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Singular elliptic problems with lack of compactness

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Abstract. We consider the following nonlinear singular elliptic equation

$$-\mathrm{div}\,(|x|^{-2a}\nabla u) = K(x)|x|^{-bp}|u|^{p-2}u + \lambda g(x) \quad \text{in } \mathbb{R}^N,$$

where g belongs to an appropriate weighted Sobolev space and p denotes the Caffarelli–Kohn–Nirenberg critical exponent associated to a, b, and N. Under some natural assumptions on the positive potential $K(x)$ we establish the existence of some $\lambda_0 > 0$ such that the above problem has at least two distinct solutions provided that $\lambda \in (0, \lambda_0)$. The proof relies on Ekeland's variational principle and on the mountain pass theorem without the Palais–Smale condition, combined with a weighted variant of the Brezis–Lieb lemma.

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Key words. singular elliptic equation – perturbation – singular minimization problem – critical point – weighted Sobolev space

1. Introduction and the main result

Many papers have been devoted in recent decades to the study of degenerate elliptic problems. We start with the following example:

$$\begin{cases} \mathrm{div}\,(a(x)\nabla u) + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1}$$

where Ω is an arbitrary domain in $\mathbb{R}^N$ $(N \geq 1)$ and a is a nonnegative function that may have "essential" zeroes at some points or may even be unbounded. The continuous function f satisfies $f(0) = 0$ and $tf(t)$ behaves like $|t|^p$ as $|t| \to \infty$, with $2 < p < 2^*$, where 2^* denotes the critical Sobolev exponent. Notice that equations of this type come from the consideration of standing waves in anisotropic Schrödinger equations (see [2,20,21,25]). Equations like (1) are also introduced as models for several physical phenomena related to equilibrium of anisotropic media that possibly are somewhere *"perfect" insulators* or *"perfect" conductors* [10, p. 79]. Problem (1) also has some interest in the framework of optimization and G-convergence (see, e.g., [14] and references therein).

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Classical results [1,17] ensure the existence and the multiplicity of positive or nodal solutions for problem (1), provided that the differential operator $Tu := \operatorname{div}(a(x)\nabla u)$ is uniformly elliptic. Several difficulties occur both in the degenerate case (if $\inf_{\Omega} a = 0$) and in the singular case (if $\sup_{\Omega} a = +\infty$). In these situations, the classical methods fail to be applied directly so that the existence and the multiplicity results (which hold in the nondegenerate case) may become a delicate matter that is closely related to some phenomena due to the degenerate character of the differential equation. These problems have been intensively studied starting with the pioneering paper by Murthy and Stampacchia [15] (we also refer the reader to [8,13,16], as well as to the monograph [22]).

In concrete applications, it is natural to want to see what happens if these elliptic (degenerate or nondegenerate) problems are affected by a certain perturbation. It is worth pointing out here that the idea of using perturbation methods in the treatment of nonlinear boundary value problems was introduced by Struwe [23]. Recently, many authors have been interested in this kind of perturbation problem involving both critical and sub- or supercritical Sobolev exponents (see, e.g., [9,18,24]).

Our aim in this paper is to study the following degenerate perturbed problem:

$$-\operatorname{div}(|x|^{-2a}\nabla u) = K(x)|x|^{-bp}|u|^{p-2}u + \lambda g(x) \quad \text{in } \mathbb{R}^N, \tag{2}$$

where

$$\begin{cases} \text{for } N \geq 3: & -\infty < a < \dfrac{N-2}{2}, \quad a < b < a+1, \\ & \text{and} \quad p = \dfrac{2N}{N-2+2(b-a)}; \\ \text{for } N = 2: & -\infty < a < 0, \quad a < b < a+1, \quad \text{and} \quad p = \dfrac{2}{b-a}; \\ \text{for } N = 1: & -\infty < a < -\dfrac{1}{2}, \quad a + \dfrac{1}{2} < b < a+1, \\ & \text{and} \quad p = \dfrac{2}{-1+2(b-a)}. \end{cases} \tag{3}$$

Equation (2) contains the critical Caffarelli–Kohn–Nirenberg exponent p, defined as in (3). In this critical case, some concentration phenomena may occur, due to the action of the noncompact group of dilations in $\mathbb{R}^N$. The lack of compactness of problem (2) is also given by the fact that we are looking for entire solutions, that is, solutions defined on the whole space.

The reason we choose the parameters a, b, and p to satisfy the assumption (3) has to do with the following inequality, due to Caffarelli et al. [6]:

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}\,|u|^p dx\right)^{1/p} \leq C_{a,b}\left(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx\right)^{1/2}, \tag{4}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$, where a, b, and p satisfy condition (3). We point out that inequality (4) also holds for $b = a+1$ (if $N \geq 1$) and $b = a$ (if $N \geq 3$), but in these cases, the best Sobolev constant $C_{a,b}$ in (4) is never achieved (see [7] for details). The Caffarelli–Kohn–Nirenberg inequality (4) contains as particular cases the classical Sobolev inequality (if $a = b = 0$) and the Hardy inequality (if $a = 0$ and $b = 1$); we refer the reader to [4, 11, 19] for further details.

The extremal functions for (4) are ground state solutions of the singular Euler equation

$$-\text{div}\,(|x|^{-2a}\,\nabla u) = |x|^{-bp}\,|u|^{p-2}u, \qquad \text{in } \mathbb{R}^N .$$

This equation has been recently studied [7, 26] in connection with a complete understanding of the best constants, the qualitative properties of extremal functions, the existence (or nonexistence) of minimizers, and the symmetry properties of minimizers.

Function K is assumed to fulfill

$(K1)$ $K \in L^\infty(\mathbb{R}^N)$,
$(K2)$ $\text{esslim}_{|x|\to 0} K(x) = \text{esslim}_{|x|\to\infty} K(x) = K_0 \in (0, \infty)$ and $K(x) \geq K_0$ a.e. in $\mathbb{R}^N$,
$(K3)$ $\text{meas}\left(\{x \in \mathbb{R}^N : K(x) > K_0\}\right) > 0$.

