QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXI, NUMBER 2
JUNE 2003, PAGES 363-385

SINGLE-POINT BLOW-UP FOR A DEGENERATE PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE

By

C. Y. CHAN AND H. Y. TIAN

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, Louisiana

Abstract. Let q be a nonnegative real number, and T be a positive real number. This article studies the following degenerate semilinear parabolic first initial-boundary value problem:

$$x^q u_t(x,t) - u_{xx}(x,t) = a^2\delta(x-b)f(u(x,t)) \quad \text{for } 0 < x < 1, 0 < t \le T,$$
$$u(x,0) = \psi(x) \quad \text{for } 0 \le x \le 1,$$
$$u(0,t) = u(1,t) = 0 \quad \text{for } 0 < t \le T,$$

where $\delta(x)$ is the Dirac delta function, and f and ψ are given functions. It is shown that the problem has a unique solution before a blow-up occurs, u blows up in a finite time, and the blow-up set consists of the single point b. A lower bound and an upper bound of the blow-up time are also given. To illustrate our main results, an example is given. A computational method is also given to determine the finite blow-up time.

1. Introduction. Let a, σ, q and β be constants with $a > 0$, $\sigma > 0$, $q \ge 0$, and $0 < \beta < a$. Let us consider the following degenerate semilinear parabolic first initial-boundary value problem,

$$\left.\begin{aligned} &\varsigma^q u_\gamma - u_{\varsigma\varsigma} = \delta(\varsigma-\beta)F(u(\varsigma,\gamma)) \quad \text{in } (0,a)\times(0,\sigma], \\ &u(\varsigma,0) = \psi(\varsigma) \quad \text{on } [0,a], \\ &u(0,\gamma) = u(a,\gamma) = 0 \quad \text{for } 0 < \gamma \le \sigma, \end{aligned}\right\} \tag{1.1}$$

where $\delta(x)$ is the Dirac delta function, and F and ψ are given functions. This model is motivated by applications in which the ignition of a combustible medium is accomplished through the use of either a heated wire or a pair of small electrodes to supply a large

Received November 18, 2001.
2000 *Mathematics Subject Classification.* Primary 35K60, 35K65, 35K57.
Key words and phrases. Degenerate semilinear parabolic first initial-boundary value problem, concentrated nonlinear source, unique continuous solution, single blow-up point.
E-mail address: chan@louisiana.edu
E-mail address: tianh@uwstout.edu
Current address: Department of Mathematics, Statistics and Computer Science, University of Wisconsin-Stout, Menomonie, Wisconsin.

amount of energy to a very confined area. When $q = 1$, the model may also be used to describe the temperature u of the channel flow of a fluid with temperature-dependent viscosity in the boundary layer (cf. Chan and Kong [2]) with a concentrated nonlinear source at β; here, ς and γ denote the coordinates perpendicular and parallel to the channel wall respectively. When $q = 0$, it can be used to describe the temperature of a one-dimensional strip of a finite width that contains a concentrated nonlinear source at β. The case $q = 0$ was studied by Olmstead and Roberts [7] by analyzing its corresponding nonlinear Volterra equation of the second kind at the site of the concentrated source. A problem due to a source with local and nonlocal features was also studied by Olmstead and Roberts [8] by analyzing a pair of coupled nonlinear Volterra equations with different kernels. When the nonlinear source term in the problem (1.1) is replaced by u^p, the blow-up of the solution was studied by Floater [4] for the case $1 < p \leq q + 1$, and by Chan and Liu [3] for the case $p > q + 1$.

Let $\varsigma = ax$, $\gamma = a^{q+2}t$, $\beta = ab$, $Lu = x^q u_t - u_{xx}$, $f(u(x,t)) = F(u(\varsigma,\gamma))$, $D = (0,1)$, $\overline{D} = [0,1]$, and $\Omega = D \times (0,T]$. Then, the above system is transformed into the following problem:

$$\left.\begin{aligned} &Lu = a^2\delta(x-b)f(u(x,t)) \quad \text{in } \Omega, \\ &u(x,0) = \psi(x) \quad \text{on } \overline{D}, \\ &u(0,t) = u(1,t) = 0 \quad \text{for } 0 < t \leq T, \end{aligned}\right\} \tag{1.2}$$

with $0 < b < 1$, and $T = \sigma/a^{q+2}$. We assume that $f(0) \geq 0$, $f(u)$ and its derivatives $f'(u)$ and $f''(u)$ are positive for $u > 0$, and $\psi(x)$ is nontrivial, nonnegative, and continuous such that $\psi(b) > 0$, $\psi(0) = 0 = \psi(1)$, and

$$\psi'' + a^2\delta(x-b)f(\psi) \geq 0 \quad \text{in } D. \tag{1.3}$$

This condition (1.3) is used to show that before u blows up, u is a nondecreasing function of t. Instead of the condition (1.3), Olmstead and Roberts [7] assumed that $h(t) = \int_0^1 g(b,t;\xi,0)\psi(\xi)d\xi$, where $g(x,t;\xi,\tau)$ denotes Green's function corresponding to the heat operator $\partial/\partial t - \partial^2/\partial x^2$ with first boundary conditions, was sufficiently smooth such that $h'(t) \geq 0$, and $0 < h_0 \leq h(t) \leq h_\infty < \infty$ for some positive constants h_0 and h_∞; these were used to show that $u(b,t)$ and its derivative with respect to t were positive for $t > 0$.

A solution of the problem (1.2) is a continuous function satisfying (1.2).

A solution u of the problem (1.2) is said to blow up at the point $(\hat{x}, t_b)$ if there exists a sequence $\{(x_n, t_n)\}$ such that $u(x_n, t_n) \to \infty$ as $(x_n, t_n) \to (\hat{x}, t_b)$.

In Sec. 2, we convert the problem (1.2) into a nonlinear integral equation. We prove that the integral equation has a unique continuous and positive solution $U(b,t)$ at the site of the concentrated source. We then show that $U(x,t)$ is a nondecreasing function of t. These are used to prove that the problem (1.2) has a unique solution u. We also show that $u(b,t)$ blows up if ψ attains its maximum at b and $u(b,t)$ ceases to exist at a finite time. In Sec. 3, we show that b is the single blow-up point. We then give a criterion for u to blow up at a finite time, and use the method of Olmstead and Roberts [7] to establish a lower bound and an upper bound for the finite blow-up time. We remark that ψ attaining its maximum at b is used as a sufficient condition for u to blow up at

b. Whether it is a necessary one remains as an open question. To illustrate our main results, an example is given in Sec. 4. We also give a computational method to find the finite blow-up time.

2. Existence and uniqueness. Green's function $G(x,t;\xi,\tau)$ corresponding to the problem (1.2) is determined by the following system: for x and ξ in D, and t and τ in $(-\infty,\infty)$,

$$LG(x,t;\xi,\tau) = \delta(x-\xi)\delta(t-\tau),$$
$$G(x,t;\xi,\tau) = 0, \quad t<\tau,$$
$$G(0,t;\xi,\tau) = G(1,t;\xi,\tau) = 0.$$

By Chan and Chan [1],

$$G(x,t;\xi,\tau) = \sum_{i=1}^{\infty} \phi_i(x)\phi_i(\xi)e^{-\lambda_i(t-\tau)}, \tag{2.1}$$

where $\lambda_i (i=1,2,3,\dots)$ are the eigenvalues of the Sturm-Liouville problem,

$$\phi'' + \lambda x^q \phi = 0, \quad \phi(0) = 0 = \phi(1), \tag{2.2}$$

and their corresponding eigenfunctions are given by

$$\phi_i(x) = (q+2)^{1/2}x^{1/2}\frac{J_{\frac{1}{q+2}}\left(\frac{2\lambda_i^{1/2}}{q+2}x^{(q+2)/2}\right)}{\left|J_{1+\frac{1}{q+2}}\left(\frac{2\lambda_i^{1/2}}{q+2}\right)\right|}$$

with $J_{1/(q+2)}$ denoting the Bessel function of the first kind of order $1/(q+2)$. From Chan and Chan [1], $0<\lambda_1<\lambda_2<\lambda_3<\cdots<\lambda_i<\lambda_{i+1}<\cdots$. The set $\{\phi_i(x)\}$ is a maximal (that is, complete) orthonormal set with the weight function x^q (cf. Gustafson [6, p. 176]).

To derive the integral equation from the problem (1.2), let us consider the adjoint operator L^*, which is given by $L^*u = -x^q u_t - u_{xx}$. Using Green's second identity, we obtain

$$U(x,t) = a^2\int_0^t G(x,t;b,\tau)f(U(b,\tau))d\tau + \int_0^1 \xi^q G(x,t;\xi,0)\psi(\xi)d\xi. \tag{2.3}$$

For ease of reference, let us state below Lemmas 1(a), 1(b), 1(d), and 4 of Chan and Chan [1] as Lemma 2.1(a), 2.1(b), 2.1(c), and 2.1(d) respectively.

LEMMA 2.1. (a) For some positive constant c_1, $|\phi_i(x)| \le c_1 x^{-q/4}$ for $x\in(0,1]$.

(b) For some positive constant c_2, $|\phi_i(x)| \le c_2 x^{1/2}\lambda_i^{1/4}$ for $x\in\overline{D}$.

