Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information*

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Abstract: This paper derives sufficient conditions for a class of games of incomplete information, such as first-price auctions, to have pure strategy Nash equilibria (PSNE). The paper treats games were each agent has private information about her own type, and the types are drawn from an atomless joint probability distribution. The previous literature on existence of PSNE requires that (i) types are independently distributed, and (ii) the set of feasible actions is countable. This paper shows that a condition we call the *single crossing condition*, which is satisfied in many applications, allows those two assumptions to be relaxed. The single crossing condition can be roughly described as follows: whenever each opponent uses a nondecreasing strategy (in the sense that higher types choose higher actions), then a player's best response strategy is also nondecreasing in her type. The first result establishes that when the single crossing condition holds, a PSNE exists in every finite-action game. The second result considers games with continuous payoffs and a continuum of actions. Existence of PSNE is established by showing that there exists a sequence of equilibria to finite-action games that converges to an equilibrium of the continuum-action game. Third, these convergence and existence results are extended to a class of games with discontinuous payoffs, including first-price auctions. Fourth, the paper provides characterizations of the single crossing condition based on properties of utility functions and probability distributions. In applications, the single crossing condition is shown to hold in a variety of private and common value first-price auction games; multi-unit and all-pay auction games with independent types; pricing games with incomplete information about costs; and a class of noisy signaling games.

Keywords: Games of incomplete information, pure strategy Nash equilibrium, auctions, pricing games, signaling games, supermodularity, log-supermodularity, single crossing, affiliation.

JEL Classifications: C62, C63, D44, D82.

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1. Introduction

This paper derives sufficient conditions for a class of games of incomplete information, such as first-price auction games, to have pure strategy Nash equilibria (PSNE). The class of games is described as follows: there are I agents, each with private information about her own type, where types are drawn from a convex subset of the real line. The joint distribution of types is atomless, and the types may be correlated. Each player takes an action after observing her type. Players may be heterogeneous in utility functions or in the distribution of types, and the utility functions may depend directly on other players' types. Thus, the formulation includes the "mineral rights" auction (Milgrom and Weber (1982)), where bidders receive a signal about the underlying value of the object, and signals and values may be correlated across players.

The goal of this paper is to dispense with many of the assumptions required in the prior literature on existence of PSNE. Instead, we explore the consequences of a restriction that arises naturally in a wide variety of economic applications, the *single crossing condition* (SCC) for games of incomplete information. The SCC can be stated as follows: for every player i, whenever each of player i's opponents uses a pure strategy where higher types choose (weakly) higher actions, player i's expected payoffs satisfy Milgrom and Shannon's (1994) single crossing property.[1] The SCC implies that in response to nondecreasing strategies by opponents, each player has a best response strategy that is nondecreasing.

The paper has four parts. The first part shows that when a game of incomplete information satisfies the SCC, but when the players are restricted to choose from a finite action set, a PSNE exists. The second part considers games with a continuum of actions. It establishes that if players' utility functions are continuous, a PSNE to the continuum-action game can be found by taking the limit of a sequence of PSNE of finite-action games. Third, the latter result is extended to allocation games, such as first-price auctions, that have a particular type of discontinuity. The fourth part of the paper builds on Athey (1998a, b) to characterize the SCC based on properties of utility functions and type distributions. A wide variety of commonly studied games satisfy the SCC. It holds in many auction games, including private-value, first-price auctions where bidders are (weakly) risk averse and the types are independent or affiliated. In the class of "mineral rights" first-price auctions, the SCC holds when there are many identical bidders, or two heterogeneous bidders, whose types are affiliated. It also holds in all-pay auctions and multi-unit first-price auctions with heterogeneous bidders and independent private values. Many other examples arise in the industrial organization literature, including noisy

[1]That is, when choosing between a low action and a high action, if a low type of player i weakly (strictly) prefers the higher action, then all higher types of agent i weakly (strictly) prefers the higher action as well.

signaling games (such as limit pricing with demand shocks), as well as oligopoly games with incomplete information about costs or demand.

The existence theorems exploit a variety of consequences of the SCC. The SCC implies that we may search for equilibria in the space of nondecreasing strategies. Indeed, the existence theorems can be thought of as fixed point theorems tailored to the special case of nondecreasing functions. We begin by observing that in the case of finite action sets, a nondecreasing strategy is a step function, and thus can be represented with a finite vector which determines the points of strict increase of the function. This vector is referred to as the vector of "jump points." Second, we establish that when a player's expected payoffs satisfy Milgrom and Shannon's (1994) single crossing property, the set of vectors of jump points representing optimal best responses for each player is convex (this is not immediate, and it is important for the argument). So long as the type distribution is atomless, Kakutani's fixed point theorem can be applied to the best-response correspondence where players choose vectors of jump points. To treat the continuum-action case, we proceed by taking the limit of a sequence of equilibria with successively finer action sets. We make use of the fact that a sequence of nondecreasing functions has a subsequence which converges almost everywhere to a function which, by virtue of its monotonicity, is continuous almost everywhere. Finally, the results on auctions make use of the special structure of the auction game to rule out discontinuities in the limit as the action set gets fine.

The seminal work on the existence of PSNE in games of incomplete information (Milgrom and Weber (1985), Radner and Rosenthal (1982)) restricts attention to finite-action games. It proceeds by proving existence of mixed strategy equilibria in a game where players choose probability distributions over the actions, and then providing purification theorems.[2] However, this approach is limited because mapping from a mixed strategy equilibrium, where players effectively choose probability distributions over the actions, to a pure strategy equilibrium requires independence (or, at best, conditional independence) of type distributions, and players' types must be restricted to directly affect only their own payoffs. Radner and Rosenthal (1982) provide several counter-examples of games which fail to have PSNE, in particular games where players' types are correlated.

Although results about existence of mixed strategy equilibria can be found when actions are chosen from a compact subset of the real line (Milgrom and Weber, 1985), there are several

[2] More precisely, Milgrom and Weber (1985) show that pure strategy equilibria exist when type spaces are atomless and players choose from a finite set of actions, types are independent conditional on some common state variable (which is finite-valued), and each player's utility function depends only on his own type, the other players' actions, and the common state variable (the utility cannot depend on the other players' types directly). They also require a condition which they call "continuity of information."

known counter-examples to existence of PSNE in this context (Khan and Sun, 1996, 7). In the case where types are independent and payoffs are continuous, Khan and Sun (1995) have recently shown that PSNE exist when the action sets are countably infinite, but not when the action sets are uncountable.[3]

An alternative approach is due to Vives (1990), who shows that a sufficient condition for existence of PSNE is that the game is supermodular in the strategies, in the following sense: if one player's strategy increases pointwise, the best response strategies of all opponents must increase pointwise. However, the strategies themselves need not be monotone in types. Vives' condition is applicable in games where each player's payoff function is supermodular in actions, but not in the auctions and log-supermodular pricing games highlighted in this paper.

Now consider the case of first-price auctions. The issue of the existence of PSNE in first-price auctions with heterogeneous agents has challenged economists for many years. Recently, several authors have made substantial progress.[4] While the SCC is satisfied in the settings considered in the literature,[5] many interesting classes of auctions with heterogeneous bidders are not treated by the existing analysis. Further, even for the auctions where existence is known, computation of equilibrium (which involves numerically solving a system of nonlinear differential equations with two boundary points) can be difficult due to pathological behavior of the system. Thus, the computational algorithm suggested by the constructive existence theorems in this paper may be of use in applications. For example, it can be used to evaluate the effects of mergers between bidders in auctions, as well as to analyze common value auctions with heterogeneous bidders, about which very little is known.

2. Finite-Action Games

Consider a game of incomplete information between I players, i=1,..,I, where each player first observes her own type $t_i \in T_i \equiv [\underline{t}_i, \bar{t}_i] \subset \Re$ and then takes an action a_i from a compact action set $\mathcal{A}_i \subset \Re$. Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_I$, $\mathbf{T} \equiv T_1 \times \cdots \times T_I$, $\underline{a}_i \equiv \min \mathcal{A}_i$, and $\bar{a}_i \equiv \max \mathcal{A}_i$. Player i's payoff is $u_i(\mathbf{a}, \mathbf{t})$. The joint density over player types is $f(\mathbf{t})$, with conditional densities $f(\mathbf{t}_{-i}|t_i)$. Given any set of

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[3] Khan and Sun (1996) show further that if the type distributions are taken to be atomless on a special class of measure spaces, called hyperfinite Loeb spaces, existence of PSNE can be obtained when actions are drawn from the continuum (again maintaining continuity of payoffs and independence of types).

[4] For asymmetric independent private values auctions, see Maskin and Riley (1993, 1996), Lebrun (1995, 1996), and Bajari (1996a); for affiliated private values or common value auctions with conditionally independent signals, see Maskin and Riley (1996). Pesendorfer and Swinkels (1997) study symmetric common value auctions for multiple units. Lizzeri and Persico (1997) have independently shown that a condition closely related to the single crossing condition is sufficient for existence and uniqueness of equilibrium in two-player mineral rights auction games with heterogeneous bidders, but their approach only extends to n players under symmetry.

[5] Weber (1994) studies mixed strategy equilibria in a class of auction games where the affiliation inequality fails.

strategies for the opponents, $\alpha_j:[\underline{t}_j,\bar{t}_j]\to\mathcal{A}_j$, $j\neq i$, player i's objective function can be written as follows (using the notation $(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}))\equiv(..,\alpha_{i-1}(t_{i-1}),a_i,\alpha_{i+1}(t_{i+1}),..)$):

$$U_i(a_i,t_i;\boldsymbol{\alpha}_{-i}(\cdot))\equiv\int_{t_{-i}}u_i((a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})),\mathbf{t})f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$$

The following basic assumptions are maintained throughout the paper.

Assumption A1 *The types have joint density with respect to Lebesgue measure,* $f(\mathbf{t})$, *which is bounded and atomless.*[6] *Further,* $\int_{\mathbf{t}_{-i}\in S}u_i((a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i})),\mathbf{t})f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ *exists and is finite for all convex S and all nondecreasing functions* $\alpha_j:[\underline{t}_j,\bar{t}_j]\to\mathcal{A}_j$, $j\neq i$.

The following definitions are needed (the strong and weak versions will be referred to in later sections).