Many authors have made contributions to the study of this problem, especially for the case $\lambda = 0$. The Palais–Smale condition PS plays a central role when variational methods are applied in the study of problem (2). In this paper, we establish the existence and the multiplicity of nontrivial solutions of (2) with $\lambda > 0$ sufficiently small, in a case where the PS condition is not assumed even for $\lambda = 0$. More precisely, we will show that there exists at least two weak solutions of (2) for $g \neq 0$ in an appropriate weighted Sobolev space and $\lambda > 0$ small enough. Our proof relies on Ekeland's variational principle [12] and on the mountain pass theorem without the Palais–Smale condition (in the sense of Brezis and Nirenberg, see [5]), combined with a weighted variant of the Brezis–Lieb Lemma [3].

The natural functional space to study problem (2) is $H_a^1(\mathbb{R}^N)$, defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 dx\right)^{1/2} . \tag{5}$$

It turns out that $H_a^1(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} |x|^{-2a}\nabla u \cdot \nabla v dx, \quad \forall\, u, v \in H_a^1(\mathbb{R}^N).$$

It follows that (4) holds for all $u \in H_a^1(\mathbb{R}^N)$. According to [7] we have

$$H_a^1(\mathbb{R}^N) = \overline{C_0^\infty\left(\mathbb{R}^N \setminus \{0\}\right)}^{\|\cdot\|}, \tag{6}$$

where $\|\cdot\|$ is given by (5). Let $\|\cdot\|_{-1}$ denote the norm in the dual space $H_a^{-1}(\mathbb{R}^N)$ of $H_a^1(\mathbb{R}^N)$.

Throughout this paper we suppose that $g \in H_a^{-1}(\mathbb{R}^N) \setminus \{0\}$.

For an arbitrary open set $\Omega \subset \mathbb{R}^N$, let $L_b^p(\Omega)$ be the space of all measurable real functions u defined on Ω such that $\int_\Omega |x|^{-bp}\,|u|^p dx$ is finite. By (4) it follows that the weighted Sobolev space $H_a^1(\Omega)$ is continuously embedded in $L_b^p(\Omega)$.

Definition 1. We say that a function $u \in H_a^1(\mathbb{R}^N)$ is a weak solution of problem (2) if

$$\int_{\mathbb{R}^N} |x|^{-2a}\nabla u \cdot \nabla v\,dx - \int_{\mathbb{R}^N} K(x)|x|^{-bp}\,|u|^{p-2}uv\,dx - \lambda \int_{\mathbb{R}^N} g(x)v\,dx = 0,$$

for all $u \in C_0^\infty(\mathbb{R}^N)$.

Obviously, the solutions of problem (2) correspond to critical points of the energy functional

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} K(x)|x|^{-bp}\,|u|^p\,dx - \lambda \int_{\mathbb{R}^N} g(x)u\,dx,$$

where $u \in H_a^1(\mathbb{R}^N)$.

Our main result is the following.

Theorem 1. *Suppose that assumptions* $(K1)$, $(K2)$, $(K3)$ *are fulfilled, and fix* $g \in H_a^{-1}(\mathbb{R}^N) \setminus \{0\}$. *Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$, *problem (2) has at least two solutions.*

Since the embedding $H_a^1(\mathbb{R}^N) \hookrightarrow L_b^p(\mathbb{R}^N)$ is not compact, the energy functional J_λ fails to satisfy the PS condition. Such a failure makes it difficult to apply a variational approach to (2). Furthermore, since $g \not\equiv 0$, then 0 is no longer a trivial solution of problem (2), and therefore the mountain pass theorem cannot be applied directly. Using ideas developed in [24], we obtain the first solution by applying Ekeland's variational principle. Then, the mountain pass theorem without the PS condition yields a bounded PS sequence whose weak limit is a critical point of J_λ. The proof is concluded by showing that these two solutions are distinct because they realize different energy levels.

The paper is organized as follows. In Section 2 we give some technical results that allow us to give a variational approach of our main result, which we prove in Section 3. We point out that since the perturbation term g is not assumed to be nonnegative, we cannot expect that the distinct solutions given by Theorem 1 will be positive. However, if $g \geq 0$ is a nontrivial perturbation, then a straightforward argument based on the maximum principle implies that the solutions of problem (2) are positive.

Notations. Throughout this paper we will denote by B_R the open ball in $H_a^1(\mathbb{R}^N)$ centered at the origin and having radius $R > 0$. We also denote by $\langle \cdot\,,\,\cdot \rangle$ the duality pairing between $H_a^1(\mathbb{R}^N)$ and $H_a^{-1}(\mathbb{R}^N)$. The notations "$\rightharpoonup$" and "$\rightarrow$" stand, respectively, for the weak and the strong convergence in an arbitrary Banach space.

2. Auxiliary results

Define the functionals $J_0, I : H_a^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} K(x)|x|^{-bp}\,|u|^p\,dx,$$
$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx - \frac{1}{p}\int_{\mathbb{R}^N} K_0|x|^{-bp}\,|u|^p\,dx.$$

The Caffarelli–Kohn–Nirenberg inequality (4) and the conditions $(K1)$, $(K2)$ imply that the functionals J_λ, J_0, and I are well defined and $J_\lambda, J_0, I \in C^1(H_a^1(\mathbb{R}^N), \mathbb{R})$.

Remark 1. If $\Omega \subset \mathbb{R}^N$ is a smooth bounded set such that $0 \notin \overline{\Omega}$, then, by the Sobolev inequality, we have

$$\left(\int_\Omega |x|^{-bp}\,|u|^p dx\right)^{1/p} \le C_1\left(\int_\Omega |u|^p dx\right)^{1/p} \le C_2\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}$$
$$\le C_3\left(\int_\Omega |x|^{-2a}|\nabla u|^2 dx\right)^{1/2},$$

for all $u \in H_a^1(\Omega)$. It follows that $H_a^1(\Omega)$ is compactly embedded in $L_b^p(\Omega)$.

