

Simultaneous well-balanced orientations of graphs

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Abstract

Nash-Williams' well-balanced orientation theorem [11] is extended for orienting several graphs simultaneously.

We prove that if G_1, \dots, G_k are pairwise edge-disjoint subgraphs of a graph G , then G has a well-balanced orientation \vec{G} such that the inherited orientations \vec{G}_i of G_i are well-balanced for all $1 \leq i \leq k$. We also have new results about simultaneous well-balanced orientations of non-disjoint subgraphs of an Eulerian graph as well as those of different contractions of a graph.

Key words: orientation, connectivity

1 Introduction

This paper concerns undirected and directed graphs, more precisely we consider orientations of undirected graphs. Multiple edges are allowed, but loops are forbidden. The starting point is Robbins' theorem [13] which states that an undirected graph G has a strongly connected orientation if and only if G is 2-edge-connected. The following generalization was proved by Nash-Williams

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in [11]: an undirected graph G has a k -arc-connected orientation if and only if G is $2k$ -edge-connected (see Theorem 2 below). This result about global edge-connectivity can be easily proved by applying Lovász' splitting off theorem [8]. Nash-Williams [11] also provided the following extension on local edge-connectivity (for a stronger statement see Theorem 3): any undirected graph G has a **well-balanced** orientation \vec{G} , that is for every ordered pair of vertices (u, v) , if the maximum number of edge disjoint (u, v) -paths was $\lambda_G(u, v)$ in G , then the maximum number of arc disjoint directed (u, v) -paths is at least $\lfloor \lambda_G(u, v)/2 \rfloor$ in the resulting directed graph \vec{G} . The well-balanced orientation may also be required to be **smooth**, that is the in-degree and the out-degree of every vertex differ by at most one. A smooth well-balanced orientation is called **best-balanced**. In fact, Nash-Williams proved an even stronger result in [11] the so-called odd vertex pairing theorem (see Theorem 5).

Nash-Williams [12] formulated an extension of his orientation theorem: for an arbitrary subgraph H of an undirected graph G there exists a best-balanced orientation of H that can be extended to a best-balanced orientation of G (see Theorem 4). He mentioned that *"Given Theorem 5, the proof of Theorem 4 is not unreasonably difficult. At a certain stage in the proof of Theorem 3 ... we had occasion to select an arbitrary di-Eulerian orientation Δ of the finite Eulerian graph $G + P$ the proof of Theorem 4 depends essentially on the idea of modifying this step ... by choosing Δ to be, not just any di-Eulerian orientation of $G + P$, but one which satisfies certain additional restrictions."*

The main contribution of the present paper is to provide a simple proof for a generalization of this result, namely if G_1, \dots, G_k are pairwise edge-disjoint subgraphs of a graph G , then G has a best-balanced orientation \vec{G} such that the inherited orientations \vec{G}_i are best-balanced orientations of G_i for all $1 \leq i \leq k$. We also have a new result about simultaneous best-balanced orientations of contractions of G .

For an Eulerian graph G we can prove more: there exists simultaneous best-balanced orientations of $G - v$ for all $v \in V$. This solves a conjecture of Frank [2], a special case of an interesting conjecture about k -vertex-connected orientations (see Conjecture 1), and generalizes a theorem of Berg and Jordán [1]. We also provide a couple of consequences of the theorems.

2 Notation, definitions

We denote a directed graph by $\vec{G} = (V, A)$ and an undirected graph by $G = (V, E)$. For a directed graph \vec{G} , a set $X \subseteq V$ and $u, v \in V$, let $\delta_{\vec{G}}(\mathbf{X}) := |\{uv \in A : u \in X, v \notin X\}|$, $\varrho_{\vec{G}}(\mathbf{X}) := \delta_{\vec{G}}(V - X)$, $f_{\vec{G}}(\mathbf{X}) := \varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)$, $\lambda_{\vec{G}}(\mathbf{u}, \mathbf{v}) := \min\{\delta_{\vec{G}}(Y) : u \in Y, v \in V - Y\}$, and $\overleftarrow{\vec{G}} := (V, \{vu : uv \in A\})$.