Definition 1 $h(x,\theta)$ *satisfies the (Milgrom-Shannon)* ***single crossing property of incremental returns (SCP-IR)*** *in* $(x;\theta)$ *if, for all* $x_H>x_L$ *and all* $\theta_H>\theta_L$, $h(x_H,\theta_L)-h(x_L,\theta_L)\geq(>)0$ *implies* $h(x_H,\theta_H)-h(x_L,\theta_H)\geq(>)0$, *and h satisfies* ***weak SCP-IR*** *if for all* $x_H>x_L$ *and all* $\theta_H>\theta_L$, $h(x_H,\theta_L)-h(x_L,\theta_L)>0$ *implies* $h(x_H,\theta_H)-h(x_L,\theta_H)\geq 0$.

The definition of SCP-IR requires that the incremental returns to x cross zero at most once, from below, as a function of θ; it implies that the set of optimizers is nondecreasing in the Strong Set Order, defined as follows.

Definition 2 *A set* $A\subseteq\Re$ *is greater than a set* $B\subseteq\Re$ *in the* ***strong set order****, written* $A\geq_S B$, *if, for any* $a\in A$ *and any* $b\in B$, $\max(a,b)\in A$ *and* $\min(a,b)\in B$. *A set-valued function* $A(\tau)$ *is nondecreasing in the strong set order if for any* $\tau_H>\tau_L$, $A(\tau_H)\geq_S A(\tau_L)$.

Lemma 1 *(Milgrom and Shannon, 1994) Let* $h:\Re^2\to\Re$. *Then h satisfies SCP-IR if and only if* $x^*(\theta,B)\equiv\arg\max_{x\in B}h(x,\theta)$ *is nondecreasing in* θ *and B in the strong set order.*

Under SCP-IR, there might be a $x'\in x^*(\theta_L)$ and a $x''\in x^*(\theta_H)$ such that $x'>x''$, so that some selection of optimizers is decreasing on a region; however, if this is true, then $x'\in x^*(\theta_H)$ as well. Definition 1 can be used to state the sufficient condition for existence of a pure strategy Nash equilibria in nondecreasing strategies.

Definition 3 *The* ***Single Crossing Condition (SCC) for games of incomplete information*** *is satisfied if for each* $i=1,..,I$, *whenever every opponent* $j\neq i$ *uses a strategy* $\alpha_j:[\underline{t}_j,\bar{t}_j]\to\mathcal{A}_j$ *that is nondecreasing, player i's objective function,* $U_i(a_i,t_i;\boldsymbol{\alpha}_{-i}(\cdot))$, *satisfies single crossing of incremental returns (SCP-IR) in* $(a_i;t_i)$.

6 In games with finite actions, Assumption A1 can be relaxed to allow for mass points at the lower end of the distribution, so long as for each player, there exists a $k>\underline{t}_j$ such that the lowest action chosen by player j is chosen throughout the region $[\underline{t}_j,k)$.

Our first observation is that when the action set is finite, any nondecreasing strategy $\alpha_i(t_i)$ is a step function, and the strategy can be described simply by naming the values of the player's type t_i at which the player "jumps" from one action to the next higher action.

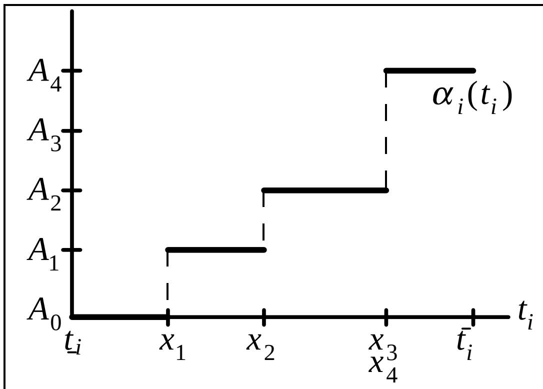

Figure 2: The elements of the vector **x** specify when the agent "jumps" to a higher action.

Consider the following representation. Let $\mathcal{A}_i=\{A_0,A_1,..,A_M\}$ be the set of potential actions, in ascending order, where M+1 is the number of potential actions (and for notational simplicity, we suppose for the moment that the action sets are the same for all players). Define $T_i^M \equiv \times_{m=1}^{M}[\underline{t}_i,\bar{t}_i]$, $\Sigma_i^M \equiv \left\{\mathbf{x}\in T_i^{M+2} \middle| x_0=\underline{t}_i,\ x_1\le x_2\le\cdot\cdot\le x_M,\ x_{M+1}=\bar{t}_i\right\}$, and let $\mathbf{\Sigma}\equiv\Sigma_1^M\times\cdot\cdot\times\Sigma_I^M$. A nondecreasing strategy for player i, $\alpha_i:[\underline{t}_i,\bar{t}_i]\rightarrow\mathcal{A}_i$, can be represented by a vector $\mathbf{x}\in\Sigma_i^M$ according to the following algorithm (illustrated in Figure 2).

Definition 4 *(i) Given a nondecreasing strategy $\alpha_i(t_i)$, we say that "the vector $\mathbf{x}\in\Sigma_i^M$ represents $\alpha_i(t_i)$" if $x_m=\inf\left\{t_i \middle| \alpha_i(t_i)\ge A_m\right\}$ whenever there is some $n\ge m$ such that $\alpha_i(t_i)=A_n$ on an open interval of T_i, and $x_m=\bar{t}_i$ otherwise.*

(ii) Given $\mathbf{x}\in\Sigma_i^M$, let $\{\mathbf{x}\}$ denote the set $\{\underline{t}_i,x_1,..,x_M,\bar{t}_i\}$, and let $m^*(t,\mathbf{x})\equiv\max\{m|x_m<t\}$. We say a nondecreasing "strategy $\alpha_i(t_i)$ is consistent with $\mathbf{x}$" if $\alpha_i(t_i)=A_{m^*(t_i,\mathbf{x})}$ for all $t_i\in T_i\backslash\{\mathbf{x}\}$.

Each component of $\mathbf{x}$ is a "jump point" of the step function described by α_i. Since $\mathbf{x}$ does not specify behavior for $t_i\in\{\mathbf{x}\}$, a given $\mathbf{x}\in\Sigma_i$ might correspond to more than one nondecreasing strategy. However, because there are no atoms in the distributions of types, a player's behavior on the set $\{\mathbf{x}\}$ (which has measure zero) will not affect the best responses of other players.

Given $\mathbf{X}=(\mathbf{x}^1,..,\mathbf{x}^I)\in\mathbf{\Sigma}$, let $V_1(a_1,t_1;\mathbf{X})$ denote the expected payoffs to player 1 with type t_1 when player 1 chooses $a_1\in\mathcal{A}_1$ and players 2,..,I use strategies consistent with $(\mathbf{x}^2,..,\mathbf{x}^I)$. Then,

$$V_1(a_1,t_1;\mathbf{X})\equiv\sum_{m_2=0}^{M}\cdot\cdot\sum_{m_I=0}^{M}\int_{t_2=x_{m_2}^2}^{x_{m_2+1}^2}\cdots\int_{t_I=x_{m_I}^I}^{x_{m_I+1}^I}u_1(a_1,A_{m_2},..,A_{m_I},\mathbf{t})\cdot f(\mathbf{t}_{-1}|t_1)d\mathbf{t}_{-1}. \tag{1}$$

Since (1) embeds the assumption that opponent strategies are nondecreasing, the SCC implies that $V_i(a_i,t_i;\mathbf{X})$ satisfies the SCP-IR in $(a_i;t_i)$ for all $\mathbf{X}\in\mathbf{\Sigma}$. Let $a_i^{BR}(t_i|\mathbf{X})=\arg\max_{a_i\in\mathcal{A}_i}V_i(a_i,t_i;\mathbf{X})$; this is nonempty for all t_i by finiteness of $\mathcal{A}_i$. By Lemma 1, $a_i^{BR}(t_i|\mathbf{X})$ is nondecreasing in the strong set order, which in turn implies (see Milgrom and Shannon, 1994) that there exists a selection, $\gamma_i(t_i)\in a_i^{BR}(t_i|\mathbf{X})$, from the set which is nondecreasing in t_i. Using Definition 3, $\gamma_i(t_i)$ can be represented by a $\mathbf{y}\in\Sigma_i$. Now define the set of all vectors that represent best response strategies:

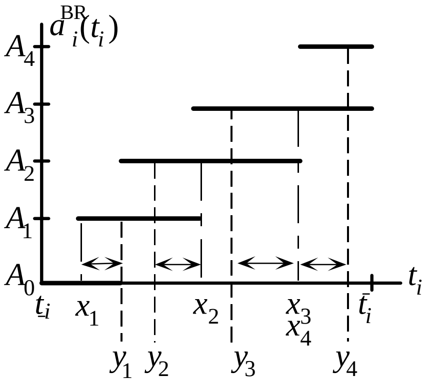

Figure 3: The set $a_i^{BR}(t_i)$ is nondecreasing in the Strong Set Order. The vectors **x** and **y** represent "jump points" corresponding to optimal strategies. The arrows indicate convex combinations of **x** and **y**.

$$\Gamma_i(\mathbf{X})=\{\mathbf{y}\in \Sigma_i^M: \ \exists \ \alpha_i(t_i) \text{ which is consistent with } \mathbf{y} \text{ such that } \alpha_i(t_i)\in a_i^{BR}(t_i|\mathbf{X})\}.$$

The existence proof proceeds by showing that a fixed point exists for this correspondence. A critical property required of $\Gamma=(\Gamma_1,..,\Gamma_I)$ for this purpose is convexity. However, establishing this property requires some additional work, since the player might be indifferent between two actions over a set of types (and this remains true even under the additional assumptions that the player's payoff function is strictly quasi-concave and that payoffs are nowhere constant in type, so long as the action set is finite).

In Figure 3, $a_i^{BR}(t_i|\mathbf{X})$ is nondecreasing in the strong set order. In the figure, **x** and **y** are both vectors of jump points representing optimal behavior; the arrows in the figure show convex combinations of x_m and y_m for m=1,..,4. Notice that any such convex combination also represents optimal behavior. The following Lemma shows that convexity of the best response correspondence is a general consequence of the strong set order.

