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Research Article

Simultaneous Unknown Input and State Estimation for the Linear System with a Rank-Deficient Distribution Matrix

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The classical recursive three-step filter can be used to estimate the state and unknown input when the system is affected by unknown input, but the recursive three-step filter cannot be applied when the unknown input distribution matrix is not of full column rank. In order to solve the above problem, this paper proposes two novel filters according to the linear minimum-variance unbiased estimation criterion. Firstly, while the unknown input distribution matrix in the output equation is not of full column rank, a novel recursive three-step filter with direct feedthrough was proposed. Then, a novel recursive three-step filter was developed when the unknown input distribution matrix in the system equation is not of full column rank. Finally, the specific recursive steps of the corresponding filters are summarized. And the simulation results show that the proposed filters can effectively estimate the system state and unknown input.

1. Introduction

The traditional Kalman filter [1] and its extension can recursively estimate the state of the linear system with process noise and measurement noise. The time-domain recursive filter brings greater convenience for continuously processing input data, so it can play a more important role in control theory and engineering. The Kalman filter requires the noise to be stationary white noise, but this supposition is sometimes not feasible because unknown input may not be white noise and cannot be measured.

In the fields of environmental monitoring [2] and disturbance suppression [3, 4], the system equation or output equation contains unknown input owing to environmental impacts and improper selection of model parameters. In recent decades, the problem of state estimation with unknown input has received extensive attention.

For continuous-time systems, the necessary and sufficient conditions for the existence of optimal state filters have been established [5–7]. Furthermore, the steps to reconstruct unknown input are also quite complete [8, 9]. For the state estimation problem of discrete-time systems, an early solution was to add an unknown input vector to the system state vector. Then, the Kalman filter was used to estimate the augmented state. However, the scenarios of using this solution are limited to that the dynamical evolution of unknown input is known [10, 11]. In order to reduce computation costs of the augmented state filter, Friedland [11] proposed the two-stage Kalman filter in which the state estimation and unknown input estimation are decoupled. Although this filter has been successfully applied in some instances, it is still limited to the requirement that the dynamic evolution of unknown input is available. When the unknown input only affects the system equation, Kitanidis [5] developed an optimal recursive state filter which can estimate the system state without prior knowledge of the unknown input. And the stability and convergence conditions of the above filter were raised by Darouach and Zasadzinski [12]. Further, Darouachet al. [13] extended this filter. So, the filter is valid when unknown input is directly feedthrough to the output equation; that is, the unknown input affects both the system equation and output equation.

Although the above methods can get the estimation value of the system state, they all ignore obtaining the estimation value of unknown input, which is necessary in some practical applications.

Hsieh [14] established a robust two-stage Kalman filter (RTSKF). For systems without direct feedthrough of unknown input to output, it can give the joint state and unknown input estimation. But the optimality of the unknown input estimation has not been proven. Furthermore, Gillijns and De Moor [15] proposed a recursive three-step filter (RTSF), which gave a proof that the unknown input estimation is optimal. And the form of the unknown input estimation obtained by RTSF is consistent with that of RTSKF. On the other hand, Gillijns and De Moor [16] extended RTSF so that it is still valid for linear discrete-time systems with direct feedthrough.

Despite the fact that the above filters can solve the problem of simultaneously estimating system state and unknown input, they are based on a precondition: the distribution matrix of unknown input must be of full column rank.

For systems with direct feedthrough, if the distribution matrix of unknown input in output equation is not of full column rank, Cheng et al. [17] presented an unbiased minimum-variance state estimation (UMVSE). This method transforms the output equation by singular value decomposition of the distribution matrix. Then UMVSE is applied to address the problem of state estimation for the new system. However, this method omitted the estimation of unknown input. Hsieh [18] used an extension of RTSF (ERTSF) to estimate the unknown input and state under the assumption that the distribution matrix is not of full column rank, but some parameters in ERTSF were obtained by experience. This paper put forward the novel recursive three-step filter with direct feedthrough (NRTSF-DF) which can give estimation value of the state and unknown input under the same assumption. And compared with ERTSF, the parameter of NRTSF-DF can be exactly obtained. For systems without direct feedthrough, the problem of filter design with unknown input still exists though there are few related literature studies about it. Similar to NRTSF-DF, the novel recursive three-step filter (NRTSF) is proposed in this paper. The novel filters can achieve a simultaneous estimation of the system state and unknown input under the condition that the unknown input distribution matrix is not of full column rank.

In recent years, the research of estimating system state with unknown input is concentrated on nonlinear systems. Based on the EKF structure, the filters estimating the state of the nonlinear system were designed in [19, 20]. Furthermore, by making some improvements on UKF [21], study [22] obtained the filter with RTSF form. And the filter can estimate the state and unknown input simultaneously.

This paper is organized as follows: in Section 2, the problem is formulated. Section 3 deals with the design of the optimal filter for the system with direct feedthrough, and the specific structure of NRTSF-DF is summarized. Next, the optimal filter for the system without feedthrough is established in Section 4. The structure of NRTSF is also obtained. Finally, Section 5 demonstrates the effectiveness of the proposed filters through simulation.

2. Problem Formulation

Consider the linear discrete-time-varying system:

$$x_k = A_{k-1}x_{k-1} + G_{k-1}d_{k-1} + w_{k-1}, \tag{1}$$

$$y_k = C_k x_k + H_k d_k + v_k, \tag{2}$$

where $x_k \in R^n$ is the state vector, $d_k \in R^m$ is an unknown input vector, and $y_k \in R^p$ is the measurement. The process noise $w_k \in R^n$ and the measurement noise $v_k \in R^p$ are assumed to be mutually uncorrelated, zero-mean, white random signals with known covariance matrices, $Q_k = E[w_k w_k^T] \geq 0$, and $R_k = E[v_k v_k^T] > 0$, respectively. The time-varying matrices A_k, G_k, C_k, and H_k are known with an appropriate dimension. Throughout the paper, the conditions that (A_k, G_k) is observable and that x_0 is independent of w_k and v_k are satisfied. And the unbiased estimate $\widehat{x}_0 = E(x_0)$ with $P_0^x = E[(x_0 - \widehat{x}_0)(x_0 - \widehat{x}_0)^T]$ is known.

