ORIENTED REAL BLOWUP

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ABSTRACT. Let X be an analytic space, $D \subseteq X$ a Cartier divisor. The (simple) oriented real blowup of X along D is a proper map of topological spaces $\operatorname{Blo}_D X \to X$ where the preimage of D is the oriented circle bundle $N^*_{D/X}/\mathbb{R}_{>0}$ associated to the normal bundle of D in X. The aim of this brief note is to give a simple explanation of this "well-known" construction and its basic properties, and to carefully explain how it is related to the "Kato-Nakayama spaces" of logarithmic geometry. We also provide a discussion of the symplectic geometry of oriented real blowups.

1. Introduction

Let X be a topological space, $\pi : L \to X$ a complex line bundle on X, $s : X \to L$ a section of π. Locally on X we can find an isomorphism $(\pi, \phi) : L \to X \times \mathbb{C}$ of line bundles over X and we can then consider the subspace $\mathrm{B}^{\phi}_{L,s} X \subseteq L$ consisting of those $l \in L$ satisfying the condition

$$\phi(l)|(\phi s\pi)(l)| \quad = \quad |\phi(l)|(\phi s\pi)(l). \tag{1}$$

If (π, ϕ') is another trivialization, then we can find a map $t : X \to \mathbb{C}^*$ making the diagram

$$\begin{tikzcd} & X \times \mathbb{C} \arrow[dd, "(x,z)\mapsto(x,t(x)z)"] \\ L \arrow[ur, "(\pi,\phi)"] \arrow[dr, "(\pi,\phi')"'] & \\ & X \times \mathbb{C} \end{tikzcd}$$

commute, so $\phi'(l) = (t\pi)(l)\phi(l)$ for all $l \in L$. Since $(t\pi)(l)|(t\pi)(l)| \in \mathbb{C}^*$ for every $l \in L$, we will have

$$\begin{array}{rrcl} & \phi(l)|(\phi s\pi)(l)| & = & |\phi(l)|(\phi s\pi)(l) \\ \text{iff} & (t\pi)(l)\phi(l)|(t\pi)(l)(\phi s\pi)(l)| & = & |(t\pi)(l)\phi(l)|(t\pi)(l)(\phi s\pi)(l) \\ \text{iff} & (t\pi)(l)\phi(l)|(t\pi s\pi)(l)(\phi s\pi)(l)| & = & |(t\pi)(l)\phi(l)|(t\pi s\pi)(l)(\phi s\pi)(l) \\ \text{iff} & \phi'(l)|(\phi' s\pi)(l)| & = & |\phi'(l)|(\phi' s\pi)(l) \end{array}$$

(note $t\pi = t\pi s\pi$ because s is a section) which says $\mathrm{B}^{\phi}_{L,s} = \mathrm{B}^{\phi'}_{L,s}$. Since the subspace $\mathrm{B}^{\phi}_{L,s}$ of L does not depend on the choice of trivialization, these subspaces defined locally on X by choosing trivializations glue to a globally defined subspace $\mathrm{B}_{L,s} \subseteq L$.

Notice that:

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1.1. The subspace $\mathrm{B}_{L,s} X \subseteq L$ is invariant under the $\mathbb{R}_{>0}$ action inherited from the $\mathbb{C}^*$ scaling action on L (though it is not invariant under the full $\mathbb{C}^*$ action, or even under the $\mathbb{R}^*$ action). Indeed, the scaling action is given in a local trivialization ϕ by scaling $\phi(l)$, and multiplying $\phi(l)$ by a positive real number preserves the condition (1).

1.2. The subspace $\mathrm{B}_{L,s} X$ contains the zero section of L because (1) is trivially satisfied when $\phi(l) = 0$. Likewise, we have $\mathrm{B}_{L,0_L} X = L$ (here $0_L : X \to L$ is the zero section) because the condition (1) is trivially satisfied when $(\phi s \pi)(l) = 0$. We let $\mathrm{B}^*_{L,s} X$ denote the complement of the zero section in $\mathrm{B}_{L,s} X$.

1.3. The subspace $\mathrm{B}_{L,s} X$ is natural under pullback of line bundles and sections: if $f : X' \to X$ is a map of topological spaces, then $\mathrm{B}_{f^*L,f^*s} X' = X' \times_X \mathrm{B}_{L,s} X$.

1.4. In particular, if we pull back to the zero locus D of s, we obtain a cartesian diagram

$$\begin{array}{ccc} L|_D & \longrightarrow & \mathrm{B}_{L,s} X \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

of topological spaces.

1.5. Similarly, if $x \in X$ is *not* in the zero locus $D \subseteq X$ of s, then the set of nonzero $l \in \pi^{-1}(x)$ satisfying (1) (in some trivialization) form a torsor under the $\mathbb{R}_{>0}$ scaling action.

1.6. The quotient $\mathrm{Blo}_{L,s} X := \mathrm{B}^*_{L,s} X / \mathbb{R}_{>0}$ is called the *simple oriented real blowup* (of X along L, s). $\mathrm{Blo}_{L,s} X$ is a closed subspace of the *oriented circle bundle* (see §1.9) $S^1 L = L^*/\mathbb{R}_{>0}$ and is therefore proper over X. Removing zero sections and taking $\mathbb{R}_{>0}$ quotients in (1.4), we obtain a cartesian diagram

$$\begin{array}{ccc} S^1 L|_D & \longrightarrow & \mathrm{Blo}_{L,s} X \\ \downarrow & & \downarrow \\ D & \longrightarrow & X \end{array}$$

of topological spaces. Using (1.5), we see similarly that $\mathrm{Blo}_{L,s} X \to X$ is an isomorphism away from D (it is a proper bijection).

1.7. Given line bundles $L_1, \dots, L_n$ and sections s_i of L_i, we can consider the tensor product L of the L_i with the section $s = s_1 \cdots s_n$. The natural map $L_1 \times_X \cdots \times_X L_n \to L$ given by $(l_1, \dots, l_n) \mapsto l_1 \otimes \cdots \otimes l_n$ is $(\mathbb{C}^*)^n$ equivariant for the $(\mathbb{C}^*)^n$ action on L given by composing the product character $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 \cdots \lambda_n$ with the usual scaling action. One checks easily that this induces a map

$$\mathrm{B}_{L_1,s_1} X \times_X \cdots \times_X \mathrm{B}_{L_n,s_n} X \quad \to \quad \mathrm{B}_{L,s} X$$

which is $\mathbb{R}^n_{>0}$ equivariant when $\mathbb{R}^n_{>0}$ acts on $\mathrm{B}_{L,s}$ through the product character. Taking quotients by $\mathbb{R}^n_{>0}$ yields a map

$$\mathrm{Blo}_{L_1,s_1} X \times_X \cdots \times_X \mathrm{Blo}_{L_n,s_n} X \quad \to \quad \mathrm{Blo}_{L,s} X.$$

1.8. We can do the same constructions in the category of analytic spaces. When $D \subseteq X$ is a Cartier divisor in an analytic space X, D is the zero locus of a tautologically defined section s of the line bundle $L = \mathcal{O}_X(D)$. We will set $\mathrm{B}_D X := \mathrm{B}_{L,s} X$ and $\mathrm{Blo}_D X := \mathrm{Blo}_{L,s} D$ to ease notation. In this case $\mathcal{O}_X(D)|_D = N_{D/X}$, so the cartesian diagram in (1.6) takes the form:

$$\begin{tikzcd} S^1 N_{D/X} \arrow[r] \arrow[d] & \mathrm{Blo}_D X \arrow[d] \\ D \arrow[r] & X \end{tikzcd}$$

The simple oriented real blowup is the basic construction discussed in this paper. We will use it to define the *oriented real blowup* in §1.13. The oriented real blowup arises naturally in log geometry, particularly in the construction of the Kato-Nakayama space associated to a log analytic space with log structure determined by a normal crossings divisor [KN]. This is discussed in §2 where we show in §2.5 that the Kato-Nakayama space associated to an analytic space X with log structure determined by a Cartier divisor D always maps to $\mathrm{Blo}_D X$. In the case of a normal crossings divisor in a smooth analytic space, this induces a map from the Kato-Nakayama space to the oriented real blowup, which is easily seen to be an isomorphism (§2.6). This section is basically my explanation of the remark (1.2.3) in [KN] and probably has significant overlap with [KN94], [P], [M].

1.9. Oriented circle bundle. If L is a complex line bundle over a topological space X, let L^* be the corresponding principal $\mathbb{C}^*$ bundle (i.e. the complement of the zero section). Let

$$S^1 L \quad := \quad L^*/\mathbb{R}_{>0}$$

denote the quotient of this principal bundle by the action of $\mathbb{R}_{>0} \subset \mathbb{C}^*$. The principal $S^1 = \mathbb{C}^*/\mathbb{R}_{>0}$ bundle $S^1 L$ is called the *oriented circle bundle* associated to L. The identification $L^* = (L^\vee)^*$ induces an orientation reversing isomorphism $S^1 L \cong S^1 L^*$. If one has a Hermitian metric $\langle\ ,\ \rangle : L \otimes \overline{L} \to \mathbb{C}$ on L, then $S^1 L$ can be identified with the locus of $v \in L$ with $\langle v, v\rangle = 1$.

For any positive integer n, there is a natural isomorphism

$$S^1(L^{\otimes n}) \quad = \quad (S^1 L)/\mu_n,$$

where $\mu_n \subseteq S^1 \subseteq \mathbb{C}$ is the group of complex n^{th} roots of unity acting on $S^1 L$ through the inclusion $\mu_n \subseteq S^1$ and the principal bundle action of S^1 on $S^1 L$. The isomorphism is given by taking $[u] \in (S^1 L)/\mu_n = L^*/(\mathbb{R}_{>0} \oplus \mu_n)$ to $[u^{\otimes n}] \in S^1 L^{\otimes n}$.