Lemma 2 *Γ_i is convex if $a_i^{BR}(t_i|\mathbf{X})$ is nondecreasing in the strong set order.*

Proof: Fix **X** and suppose that $\mathbf{w},\mathbf{y} \in\Gamma_i(\mathbf{X})$. Let $\mathbf{z}=\lambda\, \mathbf{w}+(1-\lambda)\, \mathbf{y}$ for $\lambda\in(0,1)$, and observe that $\mathbf{z}\in\Sigma_i$. Now, for m=0,..,M, we show that A_m is an optimal action on (z_m,z_{m+1}). If $w_m=w_{m+1}$ and $y_m=y_{m+1}$, then $z_m=z_{m+1}$ and there is nothing to show; so, assume that $y_m<y_{m+1}$.

Consider $t_i\in(z_m,z_{m+1})$ and a k such that $A_k\in a_i^{BR}(t_i|\mathbf{X})$. Case 1: Either $w_m<w_{m+1}$ or $y_m<w_m=w_{m+1}<y_{m+1}$. By definition of **w** and **y**, there exists a $t_i'<t_i$ and a $t_i''>t_i$ such that $A_m\in a_i^{BR}(t_i'|\mathbf{X})$ and $A_m\in a_i^{BR}(t_i''|\mathbf{X})$. If $k<m$, then the fact that $A_m\in a_i^{BR}(t_i'|\mathbf{X})$ implies that $A_m\in a_i^{BR}(t_i|\mathbf{X})$. Likewise, if $k>m$, then the fact that $A_m\in a_i^{BR}(t_i''|\mathbf{X})$ implies that $A_m\in a_i^{BR}(t_i|\mathbf{X})$. Case 2: $w_m=w_{m+1}\le y_m<y_{m+1}$. Then, there exists a $t_i'<t_i$ and an $m''>m$ such that $A_{m''}\in a_i^{BR}(t_i'|\mathbf{X})$, and a $t_i''>t_i$ such that $A_m\in a_i^{BR}(t_i''|\mathbf{X})$. If $k<m$, note that the strong set order then requires $A_{m''}\in a_i^{BR}(t_i|\mathbf{X})$. But applying the strong set order again, together with $A_m\in a_i^{BR}(t_i''|\mathbf{X})$, implies that $A_m\in a_i^{BR}(t_i|\mathbf{X})$. If $k>m$, the latter sentence applies directly. Case 3: $y_{m+1}<y_{m+1}\le w_m=w_{m+1}$. Analogous to Case 2.

Thus, $\beta_i(t_i,\mathbf{z}) = A_{m^*(t_i,\mathbf{z})}$ is a nondecreasing strategy consistent with **z** which assigns optimal actions to almost every type, implying that $\mathbf{z}\in\Gamma_i(\mathbf{X})$. □

With convexity established, it is straightforward to prove existence of a fixed point.

Lemma 3: *Suppose that A1 and the SCC hold. Then there exists a fixed point of the correspondence* $(\Gamma_1(\mathbf{X}),.., \Gamma_I(\mathbf{X})):\mathbf{\Sigma}\to\mathbf{\Sigma}$.

Proof of Lemma: Since Σ^M is a compact, convex subset of $(M+2)\cdot I$ –dimensional Euclidean space, we can apply Kakutani's fixed point theorem. We argued in the text that Γ is nonempty, and Lemma 2 established convexity. Now consider closed graph. It is clear from the definition (1) that $V_i(a_i;\mathbf{X},t_i)$ is continuous in the elements of $\mathbf{X}$ under our assumption that the type distribution is atomless. Consider a sequence $(\mathbf{X}^k, \mathbf{Y}^k)$ which converges to $(\mathbf{X},\mathbf{Y})$, such that $\mathbf{Y}^k\in\Gamma(\mathbf{X}^k)$ for all k. To see that $\mathbf{Y}\in\Gamma(\mathbf{X})$, consider player i, and a type $t_i\in T_i\backslash\{\mathbf{y}^i\}$. Then there exists an $m\in\{0,..,M\}$ such that $y^i_m<t_i<y^i_{m+1}$. Since $\mathbf{y}^{i,k}$ converges to $\mathbf{y}^i$, there must exist an K such that, for all $k>K$, $y^{i,k}_m<t_i<y^{i,k}_{m+1}$, and thus A_m is one of t_i's best responses to $\mathbf{X}^k$ since $\mathbf{Y}^k\in\Gamma(\mathbf{X}^k)$. But, since $V_i(a_i,t_i;\mathbf{X})$ is continuous in $\mathbf{X}$, if $V_i(A_m,t_i;\mathbf{X}^k)\geq V_i(A_{m'},t_i;\mathbf{X}^k)$ for all $k>K$ and all m', then $V_i(A_m,t_i;\mathbf{X})\geq V_i(A_{m'},t_i;\mathbf{X})$. □

Existence of a PSNE follows directly from this Lemma. It remains only to assign strategies to players that are consistent with a fixed point of Γ. Let $\mathbf{X}$ be such a fixed point, and let $(\beta_1(t_1),..,\beta_I(t_I))$ be a vector of nondecreasing strategies where each $\beta_i(t_i)$ is consistent with $\Gamma_i(\mathbf{X})$. Since the type distribution is atomless, a given player i does not care about the behavior of her opponents at jump points, and thus $\beta_i(t_i)$ is a best response to any set of strategies $\boldsymbol{\beta}_{-i}(\cdot)$ consistent with $\mathbf{X}_{-i}$. This implies that $(\beta_1(t_1),..,\beta_I(t_I))$ is a PSNE of the original game. Formally:

Theorem 1 *Assume A1 and the SCC hold. If $\mathcal{A}_i$ is finite for all i, this game has a PSNE, where each player's equilibrium strategy, $\beta_i(t_i)$, is a nondecreasing function of t_i.*

Before proceeding, it is useful to pause to consider the precise role of the SCC in this analysis. Can the result be extended to non-monotone strategies? In the working paper (Athey, 1997), this question is explored more fully by considering strategies of "limited complexity." The basic idea is that a PSNE exists if we can find bounds on the "complexity" (formalized as the number of times the function changes from nondecreasing to nonincreasing or vice versa) of each player's strategy, such that player i's best response stays within her specified bound whenever all opponents use strategies within their respective bounds. The main limitation of the extension to games with limited complexity is that much stronger assumptions may be required to guarantee convexity, which followed above as a consequence of the SCC: the working paper assumes that the best response action is unique for almost all types.

What is ruled out by the SCC, or more generally by a restriction like "limited complexity"? An example of a game with no PSNE, due to Radner and Rosenthal (1982), is especially instructive. The setup is as follows: the game is zero-sum, and each player can choose actions A_0 or A_1. When the players match their actions, player 2 pays \$1 to player 1, while if they do not match, the players each receive zero. The types do not directly affect payoffs, and are uniformly distributed on the triangle $0 \le t_1 \le t_2 \le 1$. Notice first that this game fails the SCC. When player 2 uses a nondecreasing strategy, player 1's best response is nondecreasing. However, when player 1 uses a nondecreasing strategy, player 2's best response is *nonincreasing*, since player 2 prefers not to match with player 1. Further, this game also fails the more general "limited complexity" condition. Intuitively, player 1's best response to any pure strategy of player 2 will essentially mirror player 2's strategy, while player 2 will wish to reverse any pure strategy of player 1.

3. Games with a Continuum of Actions and Continuous Payoffs

This section shows that the results about existence in games with a finite number of actions can be used to construct equilibria of games with a continuum of actions. The properties of the equilibrium strategies implied by the SCC play a special role in the limiting arguments used in section. While arbitrary sequences of functions need not have convergent subsequences, sequences of nondecreasing functions do have almost-everywhere convergent subsequences. Thus, all that remains is to show that the limits of these sequences are in fact equilibria to the continuous-action game.

The assumption of finite actions in Section 2 plays two roles: (i) it guarantees that an optimal action exists for every type, and (ii) it simplifies the description of strategies so that they can be represented with finite-dimensional vectors. In moving to the continuum-action case, we introduce the assumption that payoffs are continuous in actions. Continuity is a substitute for finiteness in (i), and further, it is used in showing that the limit of a sequence of equilibria in finite-action games is an equilibrium of the limiting game.

Theorem 2 *Assume A1. Suppose that (i) for all i,* $\mathcal{A}_i=[\underline{a}_i,\overline{a}_i]$*, (ii) for all i,* $u_i(\mathbf{a},\mathbf{t})$ *is continuous in* $\mathbf{a}$ *on* $[\underline{a}_i,\overline{a}_i]$*, and (iii) for any finite* $\mathcal{A}^n \subset \mathcal{A}$*, a PSNE exists in nondecreasing strategies. Then a PSNE exists in nondecreasing strategies in the game where players choose actions from* $\mathcal{A}$*.*[7]

Proof: For each player i, consider a sequence of action sets $\{\mathcal{A}_i^n\}$, where $\mathcal{A}_i^n=\left\{\underline{a}_i+\frac{m}{10^n}(\overline{a}_i-\underline{a}_i): m=0,..,10^n\right\}$. Let $\mathcal{A}^n=(\mathcal{A}_1^n,..,\mathcal{A}_I^n)$, and let $\boldsymbol{\beta}^n$ be the corresponding nondecreasing PSNE strategies. Helly's Selection Theorem (Billingsley (1968), p. 227) guarantees that a sequence of nondecreasing, bounded functions on $T_i \subseteq \Re$ has a subsequence which converges almost everywhere to a nondecreasing function (and in particular, it converges at continuity points of the limiting function). Let $\{n\}$ denote a sequence such that $\{\beta_1^n(t_1),..,\beta_I^n(t_I)\}$ converges almost everywhere to $\boldsymbol{\beta}^*(\mathbf{t})$.

