Singular and tangent slit solutions to the Löwner equation

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Abstract. We consider the Löwner differential equation generating univalent maps of the unit disk (or of the upper half-plane) onto itself minus a single slit. We prove that the circular slits, tangent to the real axis are generated by Hölder continuous driving terms with exponent 1/3 in the Löwner equation. Singular solutions are described, and the critical value of the norm of driving terms generating quasisymmetric slits in the disk is obtained.

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ be the unit disk and $\mathbb{T} := \partial\mathbb{D}$. The famous Löwner equation was introduced in 1923 [3] in order to represent a dense subclass of the whole class of univalent conformal maps $f(z) = z(1 + c_1 z + \dots)$ in $\mathbb{D}$ by the limit

$$f(z) = \lim_{t\to\infty} e^t w(z,t), \quad z \in \mathbb{D},$$

where $w(z,t) = e^{-t} z(1 + c_1(t)z + \dots)$ is a solution to the equation

$$\frac{dw}{dt} = -w\frac{e^{iu(t)} + w}{e^{iu(t)} - w}, \quad w(z,0) \equiv z, \tag{1}$$

with a continuous driving term $u(t)$ on $t \in [0,\infty)$, see [3, page 117]. All functions $w(z,t)$ map $\mathbb{D}$ onto $\Omega(t) \subset \mathbb{D}$. If $\Omega(t) = \mathbb{D} \setminus \gamma(t)$, where $\gamma(t)$ is a Jordan curve in $\mathbb{D}$ except one of its endpoints, then the driving term $u(t)$ is uniquely defined and we call the corresponding map w a *slit map*. However, from 1947 [5] it is known that solutions to (1) with continuous $u(t)$ may give non-slit maps, in particular, $\Omega(t)$ can be a family of hyperbolically convex digons in $\mathbb{D}$.

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Marshall and Rohde [4] addressed the following question: *Under which condition on the driving term $u(t)$ the solution to (1) is a slit map?* Their result states that if $u(t)$ is Lip(1/2) (Hölder continuous with exponent 1/2), and if for a certain constant $C_{\mathbb{D}} > 0$, the norm $\|u\|_{1/2}$ is bounded $\|u\|_{1/2} < C_{\mathbb{D}}$, then the solution w is a slit map, and moreover, the Jordan arc $\gamma(t)$ is s quasislit (a quasiconformal image of an interval within a Stolz angle). As they also proved, a converse statement without the norm restriction holds. The absence of the norm restriction in the latter result is essential. On one hand, Kufarev's example [5] contains $\|u\|_{1/2} = 3\sqrt{2}$, which means that $C_{\mathbb{D}} \leq 3\sqrt{2}$. On the other hand, Kager, Nienhuis, and Kadanoff [1] constructed exact slit solutions to the half-plane version of the Löwner equation with arbitrary norms of the driving term.

Let us give here the half-plane version of the Löwner equation. Let $\mathbb{H} = \{z : \text{Im } z > 0\}$, $\mathbb{R} = \partial\mathbb{H}$. The functions $h(z,t)$, normalized near infinity by $h(z,t) = z - 2t/z + b_{-2}(t)/z^2 + \dots$, solving the equation

$$\frac{dh}{dt} = \frac{-2}{h - \lambda(t)}, \quad h(z,0) \equiv z, \tag{2}$$

where $\lambda(t)$ is a real-valued continuous driving term, map $\mathbb{H}$ onto a subdomain of $\mathbb{H}$. The question about the slit mappings and the behaviour of the driving term $\lambda(t)$ in the case of the half-plane $\mathbb{H}$ was addressed by Lind [2]. The techniques used by Marshall and Rohde carry over to prove a similar result in the case of the equation (2), see [4, page 765]. Let us denote by $C_{\mathbb{H}}$ the corresponding bound for the norm $\|\lambda\|_{1/2}$. The main result by Lind is the sharp bound, namely $C_{\mathbb{H}} = 4$.

In some papers, e.g., [1, 2], the authors work with equations (1, 2) changing (−) to (+) in their right-hand sides, and with the mappings of slit domains onto $\mathbb{D}$ or $\mathbb{H}$. However, the results remain the same for both versions.