(c) For any $x_0>0$ and $x\in[x_0,1]$, there exists some positive constant c_3 depending on x_0 such that $|\phi_i'(x)| \le c_3\lambda_i^{1/2}$.

(d) In $\{(x,t;\xi,\tau)$: x and ξ are in D, $T\ge t>\tau\ge 0\}$, $G(x,t;\xi,\tau)$ is positive.

LEMMA 2.2. (a) For $(x,t;\xi,\tau)\in(\overline{D}\times(\tau,T])\times(\overline{D}\times[0,T))$, $G(x,t;\xi,\tau)$ is continuous.

(b) For each fixed $(\xi,\tau)\in\overline{D}\times[0,T)$, $G_t(x,t;\xi,\tau)\in C(\overline{D}\times(\tau,T])$.

(c) For each fixed $(\xi,\tau)\in\overline{D}\times[0,T)$, $G_x(x,t;\xi,\tau)$ and $G_{xx}(x,t;\xi,\tau)$ are in $C((0,1]\times(\tau,T])$.

(d) If $r\in C([0,T])$, then $\int_0^t G(x,t;b,\tau)r(\tau)d\tau$ is continuous for $x\in\overline{D}$ and $t\in[0,T]$.

Proof. (a) By Lemma 2.1(b),

$$|G(x,t;\xi,\tau)|\le c_2^2\sum_{i=1}^{\infty}\lambda_i^{1/2}e^{-\lambda_i(t-\tau)},$$

which converges uniformly for t in any compact subset of (τ,T). The result then follows.

(b) By Lemma 2.1(b),

$$\begin{aligned}\left|\sum_{i=1}^{\infty}\frac{\partial}{\partial t}\phi_i(x)\phi_i(\xi)e^{-\lambda_i(t-\tau)}\right| &\le\sum_{i=1}^{\infty}|\phi_i(x)|\,|\phi_i(\xi)|\lambda_i e^{-\lambda_i(t-\tau)}\\ &\le c_2^2\sum_{i=1}^{\infty}\lambda_i^{3/2}e^{-\lambda_i(t-\tau)},\end{aligned}\tag{2.4}$$

which converges uniformly with respect to $x\in\overline{D}$ and t in any compact subset of $(\tau,T]$. This proves Lemma 2.2(b).

(c) By Lemma 2.1(b) and (c),

$$\begin{aligned}\left|\sum_{i=1}^{\infty}\frac{\partial}{\partial x}\phi_i(x)\phi_i(\xi)e^{-\lambda_i(t-\tau)}\right| &\le\sum_{i=1}^{\infty}|\phi_i'(x)|\,|\phi_i(\xi)|e^{-\lambda_i(t-\tau)}\\ &\le c_2c_3\sum_{i=1}^{\infty}\lambda_i^{3/4}e^{-\lambda_i(t-\tau)},\end{aligned}\tag{2.5}$$

which converges uniformly with respect to x in any compact subset of $(0,1]$ and t in any compact subset of $(\tau,T]$.

Since ϕ_i is an eigenfunction, it follows from Lemma 2.1(b) that

$$\begin{aligned}\left|\sum_{i=1}^{\infty}\frac{\partial^2}{\partial x^2}\phi_i(x)\phi_i(\xi)e^{-\lambda_i(t-\tau)}\right| &\le\sum_{i=1}^{\infty}|\phi_i''(x)|\,|\phi_i(\xi)|e^{-\lambda_i(t-\tau)}\\ &=\sum_{i=1}^{\infty}\lambda_i x^q|\phi_i(x)|\,|\phi_i(\xi)|e^{-\lambda_i(t-\tau)}\\ &\le c_2^2\sum_{i=1}^{\infty}\lambda_i^{3/2}e^{-\lambda_i(t-\tau)},\end{aligned}\tag{2.6}$$

which converges uniformly with respect to x in any compact subset of $(0,1]$ and t in any compact subset of $(\tau,T]$.

Lemma 2.1(c) is then proved.

(d) Let ϵ be any positive number such that $t-\epsilon>0$. For any $x\in\overline{D}$, and $\tau\in[0,t-\epsilon]$, it follows from Lemma 2.1(a) and (b) that

$$\sum_{i=1}^{\infty}\phi_i(x)\phi_i(b)e^{-\lambda_i(t-\tau)}r(\tau)\le c_1c_2b^{-q/4}\left(\max_{0\le\tau\le T}r(\tau)\right)\sum_{i=1}^{\infty}\lambda_i^{1/4}e^{-\lambda_i\epsilon},$$

which converges uniformly. By the Weierstrass M-Test,

$$\int_0^{t-\epsilon} G(x,t;b,\tau)r(\tau)d\tau = \sum_{i=1}^{\infty}\int_0^{t-\epsilon} \phi_i(x)\phi_i(b)e^{-\lambda_i(t-\tau)}r(\tau)d\tau.$$

By Lemma 2.1(a) and (b),

$$\begin{aligned}\sum_{i=1}^{\infty}\int_0^{t-\epsilon} \phi_i(x)\phi_i(b)e^{-\lambda_i(t-\tau)}r(\tau)d\tau &\leq c_1c_2b^{-q/4}\left(\max_{0\leq\tau\leq T} r(\tau)\right)\sum_{i=1}^{\infty}\int_0^{t-\epsilon}\lambda_i^{1/4}e^{-\lambda_i(t-\tau)}d\tau \\ &= c_1c_2b^{-q/4}\left(\max_{0\leq\tau\leq T} r(\tau)\right)\sum_{i=1}^{\infty}\lambda_i^{-3/4}(e^{-\lambda_i\epsilon}-e^{-\lambda_i t}) \\ &\leq c_1c_2b^{-q/4}\left(\max_{0\leq\tau\leq T} r(\tau)\right)\sum_{i=1}^{\infty}\lambda_i^{-3/4},\end{aligned} \tag{2.7}$$

which converges (uniformly with respect to x, t, and ϵ) since $O(\lambda_i) = O(i^2)$ for large i (cf. Watson [12, p. 506]). Since (2.7) also holds for $\epsilon = 0$, it follows that

$$\sum_{i=1}^{\infty}\int_0^{t-\epsilon} \phi_i(x)\phi_i(b)e^{-\lambda_i(t-\tau)}r(\tau)d\tau$$

is a continuous function of x, t, and ϵ (≥ 0). Therefore,

$$\int_0^{t} G(x,t;b,\tau)r(\tau)d\tau = \lim_{\epsilon\to 0}\sum_{i=1}^{\infty}\int_0^{t-\epsilon} \phi_i(x)\phi_i(b)e^{-\lambda_i(t-\tau)}r(\tau)d\tau$$

is a continuous function of x and t. □

Let us consider the problem,

$$Lv = 0 \quad \text{in } \Omega,$$
$$v(x,0) = \psi(x) \quad \text{on } \overline{D},$$
$$v(0,t) = v(1,t) = 0 \quad \text{for } 0 < t \leq T,$$

which has a unique classical solution

$$v(x,t) = \int_0^1 \xi^q G(x,t;\xi,0)\psi(\xi)d\xi$$

(cf. Chan and Chan [1]). Since the strong maximum principle holds for the operator L (cf. Friedman [5, p. 39]), and $\psi(x)$ is nontrivial, nonnegative and continuous, it follows that $v > 0$ in Ω, and attains its maximum $\max_{x\in\overline{D}}\psi(x)$ (denoted by k_1) somewhere in $D \times \{0\}$.

From (2.3),

$$U(b,t) = a^2\int_0^t G(b,t;b,\tau)f(U(b,\tau))d\tau + \int_0^1 \xi^q G(b,t;\xi,0)\psi(\xi)d\xi. \tag{2.8}$$

By Lemma 2.2(d), we can look for a continuous function $U(b,t)$ satisfying (2.8). From Chan and Chan [1],

$$\lim_{t\to 0}\int_0^1 \xi^q G(b,t;\xi,0)\psi(\xi)d\xi = \psi(b).$$

Thus from (2.8), $U(b,0) = \psi(b) > 0$.

Let us show that there exists some t_1 such that

$$\psi(b) \leq U(b,t) \quad \text{for } 0 \leq t \leq t_1. \tag{2.9}$$

Since

$$L(\psi - u) \leq a^2\delta(x-b)(f(\psi) - f(u)) \quad \text{in } \Omega,$$

and $\psi - u = 0$ on $\partial\Omega$, it follows from (2.8) that

$$\psi(b) - U(b,t) \leq a^2 \int_0^t G(b,t;b,\tau)f'(\eta)(\psi(b) - U(b,\tau))d\tau \tag{2.10}$$

for some η between $\psi(b)$ and $U(b,t)$. Since $G(x,t;\xi,\tau)$ is nonnegative and integrable over $[0,t]$, it follows that for any t_2, there exists some ρ such that for any $t \in (t_2, t_2+\rho]$,

$$a^2 f'(\psi(b)) \int_{t_2}^t G(b,t;b,\tau)d\tau < 1.$$

We also note that $U(b,0) > 0$. Suppose there exists some t_3 such that $\psi(b) > U(b,t) \geq 0$ for $t \in (0,t_3]$. Let $t_1 = \min\{\rho, t_3\}$. From (2.10), we have

$$\psi(b) - U(b,t) \leq a^2 \left(\int_0^t G(b,t;b,\tau)f'(\eta)d\tau \right) \max_{0\leq t\leq t_1} (\psi(b) - U(b,t)).$$

This gives a contradiction. Thus, we have (2.9).