Let $\hat{Z}^i=\left\{t_i \middle| \exists n \text{ s.t. } t_i \in \{\mathbf{x}^{i,n}\}\right\}$, and note that $\hat{Z}^i$ is countable and thus has measure zero. Consider

[7] In the working paper, we show that this result can be easily extended to show that if every finite-action game has an equilibrium in strategies of bounded variation, the continuum-action game will as well.

player i, and a type $t_i \in T_i \backslash \hat{Z}^i$, such that $\beta_i^*(t_i)$ is continuous at t_i. Let $b = \beta_i^*(t_i)$; Helly's Selection Theorem implies $\beta_i^n(t_i)$ converges to b. Consider any $a' \in \mathcal{A}^i$, and consider a sequence $\{a'_n\}$ such that, for all n, $a'_n \in \mathcal{A}_i^n$, and further $a'_n \rightarrow a'$. Since $\beta_i^n(t_i)$ is an equilibrium strategy for any n, then for all n, $U_i(\beta_i^n(t_i), t_i; \boldsymbol{\beta}_{-i}^n(\cdot)) \geq U_i(a'_n, t_i; \boldsymbol{\beta}_{-i}^n(\cdot))$. Because payoffs are continuous and since $\boldsymbol{\beta}_{-i}^n(\mathbf{t}_{-i})$ converges to $\boldsymbol{\beta}_{-i}^*(\mathbf{t}_{-i})$ for almost all $\mathbf{t}_{-i}$, it follows that, for almost all $\mathbf{t}_{-i}$, $u_i(\beta_i^n(t_i), \boldsymbol{\beta}_{-i}^n(\mathbf{t}_{-i}), \mathbf{t})$ converges to $u_i(b, \boldsymbol{\beta}_{-i}^*(\mathbf{t}_{-i}), \mathbf{t})$ and $u_i(a'_n, \boldsymbol{\beta}_{-i}^n(\mathbf{t}_{-i}), \mathbf{t})$ converges to $u_i(a', \boldsymbol{\beta}_{-i}^*(\mathbf{t}_{-i}), \mathbf{t})$. As the type distribution is atomless, the expectations also converge, so that $U_i(b, t_i; \boldsymbol{\beta}_{-i}^*(\cdot)) \geq U_i(a', t_i; \boldsymbol{\beta}_{-i}^*(\cdot))$. □

Corollary 2.1 *Assume A1 and the SCC. Suppose that (i) for each i,* $\mathcal{A}_i = [\underline{a}_i, \overline{a}_i]$*, and (ii) for all i,* $u_i(\mathbf{a}, \mathbf{t})$ *is continuous in* $\mathbf{a}$ *on* $[\underline{a}_i, \overline{a}_i]$. *Then there exists a PSNE in nondecreasing strategies.*

Corollary 2.1 establishes that the assumption of finite or countable actions can be dispensed with for the class of games that satisfies the SCC. This result contrasts with the general finding (see Khan and Sun, 1996, 7) that PSNE may not exist when the action sets are uncountable and the type distribution is atomless (Lebesgue).[8] It can be readily verified that the counter-examples put forward by Khan and Sun (1996, 7) for this class of games fail the SCC.

4. Games with Discontinuities: Auctions and Pricing Games

Auctions and resource allocation games are perhaps the most widely studied applications of games of incomplete information. The problem is to allocate one or more goods to a subset of a group of agents, where each agent has private information about her value for the good. For example, in a first-price auction for a single good, each player submits a sealed bid after observing her type, and the highest bidder receives the good and pays her bid. However, Corollary 2.1 cannot be applied to auction games, because payoffs are not continuous. A player sees a discrete change in her payoffs depending on whether she is a "winner" or a "loser" in the auction. If any opponents use a given bid b with positive probability, then a player sees a discrete change in the probability of winning when she increases her bid above b. Thus, we say that the discontinuity arises as a result of "mass points" in the distribution over opponent actions.

The literature has focused on the existence question primarily for the case of first-price auctions. Two main approaches have been used: (i) establishing that a solution exists to a set of differential equations (Lebrun (1995), Bajari (1996a), Lizzeri and Persico (1997)), and (ii) establishing that an equilibrium exists when either types or actions are drawn from finite sets, and then invoking limiting arguments (Lebrun (1996), Maskin and Riley (1992)). This paper takes the second approach. Note that limiting approaches require additional work, since

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[8] However, Khan and Sun (1996, 1997) show that the use of atomless Loeb measure spaces for the types, as an alternative to Lebesgue, can restore the applicability of limiting arguments.

discontinuities can lead to a situation where there exists a PSNE for every finite action set, but not in the continuum-action case (for example, see Fullerton and McAfee (1996)). Our approach is to use properties of PSNE of finite-action games to show that in the limit, as the action set gets fine, no "mass points" arise and thus payoffs are continuous.

The setup is given as follows. Winners receive payoffs $\bar{v}_i(a_i,\mathbf{t})$, while losers receive payoffs $\underline{v}_i(a_i,\mathbf{t})$.[9] Thus, payoffs depend on the other players' actions only through the allocation decision. The allocation rule $\varphi_i(\mathbf{a})$ specifies the probability that the player wins as a function of the actions taken by all players. Thus, a player's payoff given a realization of types and actions is

$$\begin{aligned} u_i(\mathbf{a},\mathbf{t}) &= \varphi_i(\mathbf{a})\cdot\bar{v}_i(a_i,\mathbf{t}) + (1-\varphi_i(\mathbf{a}))\cdot\underline{v}_i(a_i,\mathbf{t}) \\ &= \underline{v}_i(a_i,\mathbf{t}) + \varphi_i(\mathbf{a})\cdot\Delta v_i(a_i,\mathbf{t}), \end{aligned} \tag{2}$$

where $\Delta v_i(a_i,\mathbf{t}) \equiv \bar{v}_i(a_i,\mathbf{t}) - \underline{v}_i(a_i,\mathbf{t})$, and player i's expected payoffs can be written as:

$$\begin{aligned} U_i(a_i,t_i;\boldsymbol{\alpha}_{-i}(\cdot)) &= \int u_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t}) f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \\ &= \int \underline{v}_i(a_i,\mathbf{t})\cdot f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} + \int \Delta v_i(a_i,\mathbf{t})\cdot\varphi_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}))\cdot f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i} \end{aligned}$$

There are several classes of examples with this structure. In a first-price auction, the winner receives the object and pays her bid, so that $\bar{v}_i(a_i,\mathbf{t})$ is the value of winning at bid a_i, while losers get payoffs of $\underline{v}_i(a_i,\mathbf{t})=0$. In a private values auction, $\bar{v}_i(a_i,\mathbf{t})$ does not vary with opponents' types, while in Milgrom and Weber's (1982) formulation of the mineral rights auction, $\bar{v}_i(a_i,\mathbf{t})$ represents the expected payoff to the bidder conditional on the vector of type realizations, and the vector $\mathbf{t}$ is interpreted as a vector of signals about each player's true value for the object (where signals and values may be correlated). In an all-pay auction, the player pays her bid no matter what, but the winner receives the object. In some pricing games, the lowest price (the highest action) implies that the firm captures a segment of price-sensitive consumers, while having a higher price implies that the firm only serves a set of local customers.

We further require that each player loses for certain when she chooses her lowest possible action. This can be interpreted as a "reservation price": only actions above the lowest possible price are considered for allocation. Attention is restricted to allocation rules which take the form:

$$\varphi_i(\mathbf{a}) = \sum_{\substack{\{\sigma_L,\sigma_T\}\subseteq\{1,..,I\}\setminus i \\ \text{s.t. } \sigma_L\cap\sigma_T=\varnothing}} \left[\mathbf{1}_{\{|\sigma_L|\geq I-k\}} + \mathbf{1}_{\{|\sigma_L|+|\sigma_T|\geq I-k>|\sigma_L|\}}\frac{1}{|\sigma_T|}\right]\cdot\prod_{j\in\sigma_L}\mathbf{1}_{\left\{\substack{a_j<a_i \\ \text{or } a_j=\underline{a}_j}\right\}}\cdot\prod_{j\in\sigma_T}\mathbf{1}_{\left\{\substack{a_j=a_i \\ \text{and } a_j>\underline{a}_j}\right\}}, \tag{3}$$

To interpret this, player i receives the object with probability zero if k or more opponents choose

[9] Intermediate outcomes could also be considered; the arguments used to establish existence extend naturally.

actions such that $a_j > a_i$, and with probability 1 if $I-k$ or more opponents choose actions such that $a_j < a_i$. The remaining events are "ties," resolved randomly. Consider the example of a first-price auction for a single object. Then, $k=1$, and the player wins with probability zero if 1 or more opponents place a higher bid. More general mechanism design problems also fall into this framework.[10]

We group together several regularity assumptions.

Assumption A2 *(i) For all i, all* $a_i \in [\underline{a}_i, \overline{a}_i]$ *and all* $\mathbf{t}$, $\overline{v}_i(a_i, \mathbf{t})$ *and* $\underline{v}_i(a_i, \mathbf{t})$ *are bounded and continuous in* $(a_i, \mathbf{t})$*; (ii) The support of* $F(\mathbf{t})$ *is a product set; (iii)* $\Delta v_i(\overline{a}_i, \overline{\mathbf{t}}) < 0$ *for all i; (iv) For all i, all* $a_i \in [\underline{a}_i, \overline{a}_i]$ *and all* $\mathbf{t}$, $\Delta v_i(a_i, \mathbf{t})$ *is strictly increasing in* $(-a_i, t_i)$*, and there exists a* $\lambda > 0$ *such that, for all* $(a_i, \mathbf{t})$ *and all* $\varepsilon > 0$, $\Delta v_i(a_i, \mathbf{t}_{-i}, t_i + \varepsilon) - \Delta v_i(a_i, \mathbf{t}_{-i}, t_i) \geq \lambda \varepsilon$.

Part (i) includes regularity assumptions. Part (ii) guarantees that any action used by a player with positive probability is viewed as having positive probability by all types of all opponents. Part (iii) guarantees that the players have available actions larger than any they would choose to use in equilibrium, and thus every action but the lowest action is an interior optimum if it is chosen. Part (iv) requires that choosing a higher action decreases the gain from winning. Further, the gain to winning is strictly increasing in the type, and there is a uniform lower bound on the slope. All of these assumptions, or more stringent ones, are standard (though sometimes implicit) in the literature on auctions.

Now consider an existence result for this class of games. We can construct a set of strategies, denoted $\boldsymbol{\beta}^*(\mathbf{t})$, as the limit of a sequence of equilibria to finite-action games, $\{\boldsymbol{\beta}^n(\mathbf{t})\}$, as in Theorem 2. Because of the potential discontinuity described above, Theorem 2 cannot be applied directly to establish that $\boldsymbol{\beta}^*(\mathbf{t})$ is a PSNE of the continuum-action game. However, if we can prove that there are no mass points in the limit (that is, no action is used with positive probability), continuity will be restored and the arguments of Theorem 2 can be applied.

