SINGULAR BRASCAMP-LIEB INEQUALITIES WITH CUBICAL STRUCTURE

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ABSTRACT. We prove a singular Brascamp-Lieb inequality, stated in Theorem 1, with a large group of involutive symmetries.

1. INTRODUCTION

Much research has been devoted in recent years to Brascamp-Lieb and related inequalities, we refer to [4], [1], [2], [3] and the references therein. Brascamp-Lieb inequalities are $L^p$ estimates for certain multilinear forms on functions on Euclidean spaces. The forms consist of integrating the tensor product of the input functions over a subspace of the direct sum of the domain spaces. Following general conventions, we parameterize the subspace of integration by $\mathbb{R}^m$ and write the corresponding Brascamp-Lieb inequality

$$\left| \int_{\mathbb{R}^m} \left( \prod_{i=1}^{n} F_i(\Pi_i x) \right) dx \right| \leq C \prod_{i=1}^{n} \| F_i \|_{p_i} \quad (1.1)$$

with suitable surjective linear maps

$$\Pi_i : \mathbb{R}^m \to \mathbb{R}^{k_i}.$$ 

Here the constant $C$ is independent of the measurable functions $F_i$ on $\mathbb{R}^{k_i}$, and integrability on the left-hand side being implied by finiteness of the right-hand side.

It is well understood, under which conditions the Brascamp-Lieb inequality holds. Bennett, Carbery, Christ, and Tao [1] prove a necessary and sufficient dimensional condition, namely that

$$\dim(V) \leq \sum_{i=1}^{n} \frac{1}{p_i} \dim(\Pi_i V) \quad (1.2)$$

for every subspace $V$ of $\mathbb{R}^m$, with equality if $V = \mathbb{R}^m$. Necessity of inequality (1.2) is easily seen by testing the Brascamp-Lieb inequality on certain characteristic functions $F_i$. These functions have minimal support such that the integrand on the left-hand side of (1.1) is nonzero on a one-neighborhood in $\mathbb{R}^m$ of an arbitrarily large ball in $V$. Necessity of the reverse inequality in case $V = \mathbb{R}^m$ is obtained by using similarly an arbitrarily small ball in $\mathbb{R}^m$.

In this paper, we focus on singular Brascamp-Lieb inequalities. This variant has also seen much development in recent years, but still lacks a general criterion mirroring the condition (1.2). A singular Brascamp-Lieb inequality incorporates a Calderón-Zygmund kernel on the left hand side:

$$\left| \int_{\mathbb{R}^m} \left( \prod_{i=1}^{n} F_i(\Pi_i x) \right) K(\Pi x) dx \right| \leq C \prod_{i=1}^{n} \| F_i \|_{p_i}. \quad (1.3)$$

Here $\Pi : \mathbb{R}^m \to \mathbb{R}^k$ is a surjective linear map, and by Calderón-Zygmund kernel we mean in this paper a tempered distribution $K$ on $\mathbb{R}^k$ whose Fourier transform $\hat{K}$, called the
multiplier associated with $K$, is a measurable function satisfying the symbol estimates
\[ |\partial^{\alpha} \hat{K}(\xi)| \leq |\xi|^{-|\alpha|} \] (1.4)
for all $\xi \neq 0$ and all multi-indices $\alpha$ up to suitably large order.

A necessary condition for the singular Brascamp-Lieb inequality (1.3) can be obtained by specifying $K$ to be the Dirac delta, that is $\hat{K} = 1$. In this case, (1.3) can be recognized as a classical Brascamp-Lieb inequality (1.1) with integration over the kernel of $\Pi$. Condition (1.2) then yields the necessary condition
\[ \dim(V) \leq \sum_{i=1}^{n} \frac{1}{p_i} \dim(\Pi_i|_{\ker \Pi(V)}) \] (1.5)
for all $V \subseteq \ker \Pi$, with equality if $V = \ker \Pi$.

Lacking a general necessary and sufficient condition, the theory of singular Brascamp-Lieb inequalities remains at the stage of a case-by-case study. Here, for the first time, we study a sufficiently general family to expose a non-trivial role of the condition (1.5).

We focus on a case that features the following cubical structure. For a parameter $m \geq 1$ we consider $\mathbb{R}^{2m}$ with coordinates $(x_0^1, x_0^2, \ldots, x_0^m, x_1^1, x_1^2, \ldots, x_1^m)^T$, which we also combine as pair of vectors $(x^0, x^1)^T$ or we write as vector $x$. Define the cube $Q$ to be the set of functions $j : \{1, 2, \ldots, m\} \to \{0, 1\}$.

For $j \in Q$ define the projection $\Pi_j : \mathbb{R}^{2m} \to \mathbb{R}^m$ by
\[ \Pi_j x = (x_j^{1(1)}, x_j^{2(2)}, \ldots, x_j^{m(m)})^T. \]

Our main theorem states that for these particular projections $\Pi_j$ and for the exponents $p_j = 2^m$, inequalities (1.5) provide a sufficient condition on an otherwise arbitrary surjective linear map $\Pi : \mathbb{R}^{2m} \to \mathbb{R}^m$ for the singular Brascamp-Lieb inequality to hold.

**Theorem 1.** Given $m \geq 1$, there is an $N \geq 0$ such that for all surjective linear maps $\Pi : \mathbb{R}^{2m} \to \mathbb{R}^m$ the following are equivalent.

1. For all subspaces $V \subset \ker \Pi$ we have
\[ \dim(V) \leq \sum_{j \in Q} 2^{-m} \dim(\Pi_j|_{\ker \Pi(V)}), \] (1.6)
with equality if $V = \ker \Pi$.

2. For all $j \in Q$, the composed map $\Pi \Pi_j^T$ is regular.

3. There is a constant $C$ such that for all Calderón-Zygmund kernels $K$ satisfying the symbol estimates (1.4) for all multi-indices up to degree $N$, and for all tuples of Schwartz functions $(F_j)_{j \in Q}$ we have
\[ \left| \int_{\mathbb{R}^{2m}} \left( \prod_{j \in Q} F_j(\Pi_j x) \right) K(\Pi x) \, dx \right| \leq C \prod_{j \in Q} \| F_j \|_{2^m}. \] (1.7)

Condition (1) of Theorem 1 is the necessary condition derived from that of Bennett, Carbery, Christ, and Tao. In the present setting it can immediately be simplified. For $V = \ker \Pi$, the left-hand side of (1.6) is at least $m$, while the right-hand side is at most $m$, because each summand is at most $2^{-m}m$ and there are $2^m$ summands. Assuming that inequality (1.6) holds for this $V$, we conclude actual equality for this $V$. We further conclude that the restriction of $\Pi_j$ to $\ker \Pi$ is injective for each $j$, and therefore equality in (1.6) holds for all subspaces $V$ of $\ker \Pi$. Thus condition (1) in Theorem 1 is equivalent
to the single instance with \( V = \ker \Pi \), which in turn is equivalent to all \( \Pi_j \) being injective on \( \ker \Pi \).

