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Single Server Retrial Queuing System with Two Different Vacation Policies

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Abstract

This paper deals with a single server retrial queueing model in which customers arrive according to a Markovian arrival process. An arriving customer on finding a free server enters into service immediately; otherwise the customer enters into an orbit of infinite size. An orbiting customer competes for service by sending out signals at random times until a free server is captured. The server operates under two vacation mechanisms. During service the server may go for vacation to attend an emergency call and after completing the vacation, the server continues the service for the same customer. Upon completion of a service, with a certain probability the server takes multiple vacations depending on the orbit size. Assuming the service, retrial and vacation times follow a general distribution, the joint distribution of the state of the server and the number of customer in the orbit is derived in the steady state. The explicit expressions of some performances measures are given. In addition numerical results are presented.

Keywords : M/G/1 Retrial queue, Steady state equations, Stability condition, Multiple vacation, Emergency vacation

1 INTRODUCTION

Recently, considerable attention has been paid to the analysis of queueing systems with repeated calls (or retrial queues, queues with returning customers, etc.) see, for example, the reviews by Yang and Templeton [7], Falin [3], Kulkarni and Liang [5] and the book by Falin and Templeton [4].

Over the past two decades, queueing systems with vacation have been studied by many researchers due to their wide applications in manufacturing and telecommunication systems. Some comprehensive surveys on the recent results for a variety of vacation models can be found in [1, 2, 6]. In almost all these papers the server operates under any one of the vacation policies: single vacation, multiple vacation, gated vacation and so on. In this article an M/G/1 retrial queueing system with two simultaneous vacation mechanisms is discussed. During the service the server may leave the system to attend an emergency call termed as emergency vacation and after each service completion the server may take multiple vacation.

2 MODEL DESCRIPTION

Assume that the customers arrive at the system in accordance with a Poisson process with rate λ. If an arriving customer finds the server idle, the customer enters the service immediately and leaves the system after service completion. If the server is found to be blocked, the arriving customer enters a retrial queue. The customer at the head of the retrial queue attempts to reach the server in a retrial time distributed with general distribution function $A(x)$, density function $a(x)$ and Laplace – Stieltjes transform $A^*(s)$. The service times are independent, identically distributed with common distribution function $B(x)$, density function $b(x)$ and Laplace – Stieltjes transform $B^*(s)$ with first two moments b_1 and b_2.

During the service the server may take the emergency vacation distributed as exponentially with rate β. When the server is in emergency vacation, the customer in service either remains in the service position with probability p until the completion of vacation or enters a retrial orbit with probability $1 – p$ and keeps returning at times exponentially distributed with mean $1/\theta$ until the server return. After completion of the emergency vacation, if the customer is not in service position, the server must wait for the interrupted customer to return. This time is referred as the ‘reserved time’. The server is not allowed to accept new customers until the customer in service leaves the system. The emergency vacation time has the distribution function $H(x)$, density function $h(x)$, Laplace – Stieltjes transform $H^*(s)$ with first two moments h_1 and h_2.

After each service completion, the server takes a vacation with probability r and with probability $1 – r$ the server waits for the next customer. At the end of the vacation period if the orbit is not empty, the server waits to serve a customer. Otherwise the server begins another vacation immediately and continues in this manner until he finds at least one customer in the orbit upon returning from a vacation. The distribution function, density function, Laplace – Stieltjes transform of vacation time are $V(x)$, $v(x)$, $V^*(s)$ respectively and first two moments are v_1 and v_2.

If the functions $\eta(x)$, $\mu(x)$, $\gamma(x)$, $\alpha(x)$ are the conditional completion rates (at time x) for repeated attempts, for service, for emergency vacation, for multiple vacation respectively, then

$$\eta(x) = \frac{a(x)}{1-A(x)},\quad \mu(x) = \frac{b(x)}{1-B(x)},\quad \gamma(x) = \frac{h(x)}{1-H(x)},\quad \alpha(x) = \frac{v(x)}{1-V(x)}$$