Our approach (detailed formally in the Appendix) is to rule out mass points using the fact that $\{\boldsymbol{\beta}^n(\mathbf{t})\}$ converges to $\boldsymbol{\beta}^*(\mathbf{t})$ uniformly except on a set of arbitrarily small measure, together with the fact that we can characterize certain properties of each $\boldsymbol{\beta}^n(\mathbf{t})$ based on the fact that it is a PSNE. We proceed by contradiction: suppose that two players, *i* and *j*, both use action *b* with positive probability. But this is inconsistent with equilibrium, since player *j* could increase her payoff by increasing her action just above *b*, thereby winning against instead of tying with opponent types who use *b*.

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[10] When a player's payoffs satisfy the single crossing property, only direct mechanisms in which the allocation rule is monotonic can be incentive compatible; then, we can let the player's announcement of type be her action, and redefine payoffs to incorporate the allocation rule of the mechanism.

To rule this out, consider the requirements that they place on $\{\boldsymbol{\beta}^n(\mathbf{t})\}$. The argument can be roughly sketched as follows. Note that if player i's limiting strategy requires a subset S_i of types to use action b, then given a $d>0$, all but a small subset of S_i must be using an action on $[b-d,b+d]$ for a fine enough action grid. But then we will argue that there must be some action grid fine enough, and some d small enough, such that the "cost" to player j of increasing her action by d is less than the benefit from the increase in the probability of defeating most of the types $t_j \in S_i$.

However, to complete the latter argument, it is necessary to establish that increasing the probability of winning (for example, by increasing the action from $b-d$ to $b+d$) is in fact a benefit. That requires two building blocks. First, we wish to begin from the premise that if a player is using action $b-d$, she sees a positive gain from winning at $b-d$. The issue is that it is possible that the player wins with probability 0 at $b-d$, in which case nothing about her preferences for winning at $b-d$ can be inferred. To see an example where this would be problematic, suppose that a set S_i of player i types uses action $b-\varepsilon_n$ and a set $[\underline{t}_j, t_j']$ of player j types bids b for all n, where $\varepsilon_n \to 0$, so that $b-\varepsilon_n$ wins with probability zero for player i. Then the limiting strategies require player i to choose b and win with positive probability, which may not be optimal. To rule out such a scenario, we show that for every finite-action game, there exists a PSNE that is robust to perturbations that generate a small probability that each action wins. To state the result, we use the following additional notation: let $W_i(a_i;\boldsymbol{\alpha}_{-i})$ denote the event that the realization of $\mathbf{t}_{-i}$ and the tie-breaking mechanism $\varphi_i(\mathbf{a})$ are such that action a_i wins when opponents use strategies $\boldsymbol{\alpha}_{-i}$.

Lemma 4 *For i=1,..,I, let* $\mathcal{A}_i$ *be finite. Consider an auction game with payoffs given by (2)-(3), and assume that A1 and the SCC hold. (i) For all* $\delta>0$*, there exists a PSNE in nondecreasing strategies of the modified game where each player type on* $t_i \in [\underline{t}_i, \underline{t}_i+\delta]$ *is required to use action* $\underline{a}_i$*. (ii) There exists a PSNE in nondecreasing strategies,* $\boldsymbol{\beta}^*$ *that satisfies the following: There exists a sequence* $\{\delta_k\}$*, where* $\lim_{k\to\infty}\delta_k = 0$*, and a corresponding sequence* $\{\hat{\boldsymbol{\beta}}(\cdot;\delta_k)\}$ *of PSNE strategies of the game modified as in (i), such* $\hat{\boldsymbol{\beta}}(\cdot;\delta_k)\to\boldsymbol{\beta}^*$*, and further, whenever* $\beta_i(t_i)>\underline{a}_i$*,*
$\lim_{k\to\infty} E\left[\Delta v_i(\hat{\beta}_i(t_i;\delta_k),\mathbf{t}) \middle| t_i,\ W_i(\hat{\beta}_i(t_i;\delta_k);\hat{\boldsymbol{\beta}}_{-i}(\cdot;\delta_k))\right] \geq 0.$

The second building block guarantees that player i's expected gain from winning goes up when player i increases her action:

For all i=1,..,I, all $a_i, a_i' \in [\underline{a}_i, \bar{a}_i]$, and whenever every opponent $j \neq i$ uses a strategy $\alpha_j : [\underline{t}_j, \bar{t}_j] \to \mathcal{A}_j$ that is nondecreasing, $E\left[\Delta v_i(a_i,\mathbf{t}) \middle| t_i,\ W_i(a_i';\boldsymbol{\alpha}_{-i})\right]$ is strictly increasing in t_i and nondecreasing in a_i'. (4)

This assumption is also standard in the literature, although it is in fact fairly restrictive.

Although it holds trivially for private value auctions, it requires strong assumptions on the type distribution in more general models. For single-unit auctions with symmetric bidders, Milgrom and Weber (1982) showed that when Δv_i is nondecreasing in **t**, (4) holds if the types are affiliated. Using properties of affiliated random variables, Milgrom and Weber's (1982) result can be extended to show that (4) holds for single-unit auctions with asymmetric bidders, or (as shown in Pesendorfer and Swinkels (1997)) for multi-unit auctions with symmetric bidders.

With these two building blocks in place, we can rule out the possibility that two players both use the same action with positive probability. Another case of potential concern occurs if just one player, player *i*, uses action *b* with positive probability. Although this case can be ruled out using somewhat more involved arguments, we will not undertake this exercise here because it does not affect existence of PSNE: only a countable number of actions can be used by player *i* with positive probability, and thus the set of opponent types who see a discontinuity in payoffs has measure zero. Since Bayesian Nash equilibrium only requires that almost every type chooses an optimal action, such mass points do not affect existence. The conclusion then follows:

Theorem 3 *For all i, let* $\mathcal{A}_i=[\underline{a}_i,\overline{a}_i]$. *Consider an auction game with payoffs described by (2)-(3), and assume A1, A2, (4), and that the game satisfies the SCC. Then, there exists a PSNE in nondecreasing strategies.*

Theorem 3 generalizes the best available existence results about first-price auctions. Previous studies (Maskin and Riley (1993, 1996); Lebrun (1995, 1996), Bajari (1996a)) have analyzed independent private values auctions, as well as affiliated private values auctions and common value auctions with conditionally independent signals about the object's value (Maskin and Riley (1993, 1996). The work closest to ours is Lizzeri and Persico (1997), who have independently established existence and uniqueness of equilibria in a class of games similar to the one studied above, but with the restriction to two bidders. The approach taken in this paper is different from those of the existing literature, in that it separates out the issue of monotonicity of strategies and existence, showing that monotonicity implies existence. Thus, the only role played by assumptions about the joint distribution over types is to guarantee that the single crossing property holds.

Once existence is established for the continuous-action game, standard arguments can be used to verify the usual regularity properties (including optimality of actions for every type). For example, strategies are strictly increasing on the interior of the set of actions played with positive probability, and no player sees a gap in the set of actions played with positive probability by opponents. Further, with appropriate differentiability assumptions, a differential equations approach can be used for characterizations.

It is important to highlight that Theorems 2 and 3 not only provide existence results for the continuum case; they also establish that finite-action games can be used to closely approximate continuum-action games. Thus, the continuum model is an appropriate abstraction for auctions in which fixed bid increments are specified. As well, revenue and allocation in an auction will not be very sensitive to small changes in the number of bid increments allowed.

Perhaps more importantly, the convergence results also motivate a computational algorithm. This may be particularly useful in the case of first-price auctions, since in the absence of general characterization theorems and functional form examples, computation of equilibria to first-price auctions is the main tool available to evaluate the effects of mergers (or collusion) between bidders in auctions. However, prior to Marshall et al (1994), there were no general numerical algorithms available for computing equilibria to asymmetric first-price auctions. Numerical computation of equilibria in asymmetric first-price auctions in the independent private values case is difficult due to pathological behavior of the system of differential equations at the origin. Marshall et al (1994) provide computations for the independent private values case for a particular functional form; see also Bajari (1996a, b).

Theorem 3 suggests an alternative: compute equilibria to games with successively finer action sets. The computation of a finite-action equilibrium requires searching for a fixed point to the correspondence Γ defined in Section 2, where the calculation of $\Gamma(\mathbf{X})$ is a simple exercise of calculating the best-response jump points for each player. The more difficult part of the problem is solving the nonlinear set of equations $\mathbf{X}=\Gamma(\mathbf{X})$. There are a number of standard ways to approach this problem. There is not a global "contraction mapping" theorem, and so the simplest algorithm $\mathbf{X}^{k+1}=\Gamma(\mathbf{X}^k)$ is not guaranteed to converge, and indeed it does not appear to in numerical trials. However, the working paper (Athey, 1997) provides a number of computational examples which could be computed using either variations on the algorithm $\mathbf{X}^{k+1}= \lambda\cdot\Gamma(\mathbf{X}^k) + (1-\lambda)\cdot\mathbf{X}^k)$), or quasi-Newton approaches.[11]

5. Characterizing the Single Crossing Condition in Applications

This section characterizes the single crossing condition in several classes of games of incomplete information. The results make use of the properties *supermodularity* and *log-supermodularity*.[12] A function $h:X\rightarrow\Re$ is **supermodular** if, for all $\mathbf{x},\mathbf{y}\in X$,

[11] While quasi-Newton methods might at first seem computationally burdensome, there are potentially large computational benefits to using an analytic Jacobian. In particular, the point at which player i jumps to action A_m, denoted x^i_m, affects only the following elements of the best response of opponent $j\neq i$: x^j_{m-1}, x^j_m, and x^j_{m+1}. Thus, the Jacobian (of dimension $M\cdot I\times M\cdot I$) can be computed with only $I\cdot 3$ function calls.

[12] The operations "meet" ($\vee$) and "join" ($\wedge$) are defined for product sets as follows: $\mathbf{x}\vee\mathbf{y}=(\max(x_1,y_1),\ldots,\max(x_n,y_n))$ and $\mathbf{x}\wedge\mathbf{y}=(\min(x_1,y_1),\ldots,\min(x_n,y_n))$.