For an undirected graph G , a set $X \subseteq V$ and $u, v \in V$, let $\mathbf{d}_G(\mathbf{X}) := |\{uv \in E : u \in X, v \notin X\}|$, $\mathbf{T}_G := \{v \in V : d_G(v) \text{ is odd}\}$, $\lambda_G(\mathbf{u}, \mathbf{v}) := \min\{d_G(Y) : u \in Y, v \in V - Y\}$, $\mathbf{R}_G(\mathbf{X}) := \max\{\lambda_G(x, y) : x \in X, y \in V - X\}$ (let $R_G(\emptyset) = R_G(V) = 0$), $\mathbf{b}_G(\mathbf{X}) := d_G(X) - 2 \cdot \lfloor R_G(X)/2 \rfloor$, $\mathbf{G}[\mathbf{X}] := G - (V - X)$.

Observe that for all $X \subseteq V$, we have $0 \leq b_G(X) \leq d_G(X)$ and

$$f_{\vec{G}}(X) = \sum_{v \in X} f_{\vec{G}}(v). \quad (1)$$

Let $G = (V, E)$ be an undirected graph. G is called **k -edge-connected** if $G - F$ is connected for all $F \subseteq E$ with $|F| \leq k - 1$. In this paper, if it is not explicitly stated, graphs may be disconnected, and we use the notion **Eulerian graph** for a possibly disconnected graph with all degrees even. Similarly, a directed graph is called Eulerian if at every vertex the in-degree equals the out-degree. Let $D = (V, A)$ be a directed graph. D is **strongly connected** if for every ordered pair $(u, v) \in V \times V$ of vertices there is a directed (u, v) -path in D . D is called **k -arc-connected** if $G - F$ is strongly connected for all $F \subseteq A$ with $|F| \leq k - 1$. D is called **k -vertex-connected** if $|V| > k$ and $G - X$ is strongly connected for all $X \subseteq V$ with $|X| \leq k - 1$. An orientation \vec{G} of G is called **well-balanced** if

$$\lambda_{\vec{G}}(x, y) \geq \left\lfloor \frac{\lambda_G(x, y)}{2} \right\rfloor \quad \text{for all } (x, y) \in V \times V, \quad (2)$$

and \vec{G} is called **smooth** if

$$|f_{\vec{G}}(v)| \leq 1 \quad \text{for all } v \in V. \quad (3)$$

A smooth well-balanced orientation is called **best-balanced**. Note that if \vec{G} is best-balanced then so is $\overleftarrow{\vec{G}}$.

A **pairing** M of G is a new graph on vertex set T_G in which each vertex has degree one. Let M be a pairing of G . An orientation \vec{M} of M is called **good** if

$$f_{\vec{M}}(X) \leq b_G(X) \quad \text{for all } X \subseteq V. \quad (4)$$

M is **well-orientable** if there exists a good orientation of M , M is **strong** if every orientation of M is good and M is **feasible** if

$$d_M(X) \leq b_G(X) \quad \text{for all } X \subseteq V. \quad (5)$$

Clearly an oriented pairing \vec{M} is good if and only if \overleftarrow{M} is good. We say that two arc disjoint directed graphs \vec{G} and \vec{H} on the same vertex set V are **compatible** if

$$f_{\vec{G}}(v) = f_{\vec{H}}(v) \quad \text{for all } v \in V \quad (6)$$

or equivalently if $\vec{G} + \overleftarrow{H}$ is Eulerian.

3 Eulerian graphs

The following statements are well-known and/or are easy exercises.

Proposition 1 *Every undirected Eulerian graph G has an Eulerian orientation and every Eulerian orientation of G is best-balanced.*

Proposition 2 *If \vec{G}^1 and \vec{G}^2 are Eulerian orientations of a graph G , then \vec{G}^2 can be obtained from \vec{G}^1 by reversing the orientation of some directed cycles.*

Proposition 3 *The edge-set of an undirected graph G can be partitioned into some cycles and $|T_G|/2$ paths. Hence, every graph G has a pairing M such that $d_M(X) \leq d_G(X)$ for all $X \subseteq V$.*

Proposition 4 *\vec{G} is Eulerian if and only if $f_{\vec{G}}(X) = 0$ for all $X \subseteq V$. Hence if \vec{G} is Eulerian then the contracted graph \vec{G}/X is also Eulerian for all $X \subseteq V$.*

For an Eulerian graph G , an **edge-pairing** at vertex v is an arbitrary partition of the edges incident to v into pairs. Suppose that we are given an edge-pairing at each vertex. An Eulerian orientation is called **admissible** if at each vertex every edge-pair becomes a directed path.