The oriented circle bundle $\tau : S^1 L \to X$ inherits an orientation (relative to X) from the complex orientation of L, so that $\mathrm{R}^1\, \tau_* \underline{\mathbb{Z}} = \underline{\mathbb{Z}}$. More generally, one could perform the construction of $S^1 L$ starting from any rank two real bundle L, in which case the monodromy representation $\rho : \pi_1(X) \to \operatorname{Aut} \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ corresponding to the local system $\mathrm{R}^1\, \tau_* \underline{\mathbb{Z}}$ is identified with the first Stieffel-Whitney class of L under the sequence of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}(\pi_1(X), \mathbb{Z}/2\mathbb{Z}) &= \mathrm{Hom}(\mathrm{H}_1(X,\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= \mathrm{H}^1(X, \mathbb{Z}/2\mathbb{Z}). \end{aligned}$$

Going back to the case of a complex line bundle, the E_2 term of the Leray spectral sequence for τ looks like

$$\begin{tikzcd}
\mathrm{H}^0(X,\mathbb{Z}) \arrow[drr] & \mathrm{H}^1(X,\mathbb{Z}) \arrow[drr] & \mathrm{H}^2(X,\mathbb{Z}) & \cdots \\
\mathrm{H}^0(Z,\mathbb{Z}) & \mathrm{H}^1(X,\mathbb{Z}) & \mathrm{H}^2(X,\mathbb{Z}) & \cdots
\end{tikzcd}$$

It is a well-known special case of Cartan's general results on the Leray sequence for $\mathbf{E}G \to \mathbf{B}G$ that the d_2 boundary maps here are given by cup product with $c_1(L) \in \mathrm{H}^2(X,\mathbb{Z})$, so one has a simple method of computing $\mathrm{H}^*(S^1L,\mathbb{Z})$. The universal example is

$$\begin{aligned} S^1\mathcal{O}_{\mathbb{P}^n}(-1) &= (\mathbb{C}^{n+1} \setminus \{\overline{0}\})/\mathbb{R}_{>0} \\ &= S^{2n+1}. \end{aligned}$$

1.10. Higher codimension. Having defined the oriented real blowup $\operatorname{Blo}_D X$ along a Cartier divisor D in a complex variety X, one may also wish to define the oriented real blowup $\operatorname{Blo}_Z X$ along a higher codimension closed subvariety $Z \hookrightarrow X$ (or along a non Cartier divisor). Assuming everything is smooth, one would probably expect that the preimage of Z in $\operatorname{Blo}_Z X$ should be the sphere bundle $S^{2\operatorname{codim}_{\mathbb{C}} Z/X-1}N_{Z/X} = N^*_{Z/X}/\mathbb{R}_{>0}$. At least when everything is smooth, one could presumably define something like this directly by mimicking our construction in the Cartier divisor case, but it is easiest to simply define

$$\operatorname{Blo}_Z X \quad := \quad \operatorname{Blo}_E \operatorname{Bl}_Z X,$$

where E is the exceptional (Cartier!) divisor in the usual blowup $\operatorname{Bl}_Z X$. In the smooth (or l.c.i.) situation where $E = \mathbb{P}(N_{Z/X})$, we will have $N_{E/\operatorname{Bl}_Z X} = \mathcal{O}_E(-1)$ and the total space of $\mathcal{O}_E(-1)^*$ is just $N^*_{Z/X}$ so the preimage of Z in $\operatorname{Blo}_Z X$ (which is the preimage of E in $\operatorname{Blo}_E \operatorname{Bl}_Z X$) is the expected sphere bundle $\mathcal{O}_E(-1)^*/\mathbb{R}_{>0} = N^*_{Z/X}/\mathbb{R}_{>0}$.

1.11. Example: The complex plane. The motivating example is the simple oriented real blowup of the complex plane $\mathbb{C}$ at the origin. Here we have

$$\mathrm{B}_0\,\mathbb{C} \quad = \quad \{(z,Z) \in \mathbb{C}^2 : z|Z| = |z|Z\}$$

and $\operatorname{Blo}_0 \mathbb{C} = \mathrm{B}_0^*\,\mathbb{C}/\mathbb{R}_{>0}$ is the subspace of $\mathbb{C}_z \times S^1_Z$ given by

$$\operatorname{Blo}_0 \mathbb{C} \quad = \quad \{(z,Z) : z = |z|Z\},$$

which maps to $\mathbb{C}$ by projection to the first factor.

Consider the usual $\mathbb{C}^*$ action scaling on $\mathbb{C}$. The induced $\mathbb{R}_{>0}$ action on $\mathbb{C}$ has one effective orbit for each point of $S^1 \subset \mathbb{C}$ and the origin is in the closure of each such orbit; each such orbit closure is a half infinite interval $\mathbb{R}_{\geq 0}$ with the evident $\mathbb{R}_{>0}$ action. The $\mathbb{C}^*$ action on $\mathbb{C}$ lifts naturally to an action on $\operatorname{Blo}_0 \mathbb{C}$ by the rule $\lambda \cdot (z,Z) = (\lambda z, \lambda/|\lambda|Z)$, so that the induced $\mathbb{R}_{>0}$ action is trivial on the "exceptional locus" $\{(0,Z) : Z \in S^1\}$. This has the effect of separating out the effective $\mathbb{R}_{>0}$ orbit closures, so that

$$\begin{aligned} \mathbb{R}_{\geq 0} \times S^1 &\to \operatorname{Blo}_0 \mathbb{C} \\ (r,Z) &\mapsto (rZ,Z) \end{aligned}$$

is a $\mathbb{C}^*$ equivariant isomorphism for the action $\lambda \cdot (r,Z) := (|\lambda|r, \lambda/|\lambda|Z)$ on the domain, and the effective $\mathbb{R}_{>0}$ orbit closures in $\operatorname{Blo}_0 \mathbb{C}$ are disjoint and each is isomorphic to $\mathbb{R}_{\geq 0}$.

In particular, $\mathrm{Blo}_0\,\mathbb{C} \cong \mathbb{R}_{\geq 0} \times S^1$ is a half infinite annulus and the exceptional locus is its boundary in the manifold sense.

Similarly, we can consider the *simple oriented real blowup* of $\mathbb{C}^n$ along the normal crossings divisor $D = \{z_1 \cdots z_n = 0\}$ given by the union of the coordinate hyperplanes $H_1, \ldots, H_n$. This is the subspace of $\mathbb{C}^n_{z_1,\ldots,z_n} \times S^1_Z$ given by

$$\mathrm{Blo}_D \quad = \quad \{(z_1, \ldots, z_n, Z) : z_1 \cdots z_n = |z_1 \cdots z_n| Z\}.$$

What we will later call the *oriented real blowup* $\mathrm{Blo}_{H_1,\ldots,H_n}\,\mathbb{C}^n$ of $\mathbb{C}^n$ along D is the subspace of $\mathbb{C}^n \times (S^1)^n_{Z_1,\ldots,Z_n}$ defined by

$$\mathrm{Blo}_{H_1,\ldots,H_n}\,\mathbb{C}^n \quad = \quad \{(z_1, \ldots, z_n, Z_1, \ldots, Z_n) : z_i = |z_i| Z_i \text{ for all } i\}.$$

Notice that:

1.11.1. The oriented real blowup maps to the simple oriented real blowup via

$$(z_1, \ldots, z_n, Z_1, \ldots, Z_n) \quad \mapsto \quad (z_1, \ldots, z_n, Z_1 \cdots Z_n)$$

and this map, which clearly commutes with the projections to $\mathbb{C}^n_{z_1,\ldots,z_n}$, is an isomorphism away from the set of points $(z_1, \ldots, z_n)$ contained in two or more coordinate hyperplanes.

1.11.2. The oriented real blowup

$$\mathrm{Blo}_{H_1,\ldots,H_n}\,\mathbb{C}^n \quad = \quad \pi_1^*\,\mathrm{Blo}_0\,\mathbb{C}^1 \times_{\mathbb{C}^n} \cdots \times_{\mathbb{C}^n} \pi_n^*\,\mathrm{Blo}_0\,\mathbb{C}$$

is nothing but the fibered product of the pullbacks of $\mathrm{Blo}_0\,\mathbb{C}$. As in the $\mathbb{C}$ example, the simple oriented real blowup is the quotient of (the complement of the zero section in)

$$\mathrm{B}_D\,\mathbb{C}^n \quad := \quad \{(z_1, \ldots, z_n, Z) : z_1 \cdots z_n |Z| = |z_1 \cdots z_n| Z\}$$

by the $\mathbb{R}_{>0}$ action scaling the $\mathbb{C}_Z$ factor. The oriented real blowup has a similar quotient description obtained by pulling back the quotient description of $\mathrm{Blo}_0\,\mathbb{C} = \mathrm{B}_0^*\,\mathbb{C}/\mathbb{R}_{>0}$. The map from the oriented real blowup to the simplified oriented real blowup is obtained as the quotient by $\mathbb{R}^n_{>0}$ of an obvious lifted map

$$\mathrm{B}_0\,\mathbb{C} \times_{\mathbb{C}^n} \cdots \times_{\mathbb{C}^n} \mathrm{B}_0\,\mathbb{C} \to \mathrm{B}_D\,\mathbb{C}^n$$

(after removing the zero sections). Note that this lifted map is $\mathbb{R}^n_{>0}$ equivariant for the $\mathbb{R}^n_{>0}$ action on $\mathrm{B}_D\,\mathbb{A}^n$ inherited from the $\mathbb{R}_{>0}$ action on $\mathrm{B}_D\,\mathbb{A}^n$ via the map

$$\begin{array}{rcl} \mathbb{R}^n_{>0} & \to & \mathbb{R}_{>0} \\ (\lambda_1, \ldots, \lambda_n) & \mapsto & (\lambda_1 \cdots \lambda_n). \end{array}$$

1.12. Cohomology groups. When X is a smooth analytic space and $D \subseteq X$ is a normal crossings divisor, then it is clear from the above local considerations that $\mathrm{Blo}_D\,X$ is a smooth (oriented) manifold with boundary δ given by the preimage of D. This manifold with boundary is homotopy equivalent to its interior, so we have

$$\mathrm{H}^*(\mathrm{Blo}_D\,X) \quad = \quad \mathrm{H}^*(X \setminus D).$$

There is an isomorphism of relative cohomology groups

$$\mathrm{H}^*(\mathrm{Blo}_D\,X, \delta) \quad = \quad \mathrm{H}^*(X, D)$$

because the space Y obtained from $\mathrm{Blo}_D\,X$ by contracting δ to a point is the same as the space obtained from X by contracting D to a point, and both cohomology groups in question (at least in positive degree) agree with the cohomology of Y—c.f. Theorem 2.13 and the related discussion in [H]. The map of pairs

$$\tau : (\mathrm{Blo}_D X, S^1 N_{D/X}) \quad \to \quad (X, D)$$

induces a map of long exact cohomology sequences of pairs

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}^i(X,D) \arrow[r] \arrow[d, equal] & \mathrm{H}^i(X) \arrow[r] \arrow[d] & \mathrm{H}^i(D) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \mathrm{H}^i(\mathrm{Blo}_D X, S^1 N_{D/X}) \arrow[r] & \mathrm{H}^i(\mathrm{Blo}_D X) \arrow[r] & \mathrm{H}^i(S^1 N_{D/X}) \arrow[r] & \cdots \end{tikzcd}$$

which is often useful for computations.