$h(\mathbf{x} \vee \mathbf{y}) + h(\mathbf{x} \wedge \mathbf{y}) \geq h(\mathbf{x}) + h(\mathbf{y})$. A non-negative function $h{:}X \rightarrow \Re$ is **log-supermodular** if, for all $\mathbf{x},\mathbf{y} \in X$, $h(\mathbf{x} \vee \mathbf{y}) \cdot h(\mathbf{x} \wedge \mathbf{y}) \geq h(\mathbf{x}) \cdot h(\mathbf{y})$. Recall that when $h{:}\Re^n \rightarrow \Re$, and vectors are ordered in the usual way, Topkis (1978) proves that if h is twice differentiable, h is supermodular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} h(\mathbf{x}) \geq 0$ for all $i \neq j$.

Five facts together can be used to establish our characterization theorems: (i) if $h(x,t)$ is supermodular or log-supermodular, then $h(x,t)$ satisfies SCP-IR; (ii) sums of supermodular functions are supermodular, while products of log-supermodular functions are log-supermodular; (iii) if $h(\mathbf{x})$ is supermodular (resp. log-supermodular), then so is $h(\alpha_1(x_1),..,\alpha_n(x_n))$, where $\alpha_i(\cdot)$ is nondecreasing; (iv) a density is log-supermodular almost everywhere if and only if the random variables are *affiliated* (as defined in Milgrom and Weber, 1982); (v) if $\mathbf{t}$ is affiliated, and further $h(x,\mathbf{t})$ is supermodular in (x,t_j) for all j (resp. log-supermodular in $(x,\mathbf{t})$), then $H(x,t_i) = \int_{\mathbf{a}}^{\mathbf{b}} h(x,\mathbf{t}) f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i}$ is supermodular in (x,t_i) (resp. log-supermodular in $(x,t_i,\mathbf{a},\mathbf{b})$) (see Athey 1998a, b).[13]

5.1. General Classes of Games

First consider a general formulation of games with supermodular payoffs.

Theorem 4 *Assume A1, and suppose (i) for all i,* u_i *is supermodular in* $\mathbf{a}$ *and* (a_i,t_j)*, j=1,..,I, and (ii) the types are affiliated. Then the game satisfies the SCC.*

Theorem 4 is particularly applicable in games with additively separable payoffs (for example, when an investment has an additively separable cost, as when $u_i(\mathbf{a},\mathbf{t})=h_i(\mathbf{a})-c_i(a_i,t_i)$ or $u_i(\mathbf{a},\mathbf{t})=h_i(\mathbf{a},t_i)-c_i(a_i)$), since supermodularity is preserved by sums. Many of the supermodular games that have been studied under assumptions of complete information (see Topkis (1979), Vives (1990), and Milgrom and Roberts (1990) for examples) can be extended to the case of incomplete information using Theorem 4. For example, many oligopoly games have variations where firms have incomplete information about their rivals production costs or demand elasticity. Games between two players whose choices are strategic substitutes can also be considered, such as a Cournot quantity game between two firms.[14]

However, it should be noted that the games studied in Theorem 4 also satisfy Vives' (1990) sufficient condition for existence of PSNE. The next class of games does not.

Theorem 5 *Assume A1, and suppose (i) for all i,* $u_i(\mathbf{a},\mathbf{t})$ *is nonnegative and log-supermodular,*

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[13] For the case where payoffs are supermodular, the assumption that types are affiliated can be weakened; what is actually required is that for each i, $\int_S f(\mathbf{t}_{-i}|t_i)d\mathbf{t}_{-i}$ is nondecreasing in t_i for all sets S whose indicator function is nondecreasing in $\mathbf{t}_{-i}$. See Athey (1998a).

[14] See Fudenberg and Tirole, 1991, pp. 215-216 for an example with linear demand and incomplete information about cost.

and (ii) the types are affiliated. Then the game satisfies the SCC.

Theorem 5 may be especially useful in games with multiplicatively separable payoffs (for example, when $u_i(\mathbf{a},\mathbf{t})=h_i(\mathbf{a})\cdot c_i(a_i,t_i)$ or $u_i(\mathbf{a},\mathbf{t})=h_i(\mathbf{a},t_i)\cdot c_i(a_i)$), since log-supermodularity is preserved by multiplication. For example, this result can be applied to pricing games with incomplete information (analyzed in the case of homogenous players by Spulber (1995)). Let t_i represent the (constant) marginal cost of firm i, let a_i be the price, let demand to firm i be $D^i(a_1,..,a_I)$, and suppose payoffs are given by $(a_i-t_i)\cdot D^i(a_1,..,a_I)$. Notice first that (a_i-t_i) is log-supermodular. Athey (1998b) shows that when opponents use nondecreasing pricing functions, expected demand is log-supermodular if the cost parameters are affiliated and $D^1(a_1,..,a_I)$ is log-supermodular.[15]

5.2. *Auctions*

This section studies a variety of auction games, all of which satisfy the additional conditions required to apply Theorem 3. Consider first private-value first-price auctions. Payoffs can be written in the notation of Section 4: $\bar{v}_i(a_i,\mathbf{t})=V_i(t_i-a_i)$ and $\underline{v}_i(a_i,\mathbf{t})=0$. To satisfy A2, in this subsection we maintain the assumption that each V_i is bounded, strictly increasing with non-vanishing slope, and continuous. Each player's expected utility can then be written $V_i(t_i-a_i)\cdot\Pr(\text{win with } a_i|\,\alpha_{-i},t_i)$. If $V_i(0)\geq 0$ and $\ln(V_i)$ is concave, V_i is log-supermodular in (t_i,a_i). It can also be verified (see the working paper (Athey, 1997)) that $\Pr(\text{win with } a_i|\,\alpha_{-i},t_i)$ is log-supermodular when f is log-supermodular, opponents use nondecreasing strategies, and ties are resolved randomly. This implies that expected payoffs are log-supermodular and further, given strict monotonicity of V_i, the SCC holds.

The results also apply to private values, first-price auctions for multiple units, where each agent demands a single unit. For example, in a 2-unit auction, the players with the highest two bids win an object and pay their bids. Unfortunately, this complicates the analysis of log-supermodularity of the function $\Pr(a_i \text{ wins}|t_i)$. If the types are drawn independently, then $\Pr(a_i \text{ wins}|t_i)$ does not depend on t_i, and the expected payoff function reduces to $V_i(t_i-a_i)\cdot\Pr(a_i \text{ wins})$; the properties of this objective are then the same as in the single-unit auction analyzed above.

Another example is the all-pay auction, which has been used to model activities such as lobbying. In this auction, the highest bidder receives the object, but all bidders pay their bids. In an independent private values formulation, $\bar{v}_i(a_i,\mathbf{t})=V_i(t_i-a_i)$ and $\underline{v}_i(a_i,\mathbf{t})=V_i(-a_i)$ for some nondecreasing V_i. Expected payoffs from action a_i are then $(V_i(t_i-a_i)-V_i(-a_i))\cdot\Pr(a_i$

[15] The interpretation of the latter condition is that the elasticity of demand is a non-increasing function of the other firms' prices. As discussed in Milgrom and Roberts (1990b), demand functions which satisfy this criteria include logit, CES, transcendental logarithmic, and a set of linear demand functions (see Topkis (1979)).

wins)+$V_i(-a_i)$. Since Pr(a_i wins) is nonnegative and nondecreasing in a_i, it is straightforward to verify that the second term of this expression is supermodular if $V_i(t_i-a_i)-V_i(-a_i)$ is nondecreasing in t_i (which follows since V_i is increasing) and supermodular in (a_i,t_i). In turn, $V_i(t_i-a_i)$ is supermodular if and only if it is concave (the bidder is risk averse). Supermodularity (together with our assumption that V_i is strictly monotone) implies that the SCC holds.

Finally, consider mineral rights auctions. We introduce an auxiliary result that can be used to analyze games that do not fit into the classes of supermodular or log-supermodular games analyzed in Theorems 4 and 5. The result applies if either (i) there are only two players, or if (ii) $u_i(\mathbf{a},\mathbf{t})$ takes a very special form. In particular, u_i must depend on the opponents' types and actions through a single index, denoted s_i. That is, $u_i(a_i,\boldsymbol{\alpha}_{-i}(\mathbf{t}_{-i}),\mathbf{t}) = k_i(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i})$. In a first-price mineral rights auction with identical bidders using symmetric strategies, $\alpha_j(\cdot)=\alpha_l(\cdot)$ for all $j,l\neq i$, s_i is the value of the highest opponent type, and payoffs depend on opponent types only through the realization of this type and the associated action. Then (ignoring ties for notational simplicity), $k_i(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i}) = E_{\mathbf{t}_{-i}}\left[v_i(a_i,\mathbf{t})|\max\{t_j:j\neq i\}=s_i\right]\cdot\mathbf{1}_{[a_i>\alpha_j(s_i)]}(s_i)$. In a multi-unit auction, s_i might be a different order statistic of the distribution. In other applications, s_i might be a sufficient statistic for $\mathbf{t}_{-i}$.

Theorem 6 *Suppose that for all i=1,..I, there exists a random variable s_i and a family of functions $k_i(\cdot;\boldsymbol{\alpha}_{-i}):\Re^3\rightarrow\Re$, such that (i) $U_i(a_i,\boldsymbol{\alpha}_{-i}(\cdot),t_i)=E_{s_i}[k_i(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i})|t_i]$; (ii) when $\boldsymbol{\alpha}_{-i}(\cdot)$ is nondecreasing, $k_i(a_i,s_i,t_i;\boldsymbol{\alpha}_{-i})$ is supermodular in $(a_i;t_i)$ and satisfies weak SCP-IR in $(a_i;s_i)$; and (iii) t_i and s_i are affiliated, and the support of s_i is constant in t_i. Then the game satisfies SCC. Further, (i)-(iii) imply that if $\boldsymbol{\alpha}_{-i}(\cdot)$ is nondecreasing, $U_i(a_i,\boldsymbol{\alpha}_{-i}(\cdot),t_i)$ satisfies strict SCP-MR wherever it is differentiable in a_i and $\frac{\partial}{\partial a_i}k_i$ is non-zero on a set of s_i of positive measure.*

A proof of Theorem 6 can be found in Athey (1998b). A first application these results extends Milgrom and Weber's (1982) model of a mineral rights auction, allowing for risk averse, asymmetric bidders whose utility functions are not necessarily differentiable. Athey (1998b) shows that in this context, if there are two bidders whose types are affiliated, and the utility functions $\bar{v}_i(a_i,\mathbf{t})$ are supermodular in (a_i,t_j), j=1,2, and nondecreasing in $\mathbf{t}$, the conditions of Theorem 6 are satisfied and the SCC holds. Further, if each $\bar{v}_i$ is bounded, decreasing and differentiable in a_i and increasing in t_i with non-vanishing slope, then the auction game will be admissible. The last statement of Theorem 6 can be used to establish the SCC.