The state of the system at time t can be described by the Markov process $\{N(t)\ ;\ t \geq 0\} = \{J(t), J^*(t), X(t), \xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t)\ ;\ t \geq 0\}$ where J(t) denotes the server state 0, 1, 2, 3 and 4 according as the server being idle, busy, in emergency vacation, in reserved time and in multiple vacation. At the time of emergency vacation $J^*(t) = 0$ means that the customer in service remains in service position and $J^*(t) = 1$ means that the customer enters the orbit. Let X(t) denote the number of customers in the retrial queue at time t. If $J(t) = 0$ and $X(t) > 0$, then $\xi_0(t)$ represents the elapsed retrial time; if $J(t) = 1, 2, 3$ or 4, $\xi_1(t)$ corresponds to the elapsed service time; if $J(t) = 2$, $J^*(t) = 0$ or 1, $\xi_2(t)$ represents as the elapsed emergency vacation time; if $J(t) = 3$, $\xi_3(t)$ represents the elapsed reserved time; if $J(t) = 4$, $\xi_4(t)$ represents the elapsed multiple vacation time.

3 STABILITY CONDITION

We first study the condition for system stability. The following theorem provides the necessary and sufficient condition for the system to be stable.

Theorem:

The inequality $\lambda b_1(1+\beta((1-p)/\theta+h_1))+r\lambda v_1 < A^*(\lambda)$ is a necessary and sufficient condition for the system to be stable.

Proof :

Let $S^{(K)}$ be the generalized service time of the k^{th} customer in service. Then $\{S^{(K)}\}$ are independent and identically distributed with distribution function

$$\mathcal{B}(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\int_0^{\infty}\binom{i}{j}p^j(1-p)^{i-j}\frac{(\beta y)^i}{i!}e^{-\beta y}H_{i-j}^{(2)}(x-y)\,dB(y),$$

and Laplace transform

$$\mathcal{B}^*(s) = B^*(s+\beta-\beta((ps+\theta)/(s+\theta)H^*(s))$$

and expected value

$$E(S^{(k)}) = b_1(1+\beta((1-p)/\theta+h_1))$$

where $H_{i-j}^{(2)}(x)$ represents the two fold convolution of $H^{(i)}(x)$ which is i fold convolution of H(x) and the gamma distribution with the parameters i-j and $1/\theta$.

Suppose the retrial queue has a large number of customers in the following discussion.

Arrival rate at the retrial queue during service is $\lambda b_1(1+\beta((1-p)/\theta+h_1))$. Arrival rate at the retrial queue during multiple vacation is $r\lambda v_1$. Total arrival rate at the retrial queue is $\lambda b_1(1+\beta((1-p)/\theta+h_1)) + r\lambda v_1$.

Exit rate from the retrial queue by entering service is $A^*(\lambda)$. For stability, the arrival rate should be less than the exit rate. Hence, the necessary and the sufficient condition for the system to be in equilibrium state is $\lambda b_1(1+\beta((1-p)/\theta+h_1)) + r\lambda v_1 < A^*(\lambda)$.

4 STEADY STATE DISTRIBUTION

In this section, the steady state distributions for the system under consideration are obtained. For the process $\{N(t) ; t \geq 0\}$ define the following probability densities.

For $t \geq 0$ and $x \geq 0$

$I_n(t, x)\, dx = P\{J(t) = 0 ; X(t) = n ; x \leq \xi_0(t) < x + dx\}, \quad n \geq 1$

$W_n(t, x)\, dx = P\{J(t) = 1 ; X(t) = n ; x \leq \xi_1(t) < x + dx\}, \quad n \geq 0$

For $t \geq 0$, $x \geq 0$, $y \geq 0$ and $n \geq 0$

$E_{i,n}(t, x, y)\, dx\, dy = P\{J(t) = 2 ; J^*(t) = i ; X(t) = n ; x \leq \xi_1(t) < x + dx ; y \leq \xi_2(t) < y + dy\}, i = 0,1$

$R_n(t, x, y)\, dx\, dy = P\{J(t) = 3 ; X(t) = n ; x \leq \xi_1(t) < x + dx ; y \leq \xi_3(t) < y + dy\}$

