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Singular limit of a second order nonlocal parabolic equation of conservative type arising in the micro-phase separation of diblock copolymers

(Dedicated to Professors Masayasu Mimura and Takaaki Nishida on their sixtieth birthday)

M. HENRY, D. HILHORST and Y. NISHIURA

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Abstract. We study the limiting behavior as ε tends to zero of the solution of a second order nonlocal parabolic equation of conservative type which models the micro-phase separation of diblock copolymers. We consider the case of spherical symmetry and prove that as the reaction coefficient tends to infinity the problem converges to a free boundary problem where the interface motion is partly induced by its mean curvature.

Key words: reaction-diffusion systems of conservative type, singular limits, nonlocal motion by mean curvature, asymptotic expansions.

1. Introduction

In this paper, we consider a second order nonlocal parabolic equation of conservative type proposed by Ohnishi and Nishiura [9], namely

$$(\mathcal{P}^\varepsilon)\begin{cases} u_t^\varepsilon = \Delta u^\varepsilon + \dfrac{1}{\varepsilon^2}\Big(f(u^\varepsilon) - \displaystyle⨍_\Omega f(u^\varepsilon) - \varepsilon v^\varepsilon\Big) & \text{in } \Omega\times(0,T) \quad (1.1)\\ -\Delta v^\varepsilon = u^\varepsilon - \displaystyle⨍_\Omega u^\varepsilon & \text{in } \Omega\times(0,T) \quad (1.2)\\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0 & \text{in } \partial\Omega\times(0,T) \quad (1.3)\\ \displaystyle⨍_\Omega v^\varepsilon dx = 0 & \text{for } t\in(0,T) \quad (1.4)\\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{for } x\in\Omega \quad (1.5)\end{cases}$$

where

$$f(s) := 2s(1-s^2), \quad ⨍_\Omega u\,dx := \frac{1}{|\Omega|}\int_\Omega u\,dx$$

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and where $\Omega \subset R^N$ $(N \geq 2)$ is a smooth bounded domain.

Integrating (1.1) in Ω and using (1.4) we deduce that the integral of u is conserved in time, namely

$$\int_\Omega u^\varepsilon(x,t)dx = \int_\Omega u_0^\varepsilon(x)dx =: \mathcal{M}_0^\varepsilon \text{ for all } t \in (0,T). \tag{1.6}$$

Therefore $(\mathcal{P}^\varepsilon)$ is a second order model system which conserves mass. The main feature of this equation is that it shares the same stationary solutions as the fourth order model system arising in the micro-phase separation of diblock copolymer melts (cf. [8]). Both problems are technically quite difficult: it is well-known that fourth order equations do not have a maximum principle which excludes making use of the usual techniques involving upper and lower solutions. The situation is similar for the conserved Allen-Cahn for which the maximum principle does not apply either and the only L^∞ bounds which are known for the solutions can be obtained using arguments based on invariant domain. Moreover stability properties of those solutions also coincide with each other, namely Ohnishi and Nishiura [9] proved that the sign of the real parts of the spectrum corresponding to the fourth order model and to Problem $(\mathcal{P}^\varepsilon)$ coincide with each other. In general, it is much more difficult to study the spectrum distribution of the fourth order equation compared with the second order one, hence it is more informative to investigate $(\mathcal{P}^\varepsilon)$ than attacking the fourth order problem directly. As we shall discuss below, the singular limit equation of $(\mathcal{P}^\varepsilon)$ turns out to be a mean curvature flow with nonlocal term, which is much more intuitive than a free boundary problem of Mullin-Sekerka type associated to the fourth order problem. Note that the total mass does not change in polymer problems, hence the above conservation (1.6) is a natural consequence.

We remark that the functional

$$E^\varepsilon(u^\varepsilon) = \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}F(u^\varepsilon) + \frac{1}{2}|\nabla v^\varepsilon|^2\Big)dx \tag{1.7}$$

where $F(s) = \frac{(1-s^2)^2}{2}$ is a Lyapunov functional for Problem (P^ε) and suppose that the initial value $u_0^\varepsilon \in H^2(\Omega)$ satisfies the hypothesis H_0^ε,

$$H_0^\varepsilon \begin{cases} \text{There exists a positive constant } C \text{ such that } E^\varepsilon(u_0^\varepsilon) \leq C; \\ \text{there exists } \mathcal{M}_0 \in (-|\Omega|, |\Omega|) \text{ such that} \\ \mathcal{M}_0^\varepsilon \text{ tends to } \mathcal{M}_0 \text{ as } \varepsilon \downarrow 0. \end{cases}$$

When $\varepsilon \downarrow 0$, the solution u^ε converges to a limit function $u = \pm 1$ a.e. in $\Omega \times (0,T)$ and one can formally show that the limiting problem has the form (see also Rubinstein and Sternberg [13])

$$
(P_0) \begin{cases}
V_n = -(N-1)K + (N-1) \fint_{\Gamma_t} K + \frac{3}{2}\Big(v - \fint_{\Gamma_t} v\Big) & \text{on } \Gamma_t,\ t \in (0,T) \\
-\Delta v = u - \fint_\Omega u\,dx & \text{in } \Omega \times (0,T) \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\
\fint_\Omega v\,dx = 0 & \text{for } t \in (0,T) \\
\Gamma_{t|_{t=0}} = \Gamma_0, &
\end{cases}
$$

where $u(.,t) = \pm 1$ on $\Omega_t^\pm$, $\Omega = \Omega_t^+ \cup \Omega_t^- \cup \Gamma_t$, K is the mean curvature of Γ_t taking the sign convention that convex hypersurfaces have positive mean curvature, V_n is the normal velocity of the interface taking the sign convention that the normal velocity of expanding hypersurfaces is positive. The purpose of this paper is to give a rigorous derivation of the limit problem (P_0) in the case of spherical symmetry. In this special case we rigourously show how starting from the PDE system one obtains the limiting free boundary problem. Of course the case of spherical symmetry is a stable one whereas instabilities do arise in the general case.

This paper is organized as follows.

We pesent a formal derivation of the limit equation for the interface motion in Section 2. In Section 3 we introduce an alternative formulation with a Lagrange multiplier and prove a priori estimates on the solution of Problem $(\mathcal{P}^\varepsilon)$; we then obtain a first convergence result, namely that there exists a function u taking the values ± 1 such that u^ε tends to $u = \pm 1$ in $L^1(\Omega \times (0,T))$ and a.e.

From Section 4 we assume that Ω is the unit ball in R^N and rewrite Problem $(\mathcal{P}^\varepsilon)$ in the radial variable $r = |x|$. We prove in Section 4 a key estimate, which implies in particular that far away from the origin $r = 0$, the shape of the solution u^ε is close to that of the function $\pm \tanh\big(\frac{r}{\varepsilon}\big)$.

In Section 5 we define and characterize the "jumps" of the limit function u.

In Sections 6 and 7 we approximate u^ε by a first order asymptotic expansion with respect to ε.

We deduce the interface equation in Section 8; more precisely we prove the following result.

Theorem 1.1 *Assume that H_0^ε is satisfied.*

(i) *There exist a sequence $\{\varepsilon_n\}$ and functions u, v such that*

$$u^{\varepsilon_n} \to u \text{ in } L^1(\Omega \times (0,T)) \text{ and a.e. in } \Omega \times (0,T), \text{ where } u = \pm 1 \text{ a.e.},$$
$$v^{\varepsilon_n} \rightharpoonup v \text{ in } L^2(0,T,H^1(\Omega)),$$

as ε_n tends to 0. The functions u and v are such that

$$\begin{cases} -\Delta v = u - \fint_\Omega u & a.e. \text{ in } \Omega \times (0,T) \\ \fint_\Omega v\,dx = 0 & \text{for a.e. } t \in (0,T) \\ \dfrac{\partial v}{\partial n} = 0 & a.e. \text{ on } \partial\Omega \times (0,T) \end{cases}$$

(ii) *Suppose that Ω is the unit ball in R^N and that u_0^ε is radially symmetric so that u^ε is also radially symmetric. We suppose that the number of jumps of u, which we denote by $\mathcal{N}_0$ is finite and constant in time on an interval (t_1,t_2). Let $\{\overline{r}_i\}_{i\in[1,\dots,\mathcal{N}_0]}$ be the jumps of u in $(0,1]$ on (t_1,t_2). Then for all $l \in [1,\dots,\mathcal{N}_0]$ $\overline{r}_l$ is Lipschitz continuous on $[t_1,t_2]$ and moreover $\overline{r}_l$ satisfies*

$$\partial_t\overline{r}_l(t) = -\frac{N-1}{\overline{r}_l(t)} + \nu(\overline{r}_l)\frac{3}{2}v(\overline{r}_l(t),t)$$
$$+\,\nu(\overline{r}_l)\frac{(N-1)\sum_{i=1}^{i=\mathcal{N}_0}\nu(\overline{r}_i)\overline{r}_i(t)^{N-2} - 3/2\sum_{i=1}^{i=\mathcal{N}_0}v(\overline{r}_i(t),t)\overline{r}_i^{N-1}(t)}{\sum_{i=1}^{i=\mathcal{N}_0}\overline{r}_i(t)^{N-1}}, \tag{1.8}$$

for a.e. $t \in (t_1,t_2)$, where

$$\nu(\overline{r}_i) = \begin{cases} 1 & \text{if } u \text{ jumps from } -1 \text{ to } 1 \text{ across } \overline{r}_i, \\ -1 & \text{if } u \text{ jumps from } 1 \text{ to } -1 \text{ across } \overline{r}_i. \end{cases}$$

Equation (1.8) which is written in the case of spherical symmetry corresponds to the interface motion

$$V_n = -(N-1)K + (N-1)\fint_{\Gamma_t} K + \frac{3}{2}\Big(v - \fint_{\Gamma_t} v\Big).$$

We remark that a single interface does not move when there is spherical symmetry. The complicated form of equation (1.8) is due to the fact that a number of interfaces which corresponds to the point $\overline{r}_i(t)$ are involved.

This study is mainly based upon two articles: one of them by Henry [4] also written in the case of spherical symmetry, deals with the singular limit analysis of the fourth order model corresponding to Problem $(\mathcal{P}^\varepsilon)$; however since the limiting free boundary problems are very different, the mathematical analysis necessarily also involved different arguments. Many proofs presented here extend similar ones given by Bronsard and Stoth [3] in the simpler case of the mass-conserved Allen-Cahn equation.

We should also mention a number of articles of Ren and Wei dealing with related problems, in particular [11] and [12] where they consider the corresponding minimization problem in the one-dimensional case and in the case of spherical symmetry. They study equilibrium configurations and present the connection between the local minima of the Lyapunov functional and those of its gamma-limit.

2. Formal derivation of the interface motion equation

In this section we present a formal derivation of the equation for the displacement of the interface. We consider for a given smooth function v the equation

$$u_t = \Delta u + \frac{1}{\varepsilon^2}\Big(f(u) - \fint_\Omega f(u) - \varepsilon v\Big) \quad \text{in} \quad R^N \times (0,T)$$

together with suitable initial data and denote the solution by u^ε.

We show heuristically how to derive the motion equation

$$V_n = -(N-1)K + (N-1)\fint_{\Gamma_t} K + \frac{3}{2}\Big(v - \fint_{\Gamma_t} v\Big) \quad \text{on} \quad \Gamma_t,\ t \in [0,T] \tag{2.1}$$

as ε tends to 0.

To that purpose we define the operator

$$L^\varepsilon(\psi) := \psi_t - \Delta\psi - \frac{1}{\varepsilon^2}\Big(f(\psi) - \fint_\Omega f(\psi) - \varepsilon v\Big)$$

We denote by $q(r,w)$ the travelling wave solution associated to the function

$f(s) = 2s(1-s^2)$, namely the unique solution $(q(r,w), c(w))$ of

$$\begin{cases} q_{rr} + c(w)q_r + f(q) - w = 0 \\ q(-\infty) = h_-(w), \quad q(0) = h_0(w), \quad q(+\infty) = h_+(w), \end{cases}$$

where $c(w)$ is the travelling wave velocity and $h_-(w)$, $h_0(w)$ and $h_+(w)$ are the three solutions of the equation $f(s) = w$, such that $h_-(w) < h_0(w) < h_+(w)$. We suppose that the moving boundary Γ_t, $t \in [0,T]$ is smooth enough and denote by d the signed distance function to Γ_t. In particular, $d = 0$ on Γ_t and $|\nabla d(x,t)| = 1$ in a neighborhood of Γ_t.

We make the assumption that for ε small enough, the function u^ε can be approximated in the form

$$\tilde{u}^\varepsilon(x,t) = q\left(\frac{d(x,t)}{\varepsilon}, \varepsilon v + \fint_\Omega f(u^\varepsilon)\right).$$

We obtain

$$L^\varepsilon(\tilde{u}^\varepsilon) = \frac{\bar{q}_{rr}}{\varepsilon^2}(1 - |\nabla d|^2) + \frac{\bar{q}_r}{\varepsilon}\left[d_t - \Delta d + \frac{c\Big(\varepsilon v + \fint_\Omega f(u^\varepsilon)\Big)}{\varepsilon}\right].$$

where the notation $\bar{q}$, $\bar{q}_r$, $\bar{q}_{rr}$ means that we take the values of the functions q, q_r, q_{rr} in $\Big(\frac{d(x,t)}{\varepsilon}, \varepsilon v^\varepsilon + \fint_\Omega f(u^\varepsilon)\Big)$. Since u^ε satisfies $L^\varepsilon u^\varepsilon = 0$, the idea is to consider the expression above on Γ_t and to cancel the lower order terms. Setting to 0 the coefficient of $\frac{1}{\varepsilon}$ in the second term and using the fact that $c(0) = 0$ we deduce that

$$d_t - \Delta d + c'(0)\Big(v_{|\Gamma_t} + \fint_\Omega f(u)\Big) = 0 \quad \text{on} \quad \Gamma_t,$$

which can be rewritten as

$$V_n = -(N-1)K + c'(0)\Big(v_{|\Gamma_t} + \fint_\Omega f(u)\Big), \tag{2.2}$$

after substituting $\Delta d = (N-1)K$ and $d_t = V_n$ on Γ_t. Furthermore integrating this equation on Γ_t we obtain that

$$\fint_{\Gamma_t} V_n = -(N-1)\fint_{\Gamma_t} K + c'(0)\Big(\fint_{\Gamma_t} v + \fint_\Omega f(u)\Big) \tag{2.3}$$

By the conservation law we have that

$$\begin{cases} |\Omega_t^+| - |\Omega_t^-| = \mathcal{M}_0 \\ |\Omega_t^+| + |\Omega_t^-| = \Omega, \end{cases}$$

which implies that ([13])

$$\frac{\partial}{\partial t}|\Omega_t^-| = \int_{\Gamma_t} V_n = 0.$$

Substituting this into (2.3) we deduce that

$$⨍_\Omega f(u) = \frac{1}{c'(0)}(N-1)⨍_{\Gamma_t} K - ⨍_{\Gamma_t} v,$$

which together with (2.2) gives

$$V_n = -(N-1)K + c'(0)v_{|\Gamma_t} + (N-1)⨍_{\Gamma_t} K - c'(0)⨍_{\Gamma_t} v. \qquad (2.4)$$

In the case that $f(s) = 2s(1-s^2)$ we have that $c'(0) = \frac{3}{2}$, so that (2.4) coincides with the interface motion equation (2.1).