5.3. *Noisy Signaling Games*

Our results about existence of PSNE can also be applied to games with alternative timing assumptions. For example, consider a signaling game between two players, where player 1's utility is given by $u_1(a_1,a_2,t_1)$ and player 2's utility is given by $u_2(a_1,a_2)$. After observing her type (for example, marginal cost or a parameter of demand), Player 1 takes an action which generates a noisy signal, t_2, that is observed by player 2. Player 2 then takes an action. An example is a game of limit pricing (Matthews and Mirman (1983)), where an entrant does not know the cost of the incumbent, but can draw inferences about the incumbent's cost by observing a noisy signal of the incumbent's product market decision (the noise might be due to demand shocks). In another example, Maggi (1998) examines the extent to which noise in the signaling process undermines the first mover advantage in commitment games. Different assumptions about the nature of product market competition lead to different properties of payoffs. Even if Theorems 4 and 5 do not apply, a corollary of Theorem 6 can be used:

Corollary 6.1 *Suppose that there are two players, i=1,2. Suppose that (i) for i=1,2,* $u_i(\mathbf{a},\mathbf{t})$ *satisfies weak SCP-IR in* (a_i,t_j) *and* (a_i,a_j)*, and is supermodular in* (a_i,t_i)*, and (ii) the types are affiliated with non-moving support. Then the game satisfies SCC.*

6. Conclusions

This paper has introduced a restriction on a class of games called the single crossing condition (SCC) for games of incomplete information. We have shown that PSNE exist in such games when the set of available actions is finite. Further, with appropriate continuity or in auction games, there exists a sequence of equilibria of finite-action games that converges to an equilibrium in a game with a continuum of actions. The results developed in this paper have the following advantages. First, existence of PSNE can be verified by checking general, economically interpretable conditions, conditions which are satisfied by construction in many economic applications. Second, for games where the SCC is satisfied, the results provide a significant generalization of the scenarios under which PSNE can be shown to exist. Third, the SCC is straightforward to verify, as shown in Section 4 (see Athey (1998a, b) for more details). Finally, the constructive approach to existence taken in this paper has advantages for numerical computation. The equilibria are straightforward to calculate for finite-action games, and these approximate the continuous equilibria for continuous games and auctions.

7. Appendix

Proof of Lemma 4: (i) Define the constrained best response of player i to an arbitrary (constrained or unconstrained) strategy by opponents represented by $\mathbf{X}$: $\hat{a}_i^{BR}(t_i|\mathbf{X},\delta)$ is set equal to $a_i^{BR}(t_i|\mathbf{X})$ when $t_i > \underline{t}_i + \delta$, to $\{\underline{a}_i \cup a_i^{BR}(t_i|\mathbf{X})\}$ when $t_i = \underline{t}_i + \delta$, and $\{\underline{a}_i\}$ when $t_i < \underline{t}_i + \delta$.

Using this notation, we modify our correspondence, as follows:

$$\hat{\Gamma}_i(\mathbf{X},\delta)=\{\mathbf{y}\in\Sigma^{M+2}: \exists\ \alpha_i(t_i) \text{ which is consistent with } \mathbf{y} \text{ such that } \alpha_i(t_i)\in\ \hat{a}_i^{BR}(t_i|\mathbf{X},\delta)\}.$$

The arguments of Theorem 1 can be extended in a straightforward way to derive existence of a fixed point of $\hat{\Gamma}(\mathbf{X},\delta)$ for all $\delta>0$; details can be found in the working paper (Athey, 1997). Then, consider a sequence $\mathbf{X}^k$ such that $\mathbf{X}^k \in \hat{\Gamma}(\mathbf{X}^k,1/k)$ for each k. Since each $\mathbf{X}^k$ is an element of a compact subset of finite-dimensional Euclidean space, we can find a subsequence $\{k\}$ such that $\{\mathbf{X}^k\}$ converges to a matrix $\mathbf{X}$, and we simply need to establish that $\mathbf{X}\in\Gamma(\mathbf{X})$. Consider t_i such that $t_i\in T_i\backslash\{\mathbf{x}^i\}$. Then there exists an $m\in\{0,..,M\}$ such that $x^i_m < t_i < x^i_{m+1}$. Since $\mathbf{x}^{i,k}$ converges to $\mathbf{x}^i$, there must exist an K such that, for all $k>K$, $x^{i,k}_m < t_i < x^{i,k}_{m+1}$ and $t_i > \underline{t}_i + 1/k$. Find such a $k>K$. Then $A_m\in \hat{a}^{BR}_i(t_i|\mathbf{X}^k, 1/k)$ since $\mathbf{X}^k\in\Gamma(\mathbf{X}^k)$. By definition and since $t_i>\underline{t}_i+1/k$, $A_m\in a^{BR}_i(t_i|\mathbf{X}^k)$. But, since $V_i(a_i;\mathbf{X},t_i)$ is continuous in $\mathbf{X}$, if $V_i(A_m;\mathbf{X}^k,t_i)\geq V_i(A_{m'};\mathbf{X}^k,t_i)$ for all $k>K$ and all m', then $V_i(A_m;\mathbf{X},t_i)\geq V_i(A_{m'};\mathbf{X},t_i)$. This implies $A_m\in a^{BR}_i(t_i|\mathbf{X})$, as desired. (ii) Finally, observe that for each k, each action other than $\underline{a}_i$ wins with positive probability. Since $\underline{a}_i$ is an available action that yields zero payoffs, by revealed preference, $E\left[\Delta v_i(\hat{\beta}_{i,n}(t_i;\delta_k),\mathbf{t})\middle|t_i, W_i(\hat{\beta}_{i,n}(t_i;\delta_k);\hat{\boldsymbol{\beta}}_n(\cdot;\delta_k))\right]\geq 0$ for all k. Since payoffs are continuous in δ_k, the limit as δ_k approaches zero exists. □

Lemma 5: *Consider an auction game satisfying A1, A2, (2)-(4), and the SCC. Let* $\boldsymbol{\beta}^*(\mathbf{t})$ *be the limit of a convergent subsequence of equilibrium strategies to finite games,* $\boldsymbol{\beta}^n(\mathbf{t})$*, that satisfy Lemma 4 (ii). Then the following holds:*

For all i, for all $a_i\in\mathcal{A}_i$, $a_i>\underline{a}_i$, $\Pr(\beta^*_i(t_i) = a_i)\cdot\Pr(\text{if player } i \text{ uses } a_i, \text{ she ties for winner}) = 0$. (5)

Proof of Lemma 5: To begin we introduce some notation for the event that, when players $\{I\}\backslash i$ use strategies $\boldsymbol{\beta}^n_{-i}(\cdot)$ (or, in the limit $\boldsymbol{\beta}^*(\mathbf{t})$), the realization of $\mathbf{t}_{-i}$ and the outcome of the tie-breaking mechanism are such that the action a_i produces the stated outcome:

$W^n_i(a_i)$, $W^*_i(a_i)$: Player i wins using a_i (either by winning a tie or winning outright).

$\tau^{W*}_i(a_i)$, $\tau^{L*}_i(a_i)$: Player i ties for winner at a_i and player i wins (resp. loses) the tie.

For the case where $W^n_i(\beta^n_i(t_i))$ occurs with probability 0, we use the following convention:

$$E\left[\Delta v_i(\beta^n_i(t_i),\mathbf{t})\middle|t_i, W_i(\beta^n_i(t_i))\right] = \lim_{k\to\infty} E\left[\Delta v_i(\hat{\beta}^n_i(t_i;\delta_k),\mathbf{t})\middle|t_i, W^n_i(\hat{\beta}^n_i(t_i;\delta_k))\right],$$

where $\{\delta_k\}$ and $\hat{\boldsymbol{\beta}}^n(\cdot;\delta_k)$ are defined in Lemma 4.

We proceed by contradiction. Assume (5) fails, and $\Pr(\beta^*_i(t_i) = a_i)\cdot\Pr(\tau^{L*}_i(b)) > 0$ for some i and some b. To begin the proof, let $K=\{i:\Pr(\beta^*_i(t_i) = b)>0\}$ (note $|K|\geq 2$ follows from the assumption (5) fails), and let $J=\{1,..,I\}\backslash K$. Recall that for each n, $\beta^n_i(t_i)$ is measurable, and the sequence converges almost everywhere to $\beta^*_i(t_i)$, and so the sequence converges uniformly to $\beta^*_i(t_i)$ except on a set of arbitrarily small measure (Royden, 1988, p. 73).

A little notation. Let ε_n be the minimum increment to actions (assumed for simplicity to be the same for all players and independent of the level of the action) for the action set $\mathcal{A}$. Find a set of types $\mathbf{E}$ such that $\boldsymbol{\beta}_n(\cdot)$ converges uniformly to $\boldsymbol{\beta}^*(\cdot)$ except for types $\mathbf{t}\in\mathbf{E}$, and such that for all $i\in K$, $\sup_{t_i}\{\Pr(\mathbf{t}_{-i}\in\mathbf{E}_{-i}|t_i)\}<\frac{1}{10}\Pr(\tau^{L*}_i(b))$. For $i\in K$, let $S_i=\{t_i:\beta^*_i(t_i) = b\}$, let $\tilde{S}_i = \operatorname{int}\{t_i\in S_i\backslash E_i\}$, and let $\tilde{t}_i = \inf\tilde{S}_i$.

For any d>0, define N_d as the smallest positive integer such that, for all $n>N_d$, (i) for all i and all $t_i \notin E_i$, $|\beta_i^n(t_i) - \beta_i^*(t_i)| < d$, and (ii) for all $i \in K$, $\Pr\big(\beta_j^n(t_j) \in (b-d, b+d)\ \forall j = J\big) < \frac{1}{10}\Pr(\tau_i^{L*}(b))$. This N_d exists for d small enough, since $\Pr(\tau_i^{L*}(b))>0$ by assumption for the specified players, and since $\Pr\big(\beta_j^*(t_j) = b\ \forall j = J\big) = 0$ by assumption.

We proceed in a series of claims.