1.13. Oriented real blowup. We have reserved the name *oriented real blowup* for the following construction built from the simple oriented real blowup. Suppose $D = D_1 \cup \cdots \cup D_n$ is a ("scheme theoretic") union of Cartier divisors in an analytic space X, so each D_i is the zero locus of a section s_i of a line bundle L_i and D is the zero locus of the section $s_1 \cdots s_n$ of the tensor product line bundle $L := L_1 \cdots L_n$. Let

$$\mathrm{Blo}_{D_1,\dots,D_n} X \quad := \quad \mathrm{Blo}_{D_1} X \times_X \cdots \times_X \mathrm{Blo}_{D_n} X$$

be the fibered product over X of the oriented real blowups of the D_i. The tensor product map

$$L_1 \times_X \cdots \times_X L_n \quad \to \quad L$$

is easily seen to take

$$\mathrm{B}_{D_1} X \times_X \cdots \times_X \mathrm{B}_{D_n} X$$

into $\mathrm{B}_D X$ and this map is $\mathbb{R}^n_{>0}$ equivariant for the $\mathbb{R}^n_{>0}$ action on L inherited from the scaling action on L via (**??**). Removing the zero sections and taking quotients we obtain a map

$$\mathrm{Blo}_{D_1,\dots,D_n} X \quad \to \quad \mathrm{Blo}_D X.$$

Now suppose X is a smooth analytic space and D is a simple normal crossings divisor in X. Then locally near any point of X, D is a union of smooth divisors $D_1, \dots, D_n$ meeting like the first n coordinate hyperplanes in $\mathbb{A}^{\dim X}$ and we can form the space $\mathrm{Blo}_{D_1,\dots,D_n} X$. These locally defined spaces over X glue to yield a space $\mathrm{OBl}_D X$ which we will call the *oriented real blowup* of X along D.

Remark 1.13.1. The oriented real blowup of the Deligne-Mumford stack of marked, stable nodal curves $\overline{\mathcal{M}}_{g,n}$ along the (normal crossings) divisor $D := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ parameterizing "strictly nodal" curves is called the *Harvey bordification* e.g. by Looijenga in [EL]. (Really the Harvey bordification refers to an extension of the action of the mapping class group on Teichmüller space to an action on a larger space whose quotient is the real blowup of $\overline{\mathcal{M}}_{g,n}$ along its boundary.)

1.14. Orientations. Let us make a few remarks about orientations of oriented real blowups. First of all, by an *orientation*, one always means a choice of trivialization (global section) of a principal $\mathbb{Z}_2$ bundle (double cover). Of course an orientation need not exist if the bundle is non-trivial, but if there is an orientation, then the set of orientations is a torsor under $\mathbb{Z}_2$. All other notions of orientation are defined in terms of this one. For example, an orientation of a real line bundle L is an orientation of the $\mathbb{Z}_2$ principal bundle $L^*/\mathbb{R}_{>0}$. This means that for any open subset $U \subseteq M$, there is a partition

$$\Gamma(U, L^*) \quad = \quad \Gamma(U, L^*)^+ \coprod \Gamma(U, L^*)^-$$

of the set $\Gamma(U, L^*)$ of nowhere-vanishing sections into pieces called the "positive" sections and the "negative" sections; the pieces are preserved by the restriction maps and are invariant under scaling by positive real-valued functions (and are exchanged by $s \mapsto -s$), but they may be empty. An orientation of an n-dimensional smooth manifold M is an orientation of the real line bundle $\wedge^n TM$.

Sometimes people say that an orientation of a real line bundle L over a topological space X is a nowhere-vanishing section of L up to rescaling by a positive real valued function on X—that is, an element of

$$\Gamma(X, L^*)/\Gamma(X, \mathcal{C}(\,\_\,, \mathbb{R}_{>0})).$$

Such an orientation determines an orientation in the above sense because there is a natural monomorphism

$$\Gamma(X, L^*)/\Gamma(X, \mathcal{C}(\,\_\,, \mathbb{R}_{>0})) \quad \to \quad \Gamma(X, L^*/\mathbb{R}_{>0}).$$

In situations where one has "partitions of unity," this monomorphism is bijective because the only obstruction to lifting an element of $\Gamma(X, L^*/\mathbb{R}_{>0})$ to an actual nowhere vanishing section of L lies in $\mathrm{H}^1(X, \mathcal{C}(\,\_\,, \mathbb{R}_{>0})) = 0$, but in general this map is not surjective. For example, there are real line bundles on the long line that deserve to be called orientable (that is, they are orientable in the above sense), but which have no global non-vanishing sections.

One can uniquely extend orientations under mild hypotheses:

Lemma 1.0.1. *Let $f : X \to Y$ be a double cover, $U \subseteq Y$ an open dense subset of Y with closed complement $Z = Y \setminus U$. Suppose every point $z \in Z$ has a cofinal system of neighborhoods V in Y such that $V \setminus Z$ is connected. Then any section $s : U \to X$ over U extends uniquely to a global section $\overline{s} : Y \to X$ of f.*

Proof. Fix a point $z \in Z$. By the hypothesis, we can find a neighborhood V of z in Y such that $V \setminus Z$ is connected and such that $f : f^{-1}(V) \to V$ is a trivial double cover. The topological space V is connected (because it contains the connected space $V \setminus Z$ as a dense subspace) and $f : f^{-1}(V) \to V$ is a trivial double cover, so $f^{-1}(V)$ has two connected components V_1, V_2 each isomorphic to V via f. Since $V \setminus Z$ is connected, so is $s(V \setminus Z) \subseteq f^{-1}(V)$, so it is contained in exactly one of V_1, V_2, say V_1. Since $V \setminus Z$ is dense in V and V_1 is isomorphic to V via f, $s(V \setminus Z)$ is dense in V_1, so any section of $f : f^{-1}(V) \to V$ extending $s : U \cap V \to f^{-1}(V)$ must take values in V_1, hence the inverse of the isomorphism $f|V_1 : V \to V$ is the unique such extension. Since s extends uniquely to a neighborhood of any $z \in Z$, the unique local extensions of f glue to a global section $\overline{s}$ of f. $\square$

The typical situation where the lemma does *not* apply is when Y is a circle and Z is a point. The hypotheses of the lemma certainly hold in the situation of the following

Proposition 1.0.1. *Let M be a manifold with boundary Z, $U = M \setminus Z$ the interior. Then any orientation of U extends uniquely to an orientation of M.*

If M is an oriented manifold with boundary Z, then its boundary Z inherits an orientation by the "outward normal first" convention, which we now explain. The most difficult thing to understand here is the definition of the "outward normal direction". It is more direct to work with differentials and the conormal bundle rather than vector fields and the normal bundle. A *boundary-defining function* f on a smooth manifold M with boundary Z is a smooth function $f : M \to \mathbb{R}_{\geq 0}$ to the nonnegative reals (with coordinate t) vanishing Z so that f^*dt (or rather, its restriction to Z) is in the kernel of

$$\Gamma(Z, T^*M|_Z) \quad \to \quad \Gamma(Z, T^*Z)$$

(i.e. is a section of the conormal bundle $N^\vee_{Z/M}$) and is a nowhere vanishing on Z.