It is now easy to see that conditions (1) and (2) in Theorem 1 are equivalent. Namely, let \( j \) and \( l \) be any two opposite corners of the cube. Then the range of \( \Pi_j^T \) is obviously the kernel of \( \Pi_l \) and \( 2^m \) dimensional. Hence regularity of \( \Pi \Pi_j^T \) is the same as injectivity of \( \Pi \) on the kernel of \( \Pi_l \). By the above discussion, conditions (1) and (2) are equivalent.

We have already argued that (3) implies (1), hence the main content of the Theorem is that (2) implies (3).

While the projections \( \Pi_j \) of Theorem 1 may appear rather particular, they provide no loss of generality up to change of variables after fixing their combinatorial datum, that is the set of integer tuples \((\dim(\Pi_j(V)))_{j \in Q}\) with \( V \) a subspace of \( \mathbb{R}^{2m} \). For each \( 1 \leq i \leq m \), there exist one-dimensional subspaces \( V \) and \( W \) of \( \mathbb{R}^{2m} \), each spanning a certain standard coordinate axis, such that \( \dim(\Pi_j(V)) = j(i) \) and \( \dim(\Pi_j(W)) = 1 - j(i) \) for all \( j \). Conversely, consider any collection of linear maps \((\tilde{\Pi}_j)_{j \in Q}\) defined on \( \mathbb{R}^{2m} \) with \( m \) dimensional range, such that for each \( 1 \leq i \leq m \) there are spaces \( V \) and \( W \) with combinatorial datum analogous as above. Then these spaces necessarily are one dimensional and together span \( \mathbb{R}^{2m} \). A suitable linear transformation of \( \mathbb{R}^m \) will turn these vector spaces into the standard coordinate axes. Together with a suitable choice of basis for the range of each of the maps \( \tilde{\Pi}_j \), these maps will be identified as the above maps \( \Pi_j \).

The role of the cubical structure of the form in this theorem is to allow for a symmetrization process in the tuple of functions \( F_j \). Indeed, the main Lemma 3 stated in Section 2 is an induction over the number of axis parallel symmetry planes of this cube that the tuple \( F_j \) respects, in the sense of \((2.5)\). This symmetrization procedure, sometimes called twisted technology, originates in a series of papers such as \([13]\), \([12]\), \([7]\). Theorem 1 in the case \( m = 2 \) generalizes estimates in \([6]\) and \([9]\).

Further generalizations of Theorem 1 appear desirable, but are beyond the scope of the present paper, except for a mild vector-valued generalization in Lemma 3. Most naturally, one could seek an extension to other exponents \( p_j \) and ask for an optimal range of exponents. One may also seek generalizations in which the index set is a subset of the cube. This can sometimes be achieved by setting some functions \( F_j \) constantly equal to one, provided one has bounds with \( p_j = \infty \). A further question concerns the exact dependence on \( \Pi \) of the bounds in the theorem.

To elaborate some of the difficulties in the absence of the cubical structure, we briefly discuss a singular Brascamp-Lieb integral with three input functions. We take \( m = 4 \) and \( k = 2 \), \( k_i = 2 \) and \( p_i = 3 \) for \( i = 1, 2, 3 \). The projections \( \Pi \) and \( \Pi_1 \) are then given by \( 2 \times 4 \) matrices, which we write as block matrices \((B A)\) and \((B_1 A_1)\) with quadratic blocks. Choosing coordinates suitably on domain and range of \( \Pi \), we may assume that

\[
\Pi = \begin{pmatrix}
0 & I \\
\end{pmatrix}
\]

with the identity matrix \( I \). In order to not violate \((1.5)\) with \( V \) equal to \( \ker \Pi \), the matrices \( B_i \) need to be regular. Changing coordinates on the range of \( \Pi_1 \), we may assume \( B_i = I \) for each \( 1 \leq i \leq 3 \). Warchalski, in his PhD thesis \([17]\), classifies the possibilities for the remaining parameters \( A_1, A_2, A_3 \) into nine cases. Most cases can be normalized such that \( A_1 = 0 \) and \( A_2 = I \), leaving only \( A_3 \) as indetermined matrix. A trivial case occurs if \( A_3 = 0 \) or \( A_3 = I \), this results in a reduction of the complexity of the integral by combining \( F_3 \) with one of the other functions by a pointwise product. The case that all eigenvalues of \( A_3 \) are different from 0 and 1 is the generic two dimensional version of the bilinear Hilbert transform \([15]\). The known proofs of the singular Brascamp-Lieb inequality in this case require the technique of time-frequency analysis, which is somewhat different from the technique in the present paper. The case
that one eigenvalue of $A_3$ is equal to 0 or 1 and the other eigenvalue is different from 0 and 1 is an interesting hybrid case discussed in [4]. The case when $A_3$ has both 0 and 1 as eigenvalue is called the twisted paraproduct and is an instance of the forms in Theorem 1 with $m = 2$, albeit with the fourth function set constant equal to 1. The only case in Warchalski's thesis where the singular Brascamp-Lieb inequality is not known to hold is the one where the first columns of all three matrices $A_1, A_2, A_3$ vanish, while the second columns are $(0,0)^T$, $(0,1)^T$, $(1,0)^T$, respectively. Thanks to the vanishing first columns, one variable integrates out trivially and one reduces to a one-dimensional Calderón-Zygmund kernel. The paradigmatic example in this case is the conjectured inequality

$$
\left| \int_{\mathbb{R}^3} F(x,y)G(y,z)H(z,x) \frac{1}{x+y+z} \, dx \, dy \, dz \right| \leq C\|F\|_3\|G\|_3\|H\|_3,
$$

where the left-hand side is called the triangular Hilbert transform. Proving the displayed a priori bound is one of the most intriguing open problems in the area of singular Brascamp-Lieb inequalities. Partial progress on this problem can be found in [13] based on the approach in [16], and in [10], [14].


2. Symmetry considerations and the inductive statement

Theorem 1 will be proven by induction. The inductive statement is the content of Lemma 3 below. In this section we further discuss certain symmetries of the singular Brascamp-Lieb integrals (1.7), which will be needed in the proof of the inductive statement.

For the rest of the paper, we consider a higher-dimensional generalization of the singular Brascamp-Lieb inequality (1.7), motivated by the related paper [8] on certain patterns in positive density subsets of the Euclidean space. We write vectors as column vectors and identify $x \in (\mathbb{R}^d)^{2m}$ with a vector of vectors as

$$(x_0^0, \ldots, x_0^m, x_1^0, \ldots, x_1^m)^T,$$

where $x_0^0, \ldots, x_0^m, x_1^0, \ldots, x_1^m \in \mathbb{R}^d$, which we also combine into a pair of vectors $(x^0, x^1)^T$. For $j \in \mathcal{Q}$, let $\Pi_j : (\mathbb{R}^d)^{2m} \rightarrow (\mathbb{R}^d)^m$ be given by

$$\Pi_jx = (x_j^{(1)}, x_j^{(2)}, \ldots, x_j^{(m)})^T.$$

We define an action of an $m \times m$ matrix $A$ on a vector $y = (y_1, \ldots, y_m)^T \in (\mathbb{R}^d)^m$ by the Kronecker product of the matrix $A$ with the $m \times m$ identity matrix $I$

$$Ay := (A \otimes I)y, \quad (Ay)_i = \sum_{j=1}^m a_{ij}y_j$$

for $1 \leq i \leq m$. In similar fashion, we identify $\Pi_j : (\mathbb{R}^d)^{2m} \rightarrow (\mathbb{R}^d)^m$ with $m \times 2m$ matrices. We also restrict attention to those projections $\Pi : (\mathbb{R}^d)^{2m} \rightarrow (\mathbb{R}^d)^m$ which are given as analogous block matrix product as

$$\Pi x = (B \quad A)x,$$

where $A$ and $B$ are $m \times m$ matrices and $x \in (\mathbb{R}^d)^{2m}$. This setup makes our higher-dimensional generalization a very simple extension of the one-dimensional theory.