Claim 1: There exists a D>0 such that for all $i \in K$, all $t_i \in \tilde{S}_i$, and all $d<D$:

$$\Pr\big(W_i^n(b+d) \setminus W_i^n(b-d+\varepsilon_n) \big| t_i\big) > \tfrac{1}{2}\left(\Pr\big(\tau_i^{L*}(b)\big|t_i\big) - \sup_{t_i}\{\Pr(\mathbf{t}_{-i} \in \mathbf{E}_{-i})\}\right) \text{ for } n>N_d. \tag{6}$$

Proof of Claim 1: All opponent types outside $\mathbf{E}_{-i}$ who choose action b in the limit must choose actions on $[b-d+\varepsilon_n, b+d)$ for $n>N_d$. By choosing action $b+d$ rather than $b-d+\varepsilon_n$, player i chooses a strictly higher action than those types she would lose to or tie with using action $b-d+\varepsilon_n$; at worst, all of the types who choose action b in the limit choose action $b-d+\varepsilon_n$ with grid n, and player i defeats those players she would have otherwise tied with by increasing her action to $b+d$. The coefficient $\frac{1}{2}$ creates additional "slack" that will be utilized below.

Claim 2: For all ψ>0, there exists D>0 such that, for all $i \in K$ and all $t_i \in \tilde{S}_i$ such that $t_i > \tilde{t}_i + \psi$:

$$\inf_{0>d>D}\inf_{n>N_d} E\big[\Delta v_i(b+d, \mathbf{t}) \big| t_i, W_i^n(b+d) \setminus W_i^n(b-d+\varepsilon_n)\big] > \lambda\psi/4. \tag{7}$$

Proof of Claim 2: Let $t_i' = \tilde{t}_i + \psi/2 \in \tilde{S}_i$. Then $\inf_{0>d>D}\inf_{n>N_d} E\big[\Delta v_i(\beta_i^n(t_i'), t_i', \mathbf{t}_{-i}) \big| t_i', W_i^n(b+d)\big] \geq 0$, by Lemma 4 and since winning with $b+d$, an action higher than the assigned one, increases payoffs by (4). Combining this inequality with the lower bound on the slope of Δv_i and the fact that $t_i > \tilde{t}_i + \psi$ yields $\inf_{0>d>D}\inf_{n>N_d} E\big[\Delta v_i(\beta_i^n(t_i'), t_i, \mathbf{t}_{-i}) \big| t_i, W_i^n(b+d)\big] > \lambda\psi/2$. But, since payoffs are continuous in actions, it is possible to find D>0 small enough such that $\inf_{0>d>D}\inf_{n>N_d} E\big[\Delta v_i(b+d, t_i, \mathbf{t}_{-i}) \big| t_i, W_i^n(b+d)\big] > \frac{1}{2}\lambda\psi/2$. Finally, winning with $b+d$ but not with $b-d+\varepsilon_n$ gives higher expected payoffs than winning with $b+d$ by (4). Thus, (7) holds.

Claim 3: For all ψ>0, there exists D>0 such that, for all $i \in K$, $t_i \in \tilde{S}_i$ such that $t_i > \tilde{t}_i + \psi$, and all $0<d<D$ and $n>N_d$: $\beta_i^n(t_i) > b-d+\varepsilon_n$.

Proof of Claim 3: Player i with type t_i prefers action $b+d$ to a' if:

$$\begin{aligned} &E\big[\Delta v_i(b+d, \mathbf{t}) \big| t_i, W_i^n(b+d) \setminus W_i^n(a')\big] \cdot \Pr\big(W_i^n(b+d) \setminus W_i^n(a') \big| t_i\big) \\ &\quad > \int [\underline{v}_i(a', \mathbf{t}) - \underline{v}_i(b+d, \mathbf{t})] \cdot f(\mathbf{t}_{-i}|t_i) d\mathbf{t}_{-i} + E\big[\Delta v_i(a', \mathbf{t}) - \Delta v_i(b+d, \mathbf{t}) \big| t_i, W_i^n(a')\big] \cdot \Pr\big(W_i^n(a') \big| t_i\big) \end{aligned} \tag{8}$$

But, we have already established in Claims 1 and 2 that when $a' = b-d+\varepsilon_n$, the LHS of (8) is always larger than $\frac{1}{4}\lambda\psi \cdot \left(\Pr\big(\tau_i^{L*}(b)\big|t_i\big) - \sup_{t_i}\{\Pr(\mathbf{t}_{-i} \in \mathbf{E}_{-i})\}\right) > 0$. The RHS of (8) becomes arbitrarily small as d gets small by continuity of payoffs in actions. Thus, for small enough d, (8) holds.

Claim 4: There exists a Ψ>0 such that, for all 0<ψ<Ψ, there exists a d>0 and a positive integer $n>N_d$ such that for all $i\in K$ and all $t_i\in \tilde{S}_i$ such that $t_i\geq\tilde{t}_i+\psi$, $\beta_i^n(t_i)\notin (b-d,b+d)$.

Proof of Claim 4: The logic of Claims 1-3 can be applied again. Find a d>0 and an n such that (6) and (7) hold, and such that the RHS of (8) is less than $\frac{1}{4}\lambda\psi\cdot\left(\Pr\left(\tau_i^{L*}(b)\middle|t_i\right)-\sup_{t_i}\{\Pr(\mathbf{t}_{-i}\in\mathbf{E}_{-i})\}\right)$ for all i and all $t_i\geq\tilde{t}_i+\psi$ such that $t_i\in\tilde{S}_i$. Now, replace $b-d+\varepsilon_n$ with $b-d+2\varepsilon_n$ in (6) and (7). Notice that, for ψ small enough, (6) still holds, since by Claim 3, the set of types using $b-d+\varepsilon_n$ is decreasing in ψ. (7) holds replacing $b-d+\varepsilon_n$ with any action less than b+d. Since the RHS of (8) becomes smaller when a' increases, (8) must hold for all i and all $t_i\geq\tilde{t}_i+\psi$ such that $t_i\in\tilde{S}_i$. Thus, the bidding unravels all the way up to $b+d-\varepsilon_n$.

Finally, observe that Claim 4 establishes that for small enough d and large enough n, an arbitrarily small subset of types use actions on $(b-d,b+d)$. This contradicts the hypothesis of almost-everywhere uniform convergence to strategies that yield $\Pr(\beta_i^*(t_i)=a_i)\cdot\Pr(\tau_i^{L*}(b))>0$, and the Lemma is proved. □

Lemma 6 *Consider an auction game satisfying A1, A2, (2)-(4), and the SCC. Let $\boldsymbol{\beta}^*(\mathbf{t})$ be the limit of an almost-everywhere convergent sequence of nondecreasing PSNE strategies to finite-action games, $\boldsymbol{\beta}^n(\mathbf{t})$, that satisfy Lemma 4 (ii). (i) If $a_i^n\to b$ and $U_i(a_i,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))$ is continuous at $a_i=b$, $U_i(a_i^n,t_i;\boldsymbol{\beta}_{-i}^n(\cdot))$ converges to $U_i(b,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))$. (ii) For all i and almost every t_i, $U_i(a_i,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))$ is continuous in a_i at $a_i=\beta_i^*(t_i)$.*

Proof of Lemma 6: Part (i): First, note that

$$U_i(b,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))-U_i(a_i^n\,\beta_i^n(t_i),t_i;\boldsymbol{\beta}_{-i}^n(\cdot))$$

$$=[U_i(b,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))-U_i(a_i^n,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))]+[U_i(a_i^n,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))-U_i(a_i^n,t_i;\boldsymbol{\beta}_{-i}^n(\cdot))]. \quad \text{(A5)}$$

The first term of the RHS of (A5) goes to zero as n gets large by continuity of $U_i(a_i,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))$ in a_i at b. So it remains to consider the second term of the RHS of (A5). This term converges to zero if $U_i(a_i,t_i;\boldsymbol{\beta}_{-i}^n(\cdot))$ converges uniformly (across a_i in a neighborhood of b) to $U_i(a_i,t_i;\boldsymbol{\beta}_{-i}^*(\cdot))$. But uniform convergence follows since f, $\underline{v}_i$, and Δv_i are bounded and since $\Pr(W_i^*(a_i))$ is continuous at $a_i=b$, and further, $\boldsymbol{\beta}_{-i}^n(\cdot)$ converges uniformly to $\boldsymbol{\beta}_{-i}^*(\cdot)$ except on a set of arbitrarily small measure (details are in the working paper Athey (1997)).

Part (ii): Since $\bar{v}_i$ and $\underline{v}_i$ are continuous, whenever $\Pr(W_i^*(a_i))$ is continuous at $a_i=b$, player i's expected payoffs are continuous there as well. Consider $b\in\mathcal{A}_i$. Suppose first that $\Pr(\tau_i^{L*}(b))=0$, which implies $\Pr(\tau_i^{W*}(b))=0$ as well (recall ties are broken randomly). This in turn implies that $\Pr(W_i^*(a_i))$ is continuous at $a_i=b$. Further, Lemma 5 establishes that if $\Pr(\tau_i^{L*}(b))>0$, only a single type of player i uses action b. Since $\Pr(\tau_i^{L*}(b))>0$ for only a countable number of actions b, the set of types who face discontinuities has measure zero. □

Proof of Theorem 3: Following the proof of Theorem 2, consider a sequence of games with successively finer finite action sets, indexed by $\{n\}$. Restrict attention to sequences of PSNE to these games, $\{\boldsymbol{\beta}^n(\mathbf{t})\}$, that satisfy condition (ii) of Lemma 4 and converge almost everywhere to a set of strategies denoted $\boldsymbol{\beta}^*(\mathbf{t})$. Applying the logic of Theorem 2, so long as (i) $U_i(a_i,\boldsymbol{\beta}^*_{-i}(\cdot),t_i)$ is continuous at $a_i=\beta_i^*(t_i)$, and (ii) $U_i(\beta_i^n(t_i),t_i;\boldsymbol{\beta}^n_{-i}(\cdot))$ converges to $U_i(\beta_i^*(t_i),\boldsymbol{\beta}^*_{-i}(\cdot),t_i)$, $\beta_i^*(t_i)$ is a best response of player i, type t_i when opponents use $\boldsymbol{\beta}^*_{-i}(\cdot)$. But Lemma 6 establishes conditions (i) and (ii) hold for almost every type, and we are done. □

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