Inequality (4) implies that if $\{u_n\}$ is a sequence that converges weakly to some u_0 in $H_a^1(\mathbb{R}^N)$, then $\{u_n\}$ is bounded in $L_b^p(\mathbb{R}^N)$. Therefore, we can assume (up to a sequence) that

$$u_n \rightharpoonup u_0 \ \text{ in } L^p_{b,\,\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) \quad \text{and} \quad u_n \to u_0 \ \text{a.e. in } \mathbb{R}^N. \tag{7}$$

Definition 2. Let X be a Banach space, $F : X \to \mathbb{R}$ be a C^1–functional, and c be a real number. A sequence $\{u_n\} \subset X$ is called a $(PS)_c$ sequence of F if $F(u_n) \to c$ and $\|F'(u_n)\|_{X^*} \to 0$.

Our first result shows that if a $(PS)_c$ sequence of J_λ is weakly convergent, then its limit is a solution of problem (2).

Lemma 1. *Let $\{u_n\} \subset H_a^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence of J_λ for some $c \in \mathbb{R}$. Suppose that $\{u_n\}$ converges weakly to some u_0 in $H_a^1(\mathbb{R}^N)$. Then u_0 is a solution of problem (2).*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ be an arbitrary function, and set $\Omega := \operatorname{supp}\varphi$. Since $J'_\lambda(u_n) \to 0$ in $H_a^{-1}(\mathbb{R}^N)$, we obtain $\langle J'_\lambda(u_n), \varphi\rangle \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to\infty}\left(\int_\Omega |x|^{-2a}\nabla u_n \cdot \nabla\varphi\,dx - \int_\Omega K(x)|x|^{-bp}\,|u_n|^{p-2}u_n\varphi\,dx - \lambda\int_\Omega g(x)\varphi\,dx\right) = 0. \tag{8}$$

Since $u_n \rightharpoonup u_0$ in $H^1_a(\mathbb{R}^N)$, it follows that

$$\lim_{n\to\infty}\int_\Omega |x|^{-2a}\nabla u_n\cdot\nabla\varphi\,dx=\int_\Omega |x|^{-2a}\nabla u_0\cdot\nabla\varphi\,dx. \tag{9}$$

The boundedness of $\{u_n\}$ in $H^1_a(\mathbb{R}^N)$ and the Caffarelli–Kohn–Nirenberg inequality imply that $\{|u_n|^{p-2}u_n\}$ is bounded in $L_b^{p/p-1}(\mathbb{R}^N)$. Since $|u_n|^{p-2}u_n\to |u_0|^{p-2}u_0$ a.e. in $\mathbb{R}^N$ (which is a consequence of (7)), we deduce that $|u_0|^{p-2}u_0$ is the weak limit in $L_b^{p/p-1}(\mathbb{R}^N)$ of the sequence $\{|u_n|^{p-2}u_n\}$. Therefore,

$$\lim_{n\to\infty}\int_\Omega K(x)|x|^{-bp}\,|u_n|^{p-2}u_n\varphi\,dx=\int_\Omega K(x)|x|^{-bp}\,|u_0|^{p-2}u_0\varphi\,dx. \tag{10}$$

Consequently, relations (8)–(10) yield

$$\int_\Omega |x|^{-2a}\nabla u_0\cdot\nabla\varphi\,dx-\int_\Omega K(x)|x|^{-bp}\,|u_0|^{p-2}u_0\varphi\,dx-\lambda\int_\Omega g(x)\varphi\,dx=0.$$

By virtue of (6) we deduce that the above equality holds for all $\varphi\in H^1_a(\mathbb{R}^N)$, which means that $J'_\lambda(u_0)=0$. The proof of our lemma is now complete. □

We now establish a weighted variant of the Brezis–Lieb lemma (see [3]).

Lemma 2. *Let $\{u_n\}$ be a sequence that is weakly convergent to u_0 in $H^1_a(\mathbb{R}^N)$. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|x|^{-bp}\left(|u_n|^p-|u_n-u_0|^p\right)dx=\int_{\mathbb{R}^N} K(x)|x|^{-bp}\,|u_0|^p\,dx.$$

Proof. Using the boundedness of $\{u_n\}$ in $H^1_a(\mathbb{R}^N)$ and the Caffarelli–Kohn–Nirenberg inequality, it follows that the sequence $\{u_n\}$ is bounded in $L^p_b(\mathbb{R}^N)$. Let $\varepsilon>0$ be a positive real number. By $(K1)$ and $(K2)$ we can choose $R_\varepsilon>r_\varepsilon>0$ such that

$$\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx<\varepsilon, \tag{11}$$

and

$$\int_{|x|>R_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx<\varepsilon. \tag{12}$$

Denote $\Omega_\varepsilon=\overline{B(0,R_\varepsilon)}\setminus B(0,r_\varepsilon)$. We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} K(x)|x|^{-bp}\left(|u_n|^p-|u_0|^p-|u_n-u_0|^p\right)dx\right|\\
&\le\left|\int_{\Omega_\varepsilon} K(x)|x|^{-bp}\left(|u_n|^p-|u_0|^p\right)dx\right|+\int_{\Omega_\varepsilon} K(x)|x|^{-bp}\,|u_n-u_0|^p\,dx\\
&\quad+\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx+\left|\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\left(|u_n|^p-|u_n-u_0|^p\right)dx\right|\\
&\quad+\int_{|x|>R_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx+\left|\int_{|x|>R_\varepsilon} K(x)|x|^{-bp}\left(|u_n|^p-|u_n-u_0|^p\right)dx\right|.
\end{aligned}$$