It is no restriction to assume that all functions $F_j$ in Theorem 1 are real valued. Schwartz functions in this section will map $F_j : (\mathbb{R}^d)^m$ to $\mathbb{R}$ and multipliers $\hat{K}$ will map $(\mathbb{R}^d)^m \setminus \{0\}$ to $\mathbb{C}$.

Lemma 2 (Symmetries of (1.7)). The following two statements hold.
(1) Let $D$ be an $m \times m$ diagonal matrix of rank $m$. Let $\tilde{D}$ be a $2m \times 2m$ matrix which decomposes into four blocks of size $m \times m$, the two blocks on the diagonal being $D$ and the two off-diagonal blocks being 0. Let $1 \leq p_j \leq \infty$ with $\sum_{j \in Q} \frac{1}{p_j} = 1$. Then
\[
\int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_j(\Pi_j x) \right) K(\Pi x) \, dx = \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} \tilde{F}_j(\Pi_j x) \right) \tilde{K}(\Pi x) \, dx
\] (2.2)
holds with
\[
\tilde{F}_j(y) := \det(D)^{\frac{d}{2m}} F_j(Dy), \quad \tilde{K}(y) := \det(D)^d K(Dy), \quad \tilde{\Pi} := D^{-1} \Pi \tilde{D}.
\]

(2) Let $P$ be a permutation of $m$ elements, which we also identify with the $m \times m$ matrix in which the $ij$-th entry equals $\delta_{P(i)j}$ in the Kronecker delta notation. Let $\tilde{P}$ be a $2m \times 2m$ matrix which decomposes into four blocks of size $m \times m$, the two blocks on the diagonal being $P$ and the two off-diagonal blocks being 0. Then (2.2) holds with
\[
\tilde{F}_j(y) := F_{j \circ P}(Py), \quad \tilde{K}(y) := K(Py), \quad \tilde{\Pi} := P^{-1} \Pi \tilde{P}.
\]

**Proof.** Proof of (1). Changing variables by $\tilde{D}$ we have for the left-hand side of (2.2)
\[
det(D)^d \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_j(\Pi_j \tilde{D}x) \right) K(\Pi \tilde{D}x) \, dx.
\]
Using $D \Pi_j = \Pi_j \tilde{D}$ thanks to the special structure of the projections $\Pi_j$, and using $\det(D)^d = \det(D)^{2d}$ and $\sum_{j \in Q} \frac{1}{p_j} = 1$, the previous display equals
\[
det(D)^d \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} \det(D)^{\frac{d}{2m}} F_j(D \Pi_j x) \right) K(\Pi \tilde{D}x) \, dx.
\]
With notation as in (1) of the lemma, this becomes the right-hand side of (2.2). Proof of (2). We compute similarly as above
\[
\int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_j(\Pi_j x) \right) K(\Pi x) \, dx = \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_j(\Pi_j \tilde{P}x) \right) K(\Pi \tilde{P}x) \, dx
\]
\[
= \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_j(P \Pi_j \cdot x) \right) \tilde{K}(\Pi x) \, dx = \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_{j \circ P}(P \Pi_j x) \right) \tilde{K}(\Pi x) \, dx,
\]
with notation as in (2) of the lemma. \(\square\)

To prove Theorem (1) it suffices to consider the singular Brascamp-Lieb integral
\[
\Lambda(K, A) := \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_j(\Pi_j x) \right) K((I_{A} x)) \, dx.
\] (2.3)
This is justified as follows. Note that if $K$ is a Calderón-Zygmund kernel on $\mathbb{R}^{dm}$, then so is a certain nonzero scalar multiple of $\tilde{K}$ defined by
\[
\tilde{K}(Eu) = K(u)
\]
for some regular matrix $E$. Hence
\[
\tilde{K}(\tilde{\Pi} u) = K(\Pi u)
\]
with $\tilde{\Pi} := E \Pi$. Regularity of $\tilde{\Pi} \Pi T$ is equivalent to regularity of $\Pi T$, so we may use this flexibility to replace the matrix $(B \ A)$ in (2.1) by $(EB \ EA)$ and therefore assume
that the matrix $B$ is diagonal and idempotent. Regularity of all matrices $\tilde{\Pi}^T_j$ then requires $B$ to be the identity matrix.

Let $1 \leq i \leq m$ act by reflection $j \mapsto i \ast j$ on the cube $Q$, where

$$(i \ast j)(k) := j(k)$$

if $i \neq k$ and

$$(i \ast j)(i) := 1 - j(i).$$

Denote the Gaussian on $\mathbb{R}^s$ by $g(x) := e^{-\pi|x|^2}$ and write $g_t(x) := t^{-s}g(t^{-s}x)$, where $s$ is to be understood from the context, typically $s = 1, d, md, 2md$ or $(2m - 2)d$. By $\partial_j f$ we denote the $j$-th partial derivative of a function $f$. Recall that the Hilbert-Schmidt norm $\|A\|_{HS}$ of a matrix $A$ is monotone in each of its arguments and dominates the operator norm $\|A\|$.

**Lemma 3** (The inductive statement). *Let $m \geq 1$, $d \geq 1$. Let $0 \leq l \leq m$. Let $0 < \epsilon < 1$.

There exists a constant $C$ depending on these parameters such that the following holds.

Let $A$ be an $m \times m$ matrix such that

$$|\det((I A)\Pi^2_j)| > \epsilon \quad \text{and} \quad \|A\|_{HS} \leq \epsilon^{-1}$$

for all $1 \leq j \leq m$. Assume that the first $l$ rows of $A$ coincide with the first $l$ rows of $-I$.