It follows from (2.8), $f(0) \geq 0$, and $f(u)$ being positive for $u > 0$ that $U(b,t) > v(b,t) > 0$ for $t > 0$.

Let

$$z(t) = \int_0^1 \xi^q G(b,t;\xi,0)\psi(\xi)d\xi.$$

We note that $z(t) = v(b,t)$, and hence, $z(t)$ exists for $t \geq 0$. Let k_2 denote $\min_{0\leq t\leq T} v(b,t)$. We have

$$k_2 \leq z(t) \leq k_1 \quad \text{for } 0 \leq t \leq T.$$

It follows from $\psi(b) > 0$ and $v > 0$ in Ω that $k_2 > 0$.

Let

$$w(t) = U(b,t) - z(t). \tag{2.11}$$

From (2.8),

$$w(t) = a^2 \int_0^t G(b,t;b,\tau)f(w(\tau) + z(\tau))d\tau. \tag{2.12}$$

Let

$$Rw(t) = a^2 \int_0^t G(b,t;b,\tau)f(w(\tau) + z(\tau))d\tau.$$

From (2.12), we have $w = Rw$.

LEMMA 2.3. *For some given positive constant k_3, there exists some t_4 such that (2.12) has a unique continuous and nonnegative solution $w(t) \leq k_3$ for $0 \leq t \leq t_4$.*

Proof. By Lemma 2.2(d), $G(b,t;b,\tau)$ is integrable over $[0,t]$. Since $G(b,t;b,\tau)$ is nonnegative, there exists some t_4 such that

$$a^2 f(k_3+k_1)\int_0^t G(b,t;b,\tau)d\tau \le k_3 \quad \text{for } 0 \le t \le t_4, \tag{2.13}$$

$$a^2 f'(k_3+k_1)\int_0^t G(b,t;b,\tau)d\tau < 1 \quad \text{for } 0 \le t \le t_4. \tag{2.14}$$

From (2.13) and $f'(u) > 0$ for $u > 0$,

$$Rw(t) \le a^2 f(k_3+k_1)\int_0^1 G(b,t;b,\tau)d\tau \le k_3 \quad \text{for } 0 \le t \le t_4. \tag{2.15}$$

Thus, R maps the space of continuous functions satisfying

$$0 \le w(t) \le k_3 \quad \text{for } 0 \le t \le t_4$$

into itself. For any $w_1(t)$ and $w_2(t)$ satisfying (2.12),

$$\max_{0\le t\le t_4} |Rw_1(t) - Rw_2(t)| \le a^2 f'(k_3+k_1)\left(\max_{0\le t\le t_4} |w_1(t)-w_2(t)|\right)\int_0^t G(b,t;b,\tau)d\tau.$$

By (2.14),

$$\max_{0\le t\le t_4} |Rw_1(t) - Rw_2(t)| < \max_{0\le t\le t_4} |w_1(t)-w_2(t)| \quad \text{for } 0 \le t \le t_4.$$

Thus, R is a contraction mapping, and we obtain an interval $0 \le t \le t_4$ on which a unique solution w of (2.12) exists and is continuous and nonnegative. □

By (2.11), $U(b,t)$ exists, and is unique for $0 \le t \le t_4$; $U(b,t) > 0$ for $t > 0$. Let t_b be the supremum of the interval for which the integral equation (2.8) has a unique continuous solution $U(b,t)$.

Let $\Omega_b = D \times (0,t_b)$, and $\partial\Omega_b$ denote its parabolic boundary $(\{0,1\}\times(0,t_b))\cup\overline{D}\times\{0\}$.

THEOREM 2.4. The integral equation (2.3) has a unique continuous solution $U(x,t)$ in Ω_b. Furthermore, $\psi(x) \le U(x,t)$, and U is a nondecreasing function of t.

Proof. Since the integral equation (2.8) has a unique continuous solution $U(b,t)$, it follows that the right-hand side of the integral equation (2.3) is determined uniquely, and hence, the integral equation (2.3) has a unique continuous solution $U(x,t)$. Also $U(x,t) > 0$ in Ω_b.

Let us construct a sequence $\{u_i\}$ in Ω by $u_0(x,t) = \psi(x)$, and for $i = 0,1,2,\ldots$,

$$Lu_{i+1} = a^2\delta(x-b)f(u_i) \quad \text{in } \Omega,$$

$$u_{i+1}(x,0) = \psi(x) \quad \text{on } \overline{D},\ u_{i+1}(0,t) = u_{i+1}(1,t) = 0 \quad \text{for } 0 < t < T.$$

We have

$$L(u_1-u_0) \ge a^2\delta(x-b)(f(u_0)-f(\psi)) = 0 \quad \text{in } \Omega,$$

$$u_1 - u_0 = 0 \quad \text{on } \partial\Omega.$$

By Lemma 2.1(d) and (2.3), $u_1 \ge u_0$ in Ω. Let us assume that for some positive integer j,

$$\psi \le u_1 \le u_2 \le \cdots \le u_{j-1} \le u_j \quad \text{in } \Omega.$$

Since f is an increasing function, and $u_j \geq u_{j-1}$, we have

$$L(u_{j+1} - u_j) = a^2\delta(x-b)(f(u_j) - f(u_{j-1})) \geq 0 \quad \text{in } \Omega,$$
$$u_{j+1} - u_j = 0 \quad \text{on } \partial\Omega.$$

By Lemma 2.1(d) and (2.3), $u_{j+1} \geq u_j$. By the principle of mathematical induction,

$$\psi \leq u_1 \leq u_2 \leq \cdots \leq u_{n-1} \leq u_n \quad \text{in } \Omega \tag{2.16}$$

for any positive integer n.

We would like to show that $U(x,t) \geq \psi(x)$ for $0 \leq t < t_b$. From (2.9), $\psi(b) \leq U(b,t)$ for $0 \leq t \leq t_1$, where $t_1 = \rho$. Let t_5 be the smallest t ($\geq t_1$) such that $\psi(b) \leq U(b,t)$. Since

$$L(u-\psi) \geq a^2\delta(x-b)(f(u) - f(\psi)) \quad \text{in } \Omega,$$
$$u - \psi = 0 \quad \text{on } \partial\Omega,$$

it follows from (2.3) that

$$U(x,t) - \psi(x) \geq a^2 \int_0^t G(x,t;b,\tau)(f(U(b,\tau)) - f(\psi(b)))d\tau.$$

Thus, $U \geq \psi$ on $\overline{D} \times [0,t_5]$. By starting at $t = t_5$ (instead of $t = 0$), we repeat the procedure used in proving (2.9) and the above reasoning to show that $U \geq \psi$ on $\overline{D} \times [0,t_6]$ for some $t_6 \geq t_5 + \rho$. In this way, we prove that $U(x,t) \geq \psi(x)$ for $0 \leq t < t_b$.

Since

$$L(u - u_1) = a^2\delta(x-b)(f(u) - f(\psi)) \geq 0 \quad \text{in } \Omega,$$
$$u - u_1 = 0 \quad \text{on } \partial\Omega,$$

it follows from Lemma 2.1(d) and (2.3) that $U \geq u_1$. Using mathematical induction, $U \geq u_n$ for any positive integer n.

Let $\overline{\Omega}$ denote the closure of Ω. For any $T \in (0,t_b)$, U is bounded on $\overline{\Omega}$. There exists some positive constant K such that $U \leq K$ on $\overline{\Omega}$. Since

$$u_n(x,t) = a^2 \int_0^t G(x,t;b,\tau)f(u_{n-1}(b,\tau))d\tau + \int_0^1 \xi^q G(x,t;\xi,0)\psi(\xi)d\xi, \tag{2.17}$$

it follows from the properties of f and the Monotone Convergence Theorem (cf. Royden [9, p. 87]) that $\lim_{n\to\infty} u_n$ satisfies the integral equation (2.3). From (2.17),

$$\begin{aligned} &u_{n+1}(x,t) - u_n(x,t) \\ &\quad = a^2 \int_0^t G(x,t;b,\tau)[f(u_n(b,\tau)) - f(u_{n-1}(b,\tau))]d\tau. \end{aligned} \tag{2.18}$$

Let $S_n = \max_{\overline{\Omega}}(u_n - u_{n-1})$ for any $T < t_b$. By using the mean value theorem and $f'(u) > 0$ for $u > 0$, it follows from (2.18) (as in the derivation of (2.7)) that

$$\begin{aligned} S_{n+1} &\leq a^2 f'(K) S_n c_1 c_2 b^{-q/4} \sum_{i=1}^{\infty} \lambda_i^{1/4} \int_0^t e^{-\lambda_i(t-\tau)} d\tau \\ &= a^2 f'(K) c_1 c_2 b^{-q/4} \left[\sum_{i=1}^{\infty} \lambda_i^{-3/4}(1 - e^{-\lambda_i t})\right] S_n, \end{aligned}$$

which converges since $O(\lambda_i) = (i^2)$ for large i. Let us choose some positive number σ_1 ($\leq T < t_b$) such that for $t \in [0, \sigma_1]$,

$$a^2 f'(K) c_1 c_2 b^{-q/4} \left[\sum_{i=1}^{\infty} \lambda_i^{-3/4} (1 - e^{-\lambda_i t}) \right] < 1.$$