By the Lagrange mean value theorem we have

$$\begin{aligned}\int_{|x|<r_\varepsilon} & K(x)|x|^{-bp}\left(|u_n|^p - |u_n-u_0|^p\right)dx \\ &= p\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|\theta u_0 + (u_n-u_0)|^{p-1}|u_0|\,dx\,, \qquad (13)\end{aligned}$$

where $0<\theta(x)<1$. Next, we employ the following elementary inequality: for all $s>0$, there exists a constant $c=c(s)$ such that

$$(x+y)^s \le c(x^s+y^s) \quad \text{for any} \ \ x,y\in(0,\infty).$$

Then, by Hölder's inequality and relation (11) we deduce that

$$\begin{aligned}\int_{|x|<r_\varepsilon} & K(x)|x|^{-bp}\,|\theta u_0+(u_n-u_0)|^{p-1}|u_0|\,dx \\ &\le c\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\left(|u_0|^p + |u_n-u_0|^{p-1}|u_0|\right)dx \\ &= c\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx + c\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|u_n-u_0|^{p-1}|u_0|\,dx \\ &\le c\,\varepsilon + c\left(\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|u_n-u_0|^p\,dx\right)^{(p-1)/p} \\ &\quad\times\left(\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx\right)^{1/p} \\ &\le c_1\,(\varepsilon+\varepsilon^{1/p}),\end{aligned}$$

where the constant c_1 is independent of n and ε. Using relation (13) we have

$$\begin{aligned}\int\limits_{|x|<r_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx + \left|\int_{|x|<r_\varepsilon} K(x)|x|^{-bp}\left(|u_n|^p-|u_n-u_0|^p\right)dx\right| \\ \le p\,\tilde{c}_1\,(\varepsilon+\varepsilon^{1/p}). \qquad (14)\end{aligned}$$

In a similar manner we obtain

$$\begin{aligned}\int\limits_{|x|>R_\varepsilon} K(x)|x|^{-bp}\,|u_0|^p\,dx + \left|\int_{|x|>R_\varepsilon} K(x)|x|^{-bp}\left(|u_n|^p-|u_n-u_0|^p\right)dx\right| \\ \le p\,\tilde{c}_2\,(\varepsilon+\varepsilon^{1/p}). \qquad (15)\end{aligned}$$

Since $u_n \rightharpoonup u_0$ in $H^1_a(\mathbb{R}^N)$, relation (7) implies

$$\begin{aligned}\lim_{n\to\infty}\int_{\Omega_\varepsilon} K(x)|x|^{-bp}\left(|u_n|^p-|u_0|^p\right)dx = 0, \\ \lim_{n\to\infty}\int_{\Omega_\varepsilon} K(x)|x|^{-bp}\,|u_n-u_0|^p\,dx = 0.\end{aligned} \qquad (16)$$

Now, relations (14)–(16) yield

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N} K(x)|x|^{-bp}\left(|u_n|^p-|u_0|^p-|u_n-u_0|^p\right)dx\right|\le (pC+1)\,(\varepsilon+\varepsilon^{1/p}).$$

Since $\varepsilon>0$ is arbitrary, it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|x|^{-bp}\left(|u_n|^p-|u_n-u_0|^p\right)dx=\int_{\mathbb{R}^N} K(x)|x|^{-bp}\,|u_0|^p\,dx.$$

This concludes the proof. □

Lemma 3. *Let $\{v_n\}$ be a sequence that converges weakly to 0 in $H_a^1(\mathbb{R}^N)$. Then the following properties hold:*

$$\lim_{n\to\infty}[J_\lambda(v_n)-I(v_n)]=0,$$
$$\lim_{n\to\infty}[\langle J_\lambda'(v_n),v_n\rangle-\langle I'(v_n),v_n\rangle]=0.$$

Proof. A simple computation yields

$$J_\lambda(v_n)=I(v_n)-\frac{1}{p}\int_{\mathbb{R}^N}(K(x)-K_0)|x|^{-bp}\,|v_n|^p\,dx-\lambda\int_{\mathbb{R}^N} g(x)v_n\,dx,$$
$$\langle J_\lambda'(v_n),v_n\rangle=\langle I'(v_n),v_n\rangle-\int_{\mathbb{R}^N}(K(x)-K_0)|x|^{-bp}\,|v_n|^p\,dx-\lambda\int_{\mathbb{R}^N} g(x)v_n\,dx.$$

Since $v_n\rightharpoonup 0$ in $H_a^1(\mathbb{R}^N)$, it follows from the above equalities that it suffices to prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(K(x)-K_0)|x|^{-bp}\,|v_n|^p\,dx=0. \tag{17}$$

Fix $\varepsilon>0$. By our assumptions $(K1)$ and $(K2)$, there exists $R_\varepsilon>r_\varepsilon>0$ such that

$$|K(x)-K_0|=K(x)-K_0<\varepsilon\quad\text{for a.e.}\quad x\in\mathbb{R}^N\setminus\Omega_\varepsilon,$$

where $\Omega_\varepsilon=\overline{B(0,R_\varepsilon)}\setminus B(0,r_\varepsilon)$. Next, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}(K(x)-K_0)|x|^{-bp}\,|v_n|^p\,dx\\
&\quad=\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}(K(x)-K_0)|x|^{-bp}\,|v_n|^p\,dx+\int_{\Omega_\varepsilon}(K(x)-K_0)|x|^{-bp}\,|v_n|^p\,dx\\
&\quad\le\varepsilon\int_{\mathbb{R}^N\setminus\Omega_\varepsilon}|x|^{-bp}\,|v_n|^p\,dx+(\|K\|_\infty-K_0)\int_{\Omega_\varepsilon}|x|^{-bp}\,|v_n|^p\,dx\\
&\quad\le\varepsilon\int_{\mathbb{R}^N}|x|^{-bp}\,|v_n|^p\,dx+(\|K\|_\infty-K_0)\int_{\Omega_\varepsilon}|x|^{-bp}\,|v_n|^p\,dx.
\end{aligned}$$

Since $v_n \rightharpoonup 0$ in $H^1_a(\mathbb{R}^N)$, the Caffarelli–Kohn–Nirenberg inequality implies that $\{v_n\}$ is bounded in $L^p_b(\mathbb{R}^N)$. Moreover, by (7) it follows that $v_n \to 0$ in $L^p_{b,\,\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$. The above relations yield

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} (K(x) - K_0)|x|^{-bp}\,|v_n|^p\,dx \le C\varepsilon$$

for some constant $C > 0$ independent of n and ε. Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that (17) holds, and the proof of Lemma 3 is now complete. □

Lemma 4. *There exists* $\lambda_1 > 0$ *and* $R = R(\lambda_1) > 0$ *such that for all* $\lambda \in (0, \lambda_1)$*, the functional* J_λ *admits a* $(PS)_{c_{0,\lambda}}$ *sequence with* $c_{0,\lambda} = c_{0,\lambda}(R) = \inf_{u\in \overline{B}_R} J_\lambda(u)$*. Moreover,* $c_{0,\lambda}$ *is achieved by some* $u_0 \in H^1_a(\mathbb{R}^N)$ *with* $J'_\lambda(u_0) = 0$*.*

