Research Article

Singular Integral Equations of Convolution Type with Cosecant Kernels and Periodic Coefficients

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We study singular integral equations of convolution type with cosecant kernels and periodic coefficients in class $L^2[-\pi, \pi]$. Such equations are transformed into a discrete jump problem or a discrete system of linear algebraic equations by using discrete Fourier transform. The conditions of Noethericity and the explicit solutions are obtained by means of the theory of classical boundary value problem and of the Fourier analysis theory. This paper will be of great significance for the study of improving and developing complex analysis, integral equations, and boundary value problems.

1. Introduction

It is well-known that the boundary value problems for analytic functions have been widely used in many fields, such as engineering mechanics, physics, engineering technology, and fracture mechanics. Various types of boundary value problems for analytic functions and singular integral equations have been deeply studied and widely applied to practical problem (see [1–3]). In the theory of integral equations, the convolution type integral equations and singular integral equations are two important classes of equations, which had been studied by many mathematical workers and there were already rather complete theoretical systems (see [4, 5]). These theories have been widely used in practical applications, such as engineering mechanics, fracture mechanics, and elastic mechanics (see [6, 7]).

The main aim of this paper is to extend further the theory to singular integral equations of convolution type with periodic coefficients and cosecant kernel in class $L^2[-\pi, \pi]$ (see [8] for the definition of $L^2[-\pi, \pi]$). By using discrete Fourier transform, such equations are transformed into a discrete jump problem or a discrete system of equations. The explicit expressions of general solution and the conditions of solvability are obtained in class $L^2[-\pi, \pi]$. Therefore, this paper generalizes some results for [1–5].

2. Definitions and Lemmas

In this section we present some definitions and lemmas.

Definition 1. Suppose $\varphi = \{\varphi_k\} = \{\ldots, \varphi_{-1}, \varphi_0, \varphi_1, \ldots\}$ is an infinite dimensional vector consisting of two-way infinite sequence; then all of $\sum_{k=-\infty}^{+\infty} |\varphi_k|^p$ which satisfy $\sum_{k=-\infty}^{+\infty} |\varphi_k|^p < +\infty$ constitute a linear space; we denote it as $\varphi \in l^p$.

Remark 2. From Definition 1, we know that if $\varphi \in l^p$, then $c\varphi \in l^p$ for any constant c. Moreover, if $\{\varphi_k\} \in l^p$, $\{\psi_k\} \in l^q$, then $\{\varphi_k \psi_k\} \in l^1$, where $p^{-1} + q^{-1} = 1$ $(p > 1, q > 1)$. In this paper we mainly consider the case $p = 2$.

Definition 3. Let $\Phi(\theta)$, $\Psi(\theta)$ be periodic functions with period 2π, and $\Phi(\theta)$, $\Psi(\theta) \in L^2[-\pi, \pi]$; then

$$(\Phi * \Psi)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(\theta - \nu) \Psi(\nu) \, d\nu, \quad \theta \in [-\pi, \pi] \quad (1)$$

is called the convolution of $\Phi(\theta)$ and $\Psi(\theta)$.

Remark 4. With regard to the convolution $(\Phi * \Psi)(\theta)$, we can extend the value of θ from $[-\pi, \pi]$ to $\mathbb{R}$ by Definition 3.

Definition 5. Let $\varphi = \{\varphi_k\}_{k=-\infty}^{+\infty} \in l^2$; the transforms V and V^{-1} defined by

$$V\varphi = \sum_{k=-\infty}^{+\infty} \varphi_k e^{ik\theta} = \Phi(\theta),$$

$$V^{-1}[\Phi(\theta)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(\theta) e^{-ik\theta} d\theta = \varphi_k, \tag{2}$$

$$k = 0, \pm 1, \pm 2, \ldots,$$

are called the discrete Fourier transform and the inverse transform, respectively. For simplification, in (2), we denote them as $V\varphi = \Phi(\theta)$ and $V^{-1}[\Phi(\theta)] = \varphi_k$, respectively.