Then, the sequence $\{u_n\}$ converges uniformly to $\lim_{n\to\infty} u_n(x,t)$ for $0 \leq t \leq \sigma_1$. Similarly for $\sigma_1 \leq t \leq T < t_b$, we use $\lim_{n\to\infty} u_n(\xi, \sigma_1)$ to replace $\psi(\xi)$ in (2.17); we then obtain

$$S_{n+1} \leq a^2 f'(K) c_1 c_2 b^{-q/4} \left\{ \sum_{i=1}^{\infty} \lambda_i^{-3/4} [1 - e^{-\lambda_i (t-\sigma_1)}] \right\} S_n.$$

For $t \in [\sigma_1, \min\{2\sigma_1, T\}]$,

$$a^2 f'(K) c_1 c_2 b^{-q/4} \left\{ \sum_{i=1}^{\infty} \lambda_i^{-3/4} [1 - e^{-\lambda_i (t-\sigma_1)}] \right\} < 1.$$

Thus, the sequence $\{u_n\}$ converges uniformly to $\lim_{n\to\infty} u_n(x,t)$ for $\sigma_1 \leq t \leq \min\{2\sigma_1, T\}$. By proceeding in this way, the sequence $\{u_n\}$ converges uniformly for $0 \leq t \leq T$, and hence $\lim_{n\to\infty} u_n$ is continuous. Since the integral equation (2.3) has a unique continuous solution U for $0 \leq t < t_b$, we have $U = \lim_{n\to\infty} u_n$.

To show that U is a nondecreasing function of t, let us construct a sequence $\{w_i\}$ such that for $i = 0, 1, 2, \ldots$,

$$w_i(x,t) = u_i(x, t+h) - u_i(x,t),$$

where h ($< T$) is some positive number. Then, $w_0(x,t) = 0$. We have

$$Lw_1 = 0 \quad \text{in } D \times (0, T-h].$$

From (2.16),

$$w_1(x, 0) \geq 0 \quad \text{on } \overline{D}, w_1(0,t) = w_1(1,t) = 0 \quad \text{for } 0 < t \leq T - h.$$

By (2.3), $w_1 \geq 0$ in Ω. Let us assume that for some positive integer j, $0 \leq w_j$ in Ω. Then,

$$Lw_{j+1} = a^2 \delta(x-b) f'(\xi_j) w_j \geq 0 \quad \text{in } D \times (0, T-h]$$

for some ξ_j between $u_j(x, t+h)$ and $u_j(x,t)$. Since $w_{j+1}(x, 0) \geq 0$ on $\overline{D}$, and $w_{j+1}(0,t) = w_{j+1}(1,t) = 0$ for $0 < t \leq T - h$, it follows from (2.3) that $w_{j+1} \geq 0$ in Ω. By the principle of mathematical induction, $w_n \geq 0$ in Ω for all positive integers n. Hence, U is a nondecreasing function of t. □

The next result shows that U is the solution of the problem (1.2).

THEOREM 2.5. The problem (1.2) has a unique solution $u = U$.

Proof. By Lemma 2.2(d), $\int_0^t G(x,t;b,\tau)f(U(b,\tau))d\tau$ exists for $x \in \overline{D}$ and t in any compact subset $[t_7, t_8]$ of $[0, t_b)$. Thus, for any $x \in D$ and any $t_9 \in (0,t)$,

$$\begin{aligned}
&\int_0^t G(x,t;b,\tau)f(U(b,\tau))d\tau \\
&= \lim_{n\to\infty} \int_0^{t-1/n} G(x,t;b,\tau)f(U(b,\tau))d\tau \\
&= \lim_{n\to\infty} \left[\int_{t_9}^t \frac{\partial}{\partial\zeta}\left(\int_0^{\zeta-1/n} G(x,\zeta;b,\tau)f(U(b,\tau))d\tau\right) d\zeta \right.\\
&\qquad \left. + \int_0^{t_9-1/n} G(x,t_9;b,\tau)f(U(b,\tau))d\tau\right].
\end{aligned}$$

Since by (2.4),

$$G_\zeta(x,\zeta;b,\tau)f(U(b,\tau)) \le c_2^2 \sum_{i=1}^{\infty} \lambda_i^{3/2} e^{-\lambda_i/n} f(U(b,\tau)) \quad \text{for } \zeta - \tau \ge 1/n,$$

which is integrable with respect to τ over $(0, \zeta - 1/n)$, it follows from the Leibnitz rule (cf. Stromberg [11, p. 380]) that

$$\begin{aligned}
&\frac{\partial}{\partial\zeta}\left(\int_0^{\zeta-1/n} G(x,\zeta;b,\tau)f(U(b,\tau))d\tau\right) \\
&= G\left(x,\zeta;b,\zeta-\frac{1}{n}\right) f\left(U\left(b,\zeta-\frac{1}{n}\right)\right) + \int_0^{\zeta-1/n} G_\zeta(x,\zeta;b,\tau)f(U(b,\tau))d\tau.
\end{aligned}$$

Let us consider the problem,

$$\begin{aligned}
L\omega &= 0 \quad \text{for } x \in D, 0 < \tau < t < T, \\
\omega(0,t;\xi,\tau) &= \omega(1,t;\xi,\tau) = 0 \quad \text{for } 0 < \tau < t < T, \\
\lim_{t\to\tau^+} x^q \omega(x,t;\xi,\tau) &= \delta(x-\xi).
\end{aligned}$$

From the representation formula (2.3),

$$\begin{aligned}
\omega(x,t;\xi,\tau) &= \int_0^1 \alpha^q G(x,t;\alpha,\tau)\alpha^{-q}\delta(\alpha-\xi)d\alpha \\
&= G(x,t;\xi,\tau) \quad \text{for } t \ge \tau.
\end{aligned}$$

It follows that $\lim_{t\to\tau^+} x^q G(x,t;b,\tau) = \delta(x-b)$.

Since $G(x,\zeta;b,\zeta-1/n) = G(x,1/n;b,0)$, which is independent of ζ, we have

$$\begin{aligned}
&\int_0^t x^q G(x,t;b,\tau)f(U(b,\tau))d\tau \\
&= \delta(x-b)\int_{t_9}^t f(U(b,\zeta))d\zeta + \lim_{n\to\infty}\int_{t_9}^t \int_0^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau)f(U(b,\tau))d\tau d\zeta \\
&+ \int_0^{t_9} x^q G(x,t_9;b,\tau)f(U(b,\tau))d\tau.
\end{aligned}$$

Let

$$g_n(x,t) = \int_0^{t-1/n} x^q G_t(x,t;b,\tau) f(U(b,\tau))d\tau.$$

Without loss of generality, let $n > l$. We have

$$g_n(x,\zeta) - g_l(x,\zeta) = \int_{\zeta-1/l}^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau) f(U(b,\tau))d\tau.$$

Since $x^q G_t(x,t;b,\tau) \in C(\overline{D}\times(\tau,T])$ and $f(U(b,\tau))$ is a monotone function of τ, it follows from the Second Mean Value Theorem for Integrals (cf. Stromberg [11, p. 328]) that for any $x \neq b$ and any ζ in any compact subset $[t_7, t_8]$ of $(0, t_b)$, there exists some real number ν such that $\zeta - \nu \in (\zeta - 1/l, \zeta - 1/n)$ and

$$\begin{aligned} g_n(x,\zeta) - g_l(x,\zeta) &= f\left(U\left(b,\zeta-\frac{1}{l}\right)\right)\int_{\zeta-1/l}^{\zeta-\nu} x^q G_\zeta(x,\zeta;b,\tau)d\tau \\ &\quad + f\left(U\left(b,\zeta-\frac{1}{n}\right)\right)\int_{\zeta-\nu}^{\zeta-\frac{1}{n}} x^q G_\zeta(x,\zeta;b,\tau)d\tau. \end{aligned}$$

From $G_\zeta(x,\zeta;b,\tau) = -G_\tau(x,\zeta;b,\tau)$, we have

$$\begin{aligned} &g_n(x,\zeta) - g_l(x,\zeta) \\ &= \left[f\left(U\left(b,\zeta-\frac{1}{n}\right)\right) - f\left(U\left(b,\zeta-\frac{1}{l}\right)\right)\right] x^q G(x,\zeta;b,\zeta-\nu) \\ &+ f\left(U\left(b,\zeta-\frac{1}{l}\right)\right) x^q G\left(x,\zeta;b,\zeta-\frac{1}{l}\right) - f\left(U\left(b,\zeta-\frac{1}{n}\right)\right) x^q G\left(x,\zeta;b,\zeta-\frac{1}{n}\right). \end{aligned}$$

Since for $x \neq b$,

$$x^q G(x,\zeta;b,\zeta-\epsilon) = x^q G(x,\epsilon;b,0)$$

converges to 0 uniformly with respect to ζ as $\epsilon \to 0$, it follows that for $x \neq b$, $\{g_n\}$ is a Cauchy sequence, and hence $\{g_n\}$ converges uniformly with respect to ζ in any compact subset $[t_7, t_8]$ of $(0, t_b)$. Hence for $x \neq b$,

$$\begin{aligned} &\lim_{n\to\infty}\int_{t_9}^{t}\int_0^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau) f(U(b,\tau))d\tau d\zeta \\ &\quad = \int_{t_9}^{t}\lim_{n\to\infty}\int_0^{\zeta-1/n} x^q G_\zeta(x,\zeta;b,\tau) f(U(b,\tau))d\tau d\zeta \\ &\quad = \int_{t_9}^{t}\int_0^{\zeta} x^q G_\zeta(x,\zeta;b,\tau) f(U(b,\tau))d\tau d\zeta. \end{aligned}$$