Proof. Fix $\lambda \in (0,1)$. For all $u \in H^1_a(\mathbb{R}^N)$, the assumption (K_1) and the Caffarelli–Kohn–Nirenberg inequality imply

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} K(x)|x|^{-bp}\,|u|^p\,dx - \lambda\int_{\mathbb{R}^N} g(x)u\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{\|K\|_\infty}{p}\,C^p_{a,b}\|u\|^p - \lambda\|g\|_{-1}\|u\|. \end{aligned}$$

We now apply the inequality $\alpha\beta \le \dfrac{\alpha^2+\beta^2}{2}$, for any $\alpha, \beta \ge 0$. Hence

$$J_\lambda(u) \ge \frac{1-\lambda}{2}\|u\|^2 - \frac{\|K\|_\infty}{p}\,C^p_{a,b}\,\|u\|^p - \frac{\lambda}{2}\|g\|^2_{-1}. \tag{18}$$

Since $p > 2$ and the right side of (18) is a decreasing function on λ, we find $\lambda_1 > 0$ and $R = R(\lambda_1) > 0$, $\delta = \delta(\lambda_1) > 0$ such that

$$J_\lambda(u) \ge -\frac{\lambda}{2}\|g\|^2_{-1}, \qquad \text{for all } u \in \overline{B}_R \quad \text{and } \lambda \in (0,\lambda_1) \tag{19}$$

and

$$J_\lambda(u) \ge \delta > 0, \qquad \text{for all } u \in \partial B_R \quad \text{and } \lambda \in (0,\lambda_1). \tag{20}$$

For instance, we can take

$$\lambda_1 := \min\left\{\frac{1}{2}, \frac{1}{2\|g\|^2_{-1}}\left(\frac{1}{2} - \frac{1}{p}\right) r_0^2\right\},$$

$$r_0 := \left[\frac{1}{2\|K\|_\infty C^p_{a,b}}\right]^{1/(p-2)}, \qquad R := \left[\frac{1-\lambda_1}{\|K\|_\infty C^p_{a,b}}\right]^{1/(p-2)}$$

and

$$\delta(\lambda_1) := \frac{\lambda_1}{2}\|g\|^2_{-1}.$$

Using now estimate (18) we easily deduce (19) and (20).

Next, we define $c_{0,\lambda} := c_{0,\lambda}(R) = \inf\{J_\lambda(u)\,;\ u \in \overline{B}_R\}$. We first note that $c_{0,\lambda} \le J_\lambda(0) = 0$. The set $\overline{B}_R$ becomes a complete metric space with respect to the distance

$$\mathrm{dist}(u, v) = \|u - v\|\,, \qquad \text{for any } \ u, v \in \overline{B}_R.$$

The functional J_λ is lower semicontinuous and bounded from below on $\overline{B}_R$. Then, by Ekeland's variational principle [12, Theorem 1.1], for any positive integer n there exists u_n such that

$$c_{0,\lambda} \le J_\lambda(u_n) \le c_{0,\lambda} + \frac{1}{n} \tag{21}$$

and

$$J_\lambda(w) \ge J_\lambda(u_n) - \frac{1}{n}\|u_n - w\| \quad \text{for all } \ w \in \overline{B}_R. \tag{22}$$

We first show that $\|u_n\| < R$ for n large enough. Indeed, if not, then $\|u_n\| = R$ for infinitely many n, and so (up to a subsequence) we can assume that $\|u_n\| = R$ for all $n \ge 1$. It follows that $J_\lambda(u_n) \ge \delta > 0$. Using (21) and letting $n \to \infty$, we have $0 \ge c_{0,\lambda} \ge \delta > 0$, which is a contradiction.

We now claim that $J'_\lambda(u_n) \to 0$ in $H_a^{-1}(\mathbb{R}^N)$. Fix $u \in H_a^1(\mathbb{R}^N)$ with $\|u\| = 1$ and let $w_n = u_n + tu$. For some fixed n, we have $\|w_n\| \le \|u_n\| + t < R$ if $t > 0$ is small enough. Then relation (22) yields

$$J_\lambda(u_n + tu) \ge J_\lambda(u_n) - \frac{t}{n}\|u\|\,,$$

that is,

$$\frac{J_\lambda(u_n + tu) - J_\lambda(u_n)}{t} \ge -\frac{1}{n}\|u\| = -\frac{1}{n}.$$

Letting $t \searrow 0$, it follows that $\langle J'_\lambda(u_n), u\rangle \ge -\dfrac{1}{n}$. Arguing in a similar way for $t \nearrow 0$, we obtain $\langle J'_\lambda(u_n), u\rangle \le \dfrac{1}{n}$. Since $u \in H_a^1(\mathbb{R}^N)$ with $\|u\| = 1$ has been arbitrarily chosen, we have

$$\|J'_\lambda(u_n)\| = \sup_{\substack{u \in H_a^1(\mathbb{R}^N),\\ \|u\|=1}} |\langle J'_\lambda(u_n), u\rangle| \le \frac{1}{n} \to 0 \quad \text{as } \ n \to \infty.$$

We have proved the existence of a $(PS)_{c_{0,\lambda}}$ sequence, i.e., a sequence $\{u_n\} \subset H_a^1(\mathbb{R}^N)$ with

$$J_\lambda(u_n) \to c_{0,\lambda} \quad \text{and} \quad J'_\lambda(u_n) \to 0 \ \text{ in } \ H_a^1(\mathbb{R}^N). \tag{23}$$