Proposition 5 *We are given an Eulerian graph G and an edge-pairing at every vertex. Then there exists an admissible Eulerian orientation of G .*

A nice theorem of Ford and Fulkerson [3] about Eulerian orientations of mixed graphs implies easily the following theorem that plays an important role in this paper.

Theorem 1 *Let M be a pairing of an undirected graph G and \vec{M} be a good orientation of M . Then G has an orientation \vec{G} compatible with \vec{M} .*

4 Equivalent forms

Claim 1 *An orientation \vec{G} of an undirected graph G is well-balanced if and only if*

$$f_{\vec{G}}(X) \leq b_G(X) \quad \text{for all } X \subseteq V. \quad (7)$$

Proof. Note that $b_G(X) - f_{\vec{G}}(X) = (d_G(X) - 2\lfloor R_G(X)/2 \rfloor) - (\varrho_{\vec{G}}(X) - \delta_{\vec{G}}(X)) = 2(\delta_{\vec{G}}(X) - \lfloor R_G(X)/2 \rfloor)$. If \vec{G} is well-balanced then clearly $\delta_{\vec{G}}(X) \geq \lfloor R_G(X)/2 \rfloor$, i.e. $b_G(X) \geq f_{\vec{G}}(X)$ for all X . If $\delta_{\vec{G}}(X) \geq \lfloor R_G(X)/2 \rfloor$ for all X , then, by Menger's theorem [10] and the definition of R , \vec{G} is well-balanced. \square

Claim 2 *A pairing M of G is strong if and only if M is feasible.*

Proof. If M is feasible, then for each orientation \vec{M} , $f_{\vec{M}}(X) \leq d_M(X) \leq b_G(X)$ for all X by (5), that is \vec{M} is good so M is strong. If M is not feasible, then let $X \subseteq V$ with $d_M(X) > b_G(X)$. Let \vec{M} be an orientation of M with $\delta_{\vec{M}}(X) = 0$. Then $f_{\vec{M}}(X) = d_M(X) > b_G(X)$, that is \vec{M} is not good so M is not strong. \square

Claim 3 *An undirected graph G has a best-balanced orientation if and only if there exists a well-orientable pairing M of G . If \vec{M} is a good orientation of pairing M then there exists an orientation \vec{G} compatible with \vec{M} , and every such orientation is best-balanced.*

Proof. We start by proving the second statement, in which the first part is the repetition of Theorem 1. As \vec{G} is compatible with the oriented pairing \vec{M} , it is clearly smooth. By (1), (6) and (4), $f_{\vec{G}}(X) = f_{\vec{M}}(X) \leq b_G(X)$ so \vec{G} is best-balanced by Claim 1.

To prove the first statement, suppose that \vec{G} is best-balanced. Let u_1, \dots, u_t be the vertices with $\delta_{\vec{G}}(u_i) = \varrho_{\vec{G}}(u_i) + 1$ and v_1, \dots, v_t the vertices with $\delta_{\vec{G}}(v_i) = \varrho_{\vec{G}}(v_i) - 1$. Then the oriented pairing consisting of arcs $u_i v_i$ ($1 \leq i \leq t$) is compatible with \vec{G} and is good by (1) and by Claim 1. The other direction follows from the second statement. \square

5 Theorems

The following four theorems are due to Nash-Williams [11,12].

Theorem 2 (Nash-Williams) *A graph G has a k -arc-connected orientation if and only if G is $2k$ -edge-connected.*

Theorem 3 (Nash-Williams) *Every graph has a best-balanced orientation.*

Theorem 4 (Nash-Williams) *Every subgraph H of G has a best-balanced orientation that can be extended to a best-balanced orientation of G .*

Theorem 5 (Nash-Williams) *Every graph has a feasible pairing.*

We present in the following claim the global case of the above “odd vertex pairing” theorem. A short proof of Claim 4 is given in the next section.