The basic local case is where $M = \mathbb{R}^n \times \mathbb{R}_{\geq 0}$ with coordinates $x_1, \dots, x_n, \lambda$, so that $Z = \{\lambda = 0\}$. The cotangent sheaf of M is freely generated by $dx_1, \dots, dx_n, \lambda$, while the cotangent sheaf of Z is freely generated by $dx_1, \dots, dx_n$, so the conormal sheaf of Z in M is freely generated by $d\lambda$. The function $f : M \to \mathbb{R}$ given by $f(\overline{x}, \lambda) := \lambda$ is evidently a boundary-defining function with $f^*dt = d\lambda$. The key point is that any other boundary-defining function yields the same orientation on the conormal bundle:

Lemma 1.0.2. *Suppose $f = f(\overline{x}, \lambda) : M \to \mathbb{R}_{\geq 0}$ is a boundary defining function on $M = \mathbb{R}^n \times \mathbb{R}_{\geq 0}$. Then there is a strictly positive smooth function $g : Z \to \mathbb{R}_{>0}$ so that $f^*dt = g d\lambda$ in $\Gamma(Z, N^*_{Z/M})$, hence $[f^*dt] = [d\lambda]$ in the set $\Gamma(Z, N^*_{Z/M}/\mathbb{R}_{\geq 0})$ of orientations of the conormal bundle.*

Proof. We have

$$\begin{aligned} f^*dt &= \frac{\partial f}{\partial \lambda} d\lambda + \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \\ f^*dt|_Z &= \left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=0} d\lambda + \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\lambda=0} dx_i \end{aligned}$$

So, to say that f is a boundary defining function is to say that

$$\left.\frac{\partial f}{\partial x_i}\right|_{\lambda=0} = 0$$

for $i = 1, \dots, n$ and that

$$g := \left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=0}$$

is a nowhere-vanishing smooth function of $x_1, \dots, x_n$. But actually, it must be a *positive* smooth function of the x_i, else it would be strictly negative for some $\overline{x}$, which would violate the Mean Value Theorem because $f(\overline{x}, 0) = 0$, f is a nonnegative function, and all the *other* partials of f vanish at $(\overline{x}, 0)$. $\square$

The lemma implies that any two boundary-defining functions on a manifold M with boundary Z determine the same orientation of $N^*_{Z/M}$ (because this is a local question, so we reduce to the situation of the lemma) so, since we can always find a boundary-defining function locally, we have a global orientation of $N^*_{Z/M}$ called the *inward normal orientation*. As it turns out, the more natural orientation is the opposite orientation, called the *outward normal orientation*, which we always use by default unless otherwise indicated. If M itself is oriented, then there is a unique orientation of Z so that the isomorphism

$$\begin{array}{rcl} N^\vee_{Z/M} \otimes \wedge^{n-1} T^*Z & \to & \wedge^n Z \\ \beta \otimes \alpha_1 \wedge \cdots \wedge \alpha_{n-1} & \mapsto & \beta \end{array}$$

obtained from adjunction of the exact sequence

$$0 \to N^\vee_{Z/M} \to T^*M|_Z \to T^*Z \to 0$$

is orientation-preserving (for the outward normal orientation).

This "outward normal first" convention is the one which makes Stokes' Theorem hold. That is, the diagram

$$\begin{tikzcd} \Gamma(M, \wedge^{n-1} T^*M) \arrow[r, "d"] \arrow[d] & \Gamma(M, \wedge^n T^*M) \arrow[d, "\int"] \\ \Gamma(Z, \wedge^{n-1}) \arrow[r, "\int"] & \mathbb{R} \end{tikzcd}$$

will commute if Z is given the outward normal first orientation it inherits from the compact, oriented, smooth manifold M. For example, consider the unit interval $M = [0, 1]$ with coordinate t, with its usual orientation $[dt]$. The boundary $Z = \{0, 1\}$ consists of two points. A point is canonically oriented because $\wedge^0 T(\text{Point}) = \mathbb{R}$ is naturally oriented, but the point 0 inherits the *opposite* of the canonical orientation from M via the outward normal first convention (because the outward normal direction at 0 is $-dt$ since t is a boundary-defining function at zero, so dt gives the inward normal direction) while the point 1 inherits the canonical orientation because the outward normal direction at 1 is dt (because $1 - t$ is a boundary-defining function near 1).

Now let us put all of this together and discuss the orientation of $S^1 \times \mathbb{R}_{\geq 0} = \operatorname{Blo}_0 \mathbb{C}$ (c.f. §1.11) and its boundary circle. First of all, by Proposition 1.0.1, the usual orientation $[dx \wedge dy]$ on $\mathbb{C}^*$ extends uniquely to an orientation of $\operatorname{Blo}_0 \mathbb{C}$. The orientation inherited by the boundary S^1 via the outward normal first convention is *not* the usual counterclockwise orientation, but rather the clockwise orientation, as is clear from Figure 1 below. This is because the counterclockwise orientation is the one that S^1 would inherit as the boundary of the unit disc, while our circle has the rest of the manifold to its "outside".

1.15. Example: Nodal curves. Consider the nodal curve $f : \mathbb{A}^2_{z_1,z_2} \to \mathbb{A}^1_z$ given by $z \mapsto z_1 z_2$. Then the pullback of the Cartier divisor $\{0\} \subset \mathbb{A}^1_z$ under f is the Cartier divisor $D = \{z_1 z_2 = 0\} \subset \mathbb{A}^2_{z_1 z_2}$, which is the nodal curve given by the union of the two coordinate axes H_1, H_2. The functoriality properties of (simplified) oriented real blowup

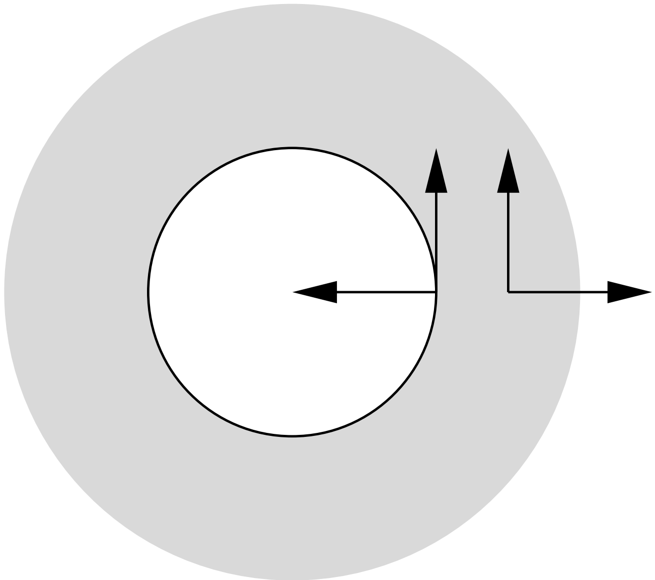

FIGURE 1. The orientation on S^1 inherited from $\mathrm{Blo}_0\,\mathbb{C}$ via the outward normal first convention is not the usual counterclockwise orientation.

yield a commutative diagram

$$\begin{tikzcd}
\mathrm{Blo}_{H_1,H_2}\,\mathbb{A}^2 \arrow[r] \arrow[dr] & \mathrm{Blo}_D\,\mathbb{A}^2 \arrow[r] \arrow[d] & \mathrm{Blo}_0\,\mathbb{A}^1 \arrow[d] \\
& \mathbb{A}^2_{z_1,z_2} \arrow[r] & \mathbb{A}^1_z.
\end{tikzcd}$$

To understand the maps in this diagram, start from the diagram

$$\begin{tikzcd}
\mathrm{B}_{H_1,H_2}\,\mathbb{A}^2 \arrow[r] \arrow[d] & \mathrm{B}_D\,\mathbb{A}^2 \arrow[r] \arrow[d] & \mathrm{B}_0\,\mathbb{A}^1 \arrow[d] \\
\mathbb{A}^2_{z_1,z_2} \times \mathbb{A}^2_{Z_1,Z_2} \arrow[r, "{(z_1,z_2,Z_1,Z_2)\mapsto(z_1,z_2,Z_1Z_2)}"] \arrow[dr] & \mathbb{A}^2_{z_1,z_2} \times \mathbb{A}^1_Z \arrow[r, "{(z_1,z_2,Z)\mapsto(z_1z_2,Z)}"] \arrow[d] & \mathbb{A}^1_z \times \mathbb{A}^1_Z \arrow[d] \\
& \mathbb{A}^2_{z_1,z_2} \arrow[r, "{(z_1,z_2)\mapsto z_1z_2}"] & \mathbb{A}^1_z,
\end{tikzcd}$$

where the lower vertical maps should be viewed as the bundle $\mathcal{O}_{\mathbb{A}^1}(0)$ over $\mathbb{A}^1$ and its pullback to $\mathbb{A}^2$, and the diagonal map should be viewed as the fibered product $\mathcal{O}_{\mathbb{A}^2}(H_1)\times_{\mathbb{A}^2}\mathcal{O}_{\mathbb{A}^2}(H_2)$ of line bundles on $\mathbb{A}^2$. The top vertical maps are the inclusions of the subspaces:

$$\begin{aligned}
\mathrm{B}_{H_1,H_2}\,\mathbb{A}^2 &= \{(z_1,z_2,Z_1,Z_2) : z_i|Z_i| = |z_i|Z_i, i=1,2\} \\
\mathrm{B}_D\,\mathbb{A}^2 &= \{(z_1,z_2,Z) : z_1z_2|Z| = |z_1z_2|Z\} \\
\mathrm{B}_0\,\mathbb{A}^1 &= \{(z,Z) : z|Z| = |z|Z\}.
\end{aligned}$$

The maps are $\mathbb{R}^2_{>0}$ equivariant for the actions

$$\begin{aligned}(\lambda_1,\lambda_2)\cdot Z &= \lambda_1\lambda_2 Z\\ (\lambda_1,\lambda_2)\cdot Z_1 &= \lambda_1 Z_1\\ (\lambda_1,\lambda_2)\cdot Z_2 &= \lambda_2 Z_2.\end{aligned}$$

Removing the locus where $Z=0$, $Z_1=0$ or $Z_2=0$ and taking the quotient by this action yields the diagram:

$$\begin{tikzcd}
\mathrm{Blo}_{H_1,H_2}\mathbb{A}^2 \arrow[r] \arrow[d] & \mathrm{Blo}_D\mathbb{A}^2 \arrow[r] \arrow[d] & \mathrm{Blo}_0\mathbb{A}^1 \arrow[d] \\
\mathbb{A}^2_{z_1,z_2}\times(S^1)^2_{Z_1,Z_2} \arrow[r, "{(z_1,z_2,Z_1,Z_2)\mapsto(z_1,z_2,Z_1Z_2)}"] \arrow[dr] & \mathbb{A}^2_{z_1,z_2}\times S^1_Z \arrow[r, "{(z_1,z_2,Z)\mapsto(z_1z_2,Z)}"] \arrow[d] & \mathbb{A}^1_z\times S^1_Z \arrow[d] \\
 & \mathbb{A}^2_{z_1,z_2} \arrow[r, "{(z_1,z_2)\mapsto z_1z_2}"] & \mathbb{A}^1_z,
\end{tikzcd}$$

where the top vertical arrows are the inclusions of the subspaces:

$$\begin{aligned}\mathrm{Blo}_{H_1,H_2}\mathbb{A}^2 &= \{(z_1,z_2,Z_1,Z_2): z_i=|z_i|Z_i, i=1,2\}\\ \mathrm{Blo}_D\mathbb{A}^2 &= \{(z_1,z_2,Z): z_1z_2=|z_1z_2|Z\}\\ \mathrm{Blo}_0\mathbb{A}^1 &= \{(z,Z): z=|z|Z\}.\end{aligned}$$