$V_n(t, x)\, dx = P\{J(t) = 4 ; X(t) = n ; x \leq \xi_4(t) < x + dx\}$

The steady state equations for the model under consideration are

$$\frac{d I_n(x)}{dx} = -(\lambda + \eta(x))\, I_n(x), \quad n \geq 1 \tag{1}$$

$$\frac{d W_n(x)}{dx} = -(\lambda + \mu(x) + \beta)\, W_n(x) + \int_0^\infty E_{0,n}(x, y)\,\gamma(y)\, dy + \int_0^\infty \theta R_n(x, y)\, dy + (1 - \delta_{0n})\,\lambda\, W_{n-1}(x), \quad n \geq 0 \tag{2}$$

$$\frac{\partial E_{i,n}(x, y)}{\partial y} = -(\lambda + \gamma(y))\, E_{i,n}(x, y) + (1-\delta_{0n})\,\lambda E_{i,n-1}(x, y), \; n \geq 0, \; i = 0,1 \tag{3}$$

$$\frac{\partial R_n(x, y)}{\partial y} = -(\lambda + \theta)\, R_n(x, y) + (1 - \delta_{0n})\,\lambda\, R_{n-1}(x, y), \quad n \geq 0 \tag{4}$$

$$\frac{d V_n(x)}{dx} = -(\lambda + \alpha(x))\, V_n(x) + (1 - \delta_{0n})\,\lambda\, V_{n-1}(x), \quad n \geq 0 \tag{5}$$

The steady state boundary conditions are

$$I_n(0) = \int_0^\infty V_n(x)\,\alpha(x)\,dx + (1-r)\int_0^\infty W_n(x)\,\mu(x)\,dx, \quad n \geq 1 \tag{6}$$

$$W_n(0) = \int_0^\infty I_{n+1}(x)\,\eta(x)\,dx + (1-\delta_{0n})\,\lambda\int_0^\infty I_n(x)\,dx, \quad n \geq 0 \tag{7}$$

$$E_{0,n}(x, 0) = p\,\beta\,W_n(x), \quad n \geq 0 \tag{8}$$

$$E_{1,n}(x, 0) = (1-p)\,\beta\,W_n(x), \quad n \geq 0 \tag{9}$$

$$R_n(x, 0) = \int_0^\infty E_{1,n}(x, y)\,\gamma(y)\,dy, \quad n \geq 0 \tag{10}$$

$$V_0(0) = \int_0^\infty W_0(x)\,\mu(x)\,dx + \int_0^\infty V_0(x)\,\alpha(x)\,dx \tag{11}$$

$$V_n(0) = r\int_0^\infty W_n(x)\,\mu(x)\,dx, \quad n \geq 1 \tag{12}$$

The normalizing condition is

$$\sum_{n=1}^{\infty}\int_0^\infty I_n(x)\,dx + \sum_{n=0}^{\infty}\int_0^\infty W_n(x)\,dx + \sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty E_{0,n}(x, y)\,dx\,dy +$$

$$\sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty E_{1,n}(x, y)dx\,dy + \sum_{n=0}^{\infty}\int_0^\infty\int_0^\infty R_n(x, y)\,dx\,dy + \sum_{n=0}^{\infty}\int_0^\infty V_n(x)\,dx = 1 \tag{13}$$

Define the probability generating function,

$P(z, \cdot) = \sum_n p_n(\cdot)\,z^n$ for any probability $p_n(\cdot)$.

Then the steady state distributions of $\{N(t)\,;\, t \geq 0\}$ are given by,

$$I(z, x) = I(z, 0)\,e^{-\lambda x}\,[1 - A(x)] \tag{14}$$

$$W(z, x) = W(z, 0)\,e^{-G(\lambda(1-z))\,x}\,[1 - B(x)] \tag{15}$$

where $G(x) = x + \beta - \beta\,((xp + \theta) / (x + \theta))\,H^*(x)$

$$E_0(z, x, y) = E_0(z, x, 0)\,e^{-\lambda(1-z)\,y}\,[1 - H(y)] \tag{16}$$

$$E_1(z, x, y) = E_1(z, x, 0)\,e^{-\lambda(1-z)\,y}\,[1 - H(y)] \tag{17}$$

$$R(z, x, y) = R(z, x, 0)\,e^{-[\lambda(1-z)+\theta]\,y}\,[1 - H(y)] \tag{18}$$

$$V(z, x) = V(z, 0)\,e^{-\lambda(1-z)\,x}\,[1 - V(x)] \tag{19}$$

$$W_0(0) = V_0(0)\,[1 - V^*(\lambda)] / B^*(G(\lambda)) \tag{20}$$

$$I(z, 0) = V_0(0)\,z\,[1 - r + r\,V^*(\lambda)]\,[1-V^*(\lambda(1-z))] / D(z) \tag{21}$$