The optimal filtering problem of the above system is to obtain the unbiased optimal filtering sequence of unknown input $\{\widehat{d}_{0|0}, \ldots, \widehat{d}_{k|k}\}$ and state $\{\widehat{x}_{0|0}, \ldots, \widehat{x}_{k|k}\}$ recursively based on the initial estimate $\widehat{x}_0$, the covariance matrix P_0^x, and the sequence of measurement $\{y_0, y_1, \ldots, y_k\}$. If $H_k = 0$, the system is transformed into a linear discrete-time-varying system without direct feedthrough of unknown input to output. Then, the corresponding optimal filtering problem is transformed to obtain the unbiased optimal filtering sequence of unknown input $\{\widehat{d}_{0|1}, \ldots, \widehat{d}_{k-1|k}\}$ and state $\{\widehat{x}_{0|0}, \ldots, \widehat{x}_{k|k}\}$ under the corresponding conditions.

3. NRTSF-DF

The RTSF proposed by Steven Gilljins in [16] can solve the state and unknown input estimation problem of linear system (1)-(2) while $\text{rank}(H_k) = m, k = 0, 1, \ldots$. When the unknown input distribution matrix in the output equation is not of full column rank, that is, $\text{rank}(H_k) = r_k < m$, we consider a NRTSF-DF design method. The following is the derivation process.

If $\text{rank}(H_k) = r_k \leq m$, perform full rank decomposition:

$$H_k = \overline{H}_k T_k, \tag{3}$$

where $\overline{H}_k \in R^{p\times r_k}, T_k \in R^{r_k\times m}$, and $\text{rank}(\overline{H}_k) = \text{rank}(T_k) = r_k$. The full rank decomposition steps are given in Appendix.

Defining the virtual unknown input by $\overline{d}_k = T_k d_k$, then $H_k d_k = \overline{H}_k \overline{d}_k$. If the estimation value of $\overline{d}_k$ is expressed as $\widehat{\overline{d}}_{k|k}$, the minimal norm estimation value of unknown input d_k is

$$\widehat{d}_{k|k} = T_k^+ \widehat{\overline{d}}_{k|k}, \tag{4}$$

where T_k^+ is the Moore–Penrose inverse of T_k. Then, original output equation (2) is rewritten as

$$y_k = C_k x_k + \overline{H}_k \overline{d}_k + v_k, \tag{5}$$

where $\overline{d}_k \in R^{r_k}$ is the virtual unknown input vector.

Based on the system state equation (1) and output equation (5), we consider NRTSF-DF of the form

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + G_{k-1}\hat{d}_{k-1|k-1}, \quad (6)$$

$$\hat{\overline{d}}_{k|k} = \overline{M}_k\left(y_k - C_k\hat{x}_{k|k-1}\right), \quad (7)$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \overline{L}_k\left(y_k - C_k\hat{x}_{k|k-1}\right), \quad (8)$$

where the matrices $\overline{M}_k \in R^{r_k\times p}$ and $\overline{L}_k \in R^{n\times p}$ still have to be determined.

3.1. Time Update. Let $\hat{x}_{k-1|k-1}$ and $\hat{d}_{k-1|k-1}$ denote the optimal unbiased estimates of x_{k-1} and d_{k-1} given measurement sequence $\{y_0, y_1, \cdots, y_{k-1}\}$; then, the time update is

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + G_{k-1}\hat{d}_{k-1|k-1}. \quad (9)$$

The error in the estimate $\hat{x}_{k|k-1}$ is given by

$$\tilde{x}_{k|k-1} \triangleq x_k - \hat{x}_{k|k-1} = A_{k-1}\tilde{x}_{k-1|k-1} + G_{k-1}\tilde{d}_{k-1} + w_{k-1}, \quad (10)$$

where $\tilde{x}_{k|k} \triangleq x_k - \hat{x}_{k|k}$ and $\tilde{d}_k \triangleq d_k - \hat{d}_{k|k}$. Consequently, the covariance matrix of $\hat{x}_{k|k-1}$ is given by

$$P^x_{k|k-1} \triangleq E\left[\tilde{x}_{k|k-1}\tilde{x}^T_{k|k-1}\right] = \begin{bmatrix} A_{k-1} & G_{k-1} \end{bmatrix} \begin{bmatrix} P^x_{k-1|k-1} & P^{xd}_{k-1|k-1} \\ P^{dx}_{k-1|k-1} & P^d_{k-1|k-1} \end{bmatrix} \begin{bmatrix} A^T_{k-1} \\ G^T_{k-1} \end{bmatrix} + Q_{k-1}, \quad (11)$$

with $P^x_{k|k} \triangleq E[\tilde{x}_{k|k}\tilde{x}^T_{k|k}], P^d_{k|k} \triangleq E[\tilde{d}_k\tilde{d}^T_k], (P^{xd}_{k|k})^T = P^{dx}_{k|k} \triangleq E[\tilde{d}_k\tilde{x}^T_{k|k}]$.

3.2. Virtual Unknown Input Estimation. In this section, the estimation of the virtual unknown input $\overline{d}_k$ is considered.

3.2.1. Unbiased Virtual Unknown Input Estimation. Defining the innovation $\tilde{y}_k \triangleq y_k - C_k\hat{x}_{k|k-1}$, it follows from (5) that

$$\tilde{y}_k = \overline{H}_k\overline{d}_k + e_k, \quad (12)$$

where e_k is given by

$$e_k = C_k\tilde{x}_{k|k-1} + v_k. \quad (13)$$

Owing to $\hat{x}_{k|k-1}$ is unbiased, $E[e_k] = 0$ and $E[\tilde{y}_k] = \overline{H}_k E[\overline{d}_k]$. So, we can obtain an unbiased estimate of the virtual unknown input $\overline{d}_k$ from $\tilde{y}_k$.

Theorem 1. *Suppose $\hat{x}_{k|k-1}$ is unbiased; then, (6) and (7) calculate the unbiased value of $\overline{d}_k$ if and only if $\overline{M}_k$ satisfies $\overline{M}_k\overline{H}_k = I_{r_k}$.*

Proof. This process is similar to the proof of Theorem 1 in [16], so it is omitted.

From Theorem 1, $\operatorname{rank}(\overline{H}_k) = r_k$ is a necessary and sufficient condition for an unbiased virtual unknown input estimator of form (7). The matrix $\overline{M}_k = (\overline{H}^T_k\overline{H}_k)^{-1}\overline{H}^T_k$ corresponding to the least-squares (LS) solution of (12) satisfies Theorem 1. But from the Gauss–Markov theorem, the LS solution is not necessarily minimum-variance as a result of

$$\tilde{R}_k \triangleq E\left[e_k e^T_k\right] = C_k P^x_{k|k-1}C^T_k + R_k \neq cI, \quad (14)$$

where c is a positive real number. □

3.2.2. MVU Virtual Unknown Input Estimation. An MVU estimate of $\overline{d}_k$ is calculated by weighted LS (WLS) estimation.