3. A priori estimates and first convergence results

First we give an equivalent form of Problem $(\mathcal{P}^\varepsilon)$, namely

$$(P^\varepsilon) \begin{cases} u_t^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2}(f(u^\varepsilon) - \varepsilon v^\varepsilon - \varepsilon\lambda^\varepsilon) & \text{in } \Omega \times (0,T) & (3.1) \\ \int_\Omega u^\varepsilon(x,t)dx = \int_\Omega u_0^\varepsilon(x)dx & \text{for } t \in (0,T) & (3.2) \\ -\Delta v^\varepsilon = u^\varepsilon - ⨍_\Omega u^\varepsilon & \text{in } \Omega \times (0,T) & (3.3) \\ \int_\Omega v^\varepsilon(x,t)dx = 0 & \text{for } t \in (0,T) & (3.4) \\ \frac{\partial u^\varepsilon}{\partial n} = \frac{\partial v^\varepsilon}{\partial n} = 0 & \text{in } \partial\Omega \times (0,T) & (3.5) \\ u^\varepsilon(x,0) = u_0^\varepsilon(x) & \text{for } x \in \Omega & (3.6) \end{cases}$$

where $\lambda^\varepsilon(t) = \frac{1}{\varepsilon}⨍_\Omega f(u^\varepsilon(x,t))dx$.

We show below some estimates, which imply in particular the compactness of $\{u^\varepsilon\}$ in $L^1(\Omega \times (0,T))$.

Lemma 3.1 *Let $(u^\varepsilon, v^\varepsilon)$ be the solution of Problem (P^ε) and suppose that u_0^ε satisfies the hypothesis H_0^ε. Let $g(s) = \int_0^s \sqrt{2F(\tau)}d\tau$ and $0 \leq \tau \leq s \leq T$, then*

$$\varepsilon \int_\tau^s \int_\Omega (u_t^\varepsilon)^2 dx\, dt + E^\varepsilon(u^\varepsilon)(s) - E^\varepsilon(u^\varepsilon)(\tau) = 0, \tag{3.7}$$

which implies that the function $t \to E^\varepsilon(u^\varepsilon)(t)$ is nonincreasing. We have that

$$\int_\Omega |\nabla g(u^\varepsilon(x,t))| dx \leq E^\varepsilon(u^\varepsilon)(t) \leq C, \quad \text{for all} \ \ t \in [0,T], \tag{3.8}$$

$$\int_\tau^s \int_\Omega |(g(u^\varepsilon))_t| dx\, dt \leq C\sqrt{s-\tau}. \tag{3.9}$$

Proof. In order to prove (3.7) we multiply (3.1) by u_t^ε and integrate on $\Omega \times (\tau, s)$. This gives

$$\int_\tau^s \int_\Omega (u_t^\varepsilon)^2 - \int_\tau^s \int_\Omega \Delta(u^\varepsilon) u_t^\varepsilon = \frac{1}{\varepsilon^2} \int_\tau^s \int_\Omega (f(u^\varepsilon) - \varepsilon v^\varepsilon - \varepsilon \lambda^\varepsilon) u_t^\varepsilon.$$

Thus we have

$$\int_\tau^s \int_\Omega (u_t^\varepsilon)^2 \, dx\, dt + \int_\Omega \Big(\frac{1}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u^\varepsilon) + \frac{1}{2\varepsilon}|\nabla v^\varepsilon|^2\Big)(x,s)\, dx$$
$$= \int_\Omega \Big(\frac{1}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u^\varepsilon) + \frac{1}{2\varepsilon}|\nabla v^\varepsilon|^2\Big)(x,\tau)\, dx,$$

which coincides with (3.7). Next we prove (3.8). We have in view of (3.7) and hypothesis H_0^ε that

$$\begin{aligned}\int_\Omega |\nabla(g(u^\varepsilon(x,t)))| dx &= \int_\Omega \sqrt{2F(u^\varepsilon(x,t))}\, |\nabla u^\varepsilon(x,t)| \\ &\leq \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon)\Big) dx \\ &\leq E^\varepsilon(u^\varepsilon)(t) \leq C.\end{aligned}$$

Finally we prove (3.9).

$$\begin{aligned}&\int_\tau^s \int_\Omega |(g(u^\varepsilon))_t| dx\, dt \\ &\quad = \int_\tau^s \int_\Omega \left| \frac{g'(u^\varepsilon)}{\sqrt{\varepsilon}} \sqrt{\varepsilon} u_t^\varepsilon \right| dx\, dt\end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{2}\left(\int_\tau^s\int_\Omega\left|\frac{F(u^\varepsilon)}{\varepsilon}\right|dx\,dt\right)^{1/2}\left(\int_\tau^s\int_\Omega\varepsilon|u_t^\varepsilon|^2dx\,dt\right)^{1/2}\\
&\leq \sqrt{2}\left(\int_\tau^s E^\varepsilon(u^\varepsilon)(t)dt\right)^{1/2}\left(E^\varepsilon(u^\varepsilon)(\tau)-E^\varepsilon(u^\varepsilon)(s)\right)^{1/2}\\
&\leq C\sqrt{s-\tau}.
\end{aligned}
$$

This completes the proof of Lemma 3.1. □

Corollary 3.2 *Suppose that the Hypothesis H_0^ε is satisfied. Then*

$$\{u^\varepsilon\} \text{ is bounded in } L^\infty(0,T,L^4(\Omega)); \tag{3.10}$$

$$\sup_{t\in[0,T]}|\lambda^\varepsilon(t)|\leq C\varepsilon^{-1/2} \tag{3.11}$$

Proof. (3.10) follows from the Hypothesis H_0^ε and (3.7). Next we prove (3.11). Integrating (3.1) on Ω we deduce that

$$\lambda^\varepsilon(t)=\frac{2}{\varepsilon|\Omega|}\int_\Omega u^\varepsilon(1-(u^\varepsilon)^2),$$

so that

$$|\lambda^\varepsilon(t)|\leq\frac{2}{\varepsilon|\Omega|}\left(\int_\Omega(u^\varepsilon)^2\right)^{1/2}\left(\int_\Omega(1-(u^\varepsilon)^2)^2\right)^{1/2}.$$

Using (3.7) we deduce that

$$|\lambda^\varepsilon(t)|\leq\frac{C}{\varepsilon}\varepsilon^{1/2}\left(\int_\Omega(u^\varepsilon)^2\right)^{1/2}\leq\tilde{C}\varepsilon^{-1/2},$$

which coincides with (3.11). □

Lemma 3.3 *There exists a positive constant C, depending on Ω, such that*

$$\sup_{t\in[0,T]}\int_\Omega\left(|v^\varepsilon|^2+|\nabla v^\varepsilon|^2\right)(x,t)dx\leq C$$

Proof. We deduce from (3.8) that

$$\sup_{t\in[0,T]}\int_\Omega|\nabla v^\varepsilon|^2(x,t)dx\leq C_1. \tag{3.12}$$

The generalized Poincaré inequality

$$\|h\|^2_{L^2(\Omega)} \le C\left(\|\nabla h\|^2_{L^2(\Omega)} + \left|\int_\Omega h\right|^2\right), \quad \text{for all} \;\; h \in H^1(\Omega),$$

together with (3.12) and the fact that $\int_\Omega v^\varepsilon = 0$ implies that

$$\int_\Omega |v^\varepsilon|^2 \, dx \le C.$$

This completes the proof of Lemma 3.3. □

Next we show the first convergence results.

Theorem 3.4 *There exists a subsequence* $\{\varepsilon_n\}$ *and functions* u, v *such that*

$$u^{\varepsilon_n} \to u \;\; \textit{in} \;\; L^1(\Omega \times (0,T)) \;\; \textit{and a.e. in} \;\; \Omega \times (0,T)$$
$$\textit{and moreover} \;\; u = \pm 1,$$
$$v^{\varepsilon_n} \rightharpoonup v \;\; \textit{weakly in} \;\; L^2(0,T,H^1(\Omega)),$$

as ε *tends to* 0.

Proof. Using the fact that

$$|g(s)| \le C_1|s|^3 + C_2 \;\; \text{for all} \;\; s \in R,$$

and (3.10) we deduce that $\{g(u^\varepsilon)\}$ is bounded in $L^1(\Omega \times (0,T))$. This together with (3.8) and (3.9) implies that there exist a subsequence $\{\varepsilon_n\}$ which we denote again by $\{\varepsilon\}$ and a function ξ such that $\{g(u^\varepsilon)\}$ tends to ξ in $L^1(\Omega \times (0,T))$ and a.e as $\varepsilon \downarrow 0$. Since the function g is continuous and strictly increasing one can define its inverse g^{-1} and deduce that as $\varepsilon \downarrow 0$

$$u^\varepsilon \;\; \text{tends to} \;\; u := g^{-1}(\xi) \;\; \text{a.e. in} \;\; \Omega \times (0,T).$$

Next we show that u^ε tends to u in $L^1(\Omega \times (0,T))$ as $\varepsilon \downarrow 0$. Let $\delta > 0$ be arbitrary. It follows from Egoroff's Lemma (see for instance Rudin [14]) that there exists $\Omega_\delta \subset \Omega$ such that

$$|\Omega \setminus \Omega_\delta| \le \delta \;\; \text{and} \;\; u^\varepsilon \to u \;\; \text{uniformly in} \;\; \Omega_\delta \;\; \text{as} \;\; \varepsilon \downarrow 0.$$

Then

$$\int_\Omega |u^\varepsilon - u| = \int_{\Omega\setminus\Omega_\delta} |u^\varepsilon - u| + \int_{\Omega_\delta} |u^\varepsilon - u| \tag{3.13}$$

and we have that

$$\int_{\Omega\setminus\Omega_\delta}|u^\varepsilon-u|\le|\Omega\setminus\Omega_\delta|^{2/3}\Big(\int_{\Omega\setminus\Omega_\delta}|u^\varepsilon-u|^3\Big)^{1/3}$$
$$\le C|\Omega\setminus\Omega_\delta|^{2/3}\Big(\int_{\Omega\setminus\Omega_\delta}|u^\varepsilon|^3+|u|^3\Big)^{1/3} \tag{3.14}$$

Note that there exist positive constants D_1 and D_2 such that

$$|s|^3\le D_1g(s)+D_2,\quad \text{for all}\ \ s\in R. \tag{3.15}$$

Therefore we deduce from (3.14) that

$$\int_{\Omega\setminus\Omega_\delta}|u^\varepsilon-u|\le C_1|\Omega\setminus\Omega_\delta|^{2/3}\Big(\int_{\Omega\setminus\Omega_\delta}|g(u^\varepsilon)|+|g(u)|+C_2\Big)^{1/3}$$
$$\le C_3|\Omega\setminus\Omega_\delta|^{2/3}\le C\delta^{2/3} \tag{3.16}$$

Let $\eta>0$ be arbitrary. Choose δ small enough such that $C\delta^{2/3}<\frac{\eta}{2}$ and choose ε small enough such that

$$\int_{\Omega_\delta}|u^\varepsilon-u|\le\frac{\eta}{2} \tag{3.17}$$

Using (3.13), (3.16) and (3.17) we deduce that $\int_\Omega|u^\varepsilon-u|\le\eta$. Next we check that $u=\pm1$ a.e. in $\Omega\times(0,T)$. In view of (3.7) and Fatou's Lemma we deduce that

$$\int_o^T\int_\Omega\liminf_{\varepsilon\to0}F(u^\varepsilon)\,dx\,dt\le\liminf_{\varepsilon\to0}\int_o^T\int_\Omega F(u^\varepsilon)\,dx\,dt\le0.$$

This implies that $F(u)=0$ and thus that $u=\pm1$ a.e. in $\Omega\times(0,T)$. Moreover we also deduce that

$$u^\varepsilon(.,t)\to u\ \text{ in }\ L^1(\Omega)\ \text{ a.e. in }\ (0,T), \tag{3.18}$$

which will be used in the sequel. Next we prove the convergence of the sequence $\{v^\varepsilon\}$. We deduce from Lemma 3.3 that $\{v^\varepsilon\}$ is bounded in $L^2(0,T;H^1(\Omega))$. This in turn implies that there exists a function v and a subsequence $\{\varepsilon_n\}$ that we denote again by $\{\varepsilon\}$ such that $\{v^\varepsilon\}$ tends to v weakly in $L^2(0,T;H^1(\Omega))$. This completes the proof of Theorem 3.4.