Claim 4 *Every $2k$ -edge-connected graph $G = (V, E)$ has a pairing M so that*

$$d_M(X) \leq d_G(X) - 2k \quad \text{for all } X \subset V, X \neq \emptyset. \quad (8)$$

By Claim 2, Theorem 5 is equivalent to Theorem 6.

Theorem 6 *Every graph has a strong pairing.*

By Theorem 3 and Claim 3, every graph has a well-orientable pairing. In the following theorem we generalize this result.

Theorem 7 *Every pairing is well-orientable.*

The main results of this paper are the following generalizations of Theorem 4 and Theorem 3.

Theorem 8 *Let $G = (V, E)$ be a graph, $\{E_1, \dots, E_k\}$ be an arbitrary partition of E and let $G_i := (V, E_i)$ $1 \leq i \leq k$. Then G has a best-balanced orientation \vec{G} such that the inherited orientation \vec{G}_i of each G_i is also best-balanced.*

Theorem 9 *For every partition $\{X_1, \dots, X_l\}$ of $V = V(G)$, G has an orientation \vec{G} such that \vec{G} and its contractions $((\vec{G}/X_1)\dots)/X_l$ and $\vec{G}/(V - X_i)$ for all $1 \leq i \leq l$ are best-balanced orientations of the corresponding graphs.*

For Eulerian graphs we have the following result.

Theorem 10 *Every Eulerian graph $G = (V, E)$ has a best-balanced orientation \vec{G} such that $\vec{G} - v$ is a best-balanced orientation of $G - v$ for all $v \in V$.*

The statement of Theorem 10 is not necessarily true for non-Eulerian graphs, as the example of K_4 shows.

6 Proofs

In this section we apply Theorem 6 (or equivalently, Theorem 5) to prove all the other results in the previous section. For a relatively simple proof for Theorem 5 see [4]. A polynomial time algorithm to find a feasible pairing can be found in [6].

First, we mention that Theorems 2, 3 and 4 are easy consequences of Theorems 3, 6 and 8, respectively. We must emphasize that we do not have a new proof neither for Theorem 5 nor for Theorem 3. However, for Claim 4 we have the following simple proof.

Proof of Claim 4: If $k = 0$ then the claim is true by Proposition 3. From now on we assume that $k \geq 1$. We prove the statement by induction on $|E|$.

Case 1 There is $s \in V$ with $d(s)$ even. Then, by Lovász' splitting off theorem [8], there exists an edge-pairing $\{u_i s, s v_i\}_{i=1}^{d(s)/2}$ at s such that replacing each non-parallel pair $u_i s, s v_i$ by a new edge $u_i v_i$ and then deleting the vertex s , the new graph G' is $2k$ -edge-connected. Note that $T_{G'} = T_G$ and $|E(G')| < |E|$ so by induction there is a pairing M of G' that satisfies (8) for G' . Then M is a pairing of G and, since $d_{G'}(X) \leq d_G(X)$ for all $X \subset V$, clearly M satisfies (8) for G as well and we are done.

Case 2 Otherwise, $T_G = V$. By a result of Mader [9], since there is no vertex v with $d(v) = 2k$, there exists an edge $uv \in E$ such that $G' := G - uv$ is $2k$ -edge-connected. Note that $T_{G'} = T_G - \{u, v\}$ and $|E(G')| < |E|$ so by induction there is a pairing M' of G' so that (8) is satisfied for G' and M' . Let $M := M' \cup uv$. Then M is a pairing of G and for all $X \subseteq V$ either $d_M(X) = d_{M'}(X)$ and $d_G(X) = d_{G'}(X)$ or $d_M(X) = d_{M'}(X) + 1$ and $d_G(X) = d_{G'}(X) + 1$ so (8) is satisfied for G and M and this completes the proof. \square