Definition 6. Let $\varphi = \{\varphi_k\}_{k=-\infty}^{+\infty}$ be a number sequence and M and η be positive constants with $0 < \eta \leq 1$. If $|k^{1+\eta}\varphi_k| \leq M$ holds for any integer k, we say that $\varphi = \{\varphi_k\}$ belongs to class $\{1\}$.

Definition 7. If $\varphi = \{\varphi_k\}_{k=-\infty}^{+\infty} \in \{1\}$, then we say $\Phi(\theta) \in \{\widetilde{0}\}$.

The following lemma plays an important role, and it is proposed firstly in this paper.

Lemma 8. Let $\Phi(\theta) \in L^2[-\pi, \pi]$, $S\Phi(\theta) = (1/\pi)\int_{-\pi}^{\pi} \Phi(\nu)\csc(\nu - \theta)d\nu$,

$$\widetilde{S}\Phi(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(\nu) \csc(\nu + \theta) d\nu. \tag{3}$$

Then

(1) $V^{-1}(S\Phi(\theta)) = 2\varphi_k i \operatorname{sgn} k\rho_k$, $V^{-1}(\widetilde{S}\Phi(\theta)) = -2\varphi_k i \operatorname{sgn} k\rho_k$;

(2) $S\Phi(\theta) \in L^2[-\pi, \pi]$, $V^{-1}(S\Phi(\theta)) \in l^2[-\pi, \pi]$,

where $S\Phi(\theta)$, $\widetilde{S}\Phi(\theta)$ are singular integrals with cosecant kernels and

$$\rho_k = \begin{cases} 1, & k = 2n + 1; \\ 0, & k = 2n, \end{cases} \quad n \in \mathbb{Z}. \tag{4}$$

Proof. (1) Since

$$S\Phi(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(\nu) \csc(\nu - \theta) d\nu$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi(\nu) \frac{2i}{e^{i(\nu-\theta)} - e^{-i(\nu-\theta)}} d\nu \tag{5}$$

$$= \frac{2i}{\pi} \int_{-\pi}^{\pi} \Phi(\nu) \frac{e^{i(\theta+\nu)}}{e^{2i\nu} - e^{2i\theta}},$$

by Definition 5, we have

$$V^{-1}(S\Phi(\theta))$$

$$= i \cdot \frac{1}{\pi^2} \int_{-\pi}^{\pi} \Phi(\nu) e^{i\nu} \left[\int_{-\pi}^{\pi} \frac{e^{-i(k-1)\theta}}{e^{2i\nu} - e^{2i\theta}} d\theta \right] d\nu. \tag{6}$$

It follows from the extended Residue theorem (see [9]) that

$$\int_{-\pi}^{\pi} \frac{e^{-i(k-1)\theta}}{e^{2i\nu} - e^{2i\theta}} d\theta = \frac{\pi}{e^{i(k+1)\nu}} \rho_k \operatorname{sgn} k. \tag{7}$$

This implies

$$V^{-1}(S\Phi(\theta)) = i\frac{1}{\pi^2}\pi\rho_k \operatorname{sgn} k \int_{-\pi}^{\pi} \frac{1}{e^{i(k+1)\nu}} \Phi(\nu) e^{i\nu} d\nu \tag{8}$$

$$= 2i\varphi_k\rho_k \operatorname{sgn} k.$$

The other equality can be proved similarly.

(2) By using Riesz's theory (see [9]), we have $\|S\Phi(\theta)\|_2 \leq C\|\Phi(\theta)\|_2$, where C is a constant. Therefore, it follows from $\Phi(\theta) \in L^2[-\pi, \pi]$ that $S\Phi(\theta) \in L^2[-\pi, \pi]$.