Marshall and Rohde [4] remarked that there exist many examples of driving terms $u(t)$ which are not Lip(1/2), but which generate slit solutions with simple arcs $\gamma(t)$. In particular, if $\gamma(t)$ is tangent to $\mathbb{T}$, then $u(t)$ is never Lip(1/2).

Our result states that if $\gamma(t)$ is a circular arc tangent to $\mathbb{R}$, then the driving term $\lambda(t) \in$Lip(1/3). Besides, we prove that $C_{\mathbb{D}} = C_{\mathbb{H}} = 4$, and consider properties of singular solutions to the one-slit Löwner equation.

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2. Circular tangent slits

We shall work with the half-plane version of the Löwner equation and with the sign (+) in the right-hand side, consequently with the maps of slit domains onto $\mathbb{H}$.

We construct a mapping of the half-plane $\mathbb{H}$ slit along a circular arc $\gamma(t)$ of radius 1 centered on i onto $\mathbb{H}$ starting at the origin directed, for example, positively. The inverse mapping we denote by $z = f(w,t) = w - 2t/w + \dots$. Then $\zeta = 1/f(w,t)$ maps $\mathbb{H}$ onto the lower half-plane slit along a ray co-directed with $\mathbb{R}^+$ and having the distance 1/2 between them. Let ζ_0 be the tip of this ray. Applying the Christoffel-Schwarz formula we find f in the form

$$\frac{1}{f(w,t)} = \int_0^{1/w} \frac{(1-\gamma w)\,dw}{(1-\alpha w)^2(1-\beta w)} = \frac{\beta-\gamma}{(\alpha-\beta)^2}\log\frac{w-\alpha}{w-\beta} + \frac{\alpha-\gamma}{\alpha-\beta}\frac{1}{w-\alpha}, \tag{3}$$

where the branch of logarithm vanishes at infinity, and $f(w,t)$ is expanded near infinity as

$$f(w,t) = w - \frac{2t}{w} + \dots$$

The latter expansion gives us two conditions: there is no constant term and the coefficient is $-2t$ at w, which implies $\gamma = 2\alpha+\beta$ and $\alpha(\alpha+2\beta) = -6t$. The condition $\operatorname{Im}\zeta_0 = -1/2$ yields

$$\frac{-2\alpha}{(\alpha-\beta)^2} = \frac{1}{2\pi}.$$

Then, $\beta = \alpha + 2\sqrt{-\alpha\pi}$, and $\alpha(3\alpha + 4\sqrt{-\alpha\pi}) = -6t$. Considering the latter equation with respect to α we expand the solution $\alpha(t)$ in powers of $t^{1/3}$. Hence,

$$\alpha(t) = -\left(\frac{9}{4\pi}\right)^{1/3} t^{2/3} + A_2 t + A_3 t^{4/3} + \dots$$

and

$$\beta(t) = (12\pi)^{1/3} t^{1/3} + B_2 t^{2/3} + \dots$$

Formula (3) in the expansion form regarding to $1/w$ gives

$$\frac{\beta-\alpha}{2\pi}\frac{1}{w} + \frac{\beta^2-\alpha^2}{4\pi}\frac{1}{w^2} + \dots + \left(1 + 2\frac{\alpha}{\beta} + 2\frac{\alpha^2}{\beta^2} + \dots\right)\left(\frac{1}{w} + \frac{\alpha}{w^2} + \dots\right) = \zeta. \tag{4}$$

Remember that this formula is obtained under the conditions $\gamma = 2\alpha+\beta$ and $(\alpha-\beta)^2 = 4\alpha\pi$. We substitute the expansions of $\alpha(t)$ and $\beta(t)$ in this formula and consider it as an equation for the implicit function $w = h(z,t)$. Calculating coefficients $B_2 \dots B_4$ in terms of $A_2, \dots, A_4$, and verifying $A_2 = -3/4\pi$ we come to the following expansion for $h(z,t)$:

$$w = h(z,t) = h(\frac{1}{\zeta}, t) = \frac{1}{\zeta} + 2\zeta t + \frac{3}{2}(12\pi)^{1/3} t^{4/3} + \dots.$$

This version of the Löwner equation admits the form

$$\frac{dh}{dt} = \frac{2}{h-\lambda(t)}, \quad h(z,0) \equiv z. \tag{5}$$

Being extended onto $\mathbb{R}\setminus\lambda(0)$ the function $h(z,t)$ satisfies the same equation. Let us consider $h(z,t)$, $z \in \widehat{\mathbb{H}}\setminus\lambda(0)$ with a singular point at $\lambda(0)$, where $\widehat{\mathbb{H}}$ is the closure of $\mathbb{H}$. Then

$$\lambda(t) = h(z,t) - \frac{2}{dh(z,t)/dt} = \lambda(0) + (12\pi)^{1/3} t^{1/3} + \dots$$

about the point $t = 0$. Thus, the driving term $\lambda(t)$ is Lip(1/3) about the point $t = 0$ and analytic for the rest of the points t.