Theorem 2. *Let $\hat{x}_{k|k-1}$ be unbiased, and let $\tilde{R}_k$ and $\overline{H}^T_k\tilde{R}^{-1}_k\overline{H}_k$ be nonsingular; then, for*

$$\overline{M}^*_k = \left(\overline{H}^T_k\tilde{R}^{-1}_k\overline{H}_k\right)^{-1}\overline{H}^T_k\tilde{R}^{-1}_k, \quad (15)$$

(4) is the MVU estimator of $\overline{d}_k$. The variance of the optimal virtual unknown input estimate is

$$P^{*\overline{d}}_{k|k} = \left(\overline{H}^T_k\tilde{R}^{-1}_k\overline{H}_k\right)^{-1}. \quad (16)$$

Proof. This process is similar to the proof of Theorem 2 in [16], so it is omitted.

We use $\hat{\overline{d}}^*_{k|k}$ to express the optimal virtual unknown input estimate corresponding to $\overline{M}^*_k$ and let $\tilde{\overline{d}}^*_k \triangleq \overline{d}_k - \hat{\overline{d}}^*_{k|k}$. Then, $\tilde{\overline{d}}^*_k$ is given by

$$\tilde{\overline{d}}^*_k = \left(I - \overline{M}^*_k\overline{H}_k\right)\overline{d}_k - \overline{M}^*_k e_k = -\overline{M}^*_k e_k. \quad (17)$$

□

3.3. Measurement Update. In the last step, we use measurement y_k to update $\hat{x}_{k|k-1}$. Using (8) and (12), we find that

$$\tilde{x}_{k|k} = (I - \overline{L}_k C_k)\tilde{x}_{k|k-1} - \overline{L}_k\overline{H}_k\overline{d}_k - \overline{L}_k v_k. \quad (18)$$

Consequently, (8) is unbiased for all $\overline{d}_k$ if and only if $\overline{L}_k$ contents

$$\overline{L}_k\overline{H}_k = 0. \quad (19)$$

Suppose $\overline{L}_k$ satisfy (19), from (18):

$$P^x_{k|k} = (I - \overline{L}_k C_k)P^x_{k|k-1}(I - \overline{L}_k C_k)^T + \overline{L}_k R_k\overline{L}^T_k. \quad (20)$$

So, we can calculate $\overline{L}_k$ by minimizing the trace of (20) under the unbiasedness constraint of (19).

Theorem 3. $\overline{L}_k$ *is given by*

$$\overline{L}_k^* = K_k^*\left(I - \overline{H}_k\overline{M}_k^*\right), \tag{21}$$

where $K_k^* = P_{k|k-1}^x C_k^T \widetilde{R}_k^{-1}$ *minimizes the trace of (20) under the constraint of (19).*

Proof. This process is similar to the proof of Theorem 3 in [16], so it is omitted.

We use $\widehat{x}_{k|k}^*$ to express the state estimate corresponding to $\overline{L}_k^*$. From (7) and (21),

$$\widehat{x}_{k|k}^* = \widehat{x}_{k|k-1} + K_k^*\left(I - \overline{H}_k\overline{M}_k^*\right)\left(y_k - C_k\widehat{x}_{k|k-1}\right) = \widehat{x}_{k|k-1} + K_k^*\left(y_k - C_k\widehat{x}_{k|k-1} - \overline{H}_k\widehat{\overline{d}}_{k|k}^*\right). \tag{22}$$

Then, we consider the expressions of $P_{k|k}^{*x} \triangleq E[\widetilde{x}_{k|k}^*\widetilde{x}_{k|k}^{*T}]$ and $P_{k|k}^{*x\overline{d}} \triangleq E[\widetilde{x}_{k|k}^*\widetilde{\overline{d}}_k^{*T}]$, where

$$\widetilde{x}_{k|k}^* \triangleq x_k - \widehat{x}_{k|k}^* = \left(I - \overline{L}_k^* C_k\right)\widetilde{x}_{k|k-1} - \overline{L}_k^* v_k. \tag{23}$$

From (20) and (21), we obtain

$$P_{k|k}^{*x} = P_{k|k-1}^x - K_k^*\left(\widetilde{R}_k - \overline{H}_k P_{k|k}^{*\overline{d}}\overline{H}_k^T\right)K_k^{*T}. \tag{24}$$

Using (23) and (17), it follows that

$$P_{k|k}^{*x\overline{d}} = -P_{k|k-1}^x C_k^T \overline{M}_k^{*T} = -K_k^*\overline{H}_k P_{k|k}^{*\overline{d}}. \tag{25}$$

Furthermore, by $\widehat{d}_{k|k} = T_k^+\widehat{\overline{d}}_{k|k}$, we can get

$$P_{k|k}^d = T_k^+ P_{k|k}^{*\overline{d}}\left(T_k^+\right)^T P_{k|k}^{xd} = \left(P_{k|k}^{dx}\right)^T = -\overline{K}_k\overline{H}_k P_{k|k}^{*\overline{d}}\left(T_k^+\right)^T. \tag{26}$$

□

3.4. Summary of NRTSF-DF Equations. In order to reflect NRTSF-DF clearly, summarize it as follows:

3.4.1. Time Update. Based on the unbiased estimates $\widehat{d}_{k-1|k-1}$ and $\widehat{x}_{k-1|k-1}$, the state estimates and corresponding variance matrix from the time instant k–1 to k are obtained:

$$\begin{aligned} \widehat{x}_{k|k-1} &= A_{k-1}\widehat{x}_{k-1|k-1} + G_{k-1}\widehat{d}_{k-1|k-1}, \\ P_{k|k-1}^x &= \begin{bmatrix} A_{k-1} & G_{k-1} \end{bmatrix}\begin{bmatrix} P_{k-1|k-1}^x & P_{k-1|k-1}^{xd} \\ P_{k-1|k-1}^{dx} & P_{k-1|k-1}^d \end{bmatrix}\begin{bmatrix} A_{k-1}^T \\ G_{k-1}^T \end{bmatrix} + Q_{k-1}. \end{aligned} \tag{27}$$