For $x = b$,

$$-G_\zeta(x,\zeta;b,\tau) f(U(b,\tau)) = \sum_{i=1}^{\infty} \phi_i^2(b)\lambda_i e^{-\lambda_i(\zeta-\tau)} f(U(b,\tau)),$$

which is positive. Thus, $\{-g_n\}$ is a nondecreasing sequence of nonnegative functions with respect to ζ. By the Monotone Convergence Theorem,

$$\begin{aligned}\lim_{n\to\infty}\int_{t_9}^{t}\int_0^{\zeta-1/n} & b^qG_\zeta(b,\zeta;b,\tau)f(U(b,\tau))d\tau d\zeta \\ &= \int_{t_9}^{t}\lim_{n\to\infty}\int_0^{\zeta-1/n} b^qG_\zeta(b,\zeta;b,\tau)f(U(b,\tau))d\tau d\zeta \\ &= \int_{t_9}^{t}\int_0^{\zeta} b^qG_\zeta(b,\zeta;b,\tau)f(U(b,\tau))d\tau d\zeta.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^t & x^qG(x,t;b,\tau)f(U(b,\tau))d\tau \\ &= \delta(x-b)f(U(b,t)) + \int_0^t x^qG_t(x,t;b,\tau)f(U(b,\tau))d\tau.\end{aligned}$$

By using (2.5), (2.6) and the Leibnitz rule, we have for any x in any compact subset of $(0,1]$ and t in any compact subset $[t_7,t_8]$ of $(0,t_b)$,

$$\begin{aligned}\frac{\partial}{\partial x}\int_0^{t-\epsilon} G(x,t;b,\tau)f(U(b,\tau))d\tau &= \int_0^{t-\epsilon} G_x(x,t;b,\tau)f(U(b,\tau))d\tau, \\ \frac{\partial}{\partial x}\int_0^{t-\epsilon} G_x(x,t;b,\tau)f(U(b,\tau))d\tau &= \int_0^{t-\epsilon} G_{xx}(x,t;b,\tau)f(U(b,\tau))d\tau.\end{aligned}$$

For any $x_1 \in D$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} & G(x,t;b,\tau)f(U(b,\tau))d\tau \\ &= \lim_{\varepsilon\to 0}\int_{x_1}^{x}\left(\frac{\partial}{\partial \eta}\int_0^{t-\varepsilon} G(\eta,t;b,\tau)f(U(b,\tau))d\tau\right)d\eta \\ &\quad + \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G(x_1,t;b,\tau)f(U(b,\tau))d\tau \\ &= \lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon} G_\eta(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\ &\quad + \int_0^{t} G(x_1,t;b,\tau)f(U(b,\tau))d\tau.\end{aligned} \tag{2.19}$$

We would like to show that

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon} & G_\eta(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\ &= \int_{x_1}^{x}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G_\eta(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta.\end{aligned} \tag{2.20}$$

By the Fubini Theorem (cf. Stromberg [11, p. 352]),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_1}^{x}\int_0^{t-\varepsilon} G_\eta(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\
&\quad=\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}\left(f(U(b,\tau))\int_{x_1}^{x} G_\eta(\eta,t;b,\tau)d\eta\right)d\tau \\
&\quad=\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} f(U(b,\tau))(G(x,t;b,\tau)-G(x_1,t;b,\tau))d\tau \\
&\quad=\int_0^{t} f(U(b,\tau))(G(x,t;b,\tau)-G(x_1,t;b,\tau))d\tau,
\end{aligned}$$

which exists by Lemma 2.2(d). Therefore,

$$\int_0^t f(U(b,\tau))(G(x,t;b,\tau)-G(x_1,t;b,\tau))d\tau=\int_{x_1}^{x}\int_0^t G_\eta(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta,$$

and we have (2.20). From (2.19),

$$\frac{\partial}{\partial x}\int_0^t G(x,t;b,\tau)f(U(b,\tau))d\tau=\int_0^t G_x(x,t;b,\tau)f(U(b,\tau))d\tau.$$

For any $x_2\in D$,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G_x(x,t;b,\tau)f(U(b,\tau))d\tau \\
&\quad=\lim_{\varepsilon\to 0}\int_{x_2}^{x}\frac{\partial}{\partial\eta}\left(\int_0^{t-\varepsilon} G_\eta(\eta,t;b,\tau)f(U(b,\tau))d\tau\right)d\eta \\
&\qquad+\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G_\eta(x_2,t;b,\tau)f(U(b,\tau))d\tau \qquad (2.21)\\
&\quad=\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} G_{\eta\eta}(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\
&\qquad+\int_0^{t} G_\eta(x_2,t;b,\tau)f(U(b,\tau))d\tau.
\end{aligned}$$

We would like to show that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} G_{\eta\eta}(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\
&\quad=\int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} G_{\eta\eta}(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta. \qquad (2.22)
\end{aligned}$$

Since $G_{xx}(x,t;\xi,\tau) = x^q G_t(x,t;\xi,\tau) - \delta(x-\xi)\delta(t-\tau)$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} G_{\eta\eta}(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\
&\quad= \lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} (\eta^q G_t(\eta,t;b,\tau) - \delta(\eta-b)\delta(t-\tau))f(U(b,\tau))d\tau d\eta \\
&\quad= \lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} \eta^q G_t(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\
&\quad= -\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} \eta^q G_\tau(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta.
\end{aligned}$$

By the Second Mean Value Theorem for Integrals, there exists some real number $\gamma \in (0, t-\varepsilon)$ such that

$$\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_0^{t-\varepsilon} \eta^q G_\tau(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta \\
&\quad= -\lim_{\varepsilon\to 0}\int_{x_2}^{x} f(U(b,0))\int_0^{\gamma} \eta^q G_\tau(\eta,t;b,\tau)d\tau d\eta \\
&\qquad -\lim_{\varepsilon\to 0}\int_{x_2}^{x} f(U(b,t-\varepsilon))\int_\gamma^{t-\varepsilon} \eta^q G_\tau(\eta,t;b,\tau)d\tau d\eta \\
&\quad= f(U(b,0))\lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q(G(\eta,t;b,0) - G(\eta,t;b,\gamma))d\eta \\
&\qquad +\lim_{\varepsilon\to 0}\int_{x_2}^{x} f(U(b,t-\varepsilon))\eta^q(G(\eta,t;b,\gamma) - G(\eta,t;b,t-\varepsilon))d\eta \qquad (2.23)\\
&\quad= f(U(b,0))\left(\int_{x_2}^{x}\eta^q G(\eta,t;b,0)d\eta - \lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,\gamma)d\eta\right) \\
&\qquad + f(U(b,t))\left(\lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,\gamma)d\eta - \lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,t-\varepsilon)d\eta\right) \\
&\quad= f(U(b,0))\left(\int_{x_2}^{x}\eta^q G(\eta,t;b,0)d\eta - \lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,\gamma)d\eta\right) \\
&\qquad + f(U(b,t))\left(\lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,\gamma)d\eta - \int_{x_2}^{x}\delta(\eta-b)d\eta\right)
\end{aligned}$$

since $\lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,t-\varepsilon)d\eta = \lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(b,t;\eta,t-\varepsilon)d\eta = \int_{x_2}^{x}\delta(\eta-b)d\eta$ (cf. Chan and Chan [1]).

Case 1: If $\lim_{\varepsilon\to 0}\gamma = t$, then $\lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,\gamma)d\eta = \int_{x_2}^{x}\delta(\eta-b)d\eta$. We have

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_{0}^{t-\varepsilon}\eta^q G_\tau(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta\\
&= f(U(b,0))\left(\int_{x_2}^{x}\eta^q G(\eta,t;b,0)d\eta - \int_{x_2}^{x}\delta(\eta-b)d\eta\right)\\
&\quad + f(U(b,t))\left(\int_{x_2}^{x}\delta(\eta-b)d\eta - \int_{x_2}^{x}\delta(\eta-b)d\eta\right)\\
&= f(U(b,0))\left(\int_{x_2}^{x}\eta^q G(\eta,t;b,0)d\eta - \int_{x_2}^{x}\lim_{\varepsilon\to 0}\eta^q G(\eta,t;b,\gamma)d\eta\right)\\
&\quad + f(U(b,t))\left(\int_{x_2}^{x}\lim_{\varepsilon\to 0}\eta^q G(\eta,t;b,\gamma)d\eta - \int_{x_2}^{x}\lim_{\varepsilon\to 0}\eta^q G(\eta,t;b,t-\varepsilon)d\eta\right)\\
&= f(U(b,0))\left(\int_{x_2}^{x}\eta^q G(\eta,t;b,0)d\eta - \int_{x_2}^{x}\lim_{\varepsilon\to 0}\eta^q G(\eta,t;b,\gamma)d\eta\right)\\
&\quad + \int_{x_2}^{x}\lim_{\varepsilon\to 0}[f(U(b,t-\varepsilon))\eta^q(G(\eta,t;b,\gamma) - G(\eta,t;b,t-\varepsilon))]d\eta\\
&= -\int_{x_2}^{x}\lim_{\varepsilon\to 0}\left[f(U(b,0))\int_{0}^{\gamma}\eta^q G_\tau(\eta,t;b,\tau)d\tau + f(U(b,t-\varepsilon))\int_{\gamma}^{t-\varepsilon}\eta^q G_\tau(\eta,t;b,\tau)d\tau\right]d\eta\\
&= -\int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon}\eta^q G_\tau(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta\\
&= \int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon}(\eta^q G_t(\eta,t;b,\tau) - \delta(\eta-b)\delta(t-\tau))f(U(b,\tau))d\tau d\eta\\
&= \int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon}G_{\eta\eta}(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta.
\end{aligned}
$$