It is also obvious for the second conclusion in (2).   □

Lemma 9. *If $\varphi = \{\varphi_k\}_{k=-\infty}^{+\infty}$ belongs to class $\{1\}$, then $\Phi(\theta) = V\varphi \in H$, where H is the class of Hölder continuous function.*

Proof. By Definition 6, we know that $\sum_{k=-\infty,k\neq0}^{+\infty}(M/|k|^{1+\eta})$ converges; therefore, $\sum_{k=-\infty}^{\infty} \varphi_k$ is convergent absolutely. For any two points $\theta_1, \theta_2 \in [-\pi, \pi]$, we have

$$|\Phi(\theta_1) - \Phi(\theta_2)| = \left| \sum_{k=-\infty}^{\infty} \varphi_k \left(e^{ik\theta_1} - e^{ik\theta_2} \right) \right|$$

$$\leq \left| \sum_{k=-\infty}^{\infty} |\varphi_k| \left(e^{ik\theta_1} - e^{ik\theta_2} \right) \right| \tag{9}$$

$$\leq \widetilde{M} |\theta_1 - \theta_2|,$$

where $\widetilde{M} = \sum_{k=-\infty}^{\infty} |\varphi_k|$; thus $\Phi(\theta) \in H$.   □

Lemma 10. *Let $\Phi(\theta) = \sum_{k=-\infty}^{\infty} \varphi_k e^{ik\theta}$ and $\varphi \in l^1$; then $\Phi(\theta) \in L^2[-\pi, \pi]$ if and only if $\varphi \in l^2$.*

Proof. Since $\Phi(\theta) \in L^2[-\pi, \pi]$ and $\varphi \in l^1$, then $\sum_{k=-\infty}^{\infty} \varphi_k$ is convergent absolutely and uniformly. It is easy to see that

$$\int_{-\pi}^{\pi} |\Phi(\theta)|^2 d\theta = \int_{-\pi}^{\pi} \Phi(\theta) \overline{\Phi(\theta)} d\theta$$

$$= \int_{-\pi}^{\pi} \sum \varphi_k e^{ik\theta} \sum \overline{\varphi_j} e^{-ij\theta} d\theta$$

$$= \int_{-\pi}^{\pi} \sum_{k,j=-\infty}^{\infty} \varphi_k \overline{\varphi_j} e^{i(k-j)\theta} d\theta \tag{10}$$

$$= 2\pi \sum_{k=-\infty}^{\infty} |\varphi_k|^2.$$

The proof of Lemma 10 is complete.   □

Lemma 11 is an obvious fact.

Lemma 11 (convolution theorem). *If $\Phi(\theta)$ and $\Psi(\theta) \in L^2[-\pi, \pi]$, then $\Phi * \Psi(\theta) \in L^2[-\pi, \pi]$ and $V^{-1}[\Phi * \Psi(\theta)] = \varphi_k\psi_k$, where $\psi_k = V^{-1}[\Psi(\theta)]$.*

3. Problem Presentation and Methods of Solution

Consider the following singular integral equation of convolution type with periodicity and cosecant kernels:

$$
\xi_1(\theta)\left[a\Phi(\theta) + bS\Phi(\theta) + K_1 * \Phi(\theta) + M_1 * S\Phi(\theta)\right.
$$
$$
\left. - cG(\theta)\right] + \xi_2(\theta)\left[a\Phi(\theta) + bS\Phi(\theta) + K_2 * \Phi(\theta)\right. \quad (11)
$$
$$
\left. + M_2 * S\Phi(\theta) - cG(\theta)\right] = 0, \quad \theta \in \mathbb{R},
$$

where

$$
\xi_1(\theta) = c_1 \cos\frac{m\theta}{2} + d_1 \sin\frac{m\theta}{2},
$$
$$
\xi_2(\theta) = a_1 \cos\frac{m\theta}{2} + b_1 \sin\frac{m\theta}{2}, \quad (12)
$$
$$
M * S\Phi(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} M(\theta - \nu) S\Phi(\nu)\, d\nu.
$$