Proof of Theorem 7: Let M_1 be an arbitrary pairing and M_2 be a strong pairing of G . M_2 exists by Theorem 6. The graph $M_1 \cup M_2$ is Eulerian so it has an Eulerian orientation $\vec{M}_1 \cup \vec{M}_2$. Then $f_{\vec{M}_1}(v) = f_{\vec{M}_2}(v)$ for all $v \in V$. Thus, by (1) and using that \vec{M}_2 is a good orientation of M_2 , $f_{\vec{M}_1}(X) = f_{\vec{M}_2}(X) \leq b_G(X)$ for all $X \subseteq V$, so \vec{M}_1 is a good orientation of M_1 . \square

By the above proof, if we know a feasible pairing, then for every pairing we can find a good orientation in polynomial time. Note that if we apply Theorem 4 with $H' = G$ and $G' = G + M$ we get another proof for Theorem 7.

Proof of Theorem 8: Let M_0 and M_i be strong pairings of G and of G_i for $1 \leq i \leq k$ provided by Theorem 6. Note that for $K := \bigcup_0^k M_i$, $d_K(v) = \sum_0^k d_{M_i}(v) \equiv d_G(v) + \sum_1^k d_{G_i}(v) = 2d_G(v)$ is even for all $v \in V$, so K has an

Eulerian orientation $\vec{K} = \cup_0^k \vec{M}_i$ that is $\cup_1^k \vec{M}_i$ and \overleftarrow{M}_0 are compatible. For $1 \leq i \leq k$, \vec{M}_i is a good orientation of M_i , so, by Claim 3, G_i has a best-balanced orientation \vec{G}_i compatible with \vec{M}_i . Let $\vec{G} := \cup_1^k \vec{G}_i$. Then \vec{G} and $\cup_1^k \vec{M}_i$ are compatible hence so are \vec{G} and \overleftarrow{M}_0 . Since the orientation \overleftarrow{M}_0 is good, \vec{G} is a best-balanced orientation of G by Claim 3. \square

Proof of Theorem 9: Let $G_0 := (((G/X_1)/X_2)/\dots)/X_l$ and $G_i := G/(V-X_i)$ for $1 \leq i \leq l$. Let M_i be a strong pairing of G_i ($0 \leq i \leq l$) provided by Theorem 6. It is easy to see that G has a unique pairing M whose restriction in G_i is M_i for all $0 \leq i \leq l$. By Theorem 7, M has a good orientation \vec{M} . By Claim 3, G has a best-balanced orientation \vec{G} compatible with \vec{M} . \vec{G} and \vec{M} define the orientations \vec{G}_i of G_i and \vec{M}_i of M_i for $0 \leq i \leq l$. Then, by Proposition 4, \vec{G}_i and \vec{M}_i are compatible. Since \vec{M}_i is a good orientation of M_i , \vec{G}_i is a best-balanced orientation of G_i by Claim 3. \square

Proof of Theorem 10: We define an edge-pairing for all $v \in V$ as follows. Take a maximum number of disjoint pairs of parallel edges incident to v . Since G is Eulerian, the other edges from v go to T_{G-v} . These edges can be naturally paired, defined by a strong pairing M_v of $G-v$, where M_v exists by Theorem 6. By Proposition 5 there is an admissible Eulerian orientation \vec{G} of G . Let \vec{M}_v be the natural orientation of M_v (for all $v \in V$) defined by \vec{G} ; as M_v is strong, \vec{M}_v is good. Now $\vec{G} - v + \vec{M}_v$ is an Eulerian orientation of $G-v + M_v$, so by Claim 3, $\vec{G} - v$ is a best-balanced orientation of $G-v$ for all $v \in V$. \square

7 Corollaries

Theorem 4 implies the following result for global edge-connectivity.

Corollary 1 *For a subgraph H of G , H has an l -arc-connected orientation that can be extended to a k -arc-connected orientation of G if and only if H is $2l$ -edge-connected and G is $2k$ -edge-connected.*

Note that the simple proof given for Claim 4, together with the short proof of Theorem 8 gives a direct proof for Corollary 1.