Let $(F_j)_{j \in Q}$ be a tuple of real valued Schwartz functions with

$$F_j = F_{i\ast j} \quad \text{and} \quad \|F_j\|_{2^m} = 1$$

for all $j \in Q$ and all $1 \leq i \leq l$. Then the following two estimates hold for (2.3):

1. Let $K$ be a kernel such that

$$|\partial^\alpha \hat{K}(\xi)| \leq |\xi|^{-|\alpha|}$$

for all multi-indices $\alpha \in \mathbb{N}^{3dm}_0$ with $|\alpha| \leq 3dm$ and

$$\hat{K}(\xi_1, \ldots, \xi_l, 0, \ldots, 0) \equiv 0,$$

that is, $\hat{K}$ vanishes for all $0 \neq (\xi_1, \ldots, \xi_m) \in (\mathbb{R}^d)^m$ with $\xi_k = 0$ for $k > l$. Then

$$|\Lambda(K, A)| \leq C.$$  

2. Let $l < i \leq m$ and $1 \leq k_1, k_2 \leq d$. Let $u \in \mathbb{R}^{dm}$ and let $c \in L^\infty(0, \infty)$ with $\|c\|_{\infty} = 1$. Let $K$ be the kernel defined by

$$\hat{K}(\xi) = \int_0^\infty c_t(u) \overline{g_{i, k_1, k_2}((I A)^T(t\xi))} e^{2\pi i u \cdot \xi} \frac{dt}{t},$$

where $g_{i, k_1, k_2} := \partial_{(i-1)d+k_1}(i+1)(i+m-1)d+k_2} g$. Then

$$|\Lambda(K, A)| \leq C(1 + \|u\|)^{2d(m-1)}.$$  

Note that the case $l = m$ of (1) is trivially true since then $K = 0$. On the other hand, (2) is void for $l = m$ since then $l < i \leq m$ does not exist. The case $l = 0$ of (1) implies the desired Theorem[1] We will therefore do an induction on $l$, proving Lemma assuming that we have already established the lemma for all $l < l'$ $\leq m$. We will reduce (1) at level $l$ to (2) at the same level $l$, and we will reduce (2) at level $l$ to (1) at the level $l + 1$. These two reductions will be performed in the following two sections. Note that in the case $m = 1$ we are dealing with a one-dimensional Calderón-Zygmund kernel and the claim follows from the standard Calderón-Zygmund theory. We shall therefore assume $m \geq 2.$
3. Proof of (1) of Lemma 3

Consider $m, d, l, \epsilon$ as in Lemma 3. We shall prove existence of a constant $C$ such that (1) holds, under the hypothesis that for the same $m, d, l, \epsilon$ there is a constant $C$ such that (2) holds.

Let $A, (F_j)_j$ and $K$ be given as in (1) of Lemma 3. Our aim is to decompose $K$ into a convergent sum and integral of kernels defined in (2) of Lemma 3.

We will perform a cone decomposition of $K$. The matrix $A$ determines certain subspaces of $(\mathbb{R}^d)^m$, and each cone will be small enough to avoid some of these subspaces, as elaborated in the following lemma. In this section we use the notational convention

$$\xi = (\xi', \xi'') \in \mathbb{R}^d \times \mathbb{R}^{d(m-l)} = \mathbb{R}^{dm}.$$

Lemma 4. There is a number $\delta > 0$ depending on $\epsilon, d,$ and $m$, such that the following holds. For $\gamma$ a unit vector in $\mathbb{R}^{d(m-l)}$ define the stick

$$S = \left\{ (0, \xi'') \in \mathbb{R}^{dm} : \frac{1}{2} \leq \|\xi''\| \leq 1, \left\| \frac{\xi''}{\|\xi''\|} - \gamma \right\| \leq \delta \right\}.$$

Then there is $l < i \leq m$ and some $1 \leq k_1, k_2 \leq d$ such that for all $\eta \in S$ we have

$$\min(|\eta_{ik_1}|,|(A^T \eta)_{ik_2}|) > \delta,$$

where we write $\eta = (\eta_1, \ldots, \eta_m)^T$, $\eta_j = (\eta_{j1}, \ldots, \eta_{jd})^T$ for $1 \leq j \leq m$, and analogously we write the coordinates of $A^T \eta$.

Proof. We first claim that $S$ contains a point $\xi$ such that there is $l < i \leq m$ with

$$\min(|\xi_i|, \|A^T \xi\|) > 4d \max(1, \|A\|) \delta. \tag{3.2}$$

Assume to get a contradiction that the claim is false. For every $\xi \in S$ we choose $j \in Q$ such that for $l < i \leq m$ the value of $j(i)$ corresponds to which term on the left hand side of (3.2) is less than or equal to the right-hand side. Hence we obtain

$$\|A_j^T \xi_i\| \leq 4d \max(1, \|A\|) \delta, \tag{3.3}$$

where we have denoted $A_j := (IA)\Pi_j^T$. By pigeonholing with respect to the $2^n$ elements of $Q$, there exists $j \in Q$ and $S' \subseteq S$ of size $|S'| \geq 2^{-m} |S|$ such that (3.3) holds for this same $j$ and all $\xi \in S'$ and $l < i \leq m$.

To obtain a contradiction, we compare the volume of $\Psi S'$, where $\Psi$ is projection onto the $d(m-l)$ dimensional space spanned by the last components, with that of the linear image

$$\Psi A_j^T \Psi S' = \Psi A_j^T S'.$$

We obtain

$$e(d, m) \delta^{d(m-l)-1} \leq |\Psi S'| = |\det(\Psi A_j^T \Psi^T)|^{-d} |\Psi A_j^T S'| \leq C(d, m, \epsilon) \delta^{d(m-l)} \tag{3.4}$$

with positive constants $c(d, m)$ and $C(d, m, \epsilon)$. On the left hand side we used the growth in $\delta$ of the volume of the stick. On the right hand side we used that the first $l$ rows of $A$ equal those of $-I$ and thus

$$\epsilon < |\det(A_j^T)| = |\det(\Psi A_j^T \Psi^T)|,$$

and we estimated the size of the ball with radius $\delta$ in $\mathbb{R}^{d(m-l)}$ that contains $\Psi A_j^T S'$ by virtue of (3.3). Choosing $0 < \delta < 0.1$ small enough depending on $d, m, \epsilon$, inequality (3.4) is a contradiction, thereby proving the claim.

By the triangle inequality, the $\xi$ obtained via the claim also satisfies

$$\min(|\xi_{ik_1}|,|(A^T \xi)_{ik_2}|) \geq 4 \max(1, \|A\|) \delta.$$
for some $1 \leq i \leq m$ and $1 \leq k_1, k_2 \leq d$. To prove the desired lower bound \([5.1]\) for every \(\eta \in S\), since \(1/2 \leq \|\eta\|, \|\xi\| \leq 1\), it suffices by scaling to show the analogous bounds with \(2d\) on the right-hand side under the assumption that \(\|\eta\| = \|\xi\|\). Then \(|\eta - \xi| \leq \delta\) and \(|A(\eta - \xi)| \leq \|A\|\delta\). Thus

\[
| (A^T \eta)_{ik_2} | \geq | (A^T \xi)_{ik_2} | - | (A^T (\eta - \xi))_{ik_2} | \geq 4 \max(1, \|A\|) \delta - \|A\| \delta > 2\delta,
\]

and similarly

\[
| \eta_{ik_2} | \geq | \xi_{ik_2} | - | \eta_{ik_2} - \xi_{ik_2} | > 2\delta.
\]

This completes the proof of Lemma 4. \(\square\)

We proceed to decompose \(K\). Let \(\delta\) be as in the above Lemma 4. Consider a maximal set \(\Gamma\) of \(\delta/6\)-separated vectors of unit length in \(\mathbb{R}^{d(m-l)}\). By volume considerations on the unit sphere, there are at most \(C(d, m)\delta^{-d(m-l)}\) elements in \(\Gamma\). The balls of radius \(\delta/2\) centered around these points cover the sphere.