$S\Phi(\theta)$ is as in Lemma 8. $a, b, c, a_1, b_1, c_1, d_1$ are real constants with $a^2 + b^2 \neq 0$, $a_1 d_1 - b_1 c_1 \neq 0$. The functions $\xi_j(\theta)$ ($j = 1, 2$) are periodic coefficients; m is a positive integer. $K_j(\theta), M_j(\theta)$, $\Phi(\theta), S\Phi(\theta)$, and $G(\theta)$ are periodic functions with period 2π. Once $\Phi(\theta)$ is obtained in $[-\pi, \pi]$, we can make the periodic extension of $\Phi(\theta)$ with period 2π and then obtain the solution $\Phi_*(\theta)$ in $\mathbb{R}$. Therefore, in the following discussion we are restricted to $\theta \in [-\pi, \pi]$.

Equation (11) is one important class of equations in the theory of integral equations, and it has important applications in physics, air dynamics, and electronic optical (e.g., see [10–14]). Hence, the study of (11) is meaningful not only in application but also in the method of solution.

Since $e^{im\theta} = \cos m\theta + i \sin m\theta$, we write (11) as

$$
A^+\Phi(\theta) + B^+ S\Phi(\theta) + \beta^+ K_1 * \Phi(\theta) + \alpha^+ K_2 * \Phi(\theta)
$$
$$
+ \beta^+ M_1 * \Phi(\theta) + \alpha^+ M_2 * S\Phi(\theta) - F^+ G(\theta)
$$
$$
+ e^{im\theta}\left[A^-\Phi(\theta) + B^- S\Phi(\theta) + \beta^- K_1 * \Phi(\theta)\right. \quad (13)
$$
$$
\left. + \alpha^- K_2 * \Phi(\theta) + \beta^- M_1 * \Phi(\theta) + \alpha^- M_2 * S\Phi(\theta)\right.
$$
$$
\left. - F^- G(\theta)\right] = 0, \quad -\pi \leq \theta \leq \pi,
$$

where

$$
\alpha^\pm = a_1 \pm ib_1,
$$
$$
\beta^\pm = c_1 \pm id_1,
$$
$$
A^\pm = a\left[(a_1 + c_1) \pm i(b_1 + d_1)\right], \quad (14)
$$
$$
B^\pm = b\left[(a_1 + c_1) \pm i(b_1 + d_1)\right],
$$
$$
F^\pm = c\left[(a_1 + c_1) \pm i(b_1 + d_1)\right],
$$

and $\xi_j(\theta)$ ($j = 1, 2$) can be rewritten as

$$
\xi_1(\theta) = \frac{1}{2}\left(\beta^- e^{im\theta/2} + \beta^+ e^{-im\theta/2}\right),
$$
$$
\xi_2(\theta) = \frac{1}{2}\left(\alpha^- e^{im\theta/2} + \alpha^+ e^{-im\theta/2}\right). \quad (15)
$$

Let

$$
A^-\Phi(\theta) + B^- S\Phi(\theta) + \beta^- K_1 * \Phi(\theta) + \alpha^- K_2 * \Phi(\theta)
$$
$$
+ \beta^- M_1 * S\Phi(\theta) + \alpha^- M_2 * S\Phi(\theta) - F^- G(\theta) \quad (16)
$$
$$
= T(\theta),
$$

then,

$$
A^+\Phi(\theta) + B^+ S\Phi(\theta) + \beta^+ K_1 * \Phi(\theta) + \alpha^+ K_2 * \Phi(\theta)
$$
$$
+ \beta^+ M_1 * S\Phi(\theta) + \alpha^+ M_2 * S\Phi(\theta) - F^+ G(\theta) \quad (17)
$$
$$
= -e^{im\theta} T(\theta);
$$