Notice that the fiber of $\mathrm{Blo}_{H_1,H_2}\mathbb{A}^2\to\mathrm{Blo}_0\mathbb{A}^1$ over a point $(0,Z)$ in the "exceptional locus" $\{0\}\times S^1_Z\subset\mathrm{Blo}_0\mathbb{A}^1$ is the locus

$$\{(z_1,z_2,Z_1,Z_2)\in\mathrm{Blo}_{H_1,H_2}\mathbb{A}^2: z_1z_2=0, Z_1Z_2=Z\},$$

which can be described as a coproduct

$$\{(z_1,0,Z_1,Z/Z_1): z_1=|z_1|Z_1\}\coprod_{\{(0,0,Z_1,Z_2):Z_1Z_2=Z\}}\{(z_1,0,Z/Z_2,Z_2): z_2=|z_2|Z_2\}.$$

This we recognize as the coproduct

$$\mathrm{Blo}_0\mathbb{A}^1_{z_1}\textstyle\coprod_{(0,Z_1)\sim(0,Z/Z_1)}\mathrm{Blo}_0\mathbb{A}^1_{z_2} \tag{1}$$

obtained by gluing the exceptional locus of $\mathrm{Blo}_0\mathbb{A}^1_{z_1}$ (which is also its boundary in the sense of manifolds) to the exceptional locus of $\mathrm{Blo}_0\mathbb{A}^1_{z_2}$ in an orientation reversing manner.

Notice that the gluing (1) depends on the parameter $Z\in S^1_Z$. More canonically, the inclusion of the axes $\mathbb{A}^1_{z_1}\coprod_0\mathbb{A}^1_{z_2}\hookrightarrow\mathbb{A}^2_{z_1,z_2}$, and the flat family f having this as fiber over 0, yield an identification $N_{0/\mathbb{A}^1_{z_1}}\otimes N_{0/\mathbb{A}^2_{z_2}}=N_{0/\mathbb{A}^1_z}$, so we see that an orientation reversing identification $S^1N_{0/\mathbb{A}^1_{z_1}}\cong S^1N_{0/\mathbb{A}^2_{z_2}}$ is the same thing as a point of $S^1N_{0/\mathbb{A}^1_z}=S^1_Z$.

Notice also that the gluing (1) of two half infinite annuli is homeomorphic to $\mathbb{C}^*$. The fiber of $\mathrm{Blo}_{H_1,H_2}\to\mathrm{Blo}_0\mathbb{A}^1$ at a point $z\neq 0$ away from the exceptional locus is just the fiber

$$f^{-1}(z) = \{(z_1,z_2): z_1z_2=z\}$$

of f, which is also a $\mathbb{C}^*$. In fact, the map $\mathrm{Blo}_{H_1,H_2}\mathbb{A}^2\to\mathrm{Blo}_0\mathbb{A}^1$ is a locally trivial $\mathbb{C}^*$ bundle over the half infinite annulus with monodromy given by a Dehn twist around $S^1\subset\mathbb{C}^*$.

2. Kato-Nakayama Spaces

2.1. Log structures. Let $X = (X, \mathcal{O}_X)$ be a locally ringed space over $\mathbb{C}$. A *prelog structure* on X is a map $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ (we often simply write $\mathcal{M}_X$ and suppress notation for α_X) of sheaves of monoids on X, where $\mathcal{O}_X$ is regarded as a monoid under multiplication. A map of prelog structures is a morphism of monoids over $\mathcal{O}_X$. A prelog structure is called a *log structure* iff the map

$$\alpha_X|\alpha_X^{-1}\mathcal{O}_X^* : \alpha_X^{-1}\mathcal{O}_X^* \quad \to \quad \mathcal{O}_X^*$$

is an isomorphism. For a log structure $\mathcal{M}_X$ we suppress notation for this isomorphism, thereby regarding $\mathcal{O}_X^*$ as a subsheaf of $\mathcal{M}_X$. To any prelog structure $\mathcal{M}_X$, we can functorially associate a log structure $\mathcal{M}_X^a$ and a morphism of prelog structures $a : \mathcal{M}_X \to \mathcal{M}_X^a$ initial among maps from $\mathcal{M}_X$ to a log structure; if $\mathcal{M}_X$ is already a log structure, this map a will be an isomorphism so we suppress it from notation and simply write $\mathcal{M}_X = \mathcal{M}_X^a$. We form $\mathcal{M}_X^a$ by setting $\mathcal{M}_X^a := \mathcal{M}_X \oplus_{\alpha_X^{-1}\mathcal{O}_X^*} \mathcal{O}_X^*$, where the pushout is taken in the category of sheaves of monoids on X. The structure map $\mathcal{M}_X^a \to \mathcal{O}_X$ is given by $[m, u] \mapsto \alpha_X(m)u$ using the universal property of the pushout, and the map $a : \mathcal{M}_X \to \mathcal{M}_X^a$ is given by $m \mapsto [m, 1]$.

There is an obvious way to pull back a log structure $\mathcal{M}_X$ on X along a map $(f, f^\sharp) : X' \to X$. We simply declare $f^*\mathcal{M}_X$ to be the log structure on X' associated to the prelog structure

$$f^\sharp f^{-1}\alpha_X : f^{-1}\mathcal{M}_X \quad \to \quad \mathcal{O}_{X'}.$$

Locally ringed spaces over $\mathbb{C}$ with log structure form a category where a map $(X', \mathcal{M}'_X) \to (X, \mathcal{M}_X)$ is a map $f : X' \to X$ of the underlying locally ringed spaces over $\mathbb{C}$ together with a map $f^*\mathcal{M}_X \to \mathcal{M}'_X$ of log structure on X'.

2.2. Divisorial log structures. Let X be an analytic space and let U be an open subset of X. The sheaf of submonoids of $(\mathcal{O}_X, \cdot)$ given by

$$\mathcal{M}_X \quad := \quad \{f \in \mathcal{O}_X : f|_U \in \mathcal{O}_U^*\}$$

is an important example of a log structure on X often called the *divisorial log structure.* (By Hartog's Theorem, this isn't a particularly interesting log structure unless $X \setminus U$ has codimension one in X.) If the complement $D := X \setminus U$ is a Cartier divisor in X, then notice that $\mathcal{M}_X$ contains the units $\mathcal{O}_X^*$ and any "local equation" f for D.

If $D \subseteq X$ is a Cartier divisor, one can also define a log structure $\mathcal{M}_X$ on X as follows. Locally, where we have an equation f for D (a nowhere vanishing section of the invertible sheaf $\mathcal{O}_X(D)$), we let $\mathcal{M}_X$ be the log structure associated to the prelog structure $\alpha_f : \underline{\mathbb{N}} \to \mathcal{O}_X$ taking 1 to f. If f' is a different local equation for D, then we can write $f' = vf$ for a unique $v \in \mathcal{O}_X^*$. The log structures associated to $\alpha_{f'}$ and α_f are then isomorphic via the map $[n, u] \mapsto [n, v^n u]$. The locally defined log structures glue to a global log structure via these natural local isomorphisms.

2.3. Aside. Suppose X, D "are" locally finite type $\mathbb{C}$ schemes, so, locally, $X = \operatorname{Spec} A$ and $D = \operatorname{Spec} A/f$. Then $X \setminus D = \operatorname{Spec} A_f$ so certainly $f \in \mathcal{M}_X(X)$ as mentioned above, but it may not be true that f and $\mathcal{O}_X^*(X) = A^*$ generate the monoid $\mathcal{M}_X(X)$ because f may be reducible: e.g. take $A = \mathbb{C}[z_1, z_2]$, $f = z_1 z_2$.

In general, this reducibility is desirable for a good theory of normal crossings divisors. For example, $f := y^2 - x^2(x-1) \in A := \mathbb{C}[x,y]$ is irreducible in the local ring $A_{(x,y)}$ of the origin, but reducible in its Henselization where there is a square root u of the unit $x-1$ and f can be factored as $(y+ux)(y-ux)$. (Since this u is analytic near the origin, one also has this reducibility in the local ring of the *analytic space* $\mathbb{A}^2$ at the origin.) For this reason, one usually works with the étale topology of X (or its associated analytic space) in these situations, even though the definition of $\mathcal{M}_X$ makes perfect sense on the Zariski site of the scheme X. In the latter example, the *characteristic monoid* $\mathcal{M}_{X,x}/\mathcal{O}_{X,x}$ at the origin x would be the free monoid on one generator f if calculated in the Zariski topology, whereas it is the free monoid on two generators $(y+ux), (y-ux)$ if calculated in the étale topology or on the analytic space. One should keep in mind in the sections that follow that even if X "is" a locally finite type $\mathbb{C}$ scheme, the sheaf of monoids $\mathcal{M}_X$ should always be thought of as a sheaf on the étale site of X or on the analytic space X.

2.4. Kato-Nakayama spaces. Let X be a locally ringed space over $\mathbb{C}$ equipped with a log structure $\mathcal{M}_X$. The *Kato-Nakayama* space $X^{\log}$ has as points pairs (x,F) where $x \in X$ and $F : \mathcal{M}_{X,x} \to S^1$ is a monoid homomorphism satisfying

$$F(u) \quad = \quad u(x)/|u(x)| \qquad \text{for all } u \in \mathcal{O}^*_{X,x} \subseteq \mathcal{M}_{X,x}. \tag{2}$$

Given an open subset $U \subseteq X$ and a section $m \in \mathcal{M}_X(U)$, we tautologically obtain a function

$$\begin{array}{rcl} \phi_m : U^{\log} & \to & S^1 \\ (u,F) & \mapsto & F(m_x). \end{array}$$

The topology on $X^{\log}$ is the coarsest topology on this set making the map

$$\begin{array}{rcl} \tau : X^{\log} & \to & X \\ (x,F) & \mapsto & x \end{array}$$

containuous and making the maps ϕ_m continuous (for the usual metric topology on S^1).