For \(\gamma \in \Gamma\), let \(\rho_\gamma\) be a smooth nonnegative bump function in \(\mathbb{R}^{d(m-l)}\) supported on a ball of radius \(\delta\) about \(\gamma\) and constant one on ball of radius \(\delta/2\) about \(\gamma\). Then evidently \(\sum_{\gamma \in \Gamma} \rho_\gamma\) is uniformly bounded below on the unit sphere and we may consider the partition of unity of \(\mathbb{R}^{md} \setminus \{\mathbb{R}^d \times \{0\}\}\) by the functions

\[
f_\gamma(\xi) := \frac{\rho_\gamma(\xi'' / \|\xi''\|)}{\sum_{\gamma' \in \Gamma} \rho_{\gamma'}(\xi'' / \|\xi''\|)},
\]

Note the derivative bounds

\[
| \partial^\alpha f_\gamma(\xi'') | \leq C_{\alpha, m} \|\xi''\|^{-|\alpha|}
\]

for all \(\xi'' \neq 0\). We write

\[
\hat{K}(\xi) = \sum_\gamma \hat{K}_\gamma(\xi) f_\gamma(\xi''), = \sum_\gamma \hat{K}_\gamma(\xi).
\]

Since the number of summands \(K_\gamma\) depends only on \(d\) and \(m\), we may restrict attention to an individual summand and prove

\[
| \Lambda(K_\gamma, A) | \leq C.
\]

Let \(\psi : \mathbb{R}^d \to \mathbb{R}\) and \(\phi : \mathbb{R}^{d(m-l)} \to \mathbb{R}\) be radial Schwartz functions supported in the annuli \(\{1/2 \leq \|\eta\| \leq 1\}\) in \(\mathbb{R}^d\) and \(\mathbb{R}^{d(m-l)}\), respectively. We normalize them such that

\[
1 = \int_0^\infty \psi(t \xi') \frac{dt}{t} = \int_0^\infty \phi(t \xi'') \frac{dt}{t} = \int_0^\infty \int_0^\infty \psi(st \xi') \phi(st \xi'') \frac{ds \, dt}{s \, t}
\]

for every \(\xi', \xi'' \neq 0\). Then for each \(\xi\) with \(\xi' \neq 0\) we decompose \(\hat{K}_\gamma(\xi)\) according to the small and large values of \(s\) as

\[
\int_0^\infty \int_1^\infty \hat{K}_\gamma(\xi) \psi(st \xi') \phi(st \xi'') \frac{ds \, dt}{s \, t} \quad \text{(3.5)}
\]

\[
\int_0^\infty \int_0^1 \hat{K}_\gamma(\xi) \psi(st \xi') \phi(st \xi'') \frac{ds \, dt}{s \, t}. \quad \text{(3.6)}
\]

We estimate the effect of the multipliers \(3.5\) and \(3.6\) separately. For \(3.5\) we integrate in \(s\) and note that

\[
\rho(\xi') := \int_1^\infty \psi(s \xi') \frac{ds}{s}
\]
extends to a smooth bump function with compact support in $\|\xi\| < 2$. We then fix $t$ and rescale the corresponding portion of the multiplier back as on the left-hand side of the following display (3.7). Moreover, we define the multiplier $\tilde{K}_t$ by

$$\tilde{K}_t(t^{-1}\xi)\rho(\xi(t))\phi(\xi(t)) =: \tilde{K}_t(\xi)\hat{g}_{i,k_1,k_2}((I A)^T \xi),$$

(3.7)

where $g_{i,k_1,k_2}$ is defined in (2) of Lemma 3 for suitable $i, k_1, k_2$. To make sure that $\tilde{K}_t$ is well defined and well behaved, we need that the second factor on the right-hand side is bounded away from 0 on the compact support of the left-hand side. By Lemma 4 there exist $l + 1 \leq i \leq m$ and $1 \leq k_1, k_2 \leq d$ such that for each $\xi$ in the support of the left-hand side of (3.7) we have

$$|\xi_{ik_1}| > \delta \quad \text{and} \quad |(A^T \xi)_{ik_2}| > \delta.$$

Since $\hat{g}_{i,k_1,k_2}$ vanishes only at $\xi_{ik_1} = 0$ and $(A^T \xi)_{ik_2} = 0$, it is bounded uniformly away from 0 on the support of the left-hand side of (3.7). Therefore, the function $\tilde{K}_t$ is well defined, smooth, and satisfies some uniform bounds

$$|\partial^\alpha \tilde{K}_t(\xi)| \leq C$$

uniformly in $t$ for all $|\alpha| \leq 3dm$. We expand it into its Fourier integral

$$\tilde{K}_t(\xi) = \int_{\mathbb{R}^{3dm}} K_t(u) e^{2\pi i u \xi} du.$$

Integrating by parts, using the derivative estimates up to order $3dm$ and bounding the size of the support of $\tilde{K}_t$ by an absolute constant times $\delta^{dm-1}$, we obtain the bound

$$|K_t(u)| \leq C(1 + \|u\|)^{-3dm}.$$  

(3.8)

Combining (3.7) and (3.8), and rescaling back, we see that it suffices to consider the multiplier

$$\int_{\mathbb{R}^{3dm}} (1 + \|u\|)^{-3dm} \left( \int_0^\infty (K_t(u)(1 + \|u\|)^{3dm})\hat{g}_{i,k_1,k_2}((I A)^T \xi) e^{2\pi i u t \xi} \frac{dt}{t} \right) du.$$

Using (2) of Lemma 3 at level $l$ to estimate the singular Brascamp-Lieb integral associated with the multiplier in the bracket for a fixed $u$ and integrating in $u$ we obtain the desired bound for (3.5).

It remains to consider the part (3.6). Here we fix $0 < s < 1$ and consider

$$\tilde{K}_s(\xi) := \int_0^\infty \tilde{K}_t(\xi) \psi(st \xi') \phi(t \xi^m) \frac{dt}{t}.$$

We will prove a bound on $\Lambda(K_s, A)$ that is proportional to $s$, so that we will be able to integrate against $ds/s$ and obtain a good bound for the form associated with (3.6).

Let $D$ be the $m \times m$ diagonal matrix with $d_{ii} = s$ for $i \leq l$ and $d_{ii} = 1$ for $i > l$. By (1) of Lemma 2 we have

$$\Lambda(K_s, A) = \Lambda(\tilde{K}_s, \tilde{A}),$$

where

$$\tilde{K}_s(\xi) = \det(D)^d K_s(D \xi), \quad \tilde{A} = D^{-1} AD.$$

Recall that the first $l$ rows of $A$ coincide with the first $l$ rows of $-I$, hence we may view $A$ as lower triangular block matrix relative to the splitting

$$\mathbb{R}^d \times \mathbb{R}^{d(m-l)}.$$

The matrix $\tilde{A}$ arises by multiplying the non-trivial off diagonal block by $s \leq 1$. Hence