Case 2: If $\lim_{\varepsilon\to 0}\gamma < t$, then $\lim_{\varepsilon\to 0}\int_{x_2}^{x}\eta^q G(\eta,t;b,\gamma)d\eta = \int_{x_2}^{x}\eta^q G(\eta,t;b,\lim_{\varepsilon\to 0}\gamma)d\eta$ since $\int_{x_2}^{x}\eta^q G(\eta,t;b,\gamma)d\eta$ is a continuous function of γ. From (2.23), we have

$$
\begin{aligned}
&-\lim_{\varepsilon\to 0}\int_{x_2}^{x}\int_{0}^{t-\varepsilon}\eta^q G_\tau(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta\\
&= f(U(b,0))\left(\int_{x_2}^{x}\eta^q G(\eta,t;b,0)d\eta - \int_{x_2}^{x}\eta^q G\left(\eta,t;b,\lim_{\varepsilon\to 0}\gamma\right)d\eta\right)\\
&\quad + f(U(b,t))\left(\int_{x_2}^{x}\eta^q G\left(\eta,t;b,\lim_{\varepsilon\to 0}\gamma\right)d\eta - \int_{x_2}^{x}\delta(\eta-b)d\eta\right)\\
&= -\int_{x_2}^{x}\lim_{\varepsilon\to 0}\left[f(U(b,0))\int_{0}^{\gamma}\eta^q G_\tau(\eta,t;b,\tau)d\tau + f(U(b,t-\varepsilon))\int_{\gamma}^{t-\varepsilon}\eta^q G_\tau(\eta,t;b,\tau)d\tau\right]d\eta\\
&= -\int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon}\eta^q G_\tau(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta\\
&= \int_{x_2}^{x}\lim_{\varepsilon\to 0}\int_{0}^{t-\varepsilon}G_{\eta\eta}(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta.
\end{aligned}
$$

In either case, we have (2.22).

From (2.21),

$$\int_0^t G_x(x,t;b,\tau)f(U(b,\tau))d\tau$$
$$= \int_{x_2}^x \int_0^t G_{\eta\eta}(\eta,t;b,\tau)f(U(b,\tau))d\tau d\eta + \int_0^t G_\eta(x_2,t;b,\tau)f(U(b,\tau))d\tau.$$

Thus,

$$\frac{\partial}{\partial x}\int_0^t G_x(x,t;b,\tau)f(U(b,\tau))d\tau = \int_0^t G_{xx}(x,t;b,\tau)f(U(b,\tau))d\tau.$$

Therefore,

$$\frac{\partial^2}{\partial x^2}\int_0^t G(x,t;b,\tau)f(U(b,\tau))d\tau = \int_0^t G_{xx}(x,t;b,\tau)f(U(b,\tau))d\tau$$

for any x in any compact subset of $(0,1]$ and t in any compact subset $[t_7,t_8]$ of $(0,t_b)$.

By the Leibnitz rule, we have for any x in any compact subset of $(0,1]$ and any t in any compact subset of $(0,t_b)$,

$$x^q\frac{\partial}{\partial t}\int_0^1 G(x,t;\xi,0)\xi^q\psi(\xi)d\xi = \int_0^1 x^q G_t(x,t;\xi,0)\xi^q\psi(\xi)d\xi,$$
$$\frac{\partial}{\partial x}\int_0^1 G(x,t;\xi,0)\xi^q\psi(\xi)d\xi = \int_0^1 G_x(x,t;\xi,0)\xi^q\psi(\xi)d\xi,$$
$$\frac{\partial^2}{\partial x^2}\int_0^1 G(x,t;\xi,0)\xi^q\psi(\xi)d\xi = \int_0^1 G_{xx}(x,t;\xi,0)\xi^q\psi(\xi)d\xi.$$

From the integral equation (2.3), we have for $x\in D$ and $0<t<t_b$,

$$LU = a^2\delta(x-b)f(U(b,t)) + a^2\int_0^t LG(x,t;b,\tau)f(U(b,\tau))d\tau$$
$$+\int_0^1 LG(x,t;\xi,0)\xi^q\psi(\xi)d\xi$$
$$= a^2\delta(x-b)f(U(b,t)) + a^2\delta(x-b)\lim_{\varepsilon\to 0}\int_0^{t-\epsilon}\delta(t-\tau)f(U(b,\tau))d\tau$$
$$+\delta(t)\int_0^1\delta(x-\xi)\xi^q\psi(\xi)d\xi$$
$$= a^2\delta(x-b)f(U(b,t)).$$

From the integral equation (2.3), we have for $x\in\overline{D}$,

$$\lim_{t\to 0}U(x,t) = \lim_{t\to 0}\int_0^1 \xi^q G(x,t;\xi,0)\psi(\xi)d\xi = \psi(x)$$

(cf. Chan and Chan [1]). Since $G(0,t;\xi,\tau) = 0 = G(1,t;\xi,\tau)$, we have $U(0,t) = 0 = U(1,t)$. Thus, the solution U of the integral equation (2.3) is a solution of the problem (1.2). Since a solution of the latter is a solution of the former, the theorem is proved. □

The next result gives a sufficient condition for u to blow up.

THEOREM 2.6. If ψ attains its maximum at b, then the solution u of the problem (1.2) attains its maximum at b. If in addition, $t_b<\infty$, then $u(b,t)$ is unbounded in $[0,t_b)$.

Proof. Let $D_{0b} = (0, b)$, $\overline{D}_{0b} = [0, b]$, $D_{b1} = (b, 1)$, $\overline{D}_{b1} = [b, 1]$, $\Omega_{0b} = D_{0b} \times (0, t_b)$, and $\Omega_{b1} = D_{b1} \times (0, t_b)$. Since $u(b, t)$ is known, let us consider the problems:

$$\left.\begin{array}{l} Lu = 0 \quad \text{in } \Omega_{0b}, u(x,0) = \psi(x) \quad \text{on } \overline{D}_{0b}, \\ u(0,t) = 0 \quad \text{and} \quad u(b,t) = u(b,t) \quad \text{for } 0 < t < t_b, \end{array}\right\} \tag{2.24}$$

$$\left.\begin{array}{l} Lu = 0 \quad \text{in } \Omega_{b1}, u(x,0) = \psi(x) \quad \text{on } \overline{D}_{b1}, \\ u(b,t) = u(b,t) \quad \text{and} \quad u(1,t) = 0 \quad \text{for } 0 < t < t_b. \end{array}\right\} \tag{2.25}$$

Because ψ attains its maximum at b, it follows from the strong maximum principle and Theorems 2.4 and 2.5 that the solution of the problem (2.24) attains its maximum at b. Similarly, the solution of the problem (2.25) attains its maximum at b.

By Theorem 2.4, u is a nondecreasing function of t. Thus, if u blows up, it is at b. If in addition, $t_b < \infty$, then let us assume that $u(b,t)$ is bounded above by some constant k_4 in $[0, t_b)$. We consider (2.8) for $t \in [t_b, T)$ with the initial condition $u(x, 0)$ replaced by $\lim_{t\to t_b^-} u(x,t)$, which we denote by $u(x, t_b)$:

$$u(b,t) = a^2 \int_{t_b}^{t} G(b,t;b,\tau) f(u(b,\tau)) d\tau + \int_0^1 \xi^q G(b,t;\xi,t_b) u(\xi,t_b) d\xi. \tag{2.26}$$

Let

$$Z(t) = \int_0^1 \xi^q G(b,t;\xi,t_b) u(\xi,t_b) d\xi,$$

and $W(t) = u(b,t) - Z(t)$. An argument analogous to the proof of Lemma 2.3 shows that there exists some t_{10} such that W exists and is unique for $t_b \le t \le t_{10}$. Thus, (2.26) has a unique solution for $t_b \le t \le t_{10}$, and hence, (2.8) has a unique solution $u(b,t)$ for $t_b \le t \le t_{10}$. This contradicts the definition of t_b, and hence the theorem is proved. □

3. Single blow-up point.

From (2.1), we obtain the following result.

LEMMA 3.1. $G(b,t;b,\tau)$ is a strictly decreasing function of t.

THEOREM 3.2. If ψ attains its maximum at b, and u blows up, then b is the single blow-up point.