3.4.2. Estimation of Virtual Unknown Input. Calculate the rank of H_k, make full rank decomposition of H_k, and calculate the virtual unknown input estimates $\widehat{\overline{d}}_{k|k}$ and corresponding variance matrix at time instant k:

$$\begin{aligned} \widetilde{R}_k &= C_k P_{k|k-1}^x C_k^T + R_k, \\ \overline{M}_k &= \left(\overline{H}_k^T\widetilde{R}_k^{-1}\overline{H}_k\right)^{-1}\overline{H}_k^T\widetilde{R}_k^{-1}, \\ \widehat{\overline{d}}_{k|k} &= \overline{M}_k\left(y_k - C_k\widehat{x}_{k|k-1}\right), \\ P_{k|k}^{\overline{d}} &= \left(\overline{H}_k^T\widetilde{R}_k^{-1}\overline{H}_k\right)^{-1}. \end{aligned} \tag{28}$$

3.4.3. Measurement Update. Calculate the state estimate $\widehat{x}_{k|k}$ and corresponding variance matrix at time instant k:

$$\begin{aligned} K_k &= P_{k|k-1}^x C_k^T\widetilde{R}_k^{-1}, \\ \overline{L}_k &= K_k\left(I - \overline{H}_k\overline{M}_k\right), \\ \widehat{x}_{k|k} &= \widehat{x}_{k|k-1} + \overline{L}_k\left(y_k - C_k\widehat{x}_{k|k-1}\right), \\ P_{k|k}^x &= P_{k|k-1}^x - K_k\left(\widetilde{R}_k - \overline{H}_k P_{k|k}^{\overline{d}}\overline{H}_k^T\right)K_k^T, \\ P_{k|k}^{x\overline{d}} &= \left(P_{k|k}^{\overline{d}x}\right)^T = -K_k\overline{H}_k P_{k|k}^{\overline{d}}, \\ \widehat{d}_{k|k} &= T_k^+\widehat{\overline{d}}_{k|k}, \\ P_{k|k}^d &= T_k^+ P_{k|k}^{\overline{d}}\left(T_k^+\right)^T, \\ P_{k|k}^{xd} &= \left(P_{k|k}^{dx}\right)^T = -K_k\overline{H}_k P_{k|k}^{\overline{d}}\left(T_k^+\right)^T. \end{aligned} \tag{29}$$

Also, note that if H_k is of full column rank, letting $\overline{H}_k = H_k$ and $T_k = I_m$, RTSF is obtained.

4. NRTSF

If $H_k = 0$, systems (1) and (2) are transformed into a linear discrete-time-varying system without direct feedthrough of unknown input to output. It can be expressed as

$$x_k = A_{k-1}x_{k-1} + G_{k-1}d_{k-1} + w_{k-1}, \tag{30}$$

$$y_k = C_k x_k + v_k. \tag{31}$$

The classical filter proposed by Gilljins and De Moor in [15] can solve the state estimation problem when $H_k = 0$, but the application conditions to use this filter are that the unknown input distribution matrix G_{k-1} in the system equation must meet $\operatorname{rank}(G_{k-1}) = m, k = 1, 2, \ldots$.

When the unknown input distribution matrix is not of full column rank, that is, $\operatorname{rank}(G_{k-1}) = r_k < m$, then the classical filter cannot be used. Similar to Section 3, we can also consider an NRTSF design method. The following is the NRTSF derivation process.

If $\operatorname{rank}(G_{k-1}) = r_k \le m$, perform full rank decomposition:

$$G_{k-1} = \overline{G}_{k-1}T_{k-1}, \tag{32}$$

where $\overline{G}_{k-1} \in R^{n\times r_k}, T_{k-1} \in R^{r_k\times m}$, and $\operatorname{rank}(\overline{G}_{k-1}) = \operatorname{rank}(T_{k-1}) = r_k$. The full rank decomposition steps are given in Appendix.

Defining the virtual unknown input by $\overline{d}_{k-1} = T_{k-1}d_{k-1}$, then $G_{k-1}d_{k-1} = \overline{G}_{k-1}\overline{d}_{k-1}$. Because the unknown input is estimated with one step delay, the estimation value of $\overline{d}_{k-1}$ is expressed as $\hat{\overline{d}}_{k-1|k}$ and the minimal norm estimation value of the unknown input d_{k-1} is

$$\hat{d}_{k-1|k} = T_{k-1}^{+}\hat{\overline{d}}_{k-1|k}, \tag{33}$$

where T_{k-1}^{+} is the Moore–Penrose inverse of T_{k-1}. Then, original system equation (30) is rewritten as

$$x_k = A_{k-1}x_{k-1} + \overline{G}_{k-1}\overline{d}_{k-1} + w_{k-1}, \tag{34}$$

where $\overline{d}_{k-1} \in R^{r_k}$ is the virtual unknown input vector.

Based on the system state equation (34) and output equation (31), we consider NRTSF of the form

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1}, \tag{35}$$

$$\hat{\overline{d}}_{k-1|k} = \overline{M}_k\left(y_k - C_k\hat{x}_{k|k-1}\right), \tag{36}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \overline{L}_k\left(y_k - C_k\hat{x}_{k|k-1}\right), \tag{37}$$

where the matrices $\overline{M}_k \in R^{r_k\times p}$ and $\overline{L}_k \in R^{n\times p}$ still have to be determined. Compared with the previous section, the obvious difference is that the second step in NRTSF calculates the value of virtual unknown input $\overline{d}_{k-1|k}$, while the previous filter yields an estimate of virtual unknown input $\overline{d}_{k|k}$.

4.1. Time Update. Let $\hat{x}_{k-1|k-1}$ express the optimal unbiased estimates of x_{k-1} given measurement sequence $\{y_0, y_1, \ldots, y_{k-1}\}$; then, the time update is

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1}. \tag{38}$$

Similarly, the covariance matrix of $\hat{x}_{k|k-1}$ is given by

$$P_{k|k-1}^{x} \triangleq A_{k-1}P_{k-1|k-1}^{x}A_{k-1}^{T} + Q_{k-1}, \tag{39}$$

with $P_{k|k}^{x} \triangleq E[\tilde{x}_{k|k}\tilde{x}_{k|k}^{T}]$, $\tilde{x}_{k|k} \triangleq x_k - \hat{x}_{k|k}$.

4.2. Virtual Unknown Input Estimation. The derivation idea in this section is the same as Section 3.2 except that the time index of unknown input is different.