$$\|\tilde{A}\|_{HS} \leq \|A\|_{HS} \leq 1/\epsilon, \quad |\det((I \tilde{A})\Pi_j)| = |\det((I A)\Pi_j)| > \epsilon.$$
We thus plan to apply (2) of Lemma 3 with the matrix $\tilde{A}$. We note
\[ \hat{K}_s(\xi) = \int_0^\infty \hat{K}_\gamma(D^{-1}\xi)\psi(t\xi')\phi(t\xi'')dt. \]
Now we fix in addition $t$ and rescale similarly to (3.7). We set
\[ \hat{K}_s(t^{-1}D^{-1}\xi)\psi(\xi')\phi(\xi'') =: \hat{K}_{t,s}(\xi)\hat{g}_{t,k_1k_2}((I \hat{A})^T \xi) \]
with some suitable $l + 1 \leq i \leq m$ and $1 \leq k_1, k_2 \leq d$ from Lemma 3. Similarly as in the discussion of (3.7), on the compact support of the left-hand side, $\xi' \sim 1$ and $\xi'' \sim 1$, the second factor on the right hand side is bounded below, so the function $\hat{K}_{t,s}$ is well defined. We now claim that
\[ |\partial^\alpha \hat{K}_{t,s}(\xi)| \leq sC \]
uniformly in $t$ for all multi-indices $\alpha$ up to order $3dm - 1$. To see this, we need to show the analogous estimate for the left hand side of (3.9). Applying a partial derivative on the left-hand side, we apply the Leibniz rule and consider the terms separately.

By $\partial_{ik}f$ we denote the $((i - 1)d + k)$-th partial derivative of a function $f$ on $\mathbb{R}^{nd}$, $1 \leq i \leq m, 1 \leq k \leq d$. If one derivative $\partial_{ik}$ with $i > l, 1 \leq k \leq d$, falls on $\hat{K}_\gamma(t^{-1}D^{-1}\xi)$, we estimate
\[ \partial_{ik}(\hat{K}_\gamma(t^{-1}D^{-1}\xi)) \leq Ct^{-1}(t^{-1}s^{-1}\|\xi'\| + t^{-1}\|\xi''\|)^{-1} \leq Cs \]
since both $\xi'$ and $\xi''$ can be assumed of unit length. Similarly we estimate if more than one derivative $\partial_{ik}$ with $i > l$ falls on $\hat{K}_\gamma(t^{-1}D^{-1}\xi)$. If no such derivative falls on $\hat{K}_\gamma(t^{-1}D^{-1}\xi)$, then only partial derivatives $\partial_{ik}$ with $i \leq l$ fall on $\hat{K}_\gamma(t^{-1}D^{-1}\xi)$. Restricting attention to one such derivative we use the vanishing condition (2.6) to obtain with the fundamental theorem of calculus
\[ \partial_{ik}(\hat{K}_\gamma(t^{-1}D^{-1}\xi)) = \partial_{ik} \int_0^1 \partial_h(\hat{K}_\gamma((ts)^{-1}\xi', ht^{-1}\xi''))dh \]
\[ = (ts)^{-1} \int_0^1 t^{-1}\xi'' \cdot (\nabla \partial_{ik} \hat{K}_\gamma)((ts)^{-1}\xi', ht^{-1}\xi'')dh, \]
where $\nabla$ denotes the gradient in the last $d(m - l)$ components. The desired estimate now follows through derivative estimates for $\hat{K}_\gamma$ with one degree higher than $|\alpha|$, note the gain of the factor $s$ comes from the length of $\xi''$ relative to the length of $\xi$ in the relevant support.

As before, we expand the Fourier integral
\[ \hat{K}_{t,s}(\xi) = \int_{\mathbb{R}^{dm}} K_{t,s}(u)e^{2\pi i u \cdot \xi}du \]
and we observe the bound
\[ |K_{t,s}(u)| \leq Cs(1 + \|u\|)^{-3dm+1}. \]
It suffices to consider the multiplier
\[ \int_{\mathbb{R}^{dm}} (1 + \|u\|)^{-3dm+1} \left( \int_0^\infty (K_{t,s}(u)(1 + \|u\|)^{3dm-1})\hat{g}_{t,k_1k_2}((I \hat{A})^T \tau \xi)e^{2\pi i u \tau \xi}d\tau \right)du. \]
We again apply (2) of Lemma 3 at level $l$ and integration in $v$ and $s$ to obtain the desired bound.
4. Proof of (2) of Lemma 3

Consider $m, d, l, \epsilon$ as in Lemma 3. We shall prove existence of a constant $C$ such that (2) holds, under the hypothesis that for the same $m, d$ but for $l$ replaced by $l+1$ and for $\epsilon$ replaced by possibly much smaller $\hat{\epsilon}$ depending on $d, m, \epsilon$, there is a constant $C$ such that (1) holds.

Let $A$ be as in Lemma 3. Recall that the first $l$ rows of $A$ coincide with the first $l$ rows of $-I$. We shall assume $l < m$ because the case $l = m$ is void. With $i, k_1, k_2$ as in (2) of Lemma 3 we need to estimate the form associated with the multiplier

$$\hat{K}(\xi) = \int_0^\infty c_t(u) g_{i,k_1,k_2}((I A)^T (t \xi)) e^{2\pi i u t \xi} \frac{dt}{t}.$$ 

Let us first compute the kernel and the form on the spatial side. We have

$$K((I A)x) = \int_{(\mathbb{R}^d)^m} \hat{K}(\xi) e^{2\pi i \xi \cdot ((I A)x)} d\xi$$

$$= \int_0^\infty c_t(u) \int_{(\mathbb{R}^d)^m} g_{i,k_1,k_2}((I A)^T (t \xi)) e^{2\pi i u t \xi} e^{2\pi i ((I A)^T \xi) \cdot x} dt d\xi,$$

where we write $f_t(\cdot) = t^{-2dm} f(t^{-1} \cdot)$ for a function $f$ in dimension $2dm$. The last equality is verified noting that the right-hand side is the integral of the function

$$y \mapsto (g_{i,k_1,k_2})_x (x + y + (ut, 0))$$

over the subspace $\{(-A^T I)^T p : p \in (\mathbb{R}^d)^m\}$, while the left-hand side is the integral of the Fourier transform of this function over the orthogonal subspace

$$\{(I A)^T \xi : \xi \in (\mathbb{R}^d)^m\}.$$

Using the definition of $g_{i,k_1,k_2}$ and Fubini, we obtain for the associated form $\Lambda(K, A)$

$$\int_0^\infty c_t(u) \int_{(\mathbb{R}^d)^m} \int_{(\mathbb{R}^d)^2m-2} \left( \prod_{j(i)=0} \left( \prod_{j(i)=1} F_j(\Pi_j x) \right) (\partial_{k_1} g)_t (x_0^i + (Ap + ut)_i) \right)$$

$$\left( \prod_{j(i)=1} F_j(\Pi_j x) \right) (\partial_{k_2} g)_t (x_1^i + p_i) \right) dx_i^0$$

$$g_t((x^0 - Ap + ut)_{h \neq i}, (x^1 + p)_{h \neq i}) \frac{dt}{t}. \quad (4.1)$$

We next prove a particular case of the desired inequality. The particular case is defined by the assumptions $1 \leq i \leq l + 1$, $k_1 = k_2 =: k$, $c_t = 1$ for all $t > 0$, $u = 0$, and in addition to the symmetries stated in the lemma, also $F_{(l+1)+j} = F_j$ for all $j \in Q$, and the $(l+1)$-st row of $A$ also coincides with the $(l+1)$-st row of $-I$. Note all assumptions are more specific than in (2) of Lemma 3 except that we on purpose allow $i \leq l$ here.