Proof. Since ψ attains its maximum at b, it follows from Theorem 2.6 that if u blows up, then it blows up at b. To show that b is the only blow-up point, let us consider the problem (2.24). By the parabolic version of Hopf's lemma (cf. Friedman [5, p. 49]), $u_x(0,t) > 0$ for any arbitrarily fixed $t \in (0, t_b)$. For any $x \in (0, b)$, $u_{xx} = x^q u_t$, which is nonnegative by Theorem 2.4. Hence, u is concave up. Similarly, for any arbitrarily fixed $t \in (0, t_b)$, $u_x(1,t) < 0$. For any $x \in (b, 1)$, $u_{xx} = x^q u_t \ge 0$, and hence u is concave up. Thus, if u blows up, then b is the single blow-up point. □

Let

$$\mu(t) = \int_0^1 x^q \phi(x) u(x,t) dx,$$

where ϕ denotes the normalized fundamental eigenfunction of the problem (2.2) with λ denoting its corresponding eigenvalue.

THEOREM 3.3. If ψ attains its maximum at b,

$$\mu(0) > \left(\frac{\lambda}{a^2}\right)^{1/(p-1)}, \tag{3.1}$$

$$\phi(b)f(u(b,t)) \geq \left(\frac{1}{q+1}\right)^{p/2} u^p(b,t), \tag{3.2}$$

where p is a real number greater than 1, then the solution u of the problem (1.2) blows up at a finite time.

Proof. Multiplying the differential equation in the problem (1.2) by ϕ, and integrating over x from 0 to 1, we obtain

$$\mu'(t) + \lambda\mu(t) = a^2\phi(b)f(u(b,t)). \tag{3.3}$$

Since $u(x,t) \leq u(b,t)$, we have

$$\mu(t) \leq \left(\int_0^1 x^q\phi(x)dx\right) u(b,t).$$

It follows from the Schwarz inequality and $\int_0^1 x^q\phi^2(x)dx = 1$ that

$$\begin{aligned}\mu(t) &\leq \left(\int_0^1 x^q\phi^2(x)dx\right)^{1/2}\left(\int_0^1 x^q dx\right)^{1/2} u(b,t)\\ &\leq \left(\frac{1}{q+1}\right)^{1/2} u(b,t).\end{aligned}$$

By (3.2),

$$\phi(b)f(u(b,t)) \geq \mu^p(t).$$

From (3.3),

$$\mu'(t) + \lambda\mu(t) \geq a^2\mu^p(t).$$

Solving this Bernoulli inequality, we obtain

$$\mu^{1-p}(t) \leq \frac{a^2}{\lambda} + \left(\mu^{1-p}(0) - \frac{a^2}{\lambda}\right)e^{\lambda(p-1)t}.$$

From (3.1), $\mu^{1-p}(0) < a^2/\lambda$. Thus, μ tends to infinity for some finite t_b. This implies $u(b,t)$ blows up at t_b. □

If $t_b < \infty$, then we use the method of Olmstead and Roberts [7] to find a lower bound t_l and an upper bound t_u for t_b. These are used later on to compute the finite blow-up time. Using (2.14), we obtain from (2.15),

$$Rw < \frac{f(k_3 + k_1)}{f'(k_3 + k_1)}.$$

Let us assume that ψ attains its maximum at b. Then, $k_1 = \psi(b)$. Thus, an appropriate k_3 is the smallest solution of

$$k_3 = \frac{f(k_3 + \psi(b))}{f'(k_3 + \psi(b))}. \tag{3.4}$$

We note that in the proof of Lemma 2.3, (2.14) implies R is a contraction mapping. This and (3.4) show that if

$$a^2 \int_0^t G(b,t;b,\tau)d\tau < \frac{k_3}{f(k_3+\psi(b))}, \tag{3.5}$$

then R is a contraction mapping, and hence u exists. From (2.13), a lower bound t_l of t_b is given by

$$a^2 \int_0^{t_l} G(b,t_l;b,\tau)d\tau = \frac{k_3}{f(k_3+\psi(b))}. \tag{3.6}$$

For some $t_{11} < t_b$, (2.12) has a continuous solution $w(t)$ for $t \in [0, t_{11}]$. From Lemma 3.1,

$$w(t) \geq s(t), 0 \leq t \leq t_{11} < t_b,$$

where

$$s(t) = a^2 \int_0^t G(b,t_{11};b,\tau) f(w(\tau)+z(\tau))d\tau.$$

For some t_u to be determined later, let $\min_{0\leq t\leq t_u} z(t)$ be denoted by k_5, which is positive. Then,

$$\begin{aligned} s'(t) &= a^2 G(b,t_{11};b,t) f(w(t)+z(t)) \\ &\geq a^2 G(b,t_{11};b,t) f(s(t)+k_5). \end{aligned}$$

We have

$$\frac{s'(t)}{f(s(t)+k_5)} \geq a^2 G(b,t_{11};b,t).$$

That is,

$$\int_{k_5}^{s(t_{11})+k_5} \frac{d\tau}{f(\tau)} \geq a^2 \int_0^{t_{11}} G(b,t_{11};b,\tau)d\tau.$$

Since (2.12) having a continuous solution $w(t)$ for $t \in [0, t_{11}]$ insures that $s(t) < \infty$, we have

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} > a^2 \int_0^{t_{11}} G(b,t_{11};b,\tau)d\tau.$$

A contradiction to existence of a continuous solution occurs if

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} < \infty, \tag{3.7}$$

and there exists some t_{12} such that

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} = a^2 \int_0^{t_{12}} G(b,t_{12};b,\tau)d\tau.$$

Thus, an upper bound t_u of t_b is determined by

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} = a^2 \int_0^{t_u} G(b,t_u;b,\tau)d\tau. \tag{3.8}$$

That is,

$$\int_{k_5}^{\infty} \frac{d\tau}{f(\tau)} = a^2 \sum_{i=1}^{\infty} \frac{\phi_i^2(b)}{\lambda_i}(1-e^{-\lambda_i t_u}). \tag{3.9}$$

Thus, we have proved the following result.

THEOREM 3.4. If $t_b < \infty$, and ψ attains its maximum at b, then a lower bound t_l of t_b is determined by (3.6). If in addition, (3.7) holds, then an upper bound t_u of t_b is determined by (3.9).

4. An example.

As an illustrative example, let $q = 0$. Then,

$$G(x, t; \xi, \tau) = 2\sum_{n=1}^{\infty} e^{-n^2\pi^2(t-\tau)} \sin(n\pi x)\sin(n\pi\xi) \quad \text{for } t > \tau.$$

From Olmstead and Roberts [7],

$$\int_0^t G(b, t; b, \tau)d\tau = b(1-b) - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin^2 n\pi b}{n^2} e^{-n^2\pi^2 t}.$$

Let

$$\psi(x) = \begin{cases} x^2 & \text{for } 0 \le x \le b, \\ \left(\frac{b}{1-b}\right)^2 (1-x)^2 & \text{for } b < x \le 1. \end{cases}$$

It is nontrivial, nonnegative and continuous such that $\psi(0) = 0 = \psi(1)$. Its generalized second derivative (cf. Stakgold [10, pp. 38–39]) with respect to x is given by

$$\psi''(x) = \begin{cases} 2 & \text{for } 0 < x < b, \\ -\frac{2b}{1-b}\delta(x-b) & \text{for } x = b, \\ 2\left(\frac{b}{1-b}\right)^2 & \text{for } b < x < 1. \end{cases}$$

Thus, the condition (1.3) is satisfied if

$$\left(a^2 f(b^2) - \frac{2b}{1-b}\right)\delta(x-b) \ge 0.$$

A sufficient condition for this to hold is

$$a^2 f(b^2) \ge \frac{2b}{1-b}. \tag{4.1}$$

Let $f(u) = u^p$ where p is any real number greater than 1. From (3.4), $k_3 = (k_3 + \psi(b))/p$, and hence, $k_3 = b^2/(p-1)$. From (3.5),

$$a^2\left[b(1-b) - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi b}{n^2}e^{-n^2\pi^2 t}\right] < \frac{(p-1)^{p-1}}{p^p b^{2(p-1)}}.$$

This is satisfied for all $t > 0$ if

$$a^2 b^{2p-1}(1-b) < \frac{(p-1)^{p-1}}{p^p}. \tag{4.2}$$

Thus, u exists for all $t > 0$ if (4.2) holds. We note that (4.2) can always be achieved by placing the concentrated source sufficiently close to the boundaries (cf. Olmstead and Roberts [7]).