The Kato-Nakayama space is an inverse limit preserving functor from log locally ringed spaces over $\mathbb{C}$ to topological spaces.

Suppose there is a *chart* for $\mathcal{M}_X$: a finitely generated submonoid P of an abelian group, generated by $p_1, \dots, p_n$ say, and a map of monoids $\alpha : P \to \mathcal{M}_X(X)$ (this is the same thing as a map of sheaves of monoids $\underline{P}_X \to \mathcal{M}_X$) such that the map from the pushout

$$\underline{P}_X \oplus_{\alpha^{-1}\mathcal{O}^*_X} \mathcal{O}^*_X \to \mathcal{M}_X$$

is an isomorphism (in the category of sheaves of monoids on X). The map

$$\begin{array}{rcl} X^{\log} & \to & X \times (S^1)^n \\ (x,F) & \mapsto & (x, F(\alpha(p_1)_x), \dots, F(\alpha(p_n)_x)) \end{array} \tag{3}$$

is clearly monic (onto a *closed* set in fact), so we can use the product topology on $X \times (S^1)^n$ to endow $X^{\log}$ with a topology making the projection $(x,F) \mapsto x$ a proper continuous map to X. It can be shown that this topology on $X^{\log}$ is the same as the one defined above.

The paradigm example is when $X = \operatorname{Spec}(P \to \mathbb{C}[P])$, meaning X is the analytic space associated to the $\mathbb{C}$ scheme $\operatorname{Spec}\mathbb{C}[P]$ and X has the log structure associated to the prelog structure $\underline{P} \to \mathcal{O}_X$. In this case, the topological space of X is the set of monoid homomorphisms $P \to \mathbb{C}$ (with the topology induced by the metric topology on $\mathbb{C}$), while the Kato-Nakayama space of X is the set of monoid homomorphisms $P \to \mathbb{R}_{\geq 0} \times S^1$. The map $\tau : X^{\log} \to X$ is the map induced by composing with the monoid homomorphism

$$\begin{array}{rcl} \mathbb{R}_{\geq 0} \times S^1 & \to & \mathbb{C} \\ (\lambda, u) & \mapsto & \lambda u. \end{array}$$

(View S^1 and $\mathbb{R}_{\geq 0}$ as submonoids of $(\mathbb{C}, \cdot)$ in the usual way.)

2.5. The map. Let D be a Cartier divisor in an analytic space X. We will now explain how to construct a natural map of topological spaces

$$X^{\log} \to \operatorname{Blo}_D X \tag{4}$$

(over X) from the Kato-Nakayama space to the simplified oriented real blowup. The ideal sheaf $\mathcal{O}_X(-D)$ of D is an invertible $\mathcal{O}_X$ module. I first claim that, for any $x \in X$, any nonzero f in the fiber $\mathcal{O}_X(-D)|_x$, and any lift $\overline{f} \in \mathcal{O}_X(-D)_x$ of f to the stalk, we have $\overline{f} \in \mathcal{M}_{X,x}$. Indeed, if $\mathfrak{m}_x \in \mathcal{O}_{X,x}$ is the maximal ideal, then we have $\mathcal{O}_X(-D)|_x = \mathcal{O}_X(-D)_x/\mathfrak{m}_x\mathcal{O}_X(-D)$. If $g \in \mathcal{O}_X(-D)_x$ is a local equation for D near x, then $\cdot g : \mathcal{O}_{X,x} \to \mathcal{O}_X(-D)_x$ is an isomorphism, so we can write $\overline{f} = gh$ for some $h \in \mathcal{O}_{X,x}$ and, in fact, we must have $h \in \mathcal{O}^*_{X,x}$ because otherwise h would be in $\mathfrak{m}_x$ and f would be zero in the fiber. Now, we already mentioned that $g \in \mathcal{M}_{X,x}$, so certainly $\overline{f} = gh \in \mathcal{M}_{X,x}$ since h is a unit, so this proves the claim.

I next claim that (2) implies that $F(\overline{f}) \in S^1$ (which we may speak of in light of the first claim) is independent of the choice of lift $\overline{f}$ of the nonzero element f of the fiber $\mathcal{O}_X(-D)|_x$ to the stalk $\mathcal{O}_X(-D)_x$. Hence F determines a map

$$F : \mathcal{O}_X(-D)|^*_x \to S^1. \tag{5}$$

Indeed, if $\overline{f}'$ is another lift, then we can write $\overline{f}' = \overline{f} + m'$ for some $m' \in \mathfrak{m}_x\mathcal{O}_X(-D)_x$. As before, if we choose a local equation $g \in \mathcal{O}_X(-D)_x$ for D near x, then we can write $\overline{f} = gh$, $m' = gm$, for $h \in \mathcal{O}^*_{X,x}$ and $m \in \mathfrak{m}_x$, hence $\overline{f}' = gh + gm$. Now we compute

$$\begin{array}{rcl} F(\overline{f}) & = & F(gh) \\ & = & F(g)F(h) \\ & = & F(g)h(x)/|h(x)| \end{array}$$

using (2). Similarly, since $h + m \in \mathcal{O}^*_{X,x}$, we find that

$$\begin{array}{rcl} F(\overline{f}') & = & F(gh + gm) \\ & = & F(g)F(h + m) \\ & = & F(g)(h(x) + m(x))/|h(x) + m(x)| \\ & = & F(g)h(x)/|h(x)| \end{array}$$

because $m \in \mathfrak{m}_x$ is zero in the fiber $\mathcal{O}_{X,x}/\mathfrak{m}_x = \mathbb{C}$. This proves the claim.

The condition (2) implies that the map (5) is $\mathbb{C}^*$ equivariant for the scaling action on the fiber and the usual action $\lambda \cdot Z = \lambda/|\lambda| Z$ of $\mathbb{C}^*$ on S^1 so it descends to an S^1 equivariant map $F : \mathcal{O}_X(-D)|^*_x/\mathbb{R}_{>0} \to S^1$. This in turn may be viewed as a point of $S^1\mathcal{O}_X(D)$ lying over the point $x \in X$, so we have produced a map (of sets over X at least)

$$X^{\log} \to S^1\mathcal{O}_X(D). \tag{6}$$

I claim this map is continuous. Since both spaces map continuously to X, the question is local on X, hence we may assume there is a chart $\alpha : P \to \mathcal{M}_X(X)$ and a function $g \in \Gamma(X, \mathcal{O}_X(-D))$ cutting out D. Let $p_1, \ldots, p_n$ be generators for P. After possibly shrinking X again, we can write $g = up_1^{e_1} \cdots p_n^{e_n}$ in $\mathcal{M}_X(X)$ for some unit $u \in \Gamma(X, \mathcal{O}_X^*)$ (dropping notation for image under α). Using the local equation g to make the identification $S^1\mathcal{O}_X(D) = X \times S^1$, the map (6) can be written $(x, F) \mapsto (x, F(g_x)) \in X \times S^1$. This fits into the commutative diagram

$$\begin{tikzcd} X^{\log} \arrow[r, "(3)"] \arrow[dr, "(6)"'] & X \times (S^1)^n \arrow[d, "(x,v_1,\ldots,v_n)\mapsto(x,(u(x)/|u(x)|)v_1\cdots v_n)"] \\ & X \times S^1, \end{tikzcd}$$

so since the topology on $X^{\log}$ is inherited from the horizontal map (3) and the vertical map is clearly continuous, we conclude that the map (6) is also continuous.

Finally, I claim that the map (6) factors through the closed subspace $\mathrm{Blo}_D X \subseteq S^1\mathcal{O}_X(D)$, which will complete the construction of (4). Again, the question is local on X, so we can assume there is a global function $g \in \Gamma(X, \mathcal{O}_X)$ cutting out D. The function g determines a trivialization of $\mathcal{O}_X(D)$, hence an identification $S^1\mathcal{O}_X(D) = X \times S^1$ of spaces over X. Under this identification, the map (6) is given by

$$\begin{array}{rcl} X^{\log} & \to & X \times S^1 \\ (x, F) & \mapsto & (x, F(g_x)) \end{array}$$

and $\mathrm{Blo}_D X \subseteq S^1\mathcal{O}_X(D)$ is identified with the subspace of $X \times S^1$ consisting of points (x, Z) satisfying $g(x) = |g(x)|Z$.

The easiest thing to do now is to appeal to continuity: Since we already showed (6) is continuous, it suffices to establish the desired factorization on a dense set, so it suffices to check that $g(x) = |g(x)|F(g_x)$ when $x \in X \setminus D$, which is clear from (2) because when $x \in X \setminus D$, we have $g_x \in \mathcal{O}_{X,x}^*$ and hence $F(g_x) = g(x)/|g(x)|$.

Example 2.5.1. Consider the case $X = \mathbb{A}^1_t$, $D = \{0\}$. Then the map $\mathbb{N} \to \Gamma(X, \mathcal{O}_X)$ mapping 1 to $t \in \Gamma(X, \mathcal{O}_X)$ is a global chart for the divisorial log structure $\mathcal{M}_X$. In this case, the map (3) used to define the topology on $X^{\log}$ is

$$\begin{array}{rcl} X^{\log} & \to & X \times S^1 \\ (z, F) & \mapsto & (z, F(t_z)) \end{array}$$

which is a closed embedding onto the set of (z, Z) with $z = Z|z|$. In fact, by following through the definitions, we see that the map (4) is an isomorphism $X^{\log} \cong \mathrm{Blo}_D X$.