We then recognize that the first bracket in the last display becomes equal to the second bracket by the conditions on $i, u, A$, and the reflection symmetries of the tuple $(F_j)_{j \in Q}$. The two brackets therefore form a square. As Gaussians are positive and $c_t(u)$ is positive, the entire form becomes non-negative. This holds for all $1 \leq i \leq l + 1$ and all $k$. Therefore, instead of proving bounds for each of these terms, it suffices to estimate the sum of all these terms over $1 \leq i \leq l + 1$ and $k$, which has better algebraic properties.
To identify the good properties of this sum, note it is associated with the multiplier
\[ \hat{K}_\Sigma(\xi) := \sum_{i=1}^{l+1} \sum_{k=1}^{d} \int_0^\infty \hat{g}_{i,k,k}(\langle I \ A \rangle^T(t\xi)) \frac{dt}{t}. \]
We will add and subtract \( \pi \) from this multiplier. We will estimate by hand the form associated with \( \pi \), and we will apply the induction hypothesis to \( \hat{K}_\Sigma(\xi) - \pi \).

The form associated with \( \pi \) on the spatial side is \( \pi \) times
\[ \Lambda(\delta_0, A) = \int_{(\mathbb{R}^d)^{2m}} \left( \prod_{j \in Q} F_j(\Pi_j x) \right) \delta_0((I \ A)x) dx, \]
where \( \delta_0 \) denotes the Dirac delta distribution. This is a standard Brascamp-Lieb integral. Applying the arithmetic-geometric mean inequality at every point \( x \) and pulling the arithmetic mean out of the integral, we bound the last display by
\[ 2^{-m} \sum_{j \in Q} \left( \int_{(\mathbb{R}^d)^{2m}} F_j(\Pi_j x)^{2m} \delta_0((I \ A)x) dx \right). \]
This is an average over \( j \in Q \), and it suffices to prove bounds for fixed \( j \) as follows
\[
\begin{align*}
\int_{(\mathbb{R}^d)^{2m}} F_j(\Pi_j(x^0,x^1)^T)^{2m} &\delta_0(x^0+Ax^1)dx^0dx^1 \\
&= \int_{(\mathbb{R}^d)^m} F_j(\Pi_j(-Ax^1,x^1)^T)^{2m} dx^1 \\
&= |\det(\Pi_j(-A^T I)^T)|^{-d} \|F_j\|_{2m}^{2m} \leq \epsilon^{-d} \|F_j\|_{2m}^{2m}.
\end{align*}
\]
In the last inequality we used the assumption on \( A \) and that the absolute value of the determinant in this display is equal to
\[ |\det(\Pi_\ell(I \ A)^T)| = |\det((I \ A)\Pi_{-\ell}^T)| \geq \epsilon, \]
where \( \ell \) is the corner of the cube opposite to \( j \), that is \( j(i) + \ell(i) = 1 \) for all \( i \). This completes the bound for the multiplier \( \pi \).

To estimate the form associated with \( \hat{K}_\Sigma - \pi \), we apply (1) of Lemma 3 for \( l+1 \) and \( \epsilon \) to a suitably normalized kernel. Most assumptions of (1) are straightforward, the main difficulty is the vanishing condition (2.6). Using
\[ \hat{g}'(-\eta) = \hat{g}'(\eta), \quad \hat{g}(0) = 1 \]
for a Gaussian \( g \) on \( \mathbb{R} \) and the assumption that the first \( l+1 \) rows of \( A \) are equal to the first \( l+1 \) rows of \(-I\), we obtain
\[ \hat{K}_\Sigma(\xi_1, \ldots, \xi_{l+1}, 0, \ldots, 0) = \sum_{i=1}^{l+1} \sum_{k=1}^{d} \int_0^\infty |\hat{\partial}_k \hat{g}(t\xi_i)|^2 \left( \prod_{j=1, j \neq i}^{l+1} |\hat{g}(t\xi_j)|^2 \right) \frac{dt}{t}. \tag{4.2} \]
Observe the elementary identity
\[ -t \hat{\partial}_i |\hat{g}(t\eta)|^2 = \frac{1}{\pi} |\hat{g}'(t\eta)|^2 \tag{4.3} \]
valid for a one-dimensional Gaussian. Since \( \hat{g}(t\xi_i) \) is a product of one-dimensional Gaussians \( \hat{g}(t\xi_{i1}) \cdots \hat{g}(t\xi_{id}) \), together with the Leibniz rule the identity (4.3) implies
\[ -t \hat{\partial}_i |\hat{g}(t\xi_i)|^2 = \frac{1}{\pi} \sum_{k=1}^{d} |\hat{\partial}_k \hat{g}(t\xi_i)|^2. \tag{4.4} \]
By (4.4), the fundamental theorem of calculus and another application of the Leibniz rule, we equate (4.2) with
\[-\pi \int_0^\infty \theta_t \left( \prod_{j=1}^{l+1} \left| \hat{g}(t\xi_j) \right|^2 \right) \, dt = \pi.\]

This completes verification of (2.0) for \( \hat{K}_\Sigma - \pi \) and establishes the desired estimate for the associated form.

To round up the discussion, we present a derivation of the elementary identity (4.3) from the heat equation
\[\partial_t g_t(s) = \frac{t}{2\pi} \partial_s^2 g_t(s)\]
and the convolution identity
\[g_{\sqrt{2}}(s_1 - s_0) = \int_{\mathbb{R}} g_t(s_1 - p)g_t(s_0 - p) \, dp.\]
Indeed, integrating by parts in \( p \) we obtain
\[
\partial_t g_{\sqrt{2}}(s_1 - s_0) = \frac{t}{2\pi} \int_{\mathbb{R}} \partial^2_p g_t(s_1 - p)g_t(s_0 - p) \, dp + \frac{t}{2\pi} \int_{\mathbb{R}} g_t(s_1 - p)\partial^2_p g_t(s_0 - p) \, dp
\]
\[= -\frac{t}{\pi} \int_{\mathbb{R}} \partial_p g_t(s_1 - p)\partial_p g_t(s_0 - p) \, dp.\]
This can be turned into (4.3) by taking the Fourier transform.

We have completed the estimate of the form associated with (4.1) in the particular case. It remains to reduce the general case to the particular case. We will reduce to the particular case with \( A \) replaced by different matrices, which may satisfy (2.4) with different \( \tilde{c} \). These different \( \tilde{c} \) however only depend on \( m, d, \epsilon \).

We shall first reduce the general case to the case \( i \leq l + 1 \). This is done by a permutation of the coordinates if needed. If \( i > l + 1 \), let \( P \) be the involution that switches \( i \) and \( l + 1 \). Applying (2) of Lemma 2 reduces the to new data which still satisfy our assumptions of (2) of Lemma 3. Henceforth we assume \( i \leq l + 1 \).