Since $\|u_n\| \le R$, it follows that $\{u_n\}$ converges weakly (up to a subsequence) in $H_a^1(\mathbb{R}^N)$ to some u_0. Moreover, relations (7) and (23) and Remark 1 yield

$$u_n \rightharpoonup u_0 \quad \text{in } \ H_a^1(\mathbb{R}^N), \qquad u_n \to u_0 \quad \text{a.e. in } \ \mathbb{R}^N \tag{24}$$

and

$$J'_\lambda(u_0) = 0. \tag{25}$$

Next, we prove that $J_\lambda(u_0) = c_{0,\lambda}$. Using relations (23) and (24) we have

$$\begin{aligned} o(1) &= \langle J'_\lambda(u_n), u_n \rangle \\ &= \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^N} K(x)|x|^{-bp} \, |u_n|^p \, dx - \lambda \int_{\mathbb{R}^N} g(x) u_n \, dx. \end{aligned}$$

Therefore,

$$J_\lambda(u_n) = \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_n|^p \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x) u_n \, dx + o(1).$$

Hence

$$J_\lambda(u_0) = \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x) u_0 \, dx + o(1).$$

Fatou's lemma and relations (23)–(25) imply

$$\begin{aligned} c_{0,\lambda} &= \liminf_{n \to \infty} J_\lambda(u_n) \geq \left(\frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^N} K(x)|x|^{-bp} |u_0|^p \, dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} g(x) u_0 \, dx \\ &= J_\lambda(u_0). \end{aligned}$$

Thus, $c_{0,\lambda} \geq J_\lambda(u_0)$. On the other hand, since $u_0 \in \overline{B}_R$, we deduce that $J_\lambda(u_0) \geq c_{0,\lambda}$, so $J_\lambda(u_0) = c_{0,\lambda}$. This concludes the proof of Lemma 4. □

3. Proof of Theorem 1

Define

$$\mathcal{S} = \left\{ u \in H^1_a(\mathbb{R}^N) \setminus \{0\} \, ; \, \langle I'(u), u \rangle = 0 \right\}.$$

We claim that $\mathcal{S} \neq \emptyset$. For this purpose we fix $u \in H^1_a(\mathbb{R}^N) \setminus \{0\}$ and set, for any $\lambda > 0$,

$$\Psi(\lambda) = \langle I'(\lambda u), \lambda u \rangle = \lambda^2 \int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx - \lambda^p \int_{\mathbb{R}^N} K_0 |x|^{-bp} \, |u|^p \, dx.$$

Since $p > 2$, it follows that $\Psi(\lambda) < 0$ for λ large enough and $\Psi(\lambda) > 0$ for λ sufficiently close to the origin. So, there exists $\lambda > 0$ such that $\Psi(\lambda) = 0$, that is, $\lambda u \in \mathcal{S}$.

Proposition 1. *Let* $I_\infty := \inf\{ I(u) \, ; \, u \in \mathcal{S}\}$. *Then there exists* $\bar{u} \in H^1_a(\mathbb{R}^N)$ *such that*

$$I_\infty = I(\bar{u}) = \sup_{t \geq 0} I(t\bar{u}). \tag{26}$$

Proof. For some fixed $\varphi \in H^1_a(\mathbb{R}^N) \setminus \{0\}$ denote

$$f(t) = I(t\varphi) = \frac{t^2}{2}\int_{\mathbb{R}^N} |x|^{-2a}|\nabla\varphi|^2\,dx - \frac{K_0}{p}\,t^p\int_{\mathbb{R}^N} |x|^{-bp}\,|\varphi|^p\,dx.$$

We have

$$f'(t) = t\int_{\mathbb{R}^N} |x|^{-2a}|\nabla\varphi|^2\,dx - K_0\,t^{p-1}\int_{\mathbb{R}^N} |x|^{-bp}\,|\varphi|^p\,dx.$$

Then f attains its maximum at

$$t_0 = t_0(\varphi) := \left\{\frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2a}|\nabla\varphi|^2\,dx}{\displaystyle\int_{\mathbb{R}^N} K_0|x|^{-bp}|\varphi|^p\,dx}\right\}^{1/(p-2)}.$$

Hence

$$f(t_0) = I(t_0\varphi) = \sup_{t\geq 0} I(t\varphi) = \left(\frac{1}{2} - \frac{1}{p}\right)\left\{\frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2a}|\nabla\varphi|^2\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} K_0|x|^{-bp}|\varphi|^p\,dx\right)^{2/p}}\right\}^{p/(p-2)}.$$

It follows that

$$\inf_{\varphi\in H^1_0(\mathbb{R}^N)\setminus\{0\}}\ \sup_{t\geq 0} I(t\varphi) = \left(\frac{1}{2} - \frac{1}{p}\right)[S(a,b)]^{p/(p-2)}, \tag{27}$$

where

$$S(a,b) = \inf_{\varphi\in H^1_0(\mathbb{R}^N)\setminus\{0\}}\left\{\frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2a}|\nabla\varphi|^2\,dx}{\left(\displaystyle\int_{\mathbb{R}^N} K_0|x|^{-bp}|\varphi|^p\,dx\right)^{2/p}}\right\}. \tag{28}$$

We now easily observe that for every $u \in \mathcal{S}$, we have $t_0(u) = 1$, so by (27) it follows that

$$I(u) = \sup_{t\geq 0} I(tu) \quad \text{for all}\ \ u \in \mathcal{S}. \tag{29}$$

According to [7, Theorems 1.2, 7.2, 7.6], the infimum in (28) is achieved by a function $U \in H^1_a(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} K_0|x|^{-bp}|U|^p dx = 1$. Letting $\bar{u} = [S(a,b)]^{1/(p-2)}\,U$, we see that $\bar{u} \in \mathcal{S}$ and

$$I(\bar{u}) = \left(\frac{1}{2} - \frac{1}{p}\right)[S(a,b)]^{p/(p-2)}. \tag{30}$$