Remark 2.1. *The radius of the circumference is not essential for the properties of $\lambda(t)$. Passing from $h(z,t)$ to the function $\frac{1}{r}h(rz,t)$ we recalculate the coefficients of the function $h(z,t)$ and the corresponding coefficients in the expansion of $\lambda(t)$ that depend continuously on r. Therefore, they stay within bounded intervals whenever r ranges within the bounded interval.*

Remark 2.2. *In particular, the expansion for $h(z,t)$ reflects the Marshall and Rohde's remark* [4, page 765] *that the tangent slits can not be generated by driving terms from Lip(1/2).*

3. Singular solutions for slit images

Suppose that the Löwner equation (5) with driving term $\lambda(t)$ generates a map $h(z,t)$ from $\Omega(t) = \mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$, where $\gamma(t)$ is a quasislit. Extending h to the boundary $\partial\Omega(t)$ we obtain a correspondence between $\gamma(t) \subset \partial\Omega(t)$ and a segment $I(t) \subset \mathbb{R}$, while the remaining boundary part $\mathbb{R} = \partial\Omega(t) \setminus \gamma(t)$ corresponds to $\mathbb{R} \setminus I(t)$. The latter mapping is described by solutions to the Cauchy problem for the differential equation (5) with the initial data $h(x,0) = x \in \mathbb{R} \setminus \lambda(0)$. The set $\{h(x,t) : x \in \mathbb{R} \setminus \lambda(0)\}$ gives $\mathbb{R} \setminus I(t)$, and $\lambda(t)$ does not catch $h(x,t)$ for all $t \geq 0$, see [2] for details.

The image $I(t)$ of $\gamma(t)$ can be also described by solutions $h(\lambda(0),t)$ to (5), but the initial data $h(\lambda(0),0) = \lambda(0)$ forces h to be singular at $t = 0$ and to possess the following properties.

(i) There are two singular solutions $h^-(\lambda(0),t)$ and $h^+(\lambda(0),t)$ such that $I(t) = [h^-(\lambda(0),t), h^+(\lambda(0),t)]$.

(ii) $h^{\pm}(\lambda(0),t)$ are continuous for $t \geq 0$ and have continuous derivatives for all $t > 0$.

(iii) $h^-(\lambda(0),t)$ is strictly decreasing and $h^+(\lambda(0),t)$ is strictly increasing, so that $h^-(\lambda(0),t) < \lambda(t) < h^+(\lambda(0),y)$.

We will focus on studying the singularity character of $h^{\pm}$ at $t = 0$.

Theorem 3.1. *Let the Löwner differential equation (5) with the driving term $\lambda \in$ Lip(1/2), $\|\lambda\|_{1/2} = c$, generate slit maps $h(z,t) : \mathbb{H} \setminus \gamma(t) \to \mathbb{H}$ where $\gamma(t)$ is a quasislit. Then $h^+(\lambda(0),t)$ satisfies the condition*

$$\lim_{t\to 0+} \sup \frac{h^+(\lambda(0),t) - h^+(\lambda(0),0)}{\sqrt{t}} \leq \frac{c + \sqrt{c^2+16}}{2},$$

and this estimate is the best possible.

Proof. Assume without loss of generality that $h^+(\lambda(0),0) = \lambda(0) = 0$. Denote $\varphi(t) := h^+(\lambda(0),t)/\sqrt{t}$, $t > 0$. This function has a continuous derivative and satisfies the differential equation

$$t\varphi'(t) = \frac{2}{\varphi(t) - \lambda(t)/\sqrt{t}} - \frac{\varphi(t)}{2}.$$

This implies together with property (iii) that $\varphi'(t) > 0$ iff

$$\frac{\lambda(t)}{\sqrt{t}} < \varphi(t) < \varphi_1(t) := \frac{\lambda(t)}{2\sqrt{t}} + \sqrt{\frac{\lambda^2(t)}{4t} + 4}.$$

Observe that $\varphi_1(t) \le A := (c + \sqrt{c^2+16})/2$.