Finally we deduce that the functions (u, v) are such that

$$\begin{cases} -\Delta v = u - \fint_\Omega u \quad \text{a.e. in } \Omega \times (0,T) \\ \fint_\Omega v\,dx = 0 \qquad \text{for a.e. } t \in (0,T) \\ \dfrac{\partial v}{\partial n} = 0 \qquad \text{a.e. on } \partial\Omega \times (0,T). \end{cases}$$

□

Theorem 3.5 *The sequence $\{t \to E^\varepsilon(u^\varepsilon)(t)\}$ is bounded in $W^{1,1}(0,T)$. Therefore there exists a function $E_0 \in BV(0,T)$ and a subsequence $\{\varepsilon_n\}$, which we denote again by $\{\varepsilon\}$ such that*

$$E^\varepsilon(u^\varepsilon)(.) \to E_0(.) \text{ in } L^1(0,T) \text{ and a.e. in } (0,T).$$

Proof. By (3.7) $\{E^\varepsilon(u^\varepsilon)(.)\}$ is bounded in $L^\infty(0,T)$. Moreover (3.7) also implies that the function $E^\varepsilon(u^\varepsilon)(.)$ is decreasing so that in fact $\{E^\varepsilon(u^\varepsilon)(.)\}$ is bounded in $BV(0,T)$. This completes the proof of Theorem 3.5. □

As it is done by L. Bronsard and B. Stoth [3] we define for any $\eta > 0$ a set $\tilde{N}(\eta) \subset (0,T)$ as the set of all jump points of E_0 with height at least η:

$$\tilde{N}(\eta) = \left\{ t,\ ess \inf_{s<t} E_0(s) - ess \sup_{s>t} E_0(s) \geq \eta \right\} \tag{3.19}$$

Then for any $\eta > 0$, the set $\tilde{N}(\eta)$ is finite since E_0 is monotone decreasing. Furthermore since E^ε and therefore E_0 are bounded, it follows that

$$\text{card}\,(\tilde{N}(\eta)) \leq \frac{C}{\eta}.$$

For $t_0 > 0$ and η not too large we define $T^\varepsilon(\eta, t_0) > 0$ by

$$\varepsilon \int_{t_0 - T^\varepsilon(\eta,t_0)}^{t_0 + T^\varepsilon(\eta,t_0)} \int_\Omega |u_t^\varepsilon|^2 dx\,dt = \eta \tag{3.20}$$

Next we state a result which implies that for any $t_0 \notin \tilde{N}(\eta)$ one can find an interval $(t_0 - T^\varepsilon(\eta, t_0), t_0 + T^\varepsilon(\eta, t_0))$ on which the energy $E^\varepsilon(u^\varepsilon)(.)$ is uniformly small in ε.

Lemma 3.6 *Let $0 < t_0 \notin \tilde{N}(\eta)$ where $\tilde{N}(\eta)$ is defined by (3.19) and let $T^\varepsilon(\eta, t_0)$ be as in (3.20). Then there exists $T_0(\eta, t_0) > 0$ such that*

$$T^\varepsilon(\eta, t_0) > T_0(\eta, t_0), \quad \text{for } \varepsilon \leq \varepsilon_0(\eta, t_0).$$

In particular we have

$$E^\varepsilon(u^\varepsilon)(t_0 - T_0) - E^\varepsilon(u^\varepsilon)(t_0 + T_0) = \varepsilon \int_{t_0 - T_0(\eta, t_0)}^{t_0 + T_0(\eta, t_0)} \int_\Omega |u_t^\varepsilon|^2 dx\, dt \leq \eta \tag{3.21}$$

Proof. Suppose on the contrary that $T^\varepsilon \to 0$ for some sequence. Then for $\tau > 0$ it follows from (3.7) that

$$\begin{aligned}
0 < \eta &= \lim_{\varepsilon \to 0} \int_{t_0 - T^\varepsilon(\eta, t_0)}^{t_0 + T^\varepsilon(\eta, t_0)} \int_\Omega |u_t^\varepsilon|^2 dx\, dt \\
&= \lim_{\varepsilon \to 0} E^\varepsilon(u^\varepsilon)(t_0 - T^\varepsilon(\eta, t_0)) - E^\varepsilon(u^\varepsilon)(t_0 + T^\varepsilon(\eta, t_0)) \\
&\leq \lim_{\varepsilon \to 0} E^\varepsilon(u^\varepsilon)(t_0 - \tau) - E^\varepsilon(u^\varepsilon)(t_0 + \tau) \\
&\leq E_0(t_0 - \tau) - E_0(t_0 + \tau)
\end{aligned}$$

Thus

$$0 < \eta \leq ess \inf_{s < t_0} E_0(s) - ess \sup_{s > t_0} E_0(s) < \eta,$$

which is impossible. This completes the proof of Lemma 3.6. □

4. The approximation in the radial case

The purpose of the next subsection is to prove a central estimate, which implies that the function u^ε can be locally approximated by a hyperbolic tangent profile in the neighborhood of each of its zeros. More precisely, we prove a result (see also Stoth [2]) that implies that far away from the origin and near each of its zeros, the solution u^ε is close to the function $\pm q(\frac{\xi}{\varepsilon})$ where q satisfies

$$\begin{cases} q_{\xi\xi} + f(q) = 0 \\ q(-\infty) = -1, \quad q(0) = 0, \quad q(+\infty) = 1, \end{cases}$$

so that $q(\xi) = \tanh(\xi)$.

4.1. The central estimate

From now on we suppose that Ω is the unit ball in R^N and that u_0^ε is spherically symmetric. We express the solutions $(u^\varepsilon, v^\varepsilon)$ in the spherical variable $r = |x|$. It follows from the regularity of u^ε and v^ε that

$$(Or) \quad u_r^\varepsilon(0,t) = v_r^\varepsilon(0,t) = 0, \quad \text{for all } \ t \in (0,T).$$

Problem (P^ε) takes the form

$$(P_r^\varepsilon)\begin{cases} \varepsilon u_t^\varepsilon - \varepsilon u_{rr}^\varepsilon - \varepsilon \dfrac{N-1}{r} u_r^\varepsilon - \dfrac{1}{\varepsilon} f(u^\varepsilon) + v^\varepsilon + \lambda^\varepsilon = 0 & \\ & \text{in } (0,1)\times(0,T) \quad (4.1) \\ -v_{rr}^\varepsilon - \dfrac{N-1}{r} v_r^\varepsilon = u^\varepsilon - \displaystyle\fint_\Omega u^\varepsilon dx & \text{in } (0,1)\times(0,T) \quad (4.2) \\ \displaystyle\int_0^1 u^\varepsilon(r,t) r^{N-1} dr = \int_0^1 u_0^\varepsilon(r) r^{N-1} dr & \text{in } (0,T) \\ \displaystyle\int_0^1 v^\varepsilon(r,t) r^{N-1} dr = 0 & \text{for } t \in (0,T) \quad (4.3) \\ u_r^\varepsilon(0,t) = v_r^\varepsilon(0,t) = 0 & \text{for } t \in (0,T) \quad (4.4) \\ u_r^\varepsilon(1,t) = v_r^\varepsilon(1,t) = 0 & \text{for } t \in (0,T) \quad (4.5) \\ u^\varepsilon(r,0) = u_0^\varepsilon(r) & \text{for } r \in (0,1) \quad (4.6) \end{cases}$$

The energy estimates becomes

$$\sup_{t\in[0,T]} \int_0^1 \Big(\frac{\varepsilon}{2}(u_r^\varepsilon)^2 + \frac{1}{\varepsilon} F(u^\varepsilon) + \frac{1}{2}(v_r^\varepsilon)^2\Big) r^{N-1} dr \le K \tag{4.7}$$

Next we give some preliminary estimates.

Lemma 4.1 *Let $0 < R_0 < 1$; there exists a function $C(R_0)$ independant of ε such that v^ε satisfy the following estimates*

$$\sup_{t\in[0,T]} \int_{R_0}^1 (|v^\varepsilon|^2 + |v_r^\varepsilon|^2)(r,t) dr \le C(R_0), \tag{4.8}$$

$$\sup_{t\in[0,T]} \|v^\varepsilon(.,t)\|_{L^\infty(R_0,1)} \le C(R_0), \tag{4.9}$$

$$\sup_{t\in[0,T]} \|u^\varepsilon(.,t)\|_{L^\infty(R_0,1)} \le C(R_0). \tag{4.10}$$

Proof. In order to prove (4.8), we note that

$$\int_{R_0}^1 |v^\varepsilon|^2 + |v_r^\varepsilon|^2 dr \leq \frac{1}{R_0^{N-1}} \int_{R_0}^1 (|v^\varepsilon|^2 + |v_r^\varepsilon|^2) r^{N-1} dr.$$

Using Lemma 3.3 we deduce (4.8). (4.9) immediately follows from (4.8). Next we prove (4.10). We have

$$\begin{aligned}
&|g(u^\varepsilon(r,t))| \\
&\quad \leq |g(u^\varepsilon(s,t))| + \int_{R_0}^1 |(g(u^\varepsilon(\xi,t))_r| d\xi \\
&\quad \leq |g(u^\varepsilon(s,t))| + \int_{R_0}^1 |u_r^\varepsilon(\xi,t)||\sqrt{2F(u^\varepsilon(\xi,t))}| d\xi \\
&\quad \leq |g(u^\varepsilon(s,t))| + \frac{1}{R_0^{N-1}} \int_{R_0}^1 \Big[\varepsilon |u_r^\varepsilon(\xi,t)|^2 + \frac{1}{2\varepsilon} F(u^\varepsilon(\xi,t))\Big] \xi^{N-1}.d\xi
\end{aligned}$$

Using (3.15) and (4.7) this implies that

$$|u^\varepsilon(r,t)|^3 \leq D_1 |g(u^\varepsilon(s,t))| + D_2 + \frac{K}{R_0^{N-1}}, \quad \text{for all} \quad r, s \in [R_0, 1]. \tag{4.11}$$

Moreover using the mean value theorem we obtain that there exists $\rho \in (R_0, 1)$ such that

$$\int_{R_0}^1 u^\varepsilon(\xi,t) d\xi = u^\varepsilon(\rho,t)(1-R_0)$$

This gives that

$$|u^\varepsilon(\rho,t)| = \frac{1}{1-R_0} \left| \int_{R_0}^1 u^\varepsilon(\xi,t) d\xi \right| \leq \frac{1}{(1-R_0) R_0^{N-1}} \int_0^1 |u^\varepsilon| r^{N-1} dr$$

In view of (3.10) we deduce that $|u^\varepsilon(\rho,t)| \leq K_1(R_0)$, which together with the fact that the function g is continuous, implies that

$$|g(u^\varepsilon(\rho,t))| \leq K_2(R_0).$$

Therefore applying (4.11) with $s = \rho$ we obtain that $\|u^\varepsilon(.,t)\|_{L^\infty(R_0,1)} \leq K_3(R_0)$ for all $t \in [0,T]$. This completes the proof of Lemma 4.1. □

Lemma 4.2 *Let $0 < R_0 < 1$ and $t_1 > t_2$; u^ε satisfies that*

$$\left\| -\frac{\varepsilon^2}{2}(u_r^\varepsilon(.,t))^2 + F(u^\varepsilon(.,t)) \right\|_{L^\infty(R_0,1)}$$
$$\leq \tilde{C}(R_0)\Big[\varepsilon^{1/2} + \Big(\varepsilon \int_{t_2}^{t_1} \int_0^1 |u_t^\varepsilon|^2(r,t) r^{N-1} dr \Big)^{1/2} + \Big(\varepsilon^3 \int_0^1 |u_t^\varepsilon|^2(r,t_2) r^{N-1} dr \Big)^{1/2} \Big],$$

for all $t \in (t_2, t_1)$.

Proof. We multiply (4.1) by $\varepsilon u_r^\varepsilon$ and integrate on $(\eta, \rho) \subset [R_0, 1]$ to obtain

$$\int_\eta^\rho \varepsilon^2 u_{rr}^\varepsilon u_r^\varepsilon + f(u^\varepsilon) u_r^\varepsilon = \int_\eta^\rho \Big(\varepsilon^2 u_t^\varepsilon - \varepsilon^2 \frac{N-1}{r} (u_r^\varepsilon)^2 + \varepsilon(v^\varepsilon + \lambda^\varepsilon) \Big) u_r^\varepsilon.$$

This in turn implies that

$$-\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\rho,t) + F(u^\varepsilon(\rho,t))$$
$$= -\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\eta,t) + F(u^\varepsilon(\eta,t)) - \varepsilon^2 \int_\eta^\rho u_t^\varepsilon u_r^\varepsilon + \varepsilon^2(N-1) \int_\eta^\rho \frac{(u_r^\varepsilon)^2}{r} - \varepsilon \int_\eta^\rho (v^\varepsilon + \lambda^\varepsilon)) u_r^\varepsilon.$$

Taking absolute values we deduce that

$$\left| -\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\rho,t) + F(u^\varepsilon(\rho,t)) \right|$$
$$\leq \left| -\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\eta,t) + F(u^\varepsilon(\eta,t)) \right| + \frac{\varepsilon^2}{R_0^{N-1}} \left| \int_\eta^\rho |u_t^\varepsilon|^2 r^{N-1} \right|^{1/2} \left| \int_\eta^\rho |u_r^\varepsilon|^2 r^{N-1} \right|^{1/2}$$
$$+ \frac{\varepsilon^2(N-1)}{R_0^N} \left| \int_\eta^\rho |u_r^\varepsilon|^2 r^{N-1} \right| + \varepsilon \left| \int_\eta^\rho |v^\varepsilon|^2 \right|^{1/2} \left| \int_\eta^\rho |u_r^\varepsilon|^2 \right|^{1/2} + \varepsilon |\lambda^\varepsilon| |u^\varepsilon(\rho,t) - u^\varepsilon(\eta,t)|$$

Using (3.7), (3.11), (4.7), (4.8) and (4.10) we deduce that

$$\begin{aligned}&\left|-\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\rho,t)+F(u^\varepsilon(\rho,t))\right|\\&\quad\le\left|-\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\eta,t)+F(u^\varepsilon(\eta,t))\right|+\frac{\varepsilon^{3/2}}{R_0^{N-1}}K\Big(\int_\eta^\rho|u_t^\varepsilon|^2r^{N-1}\Big)^{1/2}\\&\qquad+\varepsilon(N-1)C_1(R_0)+\varepsilon^{1/2}C_2(R_0)+\varepsilon^{1/2}C(R_0),\end{aligned}\tag{4.12}$$

which we integrate for $\eta\in(R_0,1)$. It follows from (4.7) that for all $t\in[t_2,t_1]$

$$\begin{aligned}&\left|-\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\rho,t)+F(u^\varepsilon(\rho,t))\right|\\&\quad\le C_3(R_0)\left[\varepsilon^{1/2}+\varepsilon^{3/2}\Big(\int_{R_0}^1|u_t^\varepsilon|^2r^{N-1}\Big)^{1/2}\right]\end{aligned}\tag{4.13}$$

This gives that

$$\left|-\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\rho,t)+F(u^\varepsilon(\rho,t))\right|\le C(R_0)\left[\varepsilon^{1/2}+\Big(\varepsilon^3\int_\Omega|u_t^\varepsilon|^2\Big)^{1/2}\right].\tag{4.14}$$