Relations (29) and (30) yield

$$I_\infty = \inf_{u\in\mathcal{S}} I(u) = \inf_{u\in\mathcal{S}} \sup_{t\geq 0} I(tu) \geq \inf_{u\in H_0^1(\mathbb{R}^N)\setminus\{0\}} \sup_{t\geq 0} I(tu)$$
$$= \left(\frac{1}{2}-\frac{1}{p}\right)[S(a,b)]^{p/(p-2)} = I(\bar{u}),$$

which concludes our proof. □

Proposition 2. *Assume that $\{u_n\}$ is a $(PS)_c$ sequence of J_λ that is weakly convergent in $H_a^1(\mathbb{R}^N)$ to some u_0. Then the following alternative holds: either $\{u_n\}$ converges strongly in $H_a^1(\mathbb{R}^N)$, or $c \geq J_\lambda(u_0) + I_\infty$.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence and $u_n \rightharpoonup u_0$ in $H_a^1(\mathbb{R}^N)$, we have

$$J_\lambda(u_n) = c + o(1) \quad \text{and} \quad \langle J_\lambda'(u_n), u_n\rangle = o(1). \tag{31}$$

Denote $v_n = u_n - u_0$. It follows that $v_n \rightharpoonup 0$ in $H_a^1(\mathbb{R}^N)$, which implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{-2a} \nabla v_n \cdot \nabla u_0 \, dx = 0,$$
$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x) v_n \, dx = 0.$$

The above relations imply

$$\begin{aligned} \|u_n\|^2 &= \|u_0\|^2 + \|v_n\|^2 + o(1) \\ J_\lambda(v_n) &= J_0(v_n) + o(1). \end{aligned} \tag{32}$$

Using Lemmas 1–3 and relations (31) and (32) we deduce that

$$o(1) + c = J_\lambda(u_n) = J_\lambda(u_0) + J_\lambda(v_n) + o(1) = J_\lambda(u_0) + I(v_n) + o(1), \tag{33}$$
$$\begin{aligned} o(1) &= \langle J_\lambda'(u_n), u_n\rangle = \langle J_\lambda'(u_0), u_0\rangle + \langle J_\lambda'(v_n), v_n\rangle + o(1) \\ &= \langle I'(v_n), v_n\rangle + o(1). \end{aligned} \tag{34}$$

If $v_n \to 0$ in $H_a^1(\mathbb{R}^N)$, then $u_n \to u_0$ in $H_a^1(\mathbb{R}^N)$. It follows that $J_\lambda(u_0) = \lim_{n\to\infty} J_\lambda(u_n)$. If $v_n \not\to 0$ in $H_a^1(\mathbb{R}^N)$, using the fact that $v_n \rightharpoonup 0$ in $H_a^1(\mathbb{R}^N)$, we can asume that $\|v_n\| \to l > 0$.

By virtue of (33), it remains only to show that $I(v_n) \geq I_\infty + o(1)$. Taking $t > 0$ we have

$$\langle I'(tv_n), tv_n\rangle = t^2 \int_{\mathbb{R}^N} |x|^{-2a} |\nabla v_n|^2 \, dx - t^p K_0 \int_{\mathbb{R}^N} |x|^{-bp} |v_n|^p \, dx.$$

If we prove the existence of a sequence $\{t_n\} \subset (0, \infty)$ with $t_n \to 1$ and $\langle I'(t_n v_n), t_n v_n\rangle = 0$, then $t_n v_n \in \mathcal{S}$. This implies that

$$\begin{aligned} I(v_n) &= I(t_n v_n) + \frac{1-t_n^2}{2}\|v_n\|^2 - \frac{1-t_n^p}{p} K_0 \int_{\mathbb{R}^N} |x|^{-bp} |v_n|^p \, dx \\ &= I(t_n v_n) + o(1) \geq I_\infty + o(1), \end{aligned}$$

and the conclusion follows. For this purpose, we denote

$$\alpha_n = \int_{\mathbb{R}^N} |x|^{-2a}|\nabla v_n|^2\,dx = \|v_n\|^2 \geq 0,$$
$$\beta_n = K_0 \int_{\mathbb{R}^N} |x|^{-bp}|v_n|^p\,dx \geq 0,$$
$$\mu_n = \alpha_n - \beta_n.$$

From (34) it follows that $\mu_n = \langle I'(v_n), v_n\rangle \to 0$ as $n \to \infty$. If $\mu_n = 0$, then we take $t_n = 1$. We next assume that $\mu_n \neq 0$. Let $\delta \in \mathbb{R}$ with $|\delta| > 0$ sufficiently small and $t = 1 + \delta$. Then

$$\begin{aligned}\langle I'(tv_n), tv_n\rangle &= (1+\delta)^2\alpha_n - (1+\delta)^p\beta_n = (1+\delta)^2\alpha_n - (1+\delta)^p(\alpha_n - \mu_n)\\ &= \alpha_n(2\delta - p\delta + o(\delta)) + (1+\delta)^p\mu_n\\ &= \alpha_n(2-p)\delta + \alpha_n o(\delta) + (1+\delta)^p\mu_n.\end{aligned}$$

Since $p > 2$, $\alpha_n \to l^2 > 0$ and $\mu_n \to 0$, for n large enough we can define $\delta_n^+ = \dfrac{2|\mu_n|}{\alpha_n(p-2)}$ and $\delta_n^- = -\dfrac{2|\mu_n|}{\alpha_n(p-2)}$. It follows that

$$\delta_n^+ \searrow 0 \quad \text{and} \quad \big\langle I'\big(\big(1+\delta_n^+\big)v_n\big), \big(1+\delta_n^+\big)v_n\big\rangle < 0,$$
$$\delta_n^- \nearrow 0 \quad \text{and} \quad \big\langle I'\big(\big(1+\delta_n^-\big)v_n\big), \big(1+\delta_n^-\big)v_n\big\rangle < 0.$$

From the above relations we deduce the existence of some $t_n \in (1+\delta_n^-, 1+\delta_n^+)$ such that $t_n \to 1$ and $\langle I'(t_n v_n), t_n v_n\rangle = 0$. This concludes the proof. □

We now fix $\bar{u} \in H_a^1(\mathbb{R}^N)$ such that (26) holds. Since $p > 2$, there exists $\bar{t}$ such that

$$I(t\bar{u}) < 0 \quad \text{for all} \quad t > \bar{t},$$
$$J_\lambda(t\bar{u}) < 0 \quad \text{for all} \quad t > \bar{t} \ \text{ and } \ \lambda > 0.$$

Set

$$\mathcal{P} = \big\{\gamma \in C\big([0,1], H_a^1(\mathbb{R}^N)\big)\,;\ \gamma(0) = 0,\ \gamma(1) = \bar{t}\bar{u}\big\}, \tag{35}$$
$$c_g = \inf_{\gamma\in\mathcal{P}} \sup_{u\in\gamma} J_\lambda(u). \tag{36}$$

Proposition 3. *There exists* $\lambda_0 > 0$, $R_0 = R_0(\lambda_0) > 0$, $\delta_0 = \delta_0(\lambda_0) > 0$ *such that* $J_\lambda\,|_{\partial B_{R_0}} \geq \delta_0$ *and* $c_g < c_{0,\lambda} + I_\infty$ *for all* $\lambda \in (0, \lambda_0)$, *where* $c_{0,\lambda} = \inf_{u\in\overline{B}_{R_0}} J_\lambda(u)$.

