi}_j = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ ,  $W_j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and  $S_j = (4)$ ; finally,  $H_i = \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix}$  and  $H_j = (9)$ . The Bhapkar bounds then are

$$2 - \sqrt{\zeta_{1-\alpha}} \leq c_1^{1/2}[\phi_i' W_i^{-1} \phi_i] \leq 3 + 2\sqrt{\zeta_{1-\alpha}}$$

and

$$3 - 2\sqrt{\zeta_{1-\alpha}} \leq c_1^{1/2}[\phi_j' W_j^{-1} \phi_j] \leq 3 + 2\sqrt{\zeta_{1-\alpha}}.$$

If, say,  $\sqrt{\zeta_{1-\alpha}} = 0.5$ , the value of the larger parametric function may be as low as  $1\frac{1}{2}$ , whereas the smaller function must be at least 2. This is clearly incoherent.

As this type of SCBs not only seems of limited practical usefulness, but also may lead to incoherent conclusions, it seems doubtful whether they may add much useful information to that given by  $c_1$ -STPs.

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