Suppose that $\lim_{t\to 0+} \sup \varphi(t) = B > A$, including the case $B = \infty$. Then there exists $t^* > 0$, such that $\varphi(t^*) > B - \epsilon > A$, for a certain $\epsilon > 0$. If $B = \infty$, then replace $B - \epsilon$ by $B' > A$. Therefore, $\varphi'(t^*) < 0$ and $\varphi(t)$ increases as t runs from t^* to 0. Thus, $\varphi(t) > B - \epsilon$ for all $t \in (0, t^*)$ and we obtain from (5) that

$$\frac{dh^+(\lambda(0),t)}{dt} \le \frac{2}{\sqrt{t}(B-\epsilon-c)},$$

for such t. Integrating this inequality we get

$$h^+(\lambda(0),t) \le \frac{4\sqrt{t}}{B-\epsilon-c} < \frac{4\sqrt{t}}{A-c},$$

that contradicts our supposition. This proves the estimate of Theorem 3.1.

In order to attain the equality sign in Theorem 3.1, one chooses $\lambda(t) = c\sqrt{t}$. Then $h^+(\lambda(0),t) = A\sqrt{t}$ solves equation (5) with singularity at $t = 0$. This completes the proof. □

Remark 3.1. *Estimates similar to Theorem 3.1 hold for the other singular solution $h^-(\lambda(0).t)$.*

Remark 3.2. *Let us compare Theorem 3.1 with the results from Section 2. The image of a circular arc $\gamma(t) \subset \mathbb{H}$ tangent to $\mathbb{R}$ is $I(t) = [h^-(\lambda(0),t), h^+(\lambda(0),t)]$, where $h^-(\lambda(0),t) = \alpha(t) = -(9/4\pi)^{1/3}t^{2/3}+\dots$, and $h^+(\lambda(0),t) = \beta(t) = (12\pi)^{1/3}t^{1/3}+\dots$, so that $h^-(\lambda(0),t) \in Lip(2/3)$ and $h^+(\lambda(0),t) \in Lip(1/3)$.*

Remark 3.3. *Singular solutions to the differential equation (5) appear not only at $t = 0$ but at any other moment $\tau > 0$. More precisely, there exist two families $h^-(\gamma(\tau),t)$ and $h^+(\gamma(\tau),t)$, $\tau \ge 0$, $t \ge \tau$, of singular solutions to (5) that describe the image of arcs $\gamma(t)$, $t \ge \tau$ under map $h(z,t)$. They correspond to the initial data $h(\gamma(\tau),\tau) = \lambda(\tau)$ in (5) and satisfy the inequalities $h^-(\gamma(\tau),t) < \lambda(t) < h^+(\gamma(\tau),t)$, $t > \tau$. These two families of singular solutions have no common inner points and fill in the set*

$$\{(x,t) : h^-(\lambda(0),t) \le x \le h^+(\lambda(0),t), 0 \le t \le t_0\},$$

for some t_0.

4. Critical norm values for driving terms

In this section we discuss the results and techniques of Marshall and Rohde [4] and Lind [2]. The authors of [4] proved the existence of $C_{\mathbb{D}}$ such that driving terms $u(t) \in \mathrm{Lip}(1/2)$ with $\|u\|_{1/2} < C_{\mathbb{D}}$ in (1) generate quasisymmetric slit maps. This result remains true for an absolute number $C_{\mathbb{H}}$ in the half-plane version of the Löwner differential equation (2), see e.g. [2].

Lind [2] claimed that the disk version (1) of the Löwner differential equation is 'more challenging', than the half-plane version (2). Working with the half-plane version she showed that $C_{\mathbb{H}} = 4$. The key result is based on the fact that if $\lambda(t) \in \mathrm{Lip}(1/2)$ in (2), and $h(x,t) = \lambda(t)$, say at $t = 1$, then $\Omega(t) = h(\mathbb{H},t)$ is not a slit domain and $\|\lambda\|_{1/2} \ge 4$. Moreover, there is an example of $\lambda(t) = 4 - 4\sqrt{1-t}$ that yields $h(2,1) = \lambda(1)$. Although there may be more obstacles for generating slit half-planes than that of the driving term λ catching up some solution h to (2), Lind showed that this is basically the only obstacle. The latter statement was proved by using techniques of [4].