4.2.1. Unbiased Virtual Unknown Input Estimation. Defining the innovation $\tilde{y}_k \triangleq y_k - C_k\hat{x}_{k|k-1}$, it follows from (31), (34), and (35) that

$$\tilde{y}_k = C_k\overline{G}_{k-1}\overline{d}_{k-1} + e_k, \tag{40}$$

where e_k is given by

$$e_k = C_k\left(A_{k-1}\tilde{x}_{k-1|k-1} + w_{k-1}\right) + v_k. \tag{41}$$

Due to the fact that $\hat{x}_{k|k-1}$ is unbiased, we can obtain an unbiased estimate of the virtual unknown input $\overline{d}_{k-1}$ from $\tilde{y}_k$.

Theorem 4. *Suppose $\hat{x}_{k-1|k-1}$ is unbiased; then, (35) and (36) calculate the unbiased value of $\overline{d}_{k-1}$ if and only if $\overline{M}_k$ satisfies $\overline{M}_kC_k\overline{G}_{k-1} = I_{r_k}$.*

Proof. This process is similar to the proof of Theorem 1, so it is omitted.

The matrix corresponding to the LS solution of (40) satisfies Theorem 4. But from the Gauss–Markov theorem, it is not necessarily minimum-variance as a result of

$$\begin{aligned}\tilde{R}_k \triangleq E\left[e_ke_k^T\right] &= C_k\left(A_{k-1}P_{k-1|k-1}^{x}A_{k-1}^{T} + Q_{k-1}\right)C_k^T + R_k \\ &= C_kP_{k|k-1}^{x}C_k^T + R_k \neq cI,\end{aligned} \tag{42}$$

where c is a positive real number. □

4.2.2. MVU Virtual Unknown Input Estimation. Through weighted LS estimation, we obtain an MVU estimate of $\overline{d}_{k-1}$.

Theorem 5. *Let $\hat{x}_{k-1|k-1}$ be unbiased and let $\tilde{R}_k$ and $F_k^T\tilde{R}_k^{-1}F_k$ be positive definite; then, for*

$$\overline{M}_k = \left(F_k^T\tilde{R}_k^{-1}F_k\right)^{-1}F_k^T\tilde{R}_k^{-1}, \tag{43}$$

where $F_k \triangleq C_k\overline{G}_{k-1}$, (33) is the MVU estimator of $\overline{d}_{k-1}$. The variance of the corresponding input estimate is

$$P_{k-1|k}^{\overline{d}} = \left(F_k^T\tilde{R}_k^{-1}F_k\right)^{-1}. \tag{44}$$

Proof. This process is similar to the proof of Theorem 2, so it is omitted. □

4.3. Measurement Update. First, the estimator must satisfy $E[\hat{x}_{k|k} - x_k] = 0$, which can be expressed as

$$E\left[\hat{x}_{k|k-1} + \overline{L}_k\left(C_kA_{k-1}x_{k-1} + C_k\overline{G}_{k-1}\overline{d}_{k-1} + C_kw_{k-1} + v_k - C_k\hat{x}_{k|k-1}\right) - A_{k-1}x_{k-1} - \overline{G}_{k-1}\overline{d}_{k-1} - w_{k-1}\right] = 0. \tag{45}$$

Consequently, (37) is unbiased for all possible $\overline{d}_{k-1}$ if and only if $\overline{L}_k$ satisfies

$$\overline{L}_kC_k\overline{G}_{k-1} - \overline{G}_{k-1} = 0. \tag{46}$$

Let $\overline{L}_k$ satisfy (46); then, $P_{k|k}^{x}$ is given by

$$\begin{aligned}P_{k|k}^{x} &= (I - \overline{L}_kC_k)\left(A_{k-1}P_{k-1|k-1}^{x}A_{k-1}^{T} + Q_{k-1}\right)(I - \overline{L}_kC_k)^T + \overline{L}_kR_k\overline{L}_k^T \\ &= \overline{L}_k\tilde{R}_k\overline{L}_k^T - P_{k|k-1}^{x}C_k^T\overline{L}_k^T - \overline{L}_kC_kP_{k|k-1}^{x} + P_{k|k-1}^{x}.\end{aligned} \tag{47}$$

So, we can calculate $\overline{L}_k$ by minimizing the trace of (47) under the unbiasedness constraint of (46).

Theorem 6. *$\overline{L}_k$ is given by*

$$\overline{L}_k = K_k + (I - K_k C_k)\overline{G}_{k-1}\overline{M}_k, \tag{48}$$

where $K_k = P^x_{k|k-1}C_k^T\widetilde{R}_k^{-1}$ minimizes the trace of (47) under the constraint of (46).

Proof. This process is similar to the proof of Theorem 3, so it is omitted.

Substituting (48) in (37) yields the equivalent state update:

$$\widehat{x}_{k|k} = \widehat{x}_{k|k-1} + \left[K_k + (I - K_k C_k)\overline{G}_{k-1}\overline{M}_k\right]\left(y_k - C_k\widehat{x}_{k|k-1}\right). \tag{49}$$

Form (47) and (48),

$$P^x_{k|k} = P^x_{k|k-1} - K_k\widetilde{R}_k K_k^T + (I - K_k C_k)\overline{G}_{k-1}P^{\overline{d}}_{k-1|k}\overline{G}_{k-1}^T(I - K_k C_k)^T. \tag{50}$$

□

4.4. Summary of NRTSF Equations

4.4.1. Time Update. Based on the unbiased estimates $\widehat{x}_{k-1|k-1}$, the state estimates and corresponding variance matrix from time instant k−1 to k are obtained:

$$\begin{aligned}\widehat{x}_{k|k-1} &= A_{k-1}\widehat{x}_{k-1|k-1},\\ P^x_{k|k-1} &= A_{k-1}P^x_{k-1|k-1}A_{k-1}^T + Q_{k-1}.\end{aligned} \tag{51}$$

4.4.2. Estimation of Virtual Unknown Input. Calculate the rank of G_{k-1}, make full rank decomposition of G_{k-1}, and calculate the virtual unknown input estimates $\widehat{\overline{d}}_{k-1|k}$ and corresponding variance matrix at time instant k:

$$\begin{aligned}\widetilde{R}_k &= C_k P^x_{k|k-1}C_k^T + R_k,\\ F_k &= C_k\overline{G}_{k-1},\\ \overline{M}_k &= \left(F_k^T\widetilde{R}_k^{-1}F_k\right)^{-1}F_k^T\widetilde{R}_k^{-1},\\ \widehat{\overline{d}}_{k-1|k} &= \overline{M}_k\left(y_k - C_k\widehat{x}_{k|k-1}\right),\\ P^{\overline{d}}_{k-1|k} &= \left(F_k^T\widetilde{R}_k^{-1}F_k\right)^{-1}.\end{aligned} \tag{52}$$