Applying V^{-1} to both sides of (16) and (17), respectively, by Lemma 8 we have

$$
\tau_k = \left(A^- + 2B^- i\, \mathrm{sgn}\, k\rho_k + \beta^- B_k^{(1)} + \alpha^- B_k^{(2)}\right.
$$
$$
\left. + 2\beta^- C_k^{(1)} i\, \mathrm{sgn}\, k\rho_k + 2\alpha^- C_k^{(2)} i\, \mathrm{sgn}\, k\rho_k\right)\varphi_k - F^- g_k;
$$
$$
- \tau_{k-m} = \left(A^+ + 2B^+ i\, \mathrm{sgn}\, k\rho_k + \beta^+ B_k^{(1)} + \alpha^+ B_k^{(2)}\right.
$$
$$
\left. + 2\beta^+ C_k^{(1)} i\, \mathrm{sgn}\, k\rho_k + 2\alpha^+ C_k^{(2)} i\, \mathrm{sgn}\, k\rho_k\right)\varphi_k - F^+ g_k,
$$
$$
\quad (18)
$$

where

$$
\tau_k = V^{-1} T,
$$
$$
\varphi_k = V^{-1}\Phi,
$$
$$
B_k^{(j)} = V^{-1} K_j, \quad (19)
$$
$$
C_k^{(j)} = V^{-1} M_j
$$
$$
(j = 1, 2),
$$
$$
g_k = V^{-1} G.
$$

Denote

$$
\pi_k^\pm = A^\pm + 2B^\pm i\, \mathrm{sgn}\, k\rho_k + \beta^\pm B_k^{(1)} + \alpha^\pm B_k^{(2)}
$$
$$
+ 2\beta^\pm C_k^{(1)} i\, \mathrm{sgn}\, k\rho_k + 2\alpha^\pm C_k^{(2)} i\, \mathrm{sgn}\, k\rho_k. \quad (20)
$$

Then (18) can be reduced as

$$
\pi_k^- \varphi_k - F^- g_k = \tau_k;
$$
$$
\pi_k^+ \varphi_k - F^+ g_k = -\tau_{k-m}. \quad (21)
$$

Since $\{B_k^{(j)}\}_{k=-\infty}^{\infty} \in l^2$, $\{C_k^{(j)}\}_{k=-\infty}^{\infty} \in l^2$ ($j = 1, 2$), then $B_k^{(j)} \to 0$, $C_k^{(j)} \to 0$, and $\pi_k^\pm \to A^\pm \pm 2iB^\pm \neq 0$ or $\pi_k^\pm \to A^\pm \neq 0$ when

$|k| \to \infty$. Therefore there exists k_0 $(|k| > k_0)$ such that $|\pi_k^{\pm}| > \mu$ when $|k| > k_0$ for certain positive constant μ $(0 < \mu < 1)$.

Consider first the case $|k| \le k_0$.

(1) When $\pi_k^{\pm} = 0$ for $k = n_1, n_2, \ldots, n_{p_1}$ $(-k_0 \le n_j \le k_0, \ 1 \le j \le p_1)$, by (21) we know that

$$\tau_{n_j} = -F^- g_{n_j},$$
$$\tau_{n_j - m} = F^+ g_{n_j} \tag{22}$$

must be fulfilled. Now φ_{n_j} $(1 \le j \le p_1)$ take arbitrary constants.

(2) When $\pi_k^+ = 0$, $\pi_k^- \ne 0$ for $k = n_1', n_2', \ldots, n_{p_2}'$ $(-k_0 \le n_j' \le k_0, \ 1 \le j \le p_2)$, by (21) we have

$$\varphi_{n_j'} = \frac{F^- g_{n_j'} + \tau_{n_j'}}{\pi_{n_j'}^-}, \tag{23}$$

where

$$\tau_{n_j'} = F^+ g_{n_j' + m}. \tag{24}$$

(3) When $\pi_k^+ \ne 0$, $\pi_k^- = 0$ for $k = n_1'', n_2'', \ldots, n_{p_3}''$ $(-k_0 \le n_j'' \le k_0, \ 1 \le j \le p_3)$, by (21) we get

$$\varphi_{n_j''} = \frac{F^+ g_{n_j''} - \tau_{n_j'' - m}}{\pi_{n_j''}^+}, \tag{25}$$

where

$$\tau_{n_j''} = -F^- g_{n_j''}. \tag{26}$$

(4) When $\pi_k^{\pm} \ne 0$ $(-k_0 \le k \le k_0)$, φ_k are given by (21) and this situation is similar to the situation $|k| > k_0$.