We will modify here the main Lind's reasonings so that they could be applied to the disk version of the Löwner equation. After that it remains to refer to [4] and [2] to state that $C_{\mathbb{D}}$ also equals 4.

Suppose that slit disks $\Omega(t)$ correspond to $u \in \mathrm{Lip}(1/2)$ in (1) with the sign '+' in its right-hand side instead of '-'. Then the maps $w(z,t)$ are extended continuously to $\mathbb{T}\setminus\{e^{iu(0)}\}$. Let $z_0 \in \mathbb{T}\setminus\{e^{iu(0)}\}$, and let $\alpha(t,\alpha_0) := \arg w(z_0,t)$ be a solution to the following real-valued initial value problem

$$\frac{d\alpha(t)}{dt} = \cot\frac{\alpha - u}{2}, \quad \alpha(0) = \alpha_0. \tag{6}$$

Similarly, suppose that slit half-planes $\Omega(t)$ correspond to $\lambda \in \mathrm{Lip}(1/2)$ in (2) with the sign '+' in its right-hand side instead of '-'. Then the maps $h(z,t)$ are extended continuously to $\mathbb{R}\setminus\lambda(0)$. Let $x_0 \in \mathbb{R}\setminus\lambda(0)$ and let $x(t,x_0) := h(x_0,t)$ be a solution to the following real-valued initial value problem

$$\frac{dx(t)}{dt} = \frac{2}{x(t)-\lambda(t)}, \quad x(t_0) = x_0. \tag{7}$$

For all $t \geq 0$, $\tan((\alpha(t)-u(t))/2) \neq 0$ in (6), and $x(t)-\lambda(t) \neq 0$ in (7) (see [2] for the half-plane version). Let us show a connection between the solutions $\alpha(t)$ to (6), and $x(t)$ to (7), where the driving terms $u(t)$ and $\lambda(t)$ correspond to each other.

Lemma 4.1. *Given $\lambda(t) \in \mathrm{Lip}(1/2)$, there exists $u(t) \in \mathrm{Lip}(1/2)$, such that equations (6) and (7) have the same solutions. Conversely, given $u(t) \in \mathrm{Lip}(1/2)$ there exists $\lambda(t) \in \mathrm{Lip}(1/2)$, such that equations (6) and (7) have the same solutions.*

Proof. Given $\lambda(t) \in \mathrm{Lip}(1/2)$, denote by $x(t,x_0)$ a solution to the initial value problem (7). Then the solution $\alpha(t,\alpha_0)$ to the initial value problem (6) is equal to $x(t,\alpha_0)$ when

$$\tan\frac{\alpha-u}{2} = \frac{x-\lambda}{2},$$

and

$$x_0 = \lambda(0) + 2\tan\frac{\alpha_0 - u(0)}{2}.$$

The function $u(t)$ is normalized by choosing

$$u(0) = x_0 - \arctan\frac{x_0 - \lambda(0)}{2}.$$

This condition makes α_0 and x_0 equal. Hence, the first part of Lemma 1 is true if we put

$$u(t) = x(t,x_0) - 2\arctan\frac{x(t,x_0)-\lambda(t)}{2}. \tag{8}$$

Obviously, (8) preserves the Lip(1/2) property.

Conversely, given $u(t) \in \mathrm{Lip}(1/2)$, a solution $x(t,x_0)$ is equal to $\alpha(t,\alpha_0)$ when

$$\lambda(t) = \alpha(t,\alpha_0) - 2\tan\frac{\alpha(t,\alpha_0)-u(t)}{2}. \tag{9}$$