$$W(z, 0) = V_0(0)\,[1 - r + r\,V^*(\lambda)]\,[1 - V^*(\lambda(1 - z))]\,[A^*(\lambda)(1 - z) + z] / D(z) \tag{22}$$

$$E_0(z, x, 0) = V_0(0)\,p\beta\,[1 - r + r\,V^*(\lambda)]\,[1 - V^*(\lambda(1 - z))]\,[A^*(\lambda)(1 - z) + z]\,e^{-G(\lambda(1-z))x}\,[1 - B(x)] / D(z) \tag{23}$$

$$E_1(z, x, 0) = V_0(0)\,(1-p)\beta\,[1 - r + r\,V^*(\lambda)]\,[1 - V^*(\lambda(1 - z))]\,[A^*(\lambda)(1 - z) + z]\,e^{-G(\lambda(1-z))x}\,[1 - B(x)] / D(z) \tag{24}$$

$$R(z, x, 0) = V_0(0)\,(1-p)\beta\,[1 - r + r\,V^*(\lambda)]\,[1 - V^*(\lambda(1 - z))]\,H^*(\lambda(1 - z))[A^*(\lambda)(1 - z) + z]\,e^{-G(\lambda(1-z))x}\,[1-B(x)] / D(z) \tag{25}$$

$$V(z, 0) = V_0(0)\,[1 - r + r\,V^*(\lambda)]\,[k(z)\,(A^*(\lambda)(1 - z) + z) - z] / D(z) \tag{26}$$

where $D(z) = k(z)\,(A^*(\lambda)(1 - z) + z)\,(1 - r + r\,V^*(\lambda(1 - z))) - z$

and $k(z) = B^*(G(\lambda(1-z)))$ (27)

Define the partial generating function $\psi(z) = \int_0^\infty \psi(z, x)\, dx$, for any generating function $\psi(z, x)$. Then we have

$$I(z) = V_0(0)\, z\, [1 - r + r\, V^*(\lambda)]\, [1 - V^*(\lambda(1 - z))]\, [1 - A^*(\lambda)] / [\lambda\, D(z)] \quad (28)$$

$$W(z) = V_0(0)\, [1 - r + r\, V^*(\lambda)]\, [1 - V^*(\lambda(1 - z))]\, [A^*(\lambda)\, (1 - z) + z]\, [1 - k(z)] / [D(z)\, G(\lambda(1 - z))] \quad (29)$$

$$E_0(z) = V_0(0)\, p\beta\, [1-r+r\, V^*(\lambda)]\, [1-V^*(\lambda(1-z))]\, [A^*(\lambda)(1-z)+z]\, [1-k(z)]\, [1 - H^*(\lambda 1 - z))] / [\lambda D(z)\, G(\lambda(1 - z))\, (1 - z)] \quad (30)$$

$$E_1(z) = V_0(0)\, (1-p)\beta\, [1-r + r\, V^*(\lambda)]\, [1 - V^*(\lambda(1-z))]\, [A^*(\lambda)(1-z) +z]\, [1-k(z)]\, [1-H^*(\lambda 1-z))] / [\lambda D(z) G(\lambda(1-z))(1-z)] \quad (31)$$

$$R(z) = V_0(0)\, (1-p)\beta\, [1 - r + r\, V^*(\lambda)]\, [1-V^*(\lambda(1-z))]\, H^*(\lambda(1-z))\, [A^*(\lambda)(1-z)+z][1-k(z)] / [D(z)G(\lambda(1-z))[\lambda\, (1-z)+\theta]] \quad (32)$$

$$V(z) = V_0(0)\, [1 - r + r\, V^*(\lambda)]\, [1 - V^*(\lambda(1 - z))]\, [k(z)\, (A^*(\lambda)\, (1 - z) + z) - z] / [\lambda\, D(z)\, (1 - z)] \quad (33)$$