Next we consider the case $|k| > k_0$. Since $|\pi_k^{\pm}| > \mu$ for $|k| > k_0$, then $\pi_k^{\pm} \ne 0$. By (21) we obtain

$$\varphi_k = \frac{1}{\pi_k^-} \left(\tau_k + F^- g_k \right) = \frac{1}{\pi_k^+} \left(-\tau_{k-m} + F^+ g_k \right); \tag{27}$$

thus

$$\tau_k + R_k \tau_{k-m} = H_k g_k, \tag{28}$$

where

$$R_k = \frac{\pi_k^-}{\pi_k^+},$$

$$H_k = \frac{2i \left(b_1 c_1 - a_1 d_1 \right)}{\pi_k^+} \left[\left(B_k^{(2)} - B_k^{(1)} \right) \right. \tag{29}$$
$$\left. + 2 \left(C_k^{(2)} - C_k^{(1)} \right) i \operatorname{sgn} k \rho_k \right].$$

Assume that

$$\delta = \frac{B^-}{B^+}, \tag{30}$$
$$R_k = \delta \left(1 + Y_k \right),$$

where

$$Y_k = \frac{2ib \left(b_1 c_1 - a_1 d_1 \right)}{B^-}$$
$$\cdot \frac{B_k^{(1)} - B_k^{(2)} + 2i \operatorname{sgn} k \rho_k \left[C_k^{(1)} - C_k^{(2)} \right]}{\pi_k^+}. \tag{31}$$

Therefore, (28) can be written as the following discrete jump problem:

$$\tau_k + \delta \left(1 + Y_k \right) \tau_{k-m} = H_k g_k. \tag{32}$$

Thus, we should only study (32) in place of (11). Since $\{H_k\} \in l^2$, $\{g_k\} \in l^2$, then $\{H_k g_k\} \in l^2$, $\{\tau_k\} \in l^2$. Once τ_k is obtained, then φ_k can be given by (21). To solve (32), we first need to analyze the structure of $1 + Y_k$. Assume that $1 + T_k$ can be factorized; we can choose r_k such that

$$1 + Y_k = \frac{r_k}{r_{k-m}}, \tag{33}$$

where $\mu < |r_k| < \mu^{-1}$; μ is as the above. Provided that μ takes sufficiently small value, we can choose r_k satisfying the above requirement (33). We now give an expression of r_k, taking logarithms for both sides of (33) and denoting

$$w_k = \ln r_k$$
$$u_k = \ln \left(1 + Y_k \right); \tag{34}$$

then, by (33) and (34) we obtain

$$w_k - w_{k-m} = u_k. \tag{35}$$

Note that we have taken a continuous branch of $\ln(1 + Y_k)$ such that $\ln(1 + Y_k) \in l^2$. Applying Fourier transform to both sides of (35), we can obtain

$$\left(1 - e^{im\theta} \right) W(\theta) = U(\theta), \tag{36}$$

where $W(\theta) = Vw$, $U(\theta) = Vu$, $w = \{w_k\}_{k=-\infty}^{+\infty}$, and $u = \{u_k\}_{k=-\infty}^{+\infty}$.