4.4.3. Measurement Update. Calculate the state estimate $\widehat{x}_{k|k}$ and corresponding variance matrix at time instant k:

$$\begin{aligned}K_k &= P^x_{k|k-1}C_k^T\widetilde{R}_k^{-1},\\ \overline{L}_k &= K_k + (I - K_k C_k)\overline{G}_{k-1}\overline{M}_k,\\ \widehat{x}_{k|k} &= \widehat{x}_{k|k-1} + \overline{L}_k\left(y_k - C_k\widehat{x}_{k|k-1}\right),\\ P^x_{k|k} &= P^x_{k|k-1} - K_k\widetilde{R}_k K_k^T + (I - K_k C_k)\overline{G}_{k-1}P^{\overline{d}}_{k-1|k}\overline{G}_{k-1}^T(I - K_k C_k)^T,\\ \widehat{d}_{k-1|k} &= T_{k-1}^{+}\widehat{\overline{d}}_{k-1|k}.\end{aligned} \tag{53}$$

5. Example

In this section, we consider the state and unknown input estimation problem when the system is interfered by d_k as well as zero-mean Gaussian white noise. Specifically, the estimation problem we consider were given in Du [23]. The parameters for the linear system are given by

$$\begin{aligned}A &= \begin{bmatrix}0.5 & 2 & 0 & 0 & 0\\ 0 & 0.2 & 1 & 0 & 1\\ 0 & 0 & 0.3 & 0 & 1\\ 0 & 0 & 0 & 0.7 & 1\\ 0 & 0 & 0 & 0 & 0.1\end{bmatrix},\\ G &= \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0.1\end{bmatrix},\\ C &= I_5,\\ H &= \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\\ R &= 10^{-2}\times\begin{bmatrix}1 & 0 & 0 & 0.5 & 0\\ 0 & 1 & 0 & 0 & 0.3\\ 0 & 0 & 1 & 0 & 0\\ 0.5 & 0 & 0 & 1 & 0\\ 0 & 0.3 & 0 & 0 & 1\end{bmatrix},\\ Q &= 10^{-4}\times\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0.5 & 0 & 0\\ 0 & 0.5 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned} \tag{54}$$

The unknown input $d_k = [d_{1k}d_{2k}d_{3k}]^T$ used in this example is given in Figure 1.

The results of using NRTSF-DF to estimate the unknown input are presented in Figure 2. From Figure 2, NERTSF-DF can estimate the unknown inputs d_{2k}, d_{3k} but has no effect on d_{1k}. The reason is that the direct feedthrough matrix H_k is not of full column rank and, therefore, there is no information about the unknown input d_{1k} in the measurement.

Since the unknown input only affects the first three elements of the system state, we only plot the true value and estimated value of the first, second, and third element of state vector x_k in Figure 3. And the estimation errors of x_{1k}, x_{2k}, and x_{3k} are shown in Figure 4. It can be seen from the figure that the state estimation value can track the true value.

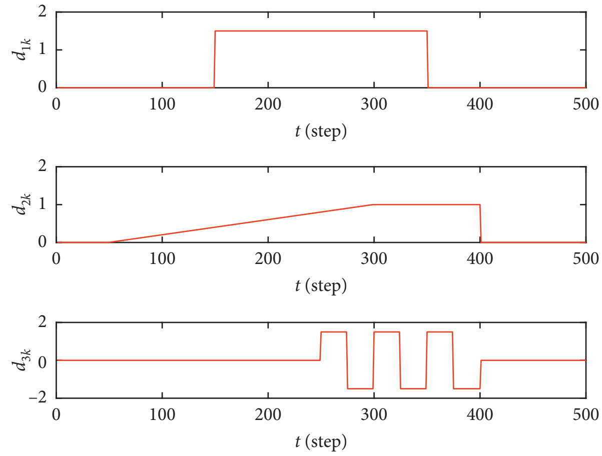


Figure 1: Actual values of the unknown input d_k.

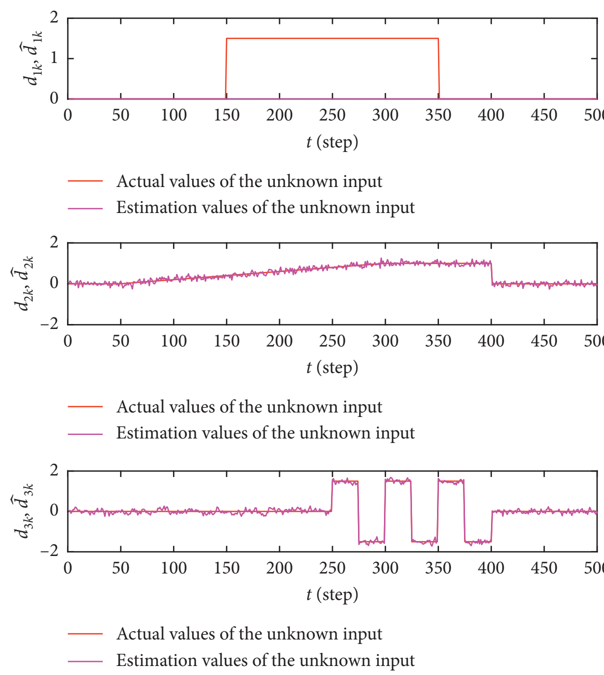
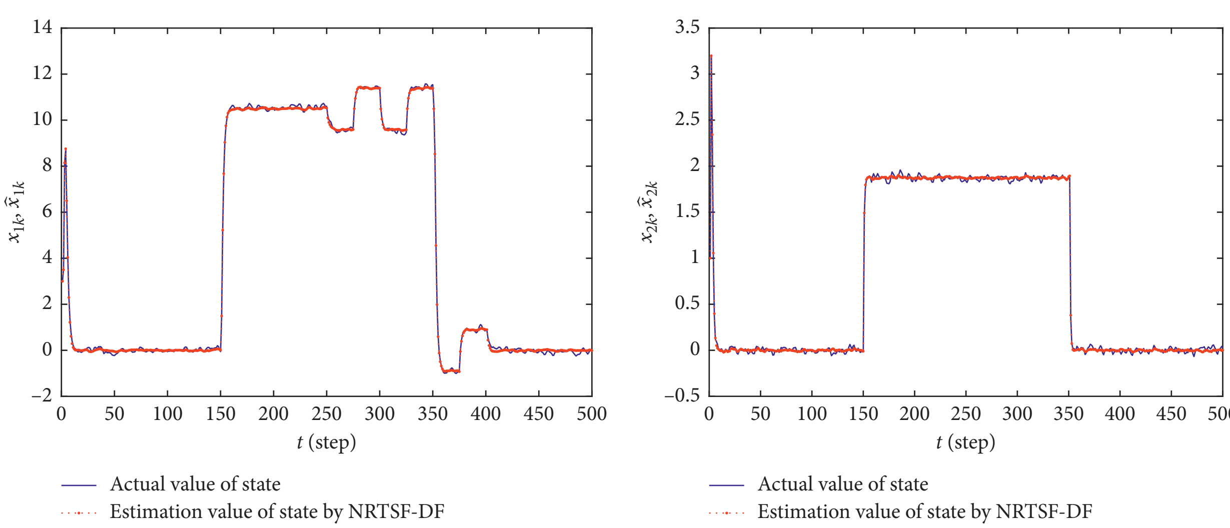
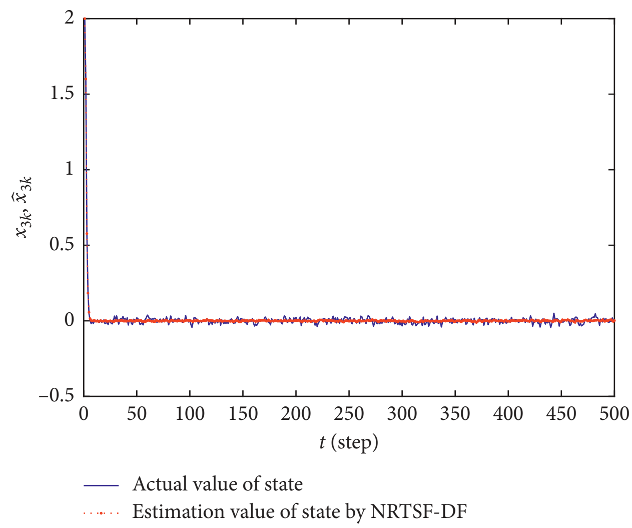