Let us differentiate (1.1) with respect to t and multiply the resulting equation by $\varepsilon^2u_t^\varepsilon$. We deduce using the conservation of mass property that

$$\varepsilon^2\int_{t_2}^\tau\int_\Omega u_{tt}^\varepsilon u_t^\varepsilon-\varepsilon^2\int_{t_2}^\tau\int_\Omega\Delta u_t^\varepsilon u_t^\varepsilon=\int_{t_2}^\tau\int_\Omega f'(u^\varepsilon)(u_t^\varepsilon)^2-\varepsilon\int_{t_2}^\tau\int_\Omega v_t^\varepsilon u_t^\varepsilon,$$

that is

$$\begin{aligned}&\frac{\varepsilon^2}{2}\int_\Omega(u_t^\varepsilon)^2(x,\tau)+\varepsilon^2\int_{t_2}^\tau\int_\Omega|\nabla u_t^\varepsilon|^2+\varepsilon\int_{t_2}^\tau\int_\Omega v_t^\varepsilon u_t^\varepsilon\\&\quad=\frac{\varepsilon^2}{2}\int_\Omega(u_t^\varepsilon)^2(x,t_2)+\int_{t_2}^\tau\int_\Omega f'(u^\varepsilon)(u_t^\varepsilon)^2.\end{aligned}\tag{4.15}$$

Next we differentiate (1.2) with respect to t and we multiply the resulting equation by v_t^ε to obtain

$$-\int_\Omega v_t^\varepsilon\Delta v_t^\varepsilon=\int_\Omega v_t^\varepsilon u_t^\varepsilon=\int_\Omega|\nabla v_t^\varepsilon|^2,$$

which we substitute in (4.15) to obtain

$$\frac{\varepsilon^2}{2}\int_\Omega (u_t^\varepsilon)^2(x,\tau) + \varepsilon^2\int_{t_2}^\tau\int_\Omega |\nabla u_t^\varepsilon|^2 + \varepsilon\int_{t_2}^\tau\int_\Omega |\nabla v_t^\varepsilon|^2$$
$$= \frac{\varepsilon^2}{2}\int_\Omega (u_t^\varepsilon)^2(x,t_2) + \int_{t_2}^\tau\int_\Omega f'(u^\varepsilon)(u_t^\varepsilon)^2.$$

Moreover using the fact that $f'(s) = 2 - 6s^2 \leq 2$ we deduce that

$$\frac{\varepsilon^2}{2}\int_\Omega (u_t^\varepsilon)^2(x,\tau)dx \leq \frac{\varepsilon^2}{2}\int_\Omega (u_t^\varepsilon)^2(x,t_2)dx + 2\int_{t_2}^\tau\int_\Omega (u_t^\varepsilon)^2 dx\,dt,$$

which we substitute into (4.14) to deduce that

$$\left|-\frac{\varepsilon^2}{2}(u_r^\varepsilon)^2(\rho,t) + F(u^\varepsilon(\rho,t))\right|$$
$$\leq C(R_0)\left[\varepsilon^{1/2} + \left(\varepsilon^3\int_\Omega |u_t^\varepsilon|^2(x,t_2)dx + \varepsilon\int_{t_2}^t\int_\Omega (u_t^\varepsilon)^2 dx\,ds\right)^{1/2}\right]$$

This completes the proof of Lemma 4.2. □

Next we show that away from the origin the solution u^ε is close to the standing wave q in $(t_0 - T_0, t_0 + T_0)$.

Theorem 4.3 *Let $R_0 \in (0,1)$, $\delta > 0$ and $\eta = \frac{\delta^4}{4\tilde{C}^2(R_0)}$ where $\tilde{C}(R_0)$ is defined in Lemma 4.2. Suppose that $t_0 \neq 0$ and that $t_0 \notin \tilde{N}(\eta) =: N(\delta, R_0)$ where $\tilde{N}$ is given by (3.19). Then there exists $T_0 = T_0(\delta, R_0, t_0) > 0$ and $\varepsilon_0 = \varepsilon_0(\delta, R_0, t_0) > 0$ such that*

$$\left\|\frac{\varepsilon^2}{2}(u_r^\varepsilon(.,t))^2 - F(u^\varepsilon(.,t))\right\|_{L^\infty((R_0,1)\times(t_0-T_0,t_0+T_0))} \leq \delta^2,$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. We choose $T_0 = T_0(\delta, R_0, t_0)$ as in Lemma 3.6. So that

$$\int_{t_0-T_0}^{t_0-T_0/2}\int_0^1 |u_t^\varepsilon|^2(r,t)r^{N-1}dr\,dt, \leq \frac{\eta}{\varepsilon}.$$

which by the mean value theorem implies that there exists $t_2 \in \left(t_0 - T_0, t_0 - \frac{T_0}{2}\right)$ such that

$$\frac{T_0}{2}\int_0^1 |u_t^\varepsilon|^2(r,t_2)r^{N-1}dr \leq \frac{\eta}{\varepsilon}. \tag{4.16}$$

Applying Lemma 4.2 with $t_1 = t_0 + \frac{T_0}{2}$ and $t_2 \in \left(t_0 - T_0, t_0 - \frac{T_0}{2}\right)$ we deduce that for all $t \in \left[t_0 - \frac{T_0}{2}, t_0 + \frac{T_0}{2}\right]$

$$\left\| -\frac{\varepsilon^2}{2}(u_r^\varepsilon(.,t))^2 + F(u^\varepsilon(.,t)) \right\|_{L^\infty(R_0,1)}$$
$$\leq \tilde{C}(R_0)\left[\varepsilon^{1/2} + \left(\varepsilon \int_{t_0-T_0}^{t_0+T_0/2} \int_0^1 |u_t^\varepsilon|^2(r,t) r^{N-1} dr\right)^{1/2} + \left(\varepsilon^3 \int_\Omega |u_t^\varepsilon|^2(x,t_2)dx\right)^{1/2}\right].$$

Using (4.16) and Lemma 3.6 we obtain for all ε small enough

$$\left\| -\frac{\varepsilon^2}{2}(u_r^\varepsilon(.,t))^2 + F(u^\varepsilon(.,t)) \right\|_{L^\infty(R_0,1)}$$
$$\leq \tilde{C}(R_0)\left[\varepsilon^{1/2} + \eta^{1/2} + \varepsilon\sqrt{\frac{2}{T_0}}\eta^{1/2}\right]$$
$$\leq \tilde{C}(R_0)\varepsilon^{1/2} + \frac{\delta^2}{2}\left(1 + \varepsilon\sqrt{\frac{2}{T_0}}\right) \leq \delta^2$$

where we have used the definition of η given in Theorem 4.3. Replacing $\frac{T_0}{2}$ by T_0 we deduce the result of Theorem 4.3. □

4.2. A new estimate for λ^ε

In this section we prove a uniform estimate for the Lagrange multiplier λ^ε.

Lemma 4.4 *There exists a positive constant C, independent of ε, such that*

$$\int_0^T |\lambda^\varepsilon(t)|^2 dt \leq C$$

Proof. We multiply the equation (4.1) by u_r^ε and integrate on (σ, ρ) with $R_0 < \sigma < \rho < 1$ to deduce that

$$\int_\sigma^\rho \left(\varepsilon u_t^\varepsilon u_r^\varepsilon - \varepsilon u_{rr}^\varepsilon u_r^\varepsilon - \varepsilon \frac{N-1}{r}(u_r^\varepsilon)^2 - \frac{1}{\varepsilon} f(u^\varepsilon) u_r^\varepsilon + v^\varepsilon u_r^\varepsilon + \lambda^\varepsilon(t) u_r^\varepsilon\right) dr = 0,$$

which we multiply by $\sigma^{N-1}\rho^{N-1}$ to obtain that

$$\begin{aligned}
&\lambda^\varepsilon(t)\,(u^\varepsilon(\rho,t) - u^\varepsilon(\sigma,t))\,\sigma^{N-1}\rho^{N-1} \\
&= \sigma^{N-1}\rho^{N-1}\Big(-\varepsilon\int_\sigma^\rho u_t^\varepsilon u_r^\varepsilon + \frac{\varepsilon}{2}[(u_r^\varepsilon(\rho,t))^2 - (u_r^\varepsilon(\sigma,t))^2] \\
&\quad + \varepsilon(N-1)\int_\sigma^\rho \frac{(u_r^\varepsilon)^2}{r} - \frac{1}{\varepsilon}\,[F(u^\varepsilon(\rho,t)) - F(u^\varepsilon(\sigma,t))] - \int_\sigma^\rho v^\varepsilon u_r^\varepsilon\Big) \\
&=: G(\sigma,\rho). \qquad (4.17)
\end{aligned}$$

In what follows we integrate (4.17) in ρ in a set where u^ε is close to 1 and we integrate it in σ in a set where u^ε is close to -1. The idea is to use the mass conservation property to show that the measure of those sets is bounded from below by a positive constant which does not depend on t. By hypothesis H_0^ε we may define $\omega > 0$ such that

$$-|\Omega| + \omega \le \mathcal{M}_0^\varepsilon \le |\Omega| - \omega, \qquad (4.18)$$

for ε small enough. For $1/2 < \eta < 1$ and close to 1 we define

$$A_1^\varepsilon := \left\{y \in \Omega \text{ such that } \eta \le u^\varepsilon(y,t) \le \frac{1}{\eta}\right\},$$

and

$$A_2^\varepsilon := \left\{y \in \Omega \text{ such that } -\frac{1}{\eta} \le u^\varepsilon(y,t) \le -\eta\right\},$$

$$B_1^\varepsilon := \{y \in \Omega \text{ such that } |u^\varepsilon(y,t)| < \eta\},$$

and

$$B_2^\varepsilon := \left\{y \in \Omega \text{ such that } |u^\varepsilon(y,t)| > \frac{1}{\eta}\right\}.$$

We have that $y \in B_1^\varepsilon$ implies that $F(u^\varepsilon(y,t)) \ge F(\eta)$ and $y \in B_2^\varepsilon$ implies that $F(u^\varepsilon(y,t)) \ge F(\frac{1}{\eta}) \ge F(\eta)$. This gives in view of (3.7) that

$$|B_1^\varepsilon \cup B_2^\varepsilon| \le \frac{\int_\Omega F(u^\varepsilon)}{F(\eta)} \le \frac{C\varepsilon}{F(\eta)}. \qquad (4.19)$$

Moreover we note that

$$|s| \le 1 + |1 - s^2| \quad \text{for all} \quad s \in R, \qquad (4.20)$$

which implies that

$$\int_{B_1^\varepsilon \cup B_2^\varepsilon} |u^\varepsilon| \le \int_{B_1^\varepsilon \cup B_2^\varepsilon} 1 + |1-(u^\varepsilon)^2|$$
$$\le |B_1^\varepsilon \cup B_2^\varepsilon| + |B_1^\varepsilon \cup B_2^\varepsilon|^{1/2}\Big(\int_\Omega |1-(u^\varepsilon)^2|^2\Big)^{1/2}.$$

In view of (3.7) and of (4.19) we deduce that $\int_{B_1^\varepsilon \cup B_2^\varepsilon} |u^\varepsilon| \le \frac{C_1 \varepsilon}{F(\eta)}$, which together with (4.19) implies that for ε small enough

$$\int_{B_1^\varepsilon \cup B_2^\varepsilon} |u^\varepsilon| \le \frac{\omega}{4} \quad \text{and} \quad |B_1^\varepsilon \cup B_2^\varepsilon| \le \frac{\omega}{4}. \tag{4.21}$$

Moreover since $\Omega \setminus \{A_1^\varepsilon \cup A_2^\varepsilon\} = B_1^\varepsilon \cup B_2^\varepsilon$ we also have

$$|A_1^\varepsilon \cup A_2^\varepsilon| \ge |\Omega| - \frac{\omega}{4} \tag{4.22}$$

for ε small enough. Next we set

$$S^\varepsilon(t) := \int_{A_1^\varepsilon \cup A_2^\varepsilon} u^\varepsilon(x,t)dx \tag{4.23}$$

Since $\int_\Omega u^\varepsilon(x,0)dx = \int_\Omega u^\varepsilon(x,t)dx$ we deduce that

$$S^\varepsilon(t) = \mathcal{M}_0^\varepsilon - \int_{B_1^\varepsilon \cup B_2^\varepsilon} u^\varepsilon(x,t)dx.$$

Using (4.18), (4.21) and (4.23) we deduce that

$$-|\Omega| + \frac{3\omega}{4} \le \mathcal{M}_0^\varepsilon - \frac{\omega}{4} \le S^\varepsilon(t) \le \mathcal{M}_0^\varepsilon + \frac{\omega}{4} \le |\Omega| - \frac{3\omega}{4} \tag{4.24}$$

Furthermore using the definition of the sets A_1^ε and A_2^ε and (4.23) we have

$$\eta|A_1^\varepsilon| - \frac{|A_2^\varepsilon|}{\eta} \le S^\varepsilon(t) \le \frac{|A_1^\varepsilon|}{\eta} - \eta\,|A_2^\varepsilon| \tag{4.25}$$

Therefore using (4.22) and (4.25) we deduce that

$$S^\varepsilon(t) \le \Big(\frac{1}{\eta} + \eta\Big)|A_1^\varepsilon| - \eta\big[|A_1^\varepsilon| + |A_2^\varepsilon|\big]$$
$$\le \frac{\eta^2+1}{\eta}|A_1^\varepsilon| - \eta\Big[|\Omega| - \frac{\omega}{4}\Big]$$
$$\le \frac{\eta^2+1}{\eta}|A_1^\varepsilon| - \eta|\Omega| + \frac{\omega}{4},$$

which together with (4.24) gives that

$$|A_1^\varepsilon| \geq \frac{\eta}{\eta^2+1}\left[\frac{\omega}{2} - (1-\eta)|\Omega|\right] \tag{4.26}$$

Similarly using (4.25) and (4.22) we deduce that

$$\begin{aligned} S^\varepsilon(t) &\geq \eta[|A_1^\varepsilon| + |A_2^\varepsilon|] - \Big(\frac{1}{\eta}+\eta\Big)|A_2^\varepsilon| \\ &\geq \eta|\Omega| - \frac{\omega}{4} - \frac{\eta^2+1}{\eta}|A_2^\varepsilon|, \end{aligned}$$

which in view of (4.24) implies that

$$|A_2^\varepsilon| \geq \frac{\eta}{\eta^2+1}\left[\frac{\omega}{2} - (1-\eta)|\Omega|\right]. \tag{4.27}$$