Since the normalized fundamental eigenfunction is given by $\phi(x) = 2^{1/2}\sin\pi x$, and its corresponding eigenvalue is $\lambda = \pi^2$, it follows from Theorem 3.3 that if

$$\frac{2^{3/2}}{(1-b)^2\pi^3}[-1+2b-2b^2+(1-2b)\cos\pi b+(1-b)\pi b\sin\pi b] > \left(\frac{\pi}{a}\right)^{2/(p-1)}, \quad (4.3)$$

$$2^{1/2}\sin\pi b \geq 1, \quad (4.4)$$

then u blows up at a finite time. A plot of the left-hand side of (4.3) as a function of b by using Mathematica® version 4.1 shows that it is positive for $0 < b < 1$. Thus for a given b, we can find a such that (4.3) is satisfied. From (3.6), a lower bound t_l for t_b is given by

$$a^2\left[b(1-b) - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi b}{n^2}e^{-n^2\pi^2 t_l}\right] = \frac{(p-1)^{p-1}}{p^p b^{2(p-1)}}. \quad (4.5)$$

We have

$$z(t) = \frac{4}{(1-b)^2\pi^3}\left\{\sum_{n=1}^{\infty}[b^2\cos n\pi + (1-2b)\cos n\pi b + (1-b)(-1+b+n\pi b\sin n\pi b)]\frac{\sin n\pi b}{n^3}e^{-n^2\pi^2 t}\right\}. \quad (4.6)$$

From (3.8), an upper bound t_u is given by

$$\frac{1}{(p-1)k_5^{p-1}} = a^2\left[b(1-b) - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi b}{n^2}e^{-n^2\pi^2 t_u}\right]. \quad (4.7)$$

Since $k_5 = \min_{0\leq t\leq t_u} z(t)$, it follows from (4.7) that an upper bound t_u may be determined by

$$\frac{1}{(p-1)k_5^{p-1}} = a^2 b(1-b). \quad (4.8)$$

As a numerical example, we further let $p = 2$ and $b = 1/2$. The sufficient condition (4.1) is satisfied if $a \geq 4\sqrt{2}$. Since (4.4) is automatically satisfied, it follows from (4.3) that u blows up in a finite time for $a > 9.74$. Thus for each value of a (> 9.74), we use (4.5) to compute a lower bound t_l by taking a finite number of terms in the infinite sum since a smaller t_l is obtained by doing so. We use (4.8) to find k_5. From (4.6),

$$\begin{aligned} z(t) &\leq \frac{4e^{-\pi^2 t}}{(1-b)^2\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}[1+b^2+(1-b)n\pi b] \\ &\leq \frac{4e^{-\pi^2 t}}{(1-b)\pi^2}\left[\frac{1+b^2}{(1-b)\pi}\left(1+\int_1^{\infty}\frac{dx}{x^3}\right) + b\sum_{n=1}^{\infty}\frac{1}{n^2}\right] \\ &= \frac{2}{1-b}\left[\frac{3(1+b^2)}{(1-b)\pi^3} + \frac{b}{3}\right]e^{-\pi^2 t}. \end{aligned}$$

Thus, an upper bound t_u may be obtained by solving

$$k_5 = \frac{2}{1-b}\left[\frac{3(1+b^2)}{(1-b)\pi^3} + \frac{b}{3}\right]e^{-\pi^2 t_u}.$$

Mathematica® is a registered trademark of Wolfram Research, Inc., Champaign, IL.

We then use the following bisection procedure with Mathematica® version 4.1 to determine the blow-up time:

Step 1. Let the lower and upper bounds $t_l^{(0)}$ and $t_u^{(0)}$ determined above be our first estimates of t_l and t_u. Then, the first estimate of t_b is $t_b^{(0)} = (t_l^{(0)} + t_u^{(0)})/2$.

Step 2. For step n, if $|t_u^{(n)} - t_l^{(n)}| < \epsilon$ (a given tolerance), then $t_b^{(n)} = (t_l^{(n)} + t_u^{(n)})/2$ is accepted as the final estimate of t_b, and we stop; otherwise, we go to the next step.

Step 3. Let $t_m = (t_l^{(n)} + t_u^{(n)})/2$, and $mh = t_m$, where m denotes the number of subdivisions of equal length h. We use the following iteration process:

$$u^{(0)}(b,t) = \psi(b),$$

and for $k = 0, 1, 2, \ldots,$

$$u^{(k+1)}(b, rh) = a^2 \int_0^{rh} G(b, rh; b, \tau) f(u^{(k)}(b, \tau)) d\tau + \int_0^1 G(b, rh; \xi, 0)\psi(\xi) d\xi,$$

where $r = 0, 1, 2, \ldots, m$. As an approximation to $G(x, t; \xi, \tau)$, we use the finite sum

$$\widetilde{G}(x, t; \xi, \tau) = 2 \sum_{n=1}^{N} e^{-n^2\pi^2(t-\tau)} \sin(n\pi x) \sin(n\pi\xi) \quad \text{for } t > \tau.$$

Using the adaptive integration procedure, we do the following calculations:

$$a^2 * N\,\text{Integrate}[\widetilde{G}(b, rh; b, \tau) f(\psi(b)), \{\tau, 0, rh\}],$$
$$N\,\text{Integrate}[\widetilde{G}(b, rh; \xi, 0)\psi(\xi), \{\xi, 0, 1\}].$$

For $r = 1, 2, 3, \ldots, m$, we obtain an approximate value $\tilde{u}^{(1)}(b, rh)$ of $u^{(1)}(b, rh)$ as

$$\tilde{u}^{(1)}(b, rh) = a^2 * N\,\text{Integrate}[\widetilde{G}(b, rh; b, \tau) f(\tilde{u}^{(0)}(b, \tau)), \{\tau, 0, rh\}] + N\,\text{Integrate}[\widetilde{G}(b, rh; \xi, 0)\psi(\xi), \{\xi, 0, 1\}],$$

where $\tilde{u}^{(0)}(b, \tau) = \psi(b)$, and $\tilde{u}^{(1)}(b, 0) = \psi(b)$.

Similarly by making use of the values,

$$\tilde{u}^{(k)}(b, 0) = \psi(b), \tilde{u}^{(k)}(b, h), \tilde{u}^{(k)}(b, 2h), \ldots, \tilde{u}^{(k)}(b, mh),$$

we obtain an approximation $\tilde{u}^{(k)}(b, t)$ of the function $u^{(k)}(b, t)$ by

$$\tilde{u}^{(k)}(b, t) = \text{Interpolation}[\{rh, \tilde{u}^{(k)}(b, rh)\}_{r=0,\ldots,m}].$$

For $r = 1, 2, 3, \ldots, m$, we perform the following calculation,

$$a^2 * N\,\text{Integrate}[\widetilde{G}(b, rh; b, \tau) f(\tilde{u}^{(k)}(b, \tau)), \{\tau, 0, rh\}],$$

to obtain an approximate value $\tilde{u}^{(k+1)}(b, rh)$ of $u^{(k+1)}(b, rh)$ as

$$\tilde{u}^{(k+1)}(b, rh) = a^2 * N\,\text{Integrate}[\widetilde{G}(b, rh; b, \tau) f(\tilde{u}^{(k)}(b, \tau)), \{\tau, 0, rh\}] + N\,\text{Integrate}[\widetilde{G}(b, rh; \xi, 0)\psi(\xi), \{\xi, 0, 1\}],$$

where $\tilde{u}^{(k+1)}(b, 0) = \psi(b)$.

For each given tolerance δ, if $|(\tilde{u}^{(k)}(b,mh) - \tilde{u}^{(k-1)}(b,mh))| < \delta$, then $t_l^{(n+1)} = t_m$, $t_u^{(n+1)} = t_u^{(n)}$, or else if $|(\tilde{u}^{(k)}(b,mh) - \tilde{u}^{(k-1)}(b,mh))| > C$ for some given positive number C, then $t_l^{(n+1)} = t_l^{(n)}$, $t_u^{(n+1)} = t_m$. We stop the iteration process and go to Step 2.

The results for t_b given in the following table were obtained by taking $N = 10$, $\varepsilon = 10^{-7}$, $\delta = 10^{-2}$, $C = 10^5$, $m = 40$, $b = 0.5$, and $f(u) = u^2$.

a	10	15	20	25	30	35	40
t_b	0.0062	0.0022	0.0012	0.00073	0.00050	0.00036	0.00027
$a^2 t_b$	0.62	0.50	0.48	0.46	0.45	0.44	0.43

The above results illustrate that the blow-up time is a decreasing function of the length a.

REFERENCES

[1] C. Y. Chan and W. Y. Chan, *Existence of classical solutions for degenerate semilinear parabolic problems*, Appl. Math. Comput. **101**, 125–149 (1999)

[2] C. Y. Chan and P. C. Kong, *Channel flow of a viscous fluid in the boundary layer*, Quart. Appl. Math. **55**, 51–56 (1997)

[3] C. Y. Chan and H. T. Liu, *Global existence of solutions for degenerate semilinear parabolic problems*, Nonlinear Anal. **34**, 617–628 (1998)

[4] M. S. Floater, *Blow-up at the boundary for degenerate semilinear parabolic equations*, Arch. Rat. Mech. Anal. **114**, 57–77 (1991)

[5] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964, pp. 39 and 49

[6] K. E. Gustafson, *Introduction to Partial Differential Equations and Hilbert Space Methods*, 2nd ed., John Wiley & Sons, New York, NY, 1987, p. 176

[7] W. E. Olmstead and C. A. Roberts, *Explosion in a diffusive strip due to a concentrated nonlinear source*, Methods Appl. Anal. **1**, 435–445 (1994)

[8] W. E. Olmstead and C. A. Roberts, *Explosion in a diffusive strip due to a source with local and nonlocal features*, Methods Appl. Anal. **3**, 345–357 (1996)

[9] H. L. Royden, *Real Analysis*, 3rd ed., Macmillan Publishing Co., New York, NY, 1988, p. 87

[10] I. Stakgold, *Boundary Value Problems of Mathematical Physics*, vol. 1, Macmillan Co., New York, NY, 1967, pp. 38–39

[11] K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth International Group, Belmont, CA, 1981, pp. 328, 352, and 380

[12] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, New York, NY, 1958, p. 506