Again (9) preserves the Lip(1/2) property. This ends the proof. □

Observe that in some extreme cases relations (8) or (9) preserve not only the Lipschitz class but also its norm. Lind [2] gave an example of the driving term $\lambda(t) = 4 - 4\sqrt{1-t}$ in (7). It is easily verified that $x(t,2) = 4 - 2\sqrt{1-t}$. If $t = 1$, then $x(1,2) = \lambda(1) = 4$, and λ cannot generate slit half-plane at $t = 1$. This implies that $C_{\mathbb{H}} \le 4$. Going from (7) to (6) we use (8) to put

$$u(t) = x(t,2) - 2\arctan\frac{x(t,2)-\lambda(t)}{2} = 4 - 2\sqrt{1-t} - 2\arctan\sqrt{1-t}.$$

From Lemma 4.1 we deduce that $\alpha(1,2) = u(1)$. Hence u cannot generate slit disk at $t = 1$, and $C_{\mathbb{D}} \le \|u\|_{1/2}$. Since

$$\sup_{0\le t<1}\frac{u(1)-u(t)}{\sqrt{1-t}} = \sup_{0\le t<1}\left(2 + 2\frac{\arctan\sqrt{1-t}}{\sqrt{1-t}}\right) = 4,$$

we have that $\|u\|_{1/2} \le 4$. It is now an easy exercise to show that $\|u\|_{1/2} = 4$. This implies that $C_{\mathbb{D}} \le 4$.

Lemma 4.2. *Let $u \in$ Lip(1/2) in (6) with $u(0) = 0$ and $\alpha_0 \in (0,\pi)$. Suppose that $\alpha(t)$ is a solution to (6) and $\alpha(1) = u(1)$. Then $\|u\|_{1/2} \ge 4$.*

Proof. Observe that $\alpha(t)$ is increasing on $[0,1]$, and $\alpha(t)-u(t) > 0$ on $(0,1)$. Let $u \in$ Lip(1/2) in (3), and $\|u\|_{1/2} = c$. Then,

$$\alpha(t) - u(t) \le \alpha(1) - u(1) + c\sqrt{1-t} = c\sqrt{1-t}. \tag{10}$$

Given $\epsilon > 0$, there exists $\delta > 0$, such that

$$\tan\frac{c\sqrt{1-t}}{2} < \frac{c\sqrt{1-t}}{2}(1+\epsilon),$$

for $1-\delta < t < 1$ and all $0 < c \le 4$. We apply this inequality to (6) and obtain that

$$\frac{d\alpha}{dt} \ge \cot\frac{c\sqrt{1-t}}{2} > \frac{2}{c\sqrt{1-t}(1+\epsilon)}.$$

Integrating gives that

$$\alpha(1) - \alpha(t) \ge \frac{4\sqrt{1-t}}{c(1+\epsilon)}.$$

This allows us to improve (10) to

$$\alpha(t) - u(t) \le \alpha(1) - \frac{4\sqrt{1-t}}{c(1+\epsilon)} - u(1) + c\sqrt{1-t} = \left(c - \frac{4}{c(1+\epsilon)}\right)\sqrt{1-t}. \tag{11}$$

Repeating these iterations we get

$$\alpha(t) - u(t) \le c_n\sqrt{1-t},$$

where $c_0 = c$, $c_{n+1} = c - 4/[(1+\varepsilon)c_n]$, and $c_n > 0$. Let g_n be recursively defined by (see Lind [2])

$$g_1(y) = y - \frac{4}{y}, \quad g_n(y) = y - \frac{4}{g_{n-1}(y)}, \quad n \ge 2.$$

It is easy to check that $c_n < g_n((1+\varepsilon)c) < (1+\varepsilon)c_n$

Lind [2] showed that $g_n(y_n) = 0$ for an increasing sequence $\{y_n\}$, and $g_{n+1}(y)$ is an increasing function from (y_n, ∞) to $\mathbb{R}$. So $c(1+\epsilon) > y_n$ for all n, and it remains to apply Lind's result [2] that $\lim_{n\to\infty} y_n = 4$. Hence, $c \geq 4/(1+\epsilon)$. The extremal estimate is obtained if $\epsilon \to 0$ which leads to $c \geq 4$. This completes the proof. □

Now Lind's reasonings in [2] based on the techniques from [4] give a proof of the following statement.

Proposition 4.1. *If $u \in \mathrm{Lip}(1/2)$ with $\|u\|_{1/2} < 4$, then the domains $\Omega(t)$ generated by the Löwner differential equation (1) are disks with quasislits.*

In other words, Proposition 4.1 states that $C_{\mathbb{D}} = C_{\mathbb{H}} = 4$.

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