From the above discussions, the following conditions of solvability $U(\theta_j) = 0$ follow and that is,

$$\sum \left(\ln \frac{1}{\delta} R_k \right) e^{ik\theta_j} = 0, \quad \left\{ \ln \frac{1}{\delta} R_k \right\} \in l^2, \tag{37}$$

where $\theta_j = (2j/m)\pi$ $(j = 0, \pm 1, \ldots, \pm [m/2]$; $[m/2]$ is the integer part of $m/2)$. Therefore,

$$W(\theta) = \frac{U(\theta)}{1 - e^{im\theta}} \quad \left(\theta \ne \theta_j, \ |j| \le 2 \left[\frac{m}{2} \right] + 1 \right). \tag{38}$$

From (38), we obtain $w_k = V^{-1} W(\theta)$ and thus $r_k = \exp w_k$. Note (33) and denote

$$e_k = \frac{\tau_k}{r_k},$$
$$\psi_k = \frac{H_k g_k}{r_k}. \tag{39}$$

Then (32) can be rewritten as

$$e_k + \delta e_{k-m} = \psi_k. \tag{40}$$

Since $\{H_k g_k\} \in l^2$, $\{r_k\} \in l^2$, and $\mu < |r_k| < \mu^{-1}$, we have $\{e_k\} \in l^2$ and $\{\psi_k\} \in l^2$. By taking Fourier transform V for both sides of (40), we have

$$\left(1 + \delta e^{im\theta}\right) E(\theta) = \Psi(\theta), \tag{41}$$

where $E(\theta) = Ve$, $\Psi(\theta) = V\psi$, $e = \{e_k\}_{k=-\infty}^{+\infty}$, and $\psi = \{\psi_k\}_{k=-\infty}^{+\infty}$.

Since $|\delta| = 1$, we know that $1 + \delta e^{im\theta}$ has finite number of zero points $\theta_1', \theta_2', \ldots, \theta_n'$ in $[-\pi, \pi]$.

Therefore the conditions of solvability $\Psi(\theta_j') = 0$, that is,

$$\sum \frac{H_k g_k}{r_k} e^{ik\theta_j'} = 0, \quad \left\{\frac{H_k g_k}{r_k}\right\} \in l^2 \quad (j = 1, 2, \ldots, n), \tag{42}$$

must be augmented. In view of (41) we get

$$E(\theta) = \frac{\Psi(\theta)}{1 + \delta e^{im\theta}} \quad \left(\theta \neq \theta_j', \ j = 1, 2, \ldots, n\right). \tag{43}$$

If (37) and (42) are satisfied, then by (43) we obtain

$$e_k = V^{-1} E(\theta) \quad (k = 0, \pm 1, \ldots). \tag{44}$$

Thus τ_k $(k = 0, \pm 1, \ldots)$ can be also obtained by (39).

Now we state our main result.

Theorem 12. *Under (37) and (42), (11) has a solution. When $\pi_k^{\pm} \neq 0$ $(k = 0, \pm 1, \ldots)$, φ_k is given by (21). When $\pi_{n_j}^{\pm} = 0$ $(j = 1, 2, \ldots, p_1)$, φ_{n_j} take arbitrary constants. When $\pi_{n_j'}^{+} = 0$, $\pi_{n_j'}^{-} \neq 0$ $(j = 1, 2, \ldots, p_2)$, $\varphi_{n_j'}$, $\tau_{n_j'}$ are given by (23) and (24), respectively. When $\pi_{n_j''}^{+} \neq 0$, $\pi_{n_j''}^{-} = 0$ $(j = 1, 2, \ldots, p_3)$, $\varphi_{n_j''}$, $\tau_{n_j''}$ are given by (25) and (26), respectively. But in (23) and (25), $\tau_{n_j'}$ and $\tau_{n_j''}$ are constants determined by (24) and (26), respectively. When $k \neq n_j, n_j', n_j''$, φ_k is given by (21). Therefore, the solution of (11) is of the form*

$$\Phi(\theta) = V\varphi = \sum_{k=-\infty}^{\infty} \varphi_k e^{ik\theta}, \tag{45}$$

and then $\Phi(\theta) \in L^2[-\pi, \pi]$.