Thus we have shown that for $\eta \geq 1 - \frac{\omega}{4|\Omega|}$

$$|A_1^\varepsilon|,\ |A_2^\varepsilon| \geq d_0 = \frac{\omega}{6} > 0. \tag{4.28}$$

Let ω_n denote the volume of the unit ball. In what follows we suppose that R_0 is a small enough positive constant such that

$$\omega_n R_0^N \leq \frac{d_0}{2}. \tag{4.29}$$

Next we return to formula (4.17) which we integrate on $A_{1,R_0}^\varepsilon := \{|y|,\ y \in A_1^\varepsilon\} \cap (R_0, 1)$ and $A_{2,R_0}^\varepsilon := \{|y|,\ y \in A_2^\varepsilon\} \cap (R_0,1)$ and apply (4.28).

$$\begin{aligned} &\left|\int_{A_{1,R_0}^\varepsilon}\int_{A_{2,R_0}^\varepsilon} G(\sigma,\rho)d\sigma\, d\rho\right| \\ &= \left|\lambda^\varepsilon(t)\int_{A_{1,R_0}^\varepsilon}\int_{A_{2,R_0}^\varepsilon}(u^\varepsilon(\rho,t)-u^\varepsilon(\sigma,t))\rho^{N-1}\sigma^{N-1}d\sigma\, d\rho\right| \\ &= |\lambda^\varepsilon(t)|\left|\int_{A_{1,R_0}^\varepsilon} u^\varepsilon(\rho,t)\rho^{N-1}d\rho\int_{A_{2,R_0}^\varepsilon}\sigma^{N-1}d\sigma \right. \\ &\qquad \left. - \int_{A_{2,R_0}^\varepsilon} u^\varepsilon(\sigma,t)\sigma^{N-1}d\sigma\int_{A_{1,R_0}^\varepsilon}\rho^{N-1}d\rho\right| \\ &\geq |\lambda^\varepsilon(t)|\frac{2\eta}{N\omega_n}\left[|A_1^\varepsilon| - \omega_n R_0^N\right]\left[|A_2^\varepsilon| - \omega_n R_0^N\right] \\ &\geq |\lambda^\varepsilon(t)|\frac{2\eta}{N\omega_n}(d_0 - \omega_n R_0^N)^2 \end{aligned} \tag{4.30}$$

Furthermore we deduce from (4.17) that

$$\begin{aligned}
&\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}} G(\sigma,\rho)d\sigma\, d\rho\right| \\
&\le \varepsilon\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}}\left(\int_\sigma^\rho u^\varepsilon_t u^\varepsilon_r dr\right)\rho^{N-1}\sigma^{N-1}d\sigma\, d\rho\right| \\
&\quad+\frac{\varepsilon}{2}\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}}((u^\varepsilon_r)^2(\rho,t)-(u^\varepsilon_r)^2(\sigma,t))\rho^{N-1}\sigma^{N-1}d\sigma\, d\rho\right| \\
&\quad+\varepsilon(N-1)\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}}\left(\int_\sigma^\rho \frac{(u^\varepsilon_r)^2}{r}dr\right)\rho^{N-1}\sigma^{N-1}d\sigma\, d\rho\right| \\
&\quad+\frac{1}{\varepsilon}\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}}[F(u^\varepsilon(\rho,t))-F(u^\varepsilon(\sigma,t))]\rho^{N-1}\sigma^{N-1}d\sigma\, d\rho\right| \\
&\quad+\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}}\left(\int_\sigma^\rho v^\varepsilon u^\varepsilon_r dr\right)\rho^{N-1}\sigma^{N-1}d\sigma\, d\rho\right| \\
&=: I_1+I_2+I_3+I_4+I_5 \qquad (4.31)
\end{aligned}$$

First we estimate I_1, we have that

$$\begin{aligned}
I_1 &\le \varepsilon\frac{(1-R_0)^2}{R_0^{N-1}}\int_0^1 |u^\varepsilon_t u^\varepsilon_r| r^{N-1}dr \\
&\le \frac{\varepsilon}{R_0^{N-1}}\left(\int_0^1 |u^\varepsilon_t|^2 r^{N-1}dr\right)^{1/2}\left(\int_0^1 |u^\varepsilon_r|^2 r^{N-1}dr\right)^{1/2} \\
&\le \sqrt{\varepsilon}C(R_0)\left(\int_0^1 |u^\varepsilon_t|^2 r^{N-1}dr\right)^{1/2}, \qquad (4.32)
\end{aligned}$$

where we have used (4.7). We also deduce from (4.7) that

$$I_2, I_3, I_4 \le C(R_0) \qquad (4.33)$$

Next we estimate I_5. We have the equality

$$\int_\sigma^\rho v^\varepsilon u^\varepsilon_r\, dr = (v^\varepsilon u^\varepsilon)(\rho,t)-(v^\varepsilon u^\varepsilon)(\sigma,t)-\int_\sigma^\rho v^\varepsilon_r u^\varepsilon\, dr,$$

which we multiply by $\sigma^{N-1}\rho^{N-1}$ and integrate in σ on A^ε_{1,R_0} and in ρ on

A^ε_{2,R_0} respectively, to deduce that

$$\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}}\left(\int_\sigma^\rho v^\varepsilon u^\varepsilon_r\,dr\right)\sigma^{N-1}\rho^{N-1}d\sigma\,d\rho\right|$$
$$\leq C(R_0)\Big(\int_0^1 |u^\varepsilon|^2 r^{N-1}dr\Big)^{1/2}$$
$$\left[\Big(\int_0^1 |v^\varepsilon|^2 r^{N-1}dr\Big)^{1/2}+\Big(\int_0^1 |v^\varepsilon_r|^2 r^{N-1}dr\Big)^{1/2}\right]$$
$$\leq C_1(R_0).$$

This together with (4.31), (4.32) and (4.33) implies that

$$\left|\int_{A^\varepsilon_{1,R_0}}\int_{A^\varepsilon_{2,R_0}} G(\sigma,\rho)d\sigma d\rho\right|$$
$$\leq C_2(R_0)\left[1+\sqrt{\varepsilon}\left(\int_0^1 |u^\varepsilon_t|^2 r^{N-1}dr\right)^{1/2}\right] \tag{4.34}$$

Then it follows from (4.29), (4.30) and (4.34) that

$$|\lambda^\varepsilon(t)|\leq C_3(R_0)\left[1+\sqrt{\varepsilon}\left(\int_0^1 |u^\varepsilon_t|^2 r^{N-1}dr\right)^{1/2}\right],$$

which in view of (3.7) yields

$$\int_0^T |\lambda^\varepsilon(t)|^2 dt \leq C_4(R_0,T)\left[1+\varepsilon\int_0^T\int_0^1 |u^\varepsilon_t|^2 r^{N-1}dr\right]\leq C_5(R_0,T).$$

This completes the proof of Lemma 4.4. □

Corollary 4.5 *There exists a sequence* $\{\varepsilon_n\}$, *which we denote again by* $\{\varepsilon\}$ *and* $\lambda_0\in L^2(0,T)$ *such that*

$$\lambda^\varepsilon \rightharpoonup \lambda_0 \ \textit{ weakly in } \ L^2(0,T) \ \textit{ as } \ \varepsilon\downarrow 0.$$

5. Definition and properties of the jumps of u

5.1. First definition and properties

Let $R_0\in(0,1)$. From now on we suppose that $\delta^2<\frac{1}{8}$, and let $Q\in(0,1)$ be such that $F(Q)\geq\frac{1}{4}$. We set

$$A_{R_0} := \bigcup_{t_0 \notin N(\delta, R_0)} (t_0 - T_0(\delta, R_0, t_0), t_0 + T_0(\delta, R_0, t_0)). \tag{5.1}$$

We remark that by definition A_{R_0} is an open set and that its complement is the finite set $N(\delta, R_0)$. We choose an increasing sequence of open sets $\{D^m_{R_0}\}$ such that $\overline{D^m_{R_0}} \subset A_{R_0}$ and $\cup D^m_{R_0} = A_{R_0}$. The set $\overline{D^m_{R_0}}$ and hence $D^m_{R_0}$ can be covered by finetely many interval of the form $(t_0 - T_0(\delta, R_0, t_0), t_0 + T_0(\delta, R_0, t_0))$. In what follows we omit the upper index m and write D_{R_0} instead of $D^m_{R_0}$.

Lemma 5.1 *There exists a real $a^\varepsilon \in (0, Q)$ and a collection of graphs $\{t \to s^\varepsilon_i(t)\}$ defined on intervals I^ε_i and taking their values in $(R_0, 1)$ such that $u^\varepsilon(s^\varepsilon_i(t), t) = a^\varepsilon$ and*

$$\int_{I^\varepsilon_i} |\partial_t s^\varepsilon_i(t)|^2 dt \le \frac{C(R_0)}{Q}. \tag{5.2}$$

Proof. In view of Theorem 4.3 we have the inequality

$$\frac{\varepsilon^2}{2}|u^\varepsilon_r|^2 \ge \frac{1}{8} \text{ on the subset } \{(r,t) \in (R_0, 1) \times D_{R_0}, |u^\varepsilon(r,t)| \le Q\}, \tag{5.3}$$

which is essential for what follows. Using the implicit function theorem we deduce that for all $b \in (-Q, Q)$ there exist functions $t \to s^\varepsilon_i(t, b)$ defined on a time interval $I^\varepsilon_i(b)$ such that $\{(r,t) \in (R_0, 1) \times D_{R_0},\ u^\varepsilon(r,t) = b\}$ consist of a collection of graphs $s^\varepsilon_i(., b)$. Moreover we have

$$\partial_t u^\varepsilon(s^\varepsilon_i(t,b), t) + \partial_t s^\varepsilon_i(t,b)\partial_r u^\varepsilon(s^\varepsilon_i(t,b), t) = 0 \tag{5.4}$$

$$\partial_b s^\varepsilon_i(t,b)\partial_r u^\varepsilon(s^\varepsilon_i(t,b), t) = 1. \tag{5.5}$$

Using the coarea formula (see for instance Theorem 2 Section 3.4.3 in [6]) we have that

$$\int_{D_{R_0}} \int_{\{r \in (R_0,1), |u^\varepsilon(r,t)| < Q\}} \frac{|\partial_t u^\varepsilon(r,t)|^2}{|u^\varepsilon_r(r,t)|} dr\, dt = \int_{-Q}^{Q} \Big(\sum_i \int_{I^\varepsilon_i} \frac{|\partial_t u^\varepsilon(r,t)|^2}{|u^\varepsilon_r(r,t)|^2} dt \Big) db$$

Thus using (5.4) we have that

$$\int_{D_{R_0}} \int_{\{r\in(R_0,1),|u^\varepsilon(r,t)|<Q\}} \frac{|\partial_t u^\varepsilon(r,t)|^2}{|u^\varepsilon_r(r,t)|} dr\,dt$$
$$= \int_{-Q}^{Q} \Big(\sum_i \int_{I_i^\varepsilon} |\partial_t s_i^\varepsilon(b,t)|^2 dt \Big) db. \tag{5.6}$$

Moreover using (3.7) and (5.3) we deduce that

$$\int_{D_{R_0}} \int_{\{r\in(R_0,1),|u^\varepsilon(r,t)|<Q\}} \frac{|\partial_t u^\varepsilon(r,t)|^2}{|u^\varepsilon_r(r,t)|} dr\,dt$$
$$\leq 2\varepsilon \int_{D_{R_0}} \int_{R_0}^{1} |\partial_t u^\varepsilon(r,t)|^2 dr\,dt \leq C(R_0). \tag{5.7}$$

Using (5.6) and (5.7) we obtain

$$\int_{-Q}^{Q} \sum_i \int_{D_{R_0}} \chi_i^\varepsilon(t) |\partial_t s_i^\varepsilon(b,t)|^2 db\,dt \leq C(R_0),$$

where χ_i^ε is the characteristic function of the interval I_i^ε. By the mean value theorem we deduce that there exists $a^\varepsilon \in (0,Q)$ such that

$$\int_{I_i^\varepsilon} |\partial_t s_i^\varepsilon(a^\varepsilon,t)|^2 dt \leq \frac{C(R_0)}{Q},$$

which coincides with (5.4). □

Moreover by Theorem 4.3 we note that $u^\varepsilon(1,t) \neq a^\varepsilon$. Indeed if at $t = t^*$, $u^\varepsilon(1,t^*) = a^\varepsilon < Q$ then $F(a^\varepsilon) > F(Q) \geq \frac{1}{4}$ so that $\varepsilon^2 u_r^\varepsilon(1,t^*) \geq \frac{1}{8}$, which contradicts the homogeneous neumann boundary condition. Thus either the function $t \to s_i^\varepsilon(t,a^\varepsilon)$ exists for all $t \in D_{R_0}$ or it stops existing by hitting the line $r = R_0$.

Definition 5.2 We call a^ε-zero of $u^\varepsilon(.,t)$ a point $r \in (R_0,1)$ such that $u^\varepsilon(r,t) = a^\varepsilon$.

Next we state a Lemma, whose proof is very similar to that of Lemma 4.3 in [4].

Lemma 5.3 *Let $t \in D_{R_0}$. There exists $\varepsilon_0(R_0,t)$ such that for all $\varepsilon < \varepsilon_0(R_0,t)$ we have,*

$$M^\varepsilon(t) := \sharp\{r \in (R_0,1],\, u^\varepsilon(r,t) = a^\varepsilon\} \leq C(R_0).$$

Thus there exists a subsequence $\{\varepsilon_n\}$, which we denote again by $\{\varepsilon\}$ such that $M^\varepsilon(t) = M(t)$ for ε small enough.

5.2. A definition of "jumps" of u and properties

We denote by

$$A_+(t) := \{r \in (0,1),\, u(r,t) = 1\}$$

$$A_-(t) := \{r \in (0,1),\, u(r,t) = -1\}$$

Definition 5.4 Let $t \in A_{R_0}$ we call $\overline{r}(t)$ a jump point of u(.,t) in $(0,1)$ if

$$\begin{cases} \big|[\overline{r}(t) - \rho, \overline{r}(t) + \rho] \cap A_+(t)\big| > 0 \\ \big|[\overline{r}(t) - \rho, \overline{r}(t) + \rho] \cap A_-(t)\big| > 0 \end{cases}$$

for all $\rho > 0$ small enough.

Next we state some preliminary results.