Corollary 2 *If H is an Eulerian subgraph of G , then any Eulerian orientation of H can be extended to a best-balanced orientation of G .*

Proof. By Theorem 4, H has a best-balanced orientation \vec{H} that can be extended to a best-balanced orientation of G . Since \vec{H} is smooth and H is Eulerian, \vec{H} is an Eulerian orientation. By Proposition 2, any other Eulerian orientation of H can be reached by reversing directed cycles, and this operation cannot make the best-balanced orientation of G wrong by Claim 1. \square

More generally, we may consider the following problem: Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $E_1 \cap E_2 \neq \emptyset$, decide whether there exist simultaneous best-balanced orientations of G_1 and G_2 . This problem is NP -complete even if both G_1 and G_2 are restricted to be Eulerian [7]. By Corollary 2, if $E_1 \cap E_2$ defines an Eulerian graph then such orientations always exist.

Corollary 3 *Let $x, y \in V(G)$ with $\lambda_G(x, y) = 2k + 1$. Then G has a best-balanced orientation \vec{G} such that $\lambda_{\vec{G}}(x, y) = k + 1$.*

Proof. Let $G' = G + xy$ and $H' = G$. Note that $\lambda_{G'}(x, y) = 2k + 2$. By applying Theorem 4 for G' and H' the corollary follows (either \vec{G} or \overleftarrow{G} is appropriate). \square

By Proposition 3 the edge-set of any undirected graph G can be decomposed into cycles and $|T_G|/2$ paths. Theorem 8 easily implies the following.

Corollary 4 *Let us fix a decomposition of the edge-set of an undirected graph G into cycles and paths. There exists a best-balanced orientation of G where all the prescribed cycles and paths become directed cycles and paths.*

As a counterpart to Theorem 9 we have the following result by Theorem 8.

Corollary 5 *For every partition $\{X_1, \dots, X_l\}$ of $V(G)$, G has an orientation \vec{G} such that \vec{G} and $\vec{G}[X_i]$ for all $1 \leq i \leq l$ are best-balanced orientations of the corresponding graphs.*

Finally we mention a conjecture on vertex-connectivity orientation (see in [5]), and prove a special case of it and some related statements.

Conjecture 1 *Let $G = (V, E)$ be an undirected graph with $|V| > k$. Then G has a k -vertex-connected orientation if and only if for all $X \subseteq V$ with $|X| < k$, $G - X$ is $(2k - 2|X|)$ -edge-connected.*

Corollary 1 implies at once the following.

Corollary 6 *Let $G = (V, E)$ be an undirected graph and $v \in V$. Then G has a k -arc-connected orientation \vec{G} such that $\vec{G} - v$ is $(k - 1)$ -arc-connected if and only if G is $2k$ -edge-connected and $G - v$ is $(2k - 2)$ -edge-connected.*

Concerning global edge-connectivity we can replace Theorem 6 by Claim 4 in the proof of Theorem 10 and hence we have short simple proofs for the following corollaries of Theorem 10.

Corollary 7 *An Eulerian graph $G = (V, E)$ has a k -arc-connected orientation \vec{G} such that $\vec{G} - v$ is $(k - 1)$ -arc-connected for all $v \in V$ if and only if \vec{G} is $2k$ -edge-connected and $G - v$ is $(2k - 2)$ -edge-connected for all $v \in V$.*

The statement of Corollary 7 is not necessarily true for non-Eulerian graphs, as an example, consider the graph obtained from K_4 by replacing each edge by three parallel edges.

The following result was conjectured by Frank in [2].

Corollary 8 *An Eulerian graph $G = (V, E)$ has an Eulerian orientation \vec{G} such that $\vec{G} - v$ is k -arc-connected for all $v \in V$ if and only if $G - v$ is $2k$ -edge-connected for all $v \in V$.*

For the special case of Conjecture 1 when the graph is Eulerian and $k = 2$, Berg and Jordán [1] provided a sophisticated proof. Their result below follows immediately from Corollary 8.

Corollary 9 (Berg-Jordán) *Let $G = (V, E)$ be a 4-edge-connected Eulerian graph such that $|V| \geq 3$ and $G - v$ is 2-edge-connected for all $v \in V$. Then G has a 2-vertex-connected Eulerian orientation.*

The interested readers may find many counter-examples for problems related to well-balanced orientations in [7].

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