2.6. An important case. Slightly generalizing this example, if X is a smooth analytic space and $D \subseteq X$ is a normal crossings divisor, then the Kato-Nakayama space $X^{\log}$ will be isomorphic to the oriented real blowup $\mathrm{OBl}_D X$. Indeed, first consider the case where D is smooth. Then $\mathrm{OBl}_D X = \mathrm{Blo}_D X$ and the claimed isomorphism is provided by the map (4), since one can reduce to the case of the above example by working locally: the local picture of $D \subseteq X$ looks like $\{0\} \subseteq \mathbb{A}^1$ times a trivial factor. In the general case, $D \subseteq X$ looks locally like a union of coordinate hyperplanes in $\mathbb{A}^n$ and the divisorial log structure splits as the direct sum of the log structures associated to each individual hyperplane.

Consequently, the Kato-Nakayama space in this local case is the fibered product over X of the Kato-Nakayama spaces of the log schemes with divisorial log structures associated to smooth divisors. But by definition, $\mathrm{OBl}_D X$ is described locally in terms of similar fibered products, so the fibered product over X of the maps (4) will provide the desired isomorphism.

3. Symplectic Geometry

This section contains some brief remarks on the symplectic geometry of oriented real blowups. Let X be a complex manifold, $L \to X$ a holomorphic line bundle equipped with a Hermitian metric

$$\langle \, , \rangle : L \otimes \overline{L} \quad \to \quad \mathbb{C}.$$

Suppose we have a holomorphic, *nonvanishing* section $s : U \to L$ of L over an open subset $U \subseteq X$. Then we obtain a smooth function $\langle s, s \rangle$ from U to $\mathbb{R}_{>0} \subseteq \mathbb{C}$, and hence a smooth function $\ln\langle s, s\rangle$ from U to $\mathbb{R}$. One checks easily[1] that the $(1,1)$-form

$$\begin{aligned} \omega_L &:= \frac{i}{2\pi} \overline{\partial}\partial \ln\langle s, s\rangle \\ &= -\frac{i}{2\pi} \partial\overline{\partial} \ln\langle s, s\rangle \\ &= \frac{i}{4\pi}(\partial + \overline{\partial})(\partial - \overline{\partial}) \ln\langle s, s\rangle \\ &= d\left(\frac{i}{4\pi}(\partial - \overline{\partial}) \ln\langle s, s\rangle \right) \end{aligned}$$

is independent of the choice of such an s. Since we can always choose such an s locally, we obtain a global $(1,1)$-form ω_L.

For any $n \in \mathbb{Z}$, if we give $L^{\otimes n}$ the natural metric $\langle \, , \rangle_n$ inherited from the metric $\langle \, , \rangle$ on L, then we have $\omega_{L^{\otimes n}} = n\omega_L$ because if s is a nowhere vanishing section of L, then s^n is a nowhere vanishing section of $L^{\otimes n}$ and we have

$$\begin{aligned} \ln\langle s^n, s^n\rangle_n &= \ln(\langle s, s\rangle^n) \\ &= n \ln\langle s, s\rangle. \end{aligned}$$

The $(1,1)$-form $\Theta_L := \overline{\partial}\partial \ln\langle s, s\rangle$ is called the *curvature form* (of L with the given metric). We prefer to keep the constant $i/(2\pi)$ around and work with ω_L instead of Θ_L for the following reasons: First of all, the differentials

$$d = \partial + \overline{\partial} \qquad \text{and} \qquad d^c = \frac{i}{4\pi}(\partial - \overline{\partial})$$

are *real* meaning that df and $d^c f$ are real-valued 1-forms for any smooth, real-valued function f (this is easy to see in local coordinates), so the form ω_L can be regarded as a real 2-form, and the form

$$\alpha \quad := \quad d^c \ln\langle s, s\rangle$$

can be regarded as a real 1-form. Furthermore, the global 2-form ω_L is closed (because locally we have the 1-form α with $d\alpha = \omega_L$), and the corresponding de Rham cohomology class $[\omega_L] \in \mathrm{H}^2_{dR}(X)$ is the the first Chern class $c_1(L)$ [GH, Page 140].

[1]c.f. the computation on Page 30 of [GH]

The form ω_L is of interest for various reasons beyond the fact that it represents the first Chern class in de Rham cohomology. For example, recall [GH, Pages 29, 148][2] that a $(1,1)$-form ω is called *positive* iff the associated pairing

$$\omega : TX \otimes \overline{TX} \quad \to \quad \mathbb{C}$$

is a Hermitian metric. Similarly, a line bundle L is called *positive* iff it admits a Hermitian metric such that ω_L is positive (it turns out that this is equivalent to the existence of a positive $(1,1)$-form β such that $[\beta] = c_1(L)$ in de Rham cohomology). The basic example is the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$, which comes with a tautological Hermitian metric via the inclusion of its dual in the trivial bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ and the usual Hermitian metric on $\mathbb{C}^{n+1}$. With this metric, the $(1,1)$-form $\omega_{\mathcal{O}(1)}$ is positive, and the corresponding Hermitian metric is called the *Fubini-Study metric*, denoted ω_{FS}.

In the rest of this section, we will consider the following situation: X is a complex manifold, D a complex submanifold of codimension one, $L \to X$ a holomorphic line bundle, s is a holomorphic section of L which vanishes to order n along D, but is nonvanishing elsewhere. We allow n to be zero or negative, in which case s has a pole of order $-n$ along D, but is nonvanishing elsewhere. In either case, s trivializes L away from D, so any 2-form ω representing $c_1(L)$ (or a multiple of it) becomes exact after restricting to $X \setminus D$. The oriented real blowup $\tau : \overline{X} \to X$ of X along D is a manifold with boundary $\overline{D} = S^1 N_{D/X}$, so it is homotopy equivalent to its interior $\overline{X} \setminus \overline{D} = X \setminus D$, hence we also know that the 2-form $\tau^*\omega$ is exact on $\overline{X}$, so we can write $\tau^*\omega = d\overline{\alpha}$ for some 1-form $\overline{\alpha}$ on $\overline{X}$. Away from D, where

$$\tau : \overline{X} \setminus \overline{D} \to X \setminus D$$

is an isomorphism, we have a natural choice for such an $\overline{\alpha}$, namely $\tau^*\alpha$, where α is defined using the nonvanishing section $s : X \setminus D \to L$ as above.

In fact, we will see momentarily that this real 1-form $\tau^*\alpha$ on $\overline{X} \setminus \overline{D}$ extends uniquely to $\overline{X}$, and the extended 1-form $\overline{\alpha}$ has some very nice properties. Let $\pi := \tau|\overline{D} : \overline{D} \to D$ be the projection for the circle bundle $S^1 N_{D/X}$, and let T^*_π be the sheaf of π-relative differentials on $\overline{D}$, defined by the exact sequence

$$0 \to \pi^* T^* D \to T^* \overline{D} \to T^*_\pi \to 0.$$

The circle bundle $\overline{D}$ is naturally oriented by the "outward normal first" convention of §1.14 in light of the fact that it is the boundary of the smooth manifold $\overline{X}$, and of course D is oriented, so π has a natural relative orientation, hence we have a well-defined map

$$\int : \Gamma(\overline{D}, T^*_\pi) \quad \to \quad \Gamma(D, \Omega^0)$$

from relative 1-forms on $\overline{D}$ to smooth functions on D given by integrating over the fibers of π.

Theorem 3.1. *Let $X, D, L, s, \overline{X}, \overline{D}$ be as above. The (real) 1-form*

$$\tau^*\alpha \quad = \quad \frac{i}{4\pi}(\partial - \overline{\partial}) \ln\langle s, s\rangle$$

extends uniquely to a global 1-form $\overline{\alpha}$ on $\overline{X}$ with $d\overline{\alpha} = \tau^*\omega_L$. In particular, $\tau^* c_1(L) = 0$ in $\mathrm{H}^2_{dR}(\overline{X})$. The image of $\overline{\alpha}|_{\overline{D}}$ in $\Gamma(\overline{D}, T^*_\pi)$ integrates to n over every fiber of π.

[2]The definition of *positive* in [GH] might differ from ours because of the point at which the constant $i/(2\pi)$ is inserted.

Proof. Since $\overline{X} \setminus \overline{D} = X \setminus D$ is dense in $\overline{X}$, such an extension $\overline{\alpha}$ is unique if it exists, so it suffices to construct it locally. The equality $d\overline{\alpha} = \tau^* \omega_L$ will then be automatic because we have the equality $d\alpha = \omega_L$ on the dense open set $X \setminus D$ by definition of ω_L. Note that formation of the oriented real blowup is local §1.3. The final statement is also local, so we can assume $X = \mathbb{C}^n$ with coordinates $z_1, \dots, z_n$ and $D = \{z_1 = 0\}$. The hypothesis on s ensures that we can write $s = z_1^n s'$ for a holomorphic *nonvanishing* section s', so α is given on $X \setminus D$ by

$$\begin{aligned} \alpha &= \frac{i}{4\pi}(\partial - \overline{\partial}) \ln \langle z_1^n s', z_1^n s' \rangle \\ &= \frac{ni}{4\pi}(\partial - \overline{\partial}) \ln(z_1 \overline{z}_1) + \frac{i}{4\pi}(\partial - \overline{\partial}) \ln \langle s', s' \rangle \\ &= \alpha_1 + \alpha_2, \end{aligned}$$

where we set

$$\begin{aligned} \alpha_1 &:= \frac{ni}{4\pi}(\partial - \overline{\partial}) \ln(z_1 \overline{z}_1) \\ &= \frac{ni}{4\pi} \frac{\overline{z}_1 dz_1 - z_1 d\overline{z}_1}{z_1 \overline{z}_1} \\ \alpha_2 &= \frac{i}{4\pi}(\partial - \overline{\partial}) \ln \langle s', s' \rangle. \end{aligned}$$