Lemma 5.5 *Let $\overline{r}(t) > R_0$ be a jump of $u(.,t)$. For all $\rho > 0$, there exists $\varepsilon_0 > 0$ such that there exists an a^ε-zero*

$$s^\varepsilon(t) \in (\overline{r}(t) - \rho, \overline{r}(t) + \rho)$$

of $u^\varepsilon(.,t)$ for all $\varepsilon \leq \varepsilon_0$.

Proof. We first set $A_+(\rho,t) := \{r \in (\overline{r}(t) - \rho, \overline{r}(t) + \rho) \cap (R_0, 1),\, u(r,t) = +1\}$, so that $|A_+(\rho,t)| > 0$. The fact that $u^\varepsilon(.,t)$ converges to $u(.,t)$ in $L^1(\Omega)$ (see (3.18)) implies that

$$\int_{R_0}^1 |(u - u^\varepsilon)(r,t)| r^{N-1} dr < C, \tag{5.8}$$

for any positive constant C and ε small enough. In particular we can choose $C = R_0^{N-1}(1-Q)|A_+(\rho,t)|$. Next we prove that there exists $r_1 \in (\overline{r}(t) - \rho, \overline{r}(t) + \rho)$ such that $u^\varepsilon(r_1,t) > a^\varepsilon$. Suppose that $u^\varepsilon(r,t) \leq a^\varepsilon$ for all $r \in (\overline{r}(t) - \rho, \overline{r}(t) + \rho)$, then we have that $(u - u^\varepsilon)(.,t) \geq 1 - a^\varepsilon$ in $A_+(\rho,t)$. This in turn implies that

$$\begin{aligned} \int_{R_0}^1 |(u - u^\varepsilon)(r,t)| r^{N-1} dr &\geq (1 - a^\varepsilon) \int_{A_+(\rho,t)} r^{N-1} dr \\ &\geq (1 - a^\varepsilon) R_0^{N-1} |A_+(\rho,t)|, \end{aligned}$$

which contradicts (5.8). Thus we deduce that there exists $r_1 \in (\overline{r}(t) -$

$\rho, \overline{r}(t)+\rho)$ such that $u^\varepsilon(r_1,t) > a^\varepsilon$. Similarly one can prove that there exists $r_2 \in (\overline{r}(t)-\rho, \overline{r}(t)+\rho)$ such that $u^\varepsilon(r_2,t) < a^\varepsilon$. Therefore we conclude that there exists $r \in (\overline{r}(t)-\rho, \overline{r}(t)+\rho)$ such that $u^\varepsilon(r,t) = a^\varepsilon$. This completes the proof of Lemma 5.5.

Next we give two results, which are proven in [4]. Their proofs are based on the fact that the Lyapunov functional $E^\varepsilon(u^\varepsilon)$ is bounded and on the central estimate in Theorem 4.3. These quantities are identical for Problem (P^ε) and for the fourth order problem consider in [4]. □

Corollary 5.6 *Let $t \in A_{R_0}$ and $\mathcal{N}_0(t)$ be the number of jumps of $u(.,t)$ in $(R_0, 1)$. This number is finite.*

Lemma 5.7 *Let $\overline{r}_1(t)$ and $\overline{r}_2(t)$ be two consecutive jumps of $u(.,t)$; then either $u(.,t) = 1$ a.e. in $(\overline{r}_1(t), \overline{r}_2(t))$ or $u(.,t) = -1$ a.e. in $(\overline{r}_1(t), \overline{r}_2(t))$.*

Definition 5.8 Let $\overline{r}(t)$ be a jump of $u(.,t)$ and η small enough such that there is no other jump of $u(.,t)$ in $[\overline{r}(t)-\eta, \overline{r}(t)+\eta]$; we set

$$\nu(\overline{r}(\overline{t})) := \begin{cases} 1 & \text{if } u(.,t) = -1 \text{ on } [\overline{r}-\eta, \overline{r}) \text{ and} \\ & \qquad u(.,t) = 1 \text{ on } (\overline{r}, \overline{r}+\eta] \\ -1 & \text{if } u(.,t) = 1 \text{ on } [\overline{r}-\eta, \overline{r}) \text{ and} \\ & \qquad u(.,t) = -1 \text{ on } (\overline{r}, \overline{r}+\eta]. \end{cases}$$

We are now in a position to make precise the convergence of the a^ε-zeros of u^ε.

Theorem 5.9 *Let $\overline{r}(\overline{t})$ be a jump of $u(.,\overline{t})$; there exists a time interval $(\overline{t}_1, \overline{t}_2)$, which contains $\overline{t}$ and there exist $\mathcal{M}(\overline{t})$ functions $t \to r_i^\varepsilon(t)$ define on $[\overline{t}_1, \overline{t}_2]$ satisfying*

$$u^\varepsilon(r_i^\varepsilon(t),t) = a^\varepsilon \quad \textit{and} \quad 1 > r_1^\varepsilon(t) > r_2^\varepsilon(t) > \cdots > r^\varepsilon_{\mathcal{M}(\overline{t})}(t) > R_0$$
$$\textit{for all } t \in [\overline{t}_1, \overline{t}_2]$$

and such that

(i) *$\mathcal{M}(\overline{t})$ is odd;*

(ii) *$\nu(\overline{r})$ and $u^\varepsilon_r\big(r^\varepsilon_{\mathcal{M}(\overline{t})}(\overline{t}), \overline{t}\big)$ have the same sign;*

(iii) *If ρ is a a^ε-zero of $u^\varepsilon(.,t)$ such that $\rho > r_1^\varepsilon(t)$ or $\rho < r^\varepsilon_{\mathcal{M}(\overline{t})}(t)$ or if ρ is equal R_0 or 1 then we have $|r_1^\varepsilon(t) - \rho| \geq \varepsilon^{1/4}$ and $\left|r^\varepsilon_{\mathcal{M}(\overline{t})}(t) - \rho\right| \geq \varepsilon^{1/4}$ for all $t \in [\overline{t}_1, \overline{t}_2]$.*

The functions $t \to r_i(t)$ *define on* $[\bar{t}_1, \bar{t}_2]$ *satisfy*

(iv) $r_i^\varepsilon \to r_i$ *uniformly on* $[\bar{t}_1, \bar{t}_2]$ *as* $\varepsilon \downarrow 0$,

(v) $\partial_t r_i^\varepsilon \rightharpoonup \partial_t r_i$ *weakly in* $L^2(\bar{t}_1, \bar{t}_2)$ *as* $\varepsilon \downarrow 0$,

(vi) $r_i(\bar{t}) = \bar{r}(\bar{t})$,

(vii) r_i *is Hölder continuous of exponent* $1/2$

for all $i \in [1, \ldots, \mathcal{M}(\bar{t})]$. *Moreover there exist* $k \geq j$ *such that* $(j, k) \in [1, \ldots, \mathcal{M}(\bar{t})]$ *and*

$$R_0 < \cdots < r_k^\varepsilon(t) - \varepsilon^{2/5} < r_k^\varepsilon(t) \leq \cdots \leq r_j^\varepsilon(t) < r_j^\varepsilon(t) + \varepsilon^{2/5} < \cdots < 1, \tag{5.9}$$

for all $t \in [\bar{t}_1, \bar{t}_2]$ *and* ε *small enough. We set*

$$m(\bar{r}(\bar{t})) := k - j + 1. \tag{5.10}$$

Proof. We first note that (i) implies (ii). Moreover we only consider the case where that $\nu(\bar{r}) = 1$ since by symmetry one can check similar properties in the case that $\nu(\bar{r}) = -1$. In view of Lemma 5.3 there exist $M(\bar{t})$ a^ε-zeros, ρ_i, of $u^\varepsilon(.,\bar{t})$ such that $R_0 < \rho_{M(\bar{t})}(\bar{t}) < \cdots < \rho_1(\bar{t}) < 1$. Extracting a subsequence, which we denote again by $\{\varepsilon\}$, we may suppose that $\rho_i(\bar{t})$ converges to a limit $\bar{\rho}_i(\bar{t})$ as $\varepsilon \downarrow 0$ for all $i \in [1, M(\bar{t})]$. We denote by $r_1^\varepsilon(\bar{t}) > \cdots > r_{\mathcal{M}(\bar{t})}^\varepsilon(\bar{t})$ the a^ε-zeros of $u^\varepsilon(.,\bar{t})$ which converge to $\bar{r}(\bar{t})$. There exists $\eta > 0$ such that $\bar{r}(\bar{t})$ is the only limit in $[\bar{r}(\bar{t}) - \eta, \bar{r}(\bar{t}) + \eta] \subset [R_0, 1]$ and such that for ε small enough

$$r_1^\varepsilon(\bar{t}), \ldots, r_{\mathcal{M}(\bar{t})}^\varepsilon(\bar{t}) \text{ are the only } a^\varepsilon\text{-zeros of } u^\varepsilon(.,\bar{t}) \quad \text{in } \Big(\bar{r}(\bar{t}) - \frac{\eta}{2}, \bar{r}(\bar{t}) + \frac{\eta}{2}\Big). \tag{5.11}$$

Let ε such that

$$\frac{\eta}{2} \geq \varepsilon^{1/4}, \tag{5.12}$$

which in particular implies that

$$r_1^\varepsilon(\bar{t}) < 1 - \varepsilon^{1/4} \quad \text{and} \quad r_{\mathcal{M}(\bar{t})}^\varepsilon(\bar{t}) > R_0 + \varepsilon^{1/4}. \tag{5.13}$$

Using the fact that $u(.,\bar{t})$ is constant between two jumps and that $\nu(\bar{r}) = 1$ we deduce that

$$u^\varepsilon(.,\bar{t}) \to -1 \quad \text{a.e. on} \quad [\bar{r}(\bar{t}) - \eta, \bar{r}(\bar{t}))$$
$$u^\varepsilon(.,\bar{t}) \to 1 \quad \text{a.e. on} \quad (\bar{r}(\bar{t}), \bar{r}(\bar{t}) + \eta].$$

By (5.11) $u^\varepsilon - a^\varepsilon$ has a constant sign on $\left(r_1^\varepsilon(\bar t), \bar r(\bar t) + \frac{\eta}{2}\right]$ and u^ε converges to 1 so that $u^\varepsilon > a^\varepsilon$ on $\left(r_1^\varepsilon(\bar t), \bar r(\bar t) + \frac{\eta}{2}\right]$. Similarly we have that $u^\varepsilon < a^\varepsilon$ on $\left[\bar r(\bar t) - \frac{\eta}{2}, r^\varepsilon_{\mathcal{M}(\bar t)}(\bar t)\right)$. This implies in particular that $\mathcal{M}(\bar t)$ is odd and that $u_r^\varepsilon(r^\varepsilon_{\mathcal{M}(\bar t)}(\bar t), \bar t) > 0$, so that (i) and (ii) are satisfied. Furthermore using the implicit function theorem we have that there exists an interval $[t_1', t_2']$, which contains $\bar t$ such that $r_i^\varepsilon \in C^1([t_1', t_2'])$. Next we prove (iii). If $\rho = R_0$ or $\rho = 1$, the result follows from (5.13) and the continuity of r_1^ε and $r_{\mathcal{M}}^\varepsilon$. Otherwise it follows from (5.11) and the continuity of r_i^ε that there exists an interval $[\bar t_1, \bar t_2]$ containing $\bar t$ such that

$$|r_1^\varepsilon(t) - \rho| \geq \frac{\eta}{2} \quad \text{and} \quad |r^\varepsilon_{\mathcal{M}(\bar t)}(t) - \rho| \geq \frac{\eta}{2}, \quad \text{for all} \;\; t \in [\bar t_1, \bar t_2].$$

So that the result of (iii) follows from (5.12). In view of (5.2) the functions r_i^ε for $i \in [1, \dots, \mathcal{M}(\bar t)]\}$ are Hölder continuous of exponent $\frac{1}{2}$ uniformly in ε; using the Ascoli theorem we deduce that there exist a subsequence ε_n, which we denote again by ε, and $\mathcal{M}(\bar t)$ functions r_i for $i \in [1, \dots, \mathcal{M}(\bar t)]$

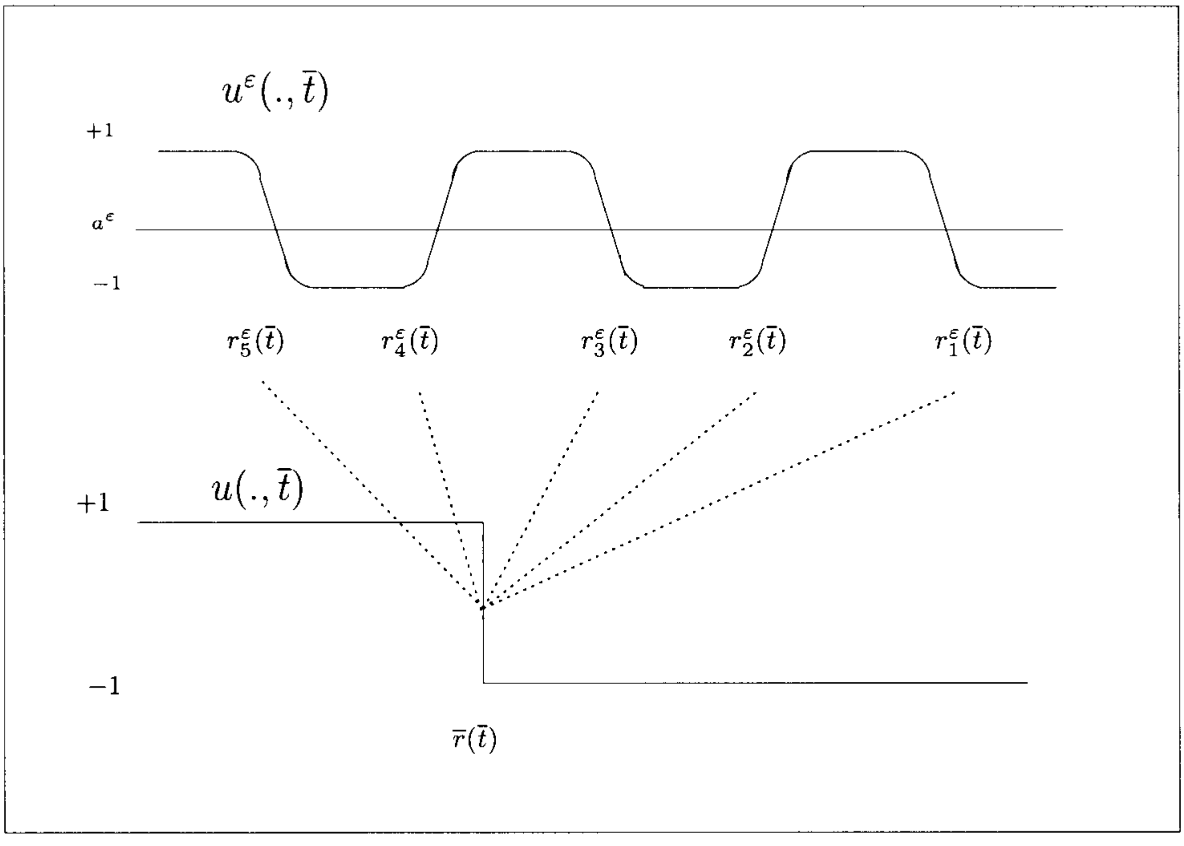

Fig. 1. A possible configuration of the zeros

define on $[\bar{t}_1, \bar{t}_2]$ such that as $\varepsilon \downarrow 0$

$$r_i^\varepsilon \to r_i \text{ uniformly on } [\bar{t}_1, \bar{t}_2] \text{ and } r_i(\bar{t}) = \bar{r}(\bar{t}),$$

for all $i \in [1, \ldots, \mathcal{M}(\bar{t})]$, which proves (iv) and (vi). Finally since by (5.2) we have that $\int_{\bar{t}_1}^{\bar{t}_2} |\partial_t r_i^\varepsilon|^2 dt$ is bounded we also have for a subsequence that $\partial_t r_i^\varepsilon \rightharpoonup \partial_t r_i$ weakly in $L^2(\bar{t}_1, \bar{t}_2)$ as $\varepsilon \downarrow 0$. This completes the proof of Theorem 5.9. □