Using the normalizing condition, the expression for $V_0(0)$ is obtained as

$$V_0(0) = [A^*(\lambda) - \lambda b_1[1 + \beta\, ((1-p) / \theta + h_1)] - r\lambda v_1] / [v_1(1 - r + r\, V^*(\lambda))] \quad (34)$$

5 PERFORMANCE MEASURES

Performance measures for the system under steady state are derived is this section.

1. The steady state probability that the server idle during the retrial time is
   $I(1) = 1 - A^*(\lambda)$
2. The steady state probability that the server busy is $W(1) = \lambda\, b_1$
3. The steady state probability that the server is in emergency vacation and the customer in service remains in the service position is
   $E_0(1) = \lambda p\beta b_1 h_1$
4. The steady state probability that the server is in emergency vacation and the customer in service enters the retrial orbit is
   $E_1(1) = \lambda(1 - p)\beta b_1 h_1$
5. The steady state probability that the server is in reserved time is
   $R(1) = \lambda(1 - p)\beta b_1 / \theta$
6. The steady state probability that the server is in vacation is
   $V(1) = A^*(\lambda) - \lambda b_1\, [1 + \beta\, ((1 - p) / \theta + h_1)$
7. The steady state probability that the system is empty while the server is on vacation is
   $V_0 = V(0) = [A^*(\lambda) - \lambda b_1\, [1 + \beta\, ((1 - p) / \theta + h_1)] - r\lambda v_1]\, [1 - V^*(\lambda)] / [\lambda v_1(1 - r + r\, V^*(\lambda))]$
8. The steady state probability that the orbit is empty while the server is busy is
   $W_0 = W(0) = [A^*(\lambda) - \lambda b_1\, [1 + \beta\, ((1 - p) / \theta + h_1)] - r\lambda v_1]$

$[1 - V^*(\lambda)]\,[1 - B^*(G(\lambda))] / [G(\lambda)v_1B^*(G(\lambda))$
$(1 - r + r\,V^*(\lambda))]$

9. The steady state probability that the orbit is empty is

$$V_0 + W_0 = [A^*(\lambda) - \lambda b_1\,[1 + \beta\,((1 - p) / \theta + h_1)] - r\lambda v_1]\;[1 - V^*(\lambda)]\,[G(\lambda)\,B^*(G(\lambda)) + \lambda\,(1 - B^*(G(\lambda)))] / [\lambda\,v_1G(\lambda)B^*(G(\lambda))\,(1 - r + r\,V^*(\lambda))]$$

10. The steady state probability that the server is idle or on emergency vacation or on multiple vacation or on reserved time is

$$I(1) + E_0(1) + E_1(1) + R(1) + V(1) = 1 - \lambda\,b_1$$

11. The probability generating function of the mean number of customer in the system is given by

$$\begin{aligned} P(z) &= I(z) + z\,W(z) + z\,E_0(z) + z\,E_1(z) + z\,R(z) + V(z) \\ &= [A^*(\lambda) - k'(1) - r\lambda v_1]\;[1 - V^*(\lambda(1-z))]\;[A^*(\lambda)\;(1-z)\;+z]\;k(z)\;/\;[p\lambda v_1 D(z)] \end{aligned}$$

The mean number of customers in the system is $L_s = P'(1)$ given by,

$$L_s = [v_1(1 - A^*(\lambda) + k'(1)) + \lambda v_2 / 2] / v_1 + T_2 / T_1$$

12. The probability generating function for the number of customers in the orbit is given by,

$$\begin{aligned} H(Z) &= I(z) + W(z) + E_0(z) + E_1(z) + R(z) + V(z) \\ &= [A^*(\lambda) - k'(1) - r\lambda v_1]\;[1 - V^*(\lambda(1-z))]\;[A^*(\lambda)(1-z) + z]\;/\;[\lambda v_1 D(z)] \end{aligned}$$