Figure 2: Estimation values of the unknown input d_k.



(a) (b)

Figure 3: Continued.



(c)

Figure 3: States and their estimations. (a) Estimation of state x_{1k}. (b) Estimation of state x_{2k}. (c) Estimation of state x_{3k}.

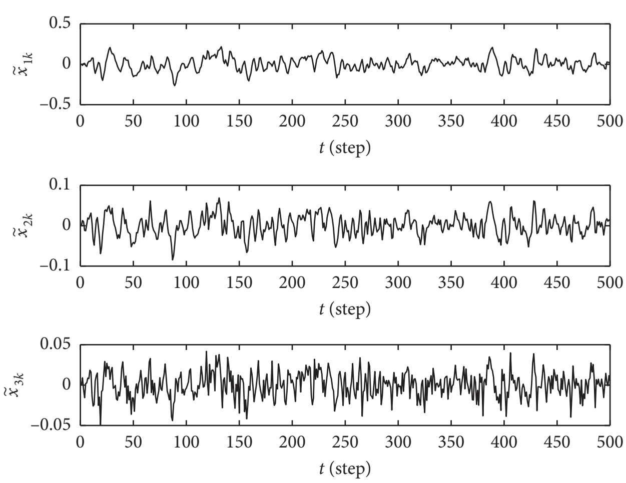


Figure 4: State estimation error.

For the linear system without direct feedthrough of unknown input to output, let $H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; then, the system is transformed into a linear discrete system without direct feedthrough.

The results of using NRTSF to estimate the unknown input are shown in Figure 5. The NRTSF has no effect on d_{3k} because G_{k-1} is not of full column rank. And there is no information about the unknown input d_{3k} in the system state. Furthermore, there is no information about the d_{3k} in measurement.

Figures 6 and 7 show the true value, estimation value, and the estimation error of the first three elements of the state vector, respectively. From the figure, NRTSF is effective.

6. Conclusion

This paper discusses the problem of joint state and unknown input estimation for linear systems with an unknown input and proposes two novel filters, respectively, in accordance with the linear minimum-variance unbiased estimation criterion. For systems with direct feedthrough, a novel recursive three-step filter with direct feedthrough is proposed. This filter can solve the problem that the classical recursive three-step filter cannot be used when the unknown input distribution matrix is not of full column rank. For the situation that unknown input only affects system equation and the distribution matrix is not of full column rank, a novel