The section s' is nowhere vanishing, so α_2 is a global 1-form on $\overline{X}$, hence $\tau^* \alpha_2$ is the pullback of a global 1-form on X. The issue is only in extending $\tau^* \alpha_1$ to a global 1-form on $\overline{X}$. By trivial special cases of the compatibility with oriented real blowup and pullback (§1.3) and the example in §1.11, we have

$$\begin{aligned} \overline{X} &= \operatorname{Blo}_D X \\ &= (\operatorname{Blo}_0 \mathbb{C}_{z_1}) \times \mathbb{C}^{n-1}_{z_2, \dots, z_n} \\ &= (\mathbb{R}_{\geq 0} \times S^1) \times \mathbb{C}^{n-1}_{z_2, \dots, z_n}. \end{aligned}$$

We give $\mathbb{R}_{\geq 0}$ and S^1 coordinates λ and t, so dt freely generates the cotangent sheaf of S^1 and a function on S^1 is a 2π-periodic function of t (t is really the coordinate on the universal cover of S^1). The map τ is given by

$$(\lambda, t, z_2, \cdots, z_n) \mapsto (\lambda e^{it}, z_2, \dots, z_n),$$

so we have:

$$\begin{aligned} \tau^* z_1 &= \lambda e^{it} \\ \tau^* dz_1 &= e^{it} d\lambda + \lambda i e^{it} dt \\ \tau^* \overline{z}_1 &= \lambda e^{-it} \\ \tau^* d\overline{z}_1 &= e^{-it} d\lambda - \lambda i e^{-it} dt \\ \tau^*(z_1 \overline{z}_1) &= \lambda^2, \end{aligned}$$

hence we compute

$$\begin{aligned} \tau^* \alpha_1 &= \frac{ni}{4\pi} \frac{\lambda d\lambda + \lambda^2 i dt - (\lambda d\lambda - \lambda^2 i dt)}{\lambda^2} \\ &= \frac{-n}{2\pi} dt. \end{aligned}$$

The fortunate pole cancellation shows that $\tau^*\alpha_1$ extends to a global 1-form $\overline{\alpha}_1 \in \Gamma(\overline{X}, T^*\overline{X})$, hence the 1-form $\tau^*\alpha$ on $\overline{X} \setminus \overline{D}$ also extends to a global 1-form $\overline{\alpha}$ on $\overline{X}$ and we have $\overline{\alpha} = \overline{\alpha}_1 + \tau^*\alpha_2$. If we restrict $\overline{\alpha}_1$ to the boundary (i.e. we set $\lambda = z_2 = \cdots = z_n = 0$) and integrate over the boundary $\overline{D}$ (i.e. integrate over t), we appear to get

$$\begin{aligned} \int \overline{\alpha}_1 &= \int_0^{2\pi} \frac{-n}{2\pi} \\ &= -n, \end{aligned}$$

but actually we get n because we got the $-n$ by integrating over the circle in the usual counter-clockwise orientation, but this is the opposite of the orientation on the boundary inherited from the natural orientation on $\overline{X}$ and the outward normal first convention (§1.14). Note that $\overline{\alpha}_1$ and $\overline{\alpha}$ have the same integral over the boundary because $\tau^*\alpha_2|\overline{D} = \pi^*(\alpha_2|D)$ is pulled back from D, hence it is zero in the relative cotangent sheaf T^*_π, hence integrates to zero over the fibers of π. □

3.1. Example: The Riemann Sphere. For the sake of concreteness and to make sure we have all of our signs, constants, and orientation issues straight, let's go through the case $X := \mathbb{P}^1$, $D = [0:1] \in \mathbb{P}^2$, $\overline{X} = \operatorname{Blo}_D X$. The space $\overline{X}$ is topologically a disc. Here our divisor (point) D is the complement of the open subset

$$\begin{aligned} U_0 = \mathbb{C} &\hookrightarrow \mathbb{P}^1 \\ w &\mapsto [1:w], \end{aligned}$$

which is the origin in the other chart

$$\begin{aligned} U_1 = \mathbb{C} &\hookrightarrow \mathbb{P}^1 \\ z &\mapsto [z:1]. \end{aligned}$$

We have the non-vanishing section $s : U_0 \to \mathcal{O}(-1)$ defined by $s(w) := (1, w)$. On $U_0 \cap U_1 \subseteq U_1$ this section is given by $s(z) = (1, z^{-1})$. On U_1, we know that the Fubini-Study form ω_{FS} is given by

$$\omega_{FS} = \frac{i}{2\pi} \partial\overline{\partial} \ln\langle t, t\rangle$$

for any non-vanishing section t of $\mathcal{O}(-1)$ over U_1. In particular, we know that the Fubini-Study form ω_{FS} is given on $U_1 \cap U_0 = \{z \neq 0\}$ by

$$\begin{aligned} \omega_{FS} &= \frac{i}{2\pi} \partial\overline{\partial} \ln\langle s, s\rangle \\ &= \frac{i}{2\pi} \partial\overline{\partial} \ln(1 + z^{-1}\overline{z}^{-1}) \\ &= \frac{i}{4\pi} (\partial + \overline{\partial})(\overline{\partial} - \partial) \ln(1 + z^{-1}\overline{z}^{-1}) \\ &= d\alpha, \end{aligned}$$

where we set

$$\alpha := \frac{i}{4\pi} (\overline{\partial} - \partial) \ln(1 + z^{-1}\overline{z}^{-1})$$

as usual. We know by general theory that the above formula for ω_{FS} on $U_1 \cap U_0$ must extend (uniquely) to the origin in U_1; we can see this very explicitly in this case by making the computation

$$\begin{aligned}
\omega_{FS} &= \frac{i}{2\pi}\partial\overline{\partial}\ln(z^{-1}\overline{z}^{-1}(z\overline{z}+1)) \\
&= \frac{i}{2\pi}\partial\overline{\partial}\ln(1+z\overline{z}) - \frac{i}{2\pi}\partial\overline{\partial}\ln(z\overline{z}^{-1}) \\
&= \frac{i}{2\pi}\partial\overline{\partial}\ln(1+z\overline{z}),
\end{aligned}$$

which is the usual expression for ω_{FS} we would get by using the nowhere-vanishing section $t(z) := (z, 1)$ of $\mathcal{O}(-1)$ over U_1. Continuing to expand this out, we get

$$\begin{aligned}
\omega_{FS} &= \frac{i}{2\pi}\partial\left(\frac{zd\overline{z}}{1+z\overline{z}}\right) \\
&= \frac{i}{2\pi}\frac{dz\wedge d\overline{z}}{(1+z\overline{z})^2} \\
&= \frac{i}{2\pi}\frac{(-2i)dx\wedge dy}{(1+x^2+y^2)^2} \\
&= \frac{1}{\pi}\frac{dx\wedge dy}{(1+x^2+y^2)^2},
\end{aligned}$$

where $z = x + iy$. We can integrate this by changing to polar coordinates:

$$\begin{aligned}
\int_X \omega_{FS} &= \int_{U_1}\omega_{FS} \\
&= \int_{\mathbb{R}^2}\frac{1}{\pi}\frac{dx\wedge dy}{(1+x^2+y^2)^2} \\
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{rdrd\theta}{(1+r^2)^2} \\
&= \int_0^{\infty}\frac{2rdr}{(1+r^2)^2} \\
&= \int_1^{\infty}\frac{du}{u^2} \\
&= 1,
\end{aligned}$$

where we made the substitution $u = 1 + r^2$. We also know from our discussion above that $\tau^*\alpha$ extends uniquely to a global 1-form $\overline{\alpha}$ on $\overline{X}$. We have $\tau^{-1}(U_1) = \operatorname{Blo}_0\mathbb{C} = \mathbb{R}_{\geq 0}\times S^1$ with coordinates λ, t, with τ given by $(\lambda, t)\mapsto \lambda e^{it}$. As usual, we view S^1 as the quotient $S^1 = \mathbb{R}_t/2\pi\mathbb{Z}$, so that the usual orientation $[dt]$ of $\mathbb{R}_t$ gives an orientation of S^1. Then we compute

$$\begin{aligned}
\tau^* z &= \lambda e^{it} \\
\tau^* dz &= e^{it}d\lambda + \lambda i e^{it}dt \\
\tau^*\overline{z} &= \lambda e^{-it} \\
\tau^* d\overline{z} &= e^{-it}d\lambda - \lambda i e^{-it}dt.
\end{aligned}$$

If we massage the formula for α a little bit to get

$$\begin{aligned}\alpha &= \frac{i}{4\pi}(\overline{\partial}-\partial)\ln(z^{-1}\overline{z}^{-1}(z\overline{z}+1)) \\ &= \frac{i}{4\pi}\frac{\overline{z}dz - zd\overline{z}}{z\overline{z}} + \frac{i}{4\pi}\frac{zd\overline{z}-\overline{z}dz}{1+z\overline{z}},\end{aligned}$$

then we easily compute

$$\begin{aligned}\tau^*\alpha &= \frac{i}{4\pi}\frac{2\lambda^2 idt}{\lambda^2} + \frac{i}{4\pi}\frac{-2\lambda^2 idt}{1+\lambda^2} \\ &= \frac{i}{4\pi}(2idt) - \frac{i}{4\pi}\frac{\lambda^2 idt}{1+\lambda^2} \\ &= \frac{-1}{2\pi}dt + \frac{1}{4\pi}\frac{\lambda^2}{1+\lambda^2}dt,\end{aligned}$$

which is the formula for our extension $\overline{\alpha}$ of $\tau^*\alpha$ to $\mathrm{Blo}_0\,\mathbb{C}$. If we now restrict this to the boundary S^1 of $\mathrm{Blo}_0\,\mathbb{C}$ (i.e. we set $\lambda = 0$) and integrate, we get -1. But we integrated over S^1 using its counterclockwise orientation, which is the opposite of the orientation it has as the boundary of $\mathrm{Blo}_0\,\mathbb{C}$, so we do indeed check that Stokes' Theorem works out here:

$$\begin{aligned}\int_X \omega_{FS} &= \int_{\overline{X}} \tau^*\omega_{FS} \\ &= \int_{\overline{X}} d\overline{\alpha} \\ &= \int_{\partial\overline{X}} \alpha \\ &= -\int_0^{2\pi} \frac{-dt}{2\pi} \\ &= 1.\end{aligned}$$

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