6. First approximation

Theorem 6.1 *Let $\delta > 0$, there exists $b_0(\delta) > 0$ and $K(\delta) > 0$ such that $b_0(\delta) \to \infty$ and $K(\delta) \to 0$ as $\delta \downarrow 0$ satisfying for $\varepsilon \le \varepsilon_0(\delta)$ the following properties*

(i) *Suppose that there exist a time interval $[t_1, t_2] \subset A_{R_0}$ and two functions $t \to r_-^\varepsilon(t)$ and $t \to r_+^\varepsilon(t)$ such that $u^\varepsilon(r_-^\varepsilon(t), t) = u^\varepsilon(r_+^\varepsilon(t), t) = a^\varepsilon$ and $r_-^\varepsilon(t)$, $r_+^\varepsilon(t)$ are two successives a^ε-zeros of $u^\varepsilon(., t)$ for all $t \in [t_1, t_2]$. Then denoting by τ^ε the sign of $u^\varepsilon - a^\varepsilon$ on $(r_-^\varepsilon, r_+^\varepsilon)$ we have that*

$$\frac{1}{\varepsilon}[r_+^\varepsilon(t) - r_-^\varepsilon(t)] \ge 2(b_0(\delta) + 1) \quad \text{for all} \quad t \in [t_1, t_2], \tag{6.1}$$

and u^ε satisfies

$$\left| u^\varepsilon(r, t) - \tanh\Big(\tau^\varepsilon\Big(\frac{r - r_-^\varepsilon(t)}{\varepsilon}\Big) + \mu^\varepsilon\Big) \right| \le K(\delta) \tag{6.2}$$

for all $(r, t) \in [r_-^\varepsilon(t), r_-^\varepsilon(t) + b_0(\delta)] \times [t_1, t_2]$,

$$\left| u^\varepsilon(r, t) + \tanh\Big(\tau^\varepsilon\Big(\frac{r - r_+^\varepsilon(t)}{\varepsilon}\Big) - \mu^\varepsilon\Big) \right| \le K(\delta) \tag{6.3}$$

for all $(r, t) \in [r_+^\varepsilon(t) - b_0(\delta), r_+^\varepsilon(t)] \times [t_1, t_2]$ and finally

$$\begin{aligned} \Bigg[\tau^\varepsilon \Bigg\{ u^\varepsilon(r,t) - \Bigg([1 - \xi^\varepsilon(r,t)] \tanh\Big(\tau^\varepsilon\Big(\frac{r - r_-^\varepsilon(t)}{\varepsilon}\Big) + \mu^\varepsilon\Big) \\ - [\xi^\varepsilon(r,t)] \tanh\Big(\tau^\varepsilon\Big(\frac{r - r_+^\varepsilon(t)}{\varepsilon}\Big) - \mu^\varepsilon\Big)\Bigg)\Bigg\}\Bigg]_- \le K(\delta) \end{aligned} \tag{6.4}$$

for all $(r, t) \in [r_-^\varepsilon, r_+^\varepsilon] \times [t_1, t_2]$, where $\mu^\varepsilon = \tanh^{-1}(a^\varepsilon)$ and ξ^ε is a smooth function on $(r_-^\varepsilon(t), r_+^\varepsilon(t))$, such that $0 \le \xi^\varepsilon \le 1$

$$\xi^\varepsilon(r,t) = \begin{cases} 0 & \textit{if } r_-^\varepsilon(t) < r < \dfrac{r_-^\varepsilon(t) + r_+^\varepsilon(t)}{2} - \varepsilon \\ 1 & \textit{if } \dfrac{r_-^\varepsilon(t) + r_+^\varepsilon(t)}{2} + \varepsilon < r < r_+^\varepsilon(t). \end{cases}$$

(ii) *Suppose that $r^\varepsilon(t)$ is a a^ε-zero of $u^\varepsilon(.,t)$ and that there exists $\alpha^\varepsilon \in (r^\varepsilon(t),1)$ be such that $\alpha^\varepsilon - r^\varepsilon(t) \geq \varepsilon^{1/4}$ and that $u^\varepsilon - a^\varepsilon$ does not vanish on the interval $(r^\varepsilon(t),\alpha^\varepsilon)$. Then denoting by τ^ε the sign of $u^\varepsilon - a^\varepsilon$ we have for $\varepsilon \leq \varepsilon_0(\delta)$ that*

$$\left[\tau^\varepsilon\left(u^\varepsilon(r,t) - \tanh\Big(\tau^\varepsilon\Big(\frac{r - r_-^\varepsilon(t)}{\varepsilon}\Big) + \mu^\varepsilon\Big)\right)\right]_- \leq K(\delta) \tag{6.5}$$

for all $(r,t) \in [r^\varepsilon(t), r^\varepsilon(t) + \varepsilon^{2/5}] \times [t_1,t_2]$.

(iii) *Suppose that $r^\varepsilon(t)$ is a a^ε-zero of $u^\varepsilon(.,t)$ and that there exists $\alpha^\varepsilon \in (R_0, r^\varepsilon(t))$ be such that $r^\varepsilon(t) - \alpha^\varepsilon \geq \varepsilon^{1/4}$ and $u^\varepsilon - a^\varepsilon$ does not vanish on the interval $(\alpha^\varepsilon, r^\varepsilon(t))$. Then denoting by τ^ε the sign of $u^\varepsilon - a^\varepsilon$ we have for $\varepsilon \leq \varepsilon_0(\delta)$ that*

$$\left[\tau^\varepsilon\left(u^\varepsilon(r,t) + \tanh\Big(\tau^\varepsilon\Big(\frac{r - r_+^\varepsilon(t)}{\varepsilon}\Big) + \mu^\varepsilon\Big)\right)\right]_- \leq K(\delta) \tag{6.6}$$

for all $(r,t) \in [r^\varepsilon(t) - \varepsilon^{2/5}, r^\varepsilon(t)] \times [t_1,t_2]$.

Note that ξ^ε is well defined since by (6.1) we have $r_+^\varepsilon - r_-^\varepsilon > 2\varepsilon$ for ε small enough.

In view of Theorem 4.3 and the definition of A_{R_0} (Definition 5.1) the function $U^\varepsilon(z,t) = u^\varepsilon(r,t)$ where $z = \frac{r}{\varepsilon}$ satisfies

$$\left\| \frac{1}{2}(U_z^\varepsilon(.,t))^2 - F(U^\varepsilon(.,t)) \right\|_{L^\infty\left(\left(\frac{R_0}{\varepsilon},\frac{1}{\varepsilon}\right)\times(t_1,t_2)\right)} \leq \delta^2$$

As in the proof of Theorem 5.1 in [4] we have to state properties of the solutions of the differential equation

$$\text{(E)} \quad (\varphi')^2 - 2F(\varphi) = g, \quad \text{in } \mathbf{R}$$

where g is a smooth function such that $\|g\|_\infty \leq \tilde{\delta}^2$. Following the proofs of Lemmas 5.2 and 5.3 in [4] one can check the two following results.

Lemma 6.2 *Let ϕ be a solution of Equation (E) such that there exist z_- and z_+ satisfying $\phi(z_-) = \phi(z_+) = a^\varepsilon$ and $\phi(z) > a^\varepsilon$ for all $z \in (z_-,z_+)$. Let $\tilde{\delta} > 0$; there exists $b_0 = b_0(\tilde{\delta}) > 0$ such that $\lim_{\tilde{\delta}\to 0} b_0(\tilde{\delta}) = +\infty$ and a* *positive constant K such that*

$$z_+ - z_- > 2(b_0 + 1) \to +\infty \quad \text{as} \quad \tilde{\delta} \downarrow 0 \tag{6.7}$$

$$|\phi(z) - \tanh(z - z_- + \mu^\varepsilon)| \le K\sqrt{\tilde{\delta}}, \quad \text{for all} \quad z \in [z_-, z_- + b_0], \tag{6.8}$$

$$|\phi(z) + \tanh(z - z_+ - \mu^\varepsilon)| \le K\sqrt{\tilde{\delta}}, \quad \text{for all} \quad z \in [z_+ - b_0, z_+]. \tag{6.9}$$

$$\phi(z) \ge 1 - K\sqrt{\tilde{\delta}}, \quad \text{for all} \quad z \in [z_- + b_0, z_+ - b_0]. \tag{6.10}$$

Moreover let $c \in (z_-, z_+)$ be such that $\phi_z(c) = 0$. Then

$$c \in [z_- + b_0, z_+ - b_0]. \tag{6.11}$$

Lemma 6.3 *Let ϕ be a solution of* (E) *and let z_- be such that $\phi(z_-) = a^\varepsilon$. Suppose that $\phi(z) > a^\varepsilon$ for all $z \in [z_-, A]$ where $A - z_- > \frac{2}{\sqrt{\tilde{\delta}}}$; then there exists a positive constant K_1 such that*

$$\sup_{\left[z_-, A - \frac{2}{\sqrt{\tilde{\delta}}}\right]} [\phi(z) - \tanh(z - z_- + \mu^\varepsilon)]_- \le K_1\sqrt{\tilde{\delta}} \tag{6.12}$$

As it is done in [4] one can deduce then Theorem 6.1 from the Lemmas 6.2 and 6.3.

Lemma 6.4 *Set $U^\varepsilon(z,t) = u^\varepsilon(r,t)$ where $z = \frac{r}{\varepsilon}$ and suppose that there exist a time interval $[t_1, t_2]$ and two functions $t \to z_-^\varepsilon(t)$ and $t \to z_+^\varepsilon(t)$ such that $U^\varepsilon(z_-^\varepsilon(t), t) = U^\varepsilon(z_+^\varepsilon(t), t) = a^\varepsilon$ and U^ε does not take the value a^ε for all $z \in [z_-^\varepsilon(t), z_+^\varepsilon(t)]$. There exists a function $b_0^\varepsilon(t)$ such that*

$$z_+^\varepsilon(t) - z_-^\varepsilon(t) > 2(b_0^\varepsilon(t) + 1) \quad \text{with} \quad \lim_{\varepsilon \to 0} b_0^\varepsilon(t) = +\infty \tag{6.13}$$

for a.e. $t \in [t_1, t_2]$. Moreover suppose that $c^\varepsilon(t) \in (z_-^\varepsilon(t), z_+^\varepsilon(t))$ satisfies $u_r^\varepsilon(c^\varepsilon(t), t) = 0$. Then $c^\varepsilon(t) \in [z_-^\varepsilon(t) + b_0^\varepsilon(t), z_+^\varepsilon(t) - b_0^\varepsilon(t)]$.

Proof. Using (3.7) we have that $\varepsilon^2 \int_0^T \int_0^1 |u_t^\varepsilon|^2 dr\, dt \le C\varepsilon$, which implies that $\{\varepsilon^2 \int_0^1 |u_t^\varepsilon|^2 dr\}$ tends to zero in $L^1(0,T)$ as $\varepsilon \downarrow 0$. Thus there exists a subsequence, which we denote again by $\{\varepsilon\}$ and a constant C_1 depending on R_0 and t such that for a.e. $t \in (0,T)$

$$\int_{R_0}^1 |u_t^\varepsilon|^2 dr \le C_1(R_0, t)\varepsilon^{-2}.$$

Substituting this into (4.13) we deduce that

$$\left\|\frac{1}{2}(U_z^\varepsilon(.,t))^2 - F(U^\varepsilon(.,t))\right\|_{L^\infty\left(\frac{R_0}{\varepsilon},\frac{1}{\varepsilon}\right)} \le C_2(R_0,t)\varepsilon^{1/2}$$

for a.e. $t \in [t_1, t_2]$. Applying (6.11) for the function U^ε and with $\tilde{\delta} = [C(R_0,t)\varepsilon^{1/2}]^{1/2}$ we deduce that there exists $b_0(\tilde{\delta})(t)$, which we denote by $b_0^\varepsilon(t)$ satisfying $z_+^\varepsilon(t) - z_-^\varepsilon(t) > 2(b_0^\varepsilon(t)+1)$ and $\lim_{\varepsilon\to 0} b_0^\varepsilon(t) = +\infty$ and such that for all points $c^\varepsilon(t) \in [z_-^\varepsilon(t), z_+^\varepsilon(t)]$ satisfying $u_r^\varepsilon(c^\varepsilon(t),t) = 0$ we have $c^\varepsilon \in [z_-^\varepsilon(t) + b_0^\varepsilon, z_+^\varepsilon(t) - b_0^\varepsilon]$, which completes the proof of Lemma 6.4. □

Following the proof of the Corollary 5.6 in [4] we obtain the following result.