Proof. From the above discussion, we only need to prove that the function $\Phi(\theta)$ obtained by (45) belongs to $L^2[-\pi, \pi]$. Obviously, (11), (13), and (21) are equivalent to each other. Since $\{g_k\}, \{\tau_k\} \in l^2$, then $\{\pi_k^{\pm}\} \in l^2$ and $\pi_k^{\pm} \neq 0$. It follows from (27) that $\{\varphi_k\} \in l^2$ is a bounded sequence, and $\sum_{k=-\infty}^{\infty} \varphi_k e^{ik\theta}$ is convergent. Thus, by Lemma 10, (11) has a unique solution $\Phi(\theta) = V\varphi$ in class $L^2[-\pi, \pi]$. □

According to Theorem 12, Definition 5, and Lemma 9, we obtain the following.

Theorem 13. *In (11), if $K_j(\theta)$, $M_j(\theta)$ $(j = 1, 2)$, $G(\theta) \in \{\tilde{0}\}$, then (11) is solvable in class $\{\tilde{0}\}$ and all conditions of solvability and its solutions are similar to those in Theorem 12.*

In order to illustrate that (11) has an explicit solution, we present an example and satisfy the above conditions (37) and (42). For example, suppose that

$$\begin{aligned} a &= b = c = 1, \\ b_1 &= c_1 = 0, \\ a_1 &= d_1 \neq 0, \\ K_1 &= K_2, \\ M_1 &= M_2; \end{aligned} \tag{46}$$

then $\xi_1(\theta) = a_1 \sin(m/2)\theta$, $\xi_2(\theta) = a_1 \cos(m/2)\theta$, and (11) can be transformed into

$$\Phi + S\Phi + K_1 * \Phi + M_1 * S\Phi = G; \tag{47}$$

it is easy to prove that (47) satisfies conditions (37) and (42), and the solution of (47) was obtained in the literature [1]. Therefore, we can conclude that a solution set of (11) is not empty.

Remark 14. If $a_1 d_1 - b_1 c_1 = 0$, $\xi_2(\theta)/\xi_1(\theta)$ is a constant. This case is simple, and we do not discuss it here.

4. Results and Discussion

In this paper, we first proposed one class of singular integral equations of convolution type with cosecant kernels and periodic coefficients. By applying discrete Fourier transform and its properties, such equation can be transformed into a discrete jump problem depending on some parameter. Here, our method is different from the ones of the classical boundary value problem, and it is novel and simple. The exact solution, denoted by series, of (11) and the conditions of solvability are obtained in $L^2[-\pi, \pi]$. We remark that our approach is also effective in other classes of equations, such as the equations of dual type with periodicity and cosecant kernel and Wiener-Hopf type equations. Thus, this paper generalizes the classical theory of boundary value problems and singular integral equations.

In this paper, we solved (11) in $L^2[-\pi, \pi]$. Indeed, this class of equations can be also solved in Clifford analysis, which is similar to that in [15–18]. Further discussion is omitted here.

5. Conclusions

Equation (11) has important applications in practical problems, such as elastic mechanics, heat conduction, and electrostatics. Many problems, such as piezoelectric material, voltage magnetic materials, and functional gradient materials, can often attribute the problem to finding their solutions to (11). For the study of such equations, the present result is still rare due to lack of effective approaches. Our approach of solving

the equations is novel, different from the ones in classical cases, and it is converted by using discrete Fourier transform into a discrete boundary value problem depending on some parameter; here we call it "a discrete jump problem." The exact solutions of (11) and the conditions of solvability are obtained in class $L^2[-\pi, \pi]$.
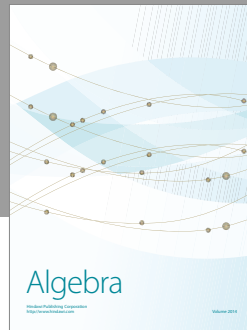
Conflicts of Interest

The author declares no conflicts of interest.

Acknowledgments

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