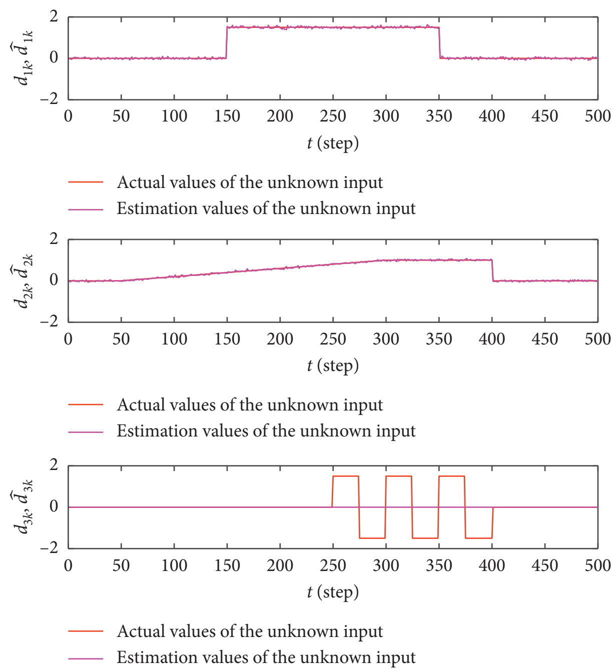
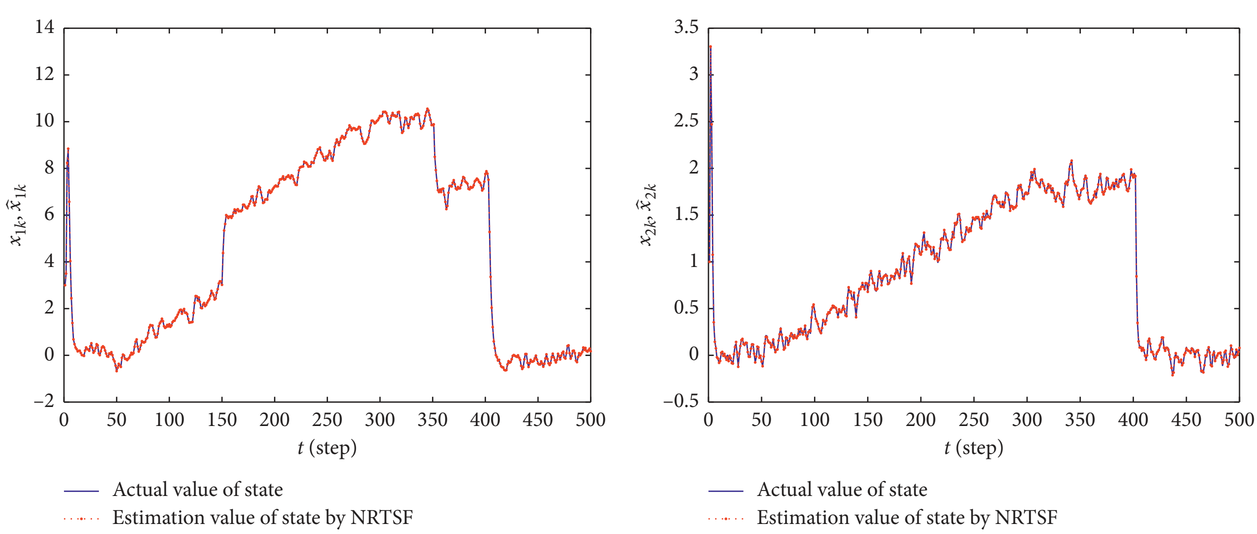
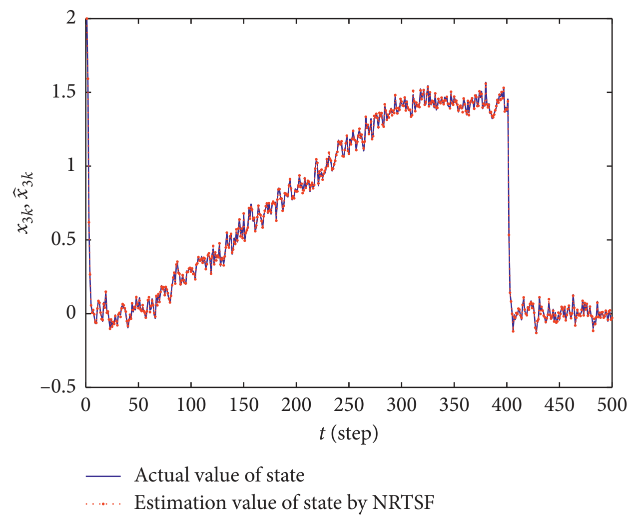


FIGURE 5: Estimation values of the unknown input d_k.



(a) (b)

FIGURE 6: Continued.



(c)

FIGURE 6: States and their estimations. (a) Estimation of state x_{1k}. (b) Estimation of state x_{2k}. (c) Estimation of state x_{3k}.

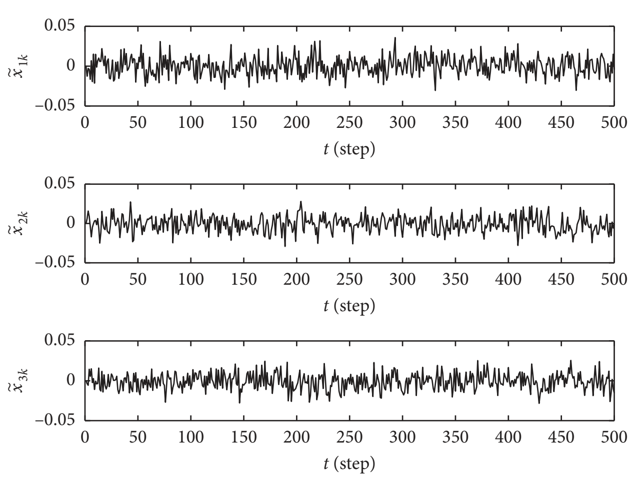


FIGURE 7: State estimation error.

recursive three-step filter is proposed. The simulation results show that both of the proposed filters can effectively estimate the unknown input and system state.

Appendix

There is a known matrix $H_k \in R^{p\times m}$. If rank$(H_k) = r_k < m$, then there is r_k order subformula in H_k, and the determinant of the subformula is nonzero. By swapping rows and columns of the matrix H_k, the subformula is located at the subblock $\widetilde{H}_{11}$ of $\widetilde{H}_k = \begin{bmatrix} \widetilde{H}_{11} & \widetilde{H}_{12} \\ \widetilde{H}_{21} & \widetilde{H}_{22} \end{bmatrix}$, while the rank of $\widetilde{H}_k$ is equal to the rank of H_k. Since the r_k order subblock is nonsingular, the matrix $\widetilde{H}_k$ can be expressed as

$$\widetilde{H}_k = \begin{bmatrix} \widetilde{H}_{11} \\ \widetilde{H}_{21} \end{bmatrix} \begin{bmatrix} I & \widetilde{H}_{11}^{-1}\widetilde{H}_{12} \end{bmatrix}. \quad (A.1)$$

In other words, there are nonsingular matrices $D_k \in R^{p\times p}$ and $F_k \in R^{m\times m}$ satisfying

$$D_k H_k F_k = \begin{bmatrix} \widetilde{H}_{11} \\ \widetilde{H}_{21} \end{bmatrix} \begin{bmatrix} I & \widetilde{H}_{11}^{-1}\widetilde{H}_{12} \end{bmatrix}. \quad (A.2)$$

And then,

$$H_k = D_k^{-1} \begin{bmatrix} \widetilde{H}_{11} \\ \widetilde{H}_{21} \end{bmatrix} \begin{bmatrix} I & \widetilde{H}_{11}^{-1}\widetilde{H}_{12} \end{bmatrix} F_k^{-1} = \overline{H}_k T_k, \quad (A.3)$$

where $\overline{H}_k = D_k^{-1} \begin{bmatrix} \widetilde{H}_{11} \\ \widetilde{H}_{21} \end{bmatrix}$ and $T_k = \begin{bmatrix} I & \widetilde{H}_{11}^{-1}\widetilde{H}_{12} \end{bmatrix} F_k^{-1}$ are nonsingular matrices of rank r_k.

Data Availability

The data used to support the findings of this study are given in [23].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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