Proof. By our hypothesis $(K3)$ and the definition of I we can assume that

$$J_0(t\bar{u}) < I(t\bar{u}) \quad \text{for all } \ t > 0.$$

An elementary computation implies the existence of some $t_0 \in (0, \bar{t})$ such that

$$\sup_{t\geq 0} J_0(t\bar{u}) = J_0(t_0\bar{u}) < I(t_0\bar{u}) \leq \sup_{t\geq 0} I(t\bar{u}) = I_\infty.$$

Thus, we can choose $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\geq 0} J_0(t\bar{u}) < I_\infty - \varepsilon_0. \tag{37}$$

Set

$$\lambda_0 := \min\left\{\lambda_1,\ \frac{\varepsilon_0}{2\bar{t}\,\|\bar{u}\|\,\|g\|_{-1}},\ \frac{\varepsilon_0}{2\|g\|_{-1}^2}\right\}. \tag{38}$$

Applying Lemma 4 it follows that there exists $R_0 = R_0(\lambda_0) > 0$ such that for all $\lambda \in (0,\lambda_0)$, the conclusion of Lemma 4 holds. Moreover, by virtue of its proof, there exists $\delta_0 = \delta(\lambda_0) > 0$ such that $J_\lambda\mid_{\partial \overline{B}_{R_0}} \geq \delta_0$. Then relations (38) and (19) yield

$$c_{0,\lambda} = \inf_{u\in\overline{B}_{R_0}} J_\lambda(u) \geq -\frac{\lambda}{2}\|g\|_{-1}^2 > -\frac{\varepsilon_0}{2}, \quad \text{for all } \lambda\in(0,\lambda_0). \tag{39}$$

For $u \in \gamma_0 = \{t\bar{t}\bar{u}\,;\ 0\leq t\leq 1\} \in \mathcal{P}$ we have

$$|J_\lambda(u) - J_0(u)| = \lambda\left|\int_{\mathbb{R}^N} g(x)u\,dx\right| \leq \lambda\,\bar{t}\,\|\bar{u}\|\,\|g\|_{-1} \leq \frac{\varepsilon_0}{2} \quad \text{for all } \lambda\in(0,\lambda_0).$$

Therefore,

$$J_\lambda(u) \leq J_0(u) + \frac{\varepsilon_0}{2}, \quad \text{for all } \lambda\in(0,\lambda_0). \tag{40}$$

Using relations (37), (39), and (40) we obtain

$$\begin{aligned} c_g &= \inf_{\gamma\in\mathcal{P}}\sup_{u\in\gamma} J_\lambda(u) \leq \sup_{u\in\gamma_0} J_\lambda(u) \\ &\leq \sup_{u\in\gamma_0} J_0(u) + \frac{\varepsilon_0}{2} \leq \sup_{t\geq 0} J_0(t\bar{u}) + \frac{\varepsilon_0}{2} < I_\infty - \frac{\varepsilon_0}{2} < I_\infty + c_{0,\lambda}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 1 concluded. Consider $R_0 > 0$, $\delta_0 > 0$ given by Proposition 3. In view of its proof, we deduce that for all $\lambda \in (0,\lambda_0)$, the conclusion of Lemma 4 holds. Therefore, we obtain the existence of a solution u_0 of problem (2) such that $J_\lambda(u_0) = c_{0,\lambda}$.

On the other hand, applying the mountain pass theorem without the Palais–Smale condition (see [5, Theorem 2.2]), it follows that there exists a $(PS)_{c_g}$ sequence $\{u_n\}$ of J_λ, that is,

$$J_\lambda(u_n) = c_g + o(1) \quad \text{and} \quad J_\lambda'(u_n) \to 0 \ \text{ in } \ H_a^{-1}(\mathbb{R}^N).$$

Therefore,

$$\begin{aligned} c_g + o(1) + \frac{1}{p}\|J_\lambda'(u_n)\|_{-1}\|u_n\| &\geq J_\lambda(u_n) - \frac{1}{p}\langle J_\lambda'(u_n), u_n\rangle \\ &\geq \left(\frac{1}{2}-\frac{1}{p}\right)\|u_n\|^2 - \lambda\left(1-\frac{1}{p}\right)\|g\|_{-1}\|u_n\|. \end{aligned}$$

This inequality shows that $\{u_n\}$ is bounded in $H^1_a(\mathbb{R}^N)$. Thus we can assume (up to a subsequence) that $u_n \rightharpoonup u_1$ in $H^1_a(\mathbb{R}^N)$. By Lemma 1 it follows that u_1 is a weak solution of problem (2).

We claim that $u_0 \neq u_1$. Indeed, by Proposition 2, the following alternative holds: either $u_n \to u_1$ in $H^1_a(\mathbb{R}^N)$, which gives

$$J_\lambda(u_1) = \lim_{n\to\infty} J_\lambda(u_n) = c_g > 0 \geq c_{0,\lambda} = J_\lambda(u_0),$$

and the conclusion follows; or

$$c_g = \lim_{n\to\infty} J_\lambda(u_n) \geq J_\lambda(u_1) + I_\infty.$$

In the last case, if we suppose that $u_1 = u_0$, then $J_\lambda(u_1) = J_\lambda(u_0) = c_{0,\lambda}$, and so $c_g \geq c_{0,\lambda} + I_\infty$, which contradicts Proposition 3. The proof of Theorem 1 is now complete. □

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