Next, we symmetrize the tuple \( F_j \) and the pair \( k_1, k_2 \). We pull \( c_t(u) \) into one of the brackets, apply Cauchy-Schwarz, and then estimate \( c_t(u) \) by a constant. This bounds (4.1) by the geometric mean of
\[\int_0^\infty \int_{[\mathbb{R}^d]^m} \int_{[\mathbb{R}^d]^{2m-2}} \left( \int_{\mathbb{R}^d} \left( \prod_{j(i)=0} F_j(\Pi_j x) \right) (\partial_{k_1} g)_t(x^0_i + (-A p + u)t_i) \, dx^0_i \right)^2 \, dt \, dp \, dx^1_i\]
\[g_t((x^0 - Ap + ut)_{h \neq i}, (x^1 + p)_{h \neq i}) \, d((x^0)_{h \neq i}, (x^1)_{h \neq i}) \, dp \, dt / t \]
(4.5)
and
\[\int_0^\infty \int_{[\mathbb{R}^d]^m} \int_{[\mathbb{R}^d]^{2m-2}} \left( \int_{\mathbb{R}^d} \left( \prod_{j(i)=1} F_j(\Pi_j x) \right) (\partial_{k_2} g)_t(x^1_i + p_i) \, dx^1_i \right)^2 \, dt \, dp \, dx^0_i\]
\[g_t((x^0 - Ap + ut)_{h \neq i}, (x^1 + p)_{h \neq i}) \, d((x^0)_{h \neq i}, (x^1)_{h \neq i}) \, dp \, dt / t \]
(4.6)

It suffices to bound both terms separately and we begin with (4.6). To get rid of \( u \), we dominate a non-centered Gaussian by a centered Gaussian as in
\[g(s + v) \leq 10g \left( \frac{s}{2 + 2\|v\|} \right)\].
Let $v$ be the vector $u$ with the $i$-th $d$-dimensional component replaced by 0. Let $D$ the $m \times m$ diagonal matrix with $d_{hh} = 2(1 + \|v\|)$ for $h \neq i$, and $d_{ii} = 1$. Using the above domination we estimate (4.6) by
\[
\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left( \prod_{j(i)=0} F_j(\Pi_j x) \frac{1}{t} (D^{-1}x_1 + p_i) \, dx_0^0 \right)^2 g_t((D^{-1}x_0 - D^{-1}Ap)_{h\neq i}, (D^{-1}x_1 + D^{-1}p)_{h\neq i}) \, d((x_0)_{h\neq i}, (x_1)_{h\neq i}) \, dp \, dt \right).
\]
Replacing variables $p$ by $Dp$, $x^0$ by $Dx^0$, $x^1$ by $Dx^1$ and using $\tilde{F}_j$ as in (1) of Lemma 2 turns this into
\[
det(D)^{2d} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left( \prod_{j(i)=0} \tilde{F}_j(\Pi_j x) \frac{1}{t} (x_1^0 + p_i) \, dx_1^0 \right)^2 g_t((x_0^0 - D^{-1}Adp)_{h\neq i}, (x_1^0 + p)_{h\neq i}) \, d((x_0)_{h\neq i}, (x_1)_{h\neq i}) \, dp \, dt \right).
\]
To obtain the desired bound, it suffices to apply the particular case of (4.1) with the matrix $D^{-1}\tilde{A}D$ in place of $A$, where $\tilde{A}$ is the matrix whose $i$-th row is that of $-I$ and whose other rows equal those of $A$. In particular, the first $l + 1$ rows of the matrix $D^{-1}\tilde{A}D$ coincide with the first $l + 1$ rows of $-I$, and we have
\[
\|D^{-1}\tilde{A}D\|_{HS} \leq \|\tilde{A}\|_{HS} \leq \|A\|_{HS} + 1 \leq \epsilon^{-1} + 1,
\]
and
\[
|\det((I D^{-1}AD)\Pi_j^T)| \geq \inf_j |\det((I D^{-1}AD)\Pi_j^T)| > \epsilon.
\]
Note that we have the upper bound
\[
det(D)^{2d} \leq (1 + \|u\|)^{2d(m-1)},
\]
which is the additional factor in (2) of Lemma 3. This concludes the estimate of the term (4.6).

It remains to estimate the term (4.5). We reduce it to the previous case (4.6) by a $t$-dependent affine linear change of variables $\tilde{p} = -Ap + ut$.

This reduces (4.5) to
\[
|\det(A)|^{-d} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left( \prod_{j(i)=0} F_j(\Pi_j x) \frac{1}{t} (x_0^0 + \tilde{p}_i) \, dx_0^0 \right)^2 g_t((x_0^0 + \tilde{p})_{h\neq i}, (x_1^0 - A^{-1}\tilde{p} + A^{-1}ut)_{h\neq i}) \, d((x_0)_{h\neq i}, (x_1)_{h\neq i}) \, dp \, dt \right).
\]
Interchanging the roles of 0 and 1 in the range of $j$ reduces this to the previous case with an additional factor $|\det(A)|^{-d}$, $A$ replaced by $A^{-1}$ and with $u$ replaced by $A^{-1}u$. As the first $l + 1$ rows of $A^{-1}$ coincide with the first $l + 1$ rows of $-I$, it remains to show the conditions (2.3) for $A^{-1}$ for some $\epsilon$ depending on $\epsilon, m, d$.

The entries of $A^{-1}$ can be estimated by Cramer’s rule by
\[
\|A\|_{HS}^{m-1} \det(A)^{-1} \leq \epsilon^{-m}
\]
and hence
\[
\|A^{-1}\|_{HS} \leq m\epsilon^{-m}.
\]
Estimating the determinants of $(I A^{-1})\Pi_j^T$ in absolute value from below is tantamount to estimating determinants of submatrices of $A^{-1}$ obtained by deleting any number of
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pairs of matching rows and columns. Considering block decompositions with squares on the diagonal

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},
\]

we will show a lower bound on \( \det(X_{11}) \). The general case, when we delete arbitrary rows and columns of \( A^{-1} \) can be deduced similarly after permuting rows and columns.

Note that \( A_{22} \) is invertible, since (2.4) gives a lower bound on its determinant when choosing suitable \( \Pi_j \). We successively compute

\[
A_{21}X_{11} + A_{22}X_{21} = 0, \\
A_{12}A_{22}^{-1}A_{21}X_{11} + A_{12}X_{21} = 0, \\
A_{12}A_{22}^{-1}A_{21}X_{11} - A_{11}X_{11} = -I.
\]

A lower bound on \( \det(X_{11}) \) follows from an upper bound on the determinant of

\[
A_{12}A_{22}^{-1}A_{21} - A_{11}.
\]

Such bound follows from an upper bound on the norm of this matrix. Upper bounds on the norms of \( A_{12}, A_{22}, A_{11} \) are obtained using the bound on the Hilbert Schmidt norm of \( A \), while the bound on the norm of \( A_{22}^{-1} \) uses Cramer’s rule as above and the lower bound on the determinant of \( A_{22} \). Note finally that

\[
(1 + \|A^{-1}u\|)^{2d(m-1)} \leq (m^d\epsilon^{-md})^{2dm}(1 + \|u\|)^{2d(m-1)},
\]

which is up to a constant dominated by the factor in (2) of Lemma 3.

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REFERENCES


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