13. The mean number of customers in the orbit is $L_q = H'(1)$, given by

$$L_q = [v_1(1 - A^*(x) + \lambda v_2 / 2] / v_1 + T_2 / T_1$$

Where,

$$\begin{aligned} k'(1) &= \lambda b_1\,[1 + \beta\,((1 - p) / \theta + h_1)] \\ k''(1) &= \lambda^2\,\{b_2\,[1+\beta((1-p) / \theta+h_1)]^2 + 2\alpha b_1[(1-p)\,(1 / \theta+h_1) / \theta+h_2 / 2]\} \\ T_1 &= A^*(\lambda) - k'(1) - r\lambda v_1 \\ T_2 &= k'(1)\,[1 - A^*(\lambda) + r\lambda v_1] + r\lambda v_1[1 - A^*(\lambda)] + [k''(1) + r\lambda^2 v_2] / 2 \end{aligned}$$

6 NUMERICAL RESULTS

In this section we calculate the numerical results based on the cases of exponential, Erlangian of order 2 and hyper exponential distributions for the retrial, service, emergency vacation, multiple vacation time distributions having rates η, μ, γ and α respectively. The graphs illustrated in figure. (a) – (c) compare the behaviour of L_s against the parameters, i) η, the retrial rate ii) α, the multiple vacation rate iii) p, the probability of customer remains in the system during emergency vacation for the above three distributions.

(a)

(b)

(c)

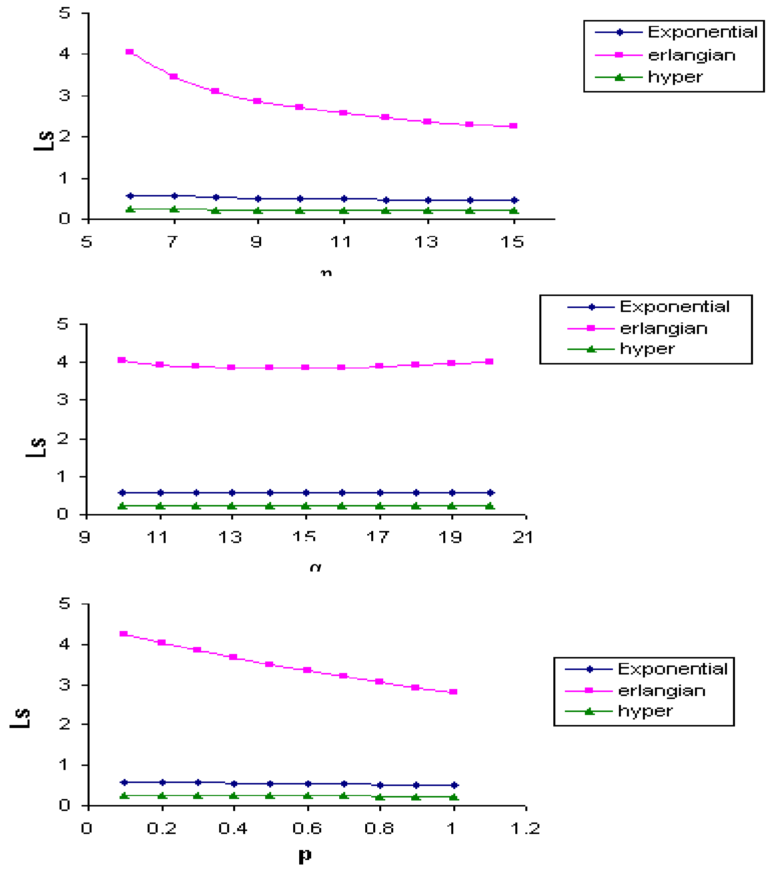

Figure (a) L_s versus η with parameters $(\lambda,\alpha,\beta,\theta,p,r,\gamma,\mu)$ = (1,10,5,10,0.2,0.5,5,15)
(b) L_s versus α with parameters $(\lambda,\eta,\beta,\theta,p,r,\gamma,\mu)$ = (1,6,5,10,0.2,0.5,5,15)
(c) L_s versus p with parameters $(\lambda,\alpha,\beta,\theta,\eta,r,\gamma,\mu)$ = (1,10,5,10,6,0.5,5,15)

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