Lemma 6.5 *Suppose the hypotheses of Theorem* 6.1 (i) *hold. Then* u_0^ε *is such that*

$$\lim_{\varepsilon\to 0} \sup_{\left[\frac{r_-^\varepsilon(t)+r_+^\varepsilon(t)}{2}-\varepsilon,\frac{r_-^\varepsilon(t)+r_+^\varepsilon(t)}{2}+\varepsilon\right]\times[t_1,t_2]} |u_0^\varepsilon - 1| = 0, \tag{6.14}$$

$$\lim_{\varepsilon\to 0} \sup_{[r_-^\varepsilon(t)+\varepsilon b_0, r_+^\varepsilon(t)-\varepsilon b_0]\times[t_1,t_2]} |(u_0^\varepsilon)_r| = 0. \tag{6.15}$$

Consequently we also have

$$\lim_{\varepsilon\to 0} \sup_{\left[\frac{r_-^\varepsilon(t)+r_+^\varepsilon(t)}{2}-\varepsilon,\frac{r_-^\varepsilon(t)+r_+^\varepsilon(t)}{2}+\varepsilon\right]\times[t_1,t_2]} |(u_0^\varepsilon)_r| = 0. \tag{6.16}$$

7. Approximation of u^ε

Introduction and preliminary definitions

Let $\overline{r}(\overline{t})$ be a jump of $u(.,\overline{t})$ and $R_0 \in (0,1)$ such that $\overline{r}(\overline{t}) > R_0$. In view of Theorem 5.9 there exists a time interval $(\overline{t}_1, \overline{t}_2)$, which contains $\overline{t}$ and there exist $m(\overline{r}(\overline{t}))$ (see (5.10) for the definition of $m(\overline{r}(\overline{t}))$) functions $t \to r_i^\varepsilon(t)$ define on $[\overline{t}_1, \overline{t}_2]$ satisfying

$$\begin{aligned} &u^\varepsilon(r_i^\varepsilon(t),t) = a^\varepsilon \quad \text{and} \\ &R_0 < \cdots < r_k^\varepsilon(t) - \varepsilon^{2/5} < r_k^\varepsilon(t) \le \cdots \le r_j^\varepsilon(t) < r_j^\varepsilon(t) + \varepsilon^{2/5} < \cdots < 1, \\ &r_i^\varepsilon - r_{i+1}^\varepsilon \ge 2\varepsilon \quad \text{for all} \quad 1 \le i \le m(\overline{r}(\overline{t})) - 1, \end{aligned} \tag{7.1}$$

and such that (i)–(vi) of are satisfied. We first state some notations and definitions. In order to simplify the notation we replace $m(\overline{r}(\overline{t}))$ by m in all the proofs of Section 6. We introduce a new variable namely,

$$z(t) := \frac{r - r_j^\varepsilon(t)}{\varepsilon},$$

moreover we set for all $t \in [\bar{t}_1, \bar{t}_2]$

$$z_i^\varepsilon(t) := \frac{r_{i+j-1}^\varepsilon(t) - r_j^\varepsilon(t)}{\varepsilon}, \quad \text{for all} \quad i \in [1, m]$$

and also

$$z_0^\varepsilon(t) := \frac{r_{j-1}^\varepsilon - r_j^\varepsilon}{\varepsilon}, \quad z_{m+1}^\varepsilon(t) := \frac{r_{k+1}^\varepsilon - r_j^\varepsilon}{\varepsilon}$$
$$z_-^\varepsilon(t) := \frac{R_0 - r_j^\varepsilon}{\varepsilon}, \quad z_+^\varepsilon(t) := \frac{1 - r_j^\varepsilon}{\varepsilon}$$

Thus on $[\bar{t}_1, \bar{t}_2]$ we have that

$$z_-^\varepsilon < z_{m+1}^\varepsilon < -\varepsilon^{-3/5} + z_m^\varepsilon < z_m^\varepsilon < z_{m-1}^\varepsilon < \cdots$$
$$\cdots < z_1^\varepsilon = 0 < z_1^\varepsilon + \varepsilon^{-3/5} < z_0^\varepsilon < z_+^\varepsilon,$$

From now on we use capital letters for functions defined in the new variable z, so that

$$U^\varepsilon(z, t) = u^\varepsilon(r, t),$$
$$V^\varepsilon(z, t) = v^\varepsilon(r, t).$$

In the z-variable equation (4.1) becomes

$$-\varepsilon u_t^\varepsilon(\varepsilon z + r_1^\varepsilon(t), t) + \frac{1}{\varepsilon} U_{zz}^\varepsilon(z, t) + \frac{(N-1)}{\varepsilon z + r_1^\varepsilon} U_z^\varepsilon(z, t)$$
$$+ \frac{1}{\varepsilon} f(U^\varepsilon(z, t)) - V^\varepsilon(z, t) - \lambda^\varepsilon(t) = 0. \tag{7.2}$$

Next we give some definitions

1. Let $\{E_i^\varepsilon(z, t)\}_{1 \le i \le m}$ be a partition of unity, such that the function $E_i^\varepsilon(z, t)$ has support in $\left(\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} - 1, \frac{z_i^\varepsilon + z_{i-1}^\varepsilon}{2} + 1\right)$ and $E_i^\varepsilon(z, t) = 1$ on $\left[\frac{z_{i+1}^\varepsilon + z_i^\varepsilon}{2} + 1, \frac{z_i^\varepsilon + z_{i-1}^\varepsilon}{2} - 1\right]$ for all $i \in [2, m-1]$, while $E_1^\varepsilon(z, t)$ has support in $\left(\frac{z_2}{2} - 1, \infty\right)$ and $E_1^\varepsilon(z, t) = 1$ on $\left(\frac{z_2}{2} + 1, \infty\right)$; $E_m(z, t)$ has support in $\left(-\infty, \frac{z_m^\varepsilon + z_{m-1}^\varepsilon}{2} + 1\right)$ and $E_m(z, t) = 1$ on $\left(-\infty, \frac{z_m^\varepsilon + z_{m-1}^\varepsilon}{2} - 1\right)$. Moreover we suppose that $(E_i^\varepsilon)_z$, and $(E_i^\varepsilon)_{zz}$ are uniformly bounded in ε. By (7.1) this

construction makes sense; similar computations have already been made in Section 5.

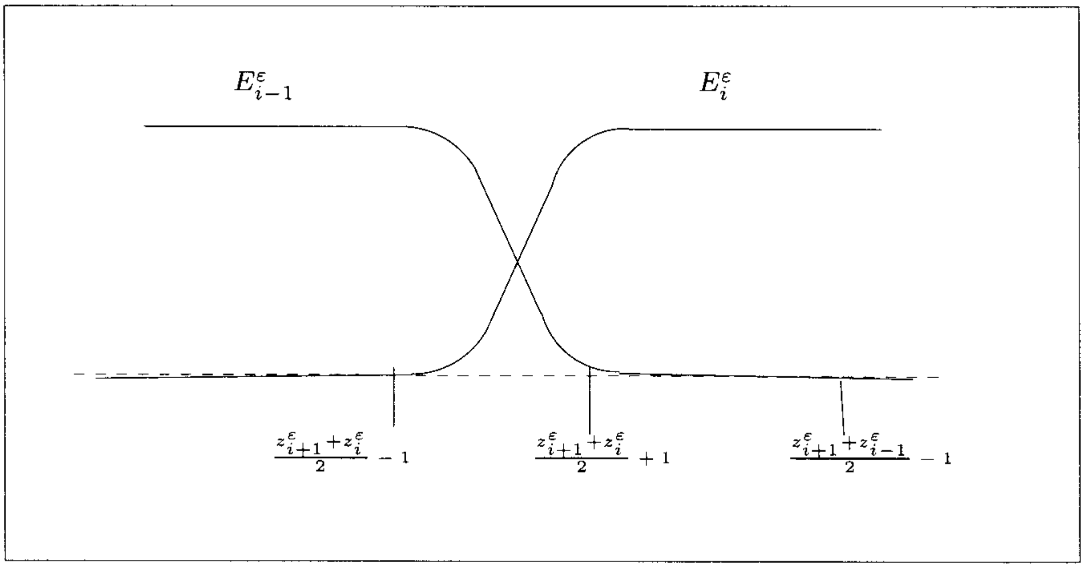

Fig. 2. The partition of unity

2. We set

$$U_0^\varepsilon(z,t) := \sum_{i=1}^{i=m} E_i^\varepsilon(z,t)U_{0i}^\varepsilon(z,t),$$

where $U_{0i}^\varepsilon(z,t) = \tanh[(-1)^{i+1}(z - z_i^\varepsilon(t)) + \mu^\varepsilon]$ where $\mu^\varepsilon = \tanh^{-1} a^\varepsilon$ and

$$U_1^\varepsilon(z,t) := \sum_{i=1}^{i=m} E_i^\varepsilon(z,t)U_{1i}^\varepsilon(z,t),$$

where U_{1i}^ε is a solution of the system

$$\begin{cases} (U_{1i}^\varepsilon)_{zz} + f'(U_{0i}^\varepsilon)U_{1i}^\varepsilon = V^\varepsilon + \lambda^\varepsilon \\ U_{1i}^\varepsilon(z_i^\varepsilon(t),t) = 0 \end{cases}$$

3. Finally we set for all $z \in [z_-^\varepsilon, z_+^\varepsilon]$

$$\Theta^\varepsilon(z,t) := \nu(\overline{r})(-1)^j(U_0^\varepsilon(z,t) + \varepsilon U_1^\varepsilon(z,t)),$$

where $\nu(\overline{r})$ is defined by Definition 5.8 and

$$\Psi^\varepsilon(z,t) := U^\varepsilon(z,t) - \Theta^\varepsilon(z,t).$$

Construction of the approximation In this Section we prove that U^ε is well approximated by Θ^ε; more precisely we prove the following result.

Theorem 7.1 *Let ξ^ε be a smooth function with values in* $[0,1]$ *such that*

$$\xi^\varepsilon(z,t) := \begin{cases} 1 \ \textit{in} \ \left(-\varepsilon^{-\frac{1}{2}} + z^\varepsilon_{m(\overline{r}(\overline{t}))}(t), \varepsilon^{-\frac{1}{2}}\right) \\ 0 \ \textit{in} \ R \setminus \left(-\varepsilon^{-3/5} + z^\varepsilon_{m(\overline{r}(\overline{t}))}(t), \varepsilon^{-3/5}\right), \end{cases}$$

$0 \le \xi^\varepsilon \le 1$ *and* $\|\xi^\varepsilon_z\|_{L^\infty} \le C\varepsilon^{3/5}$. *Then we have*

$$\int_{\overline{t}_1}^{\overline{t}_2} \int_{(-\infty, z^\varepsilon_{m(\overline{r}(\overline{t}))}) \cup (0,\infty)} (|\Psi^\varepsilon_z|^2 + |\Psi^\varepsilon|^2)(\xi^\varepsilon)^2 \, dz \, dt \le C\varepsilon^{6/5}$$

and

$$\lim_{\varepsilon\to 0} \int_{\overline{t}_1}^{\overline{t}_2} \int_{(z^\varepsilon_{m(\overline{r}(\overline{t}))}, 0)} (|\Psi^\varepsilon_z|^2 + |\Psi^\varepsilon|^2) \, dz \, dt = 0.$$

We first note that

$$z^\varepsilon_- < z^\varepsilon_m - \varepsilon^{-3/5} < z^\varepsilon_m - \varepsilon^{-1/2} < z^\varepsilon_m < z^\varepsilon_{m-1} < \cdots$$
$$\cdots < z^\varepsilon_1 = 0 < \varepsilon^{-1/2} < \varepsilon^{-3/5} < z^\varepsilon_+$$

and that $U^\varepsilon(.,t)$ does not take the value a^ε on $(z^\varepsilon_m(t) - \varepsilon^{-3/5}, z^\varepsilon_m(t))$ and in $(z^\varepsilon_1 = 0, \varepsilon^{-3/5})$ for all $t \in [\overline{t}_1, \overline{t}_2]$. Next we prove preliminary lemmas, which will be useful to prove Theorem 7.1. In each proof we only consider the case where $\nu(\overline{r})(-1)^j = 1$ since by symmetry one can obtain the same result in the case that $\nu(\overline{r})(-1)^j = -1$. In the following lemma, we give the equation satisfied by Ψ^ε and we refer to [4] for the proof.

Lemma 7.2 *Setting*

$$H^\varepsilon(z,t) := -\Theta^\varepsilon_{zz}(z,t) - f(\Theta^\varepsilon(z,t)) + \varepsilon V^\varepsilon(z,t) + \varepsilon\lambda^\varepsilon(t) \tag{7.3}$$

and

$$\begin{aligned} G^\varepsilon(z,t) &:= -\varepsilon^2 u^\varepsilon_t(\varepsilon z + r^\varepsilon_j(t), t) + (N-1)\frac{\varepsilon}{\varepsilon z + r^\varepsilon_j} U^\varepsilon_z(z,t) \\ &= -U^\varepsilon_{zz}(z,t) - f(U^\varepsilon(z,t)) + \varepsilon V^\varepsilon(z,t) + \varepsilon\lambda^\varepsilon(t), \end{aligned} \tag{7.4}$$

we have

$$H^\varepsilon = \varepsilon^2 \sum_{i=1}^{i=m(\bar{r}(\bar{t}))} E_i^\varepsilon \left(6U_{0i}^\varepsilon (U_{1i}^\varepsilon)^2 + 2\varepsilon (U_{1i}^\varepsilon)^3\right)$$
$$+ \sum_{i=1}^{i=m(\bar{r}(\bar{t}))-1} (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)\left[(\zeta_i^\varepsilon)_{zz} + 2E_i^\varepsilon E_{i+1}^\varepsilon \left((1+E_i^\varepsilon)(\Theta_i^\varepsilon)^2 + (E_i^\varepsilon - 2)(\Theta_{i+1}^\varepsilon)^2 + (1-2E_i^\varepsilon)\Theta_i^\varepsilon \Theta_{i+1}^\varepsilon\right)\right]$$
$$+ \sum_{i=1}^{i=m(\bar{r}(\bar{t}))-1} (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)_z (\zeta_i^\varepsilon)_z \tag{7.5}$$

where $\zeta_i^\varepsilon = \sum_{k=1}^{k=i} E_k^\varepsilon$. *Moreover* Ψ_ε *satisfies*

$$-\Psi_{zz}^\varepsilon(z,t) = f'(\Theta^\varepsilon(z,t))\Psi^\varepsilon(z,t) - 2\left(3\Theta^\varepsilon(\Psi^\varepsilon)^2 + (\Psi^\varepsilon)^3\right)(z,t) + G^\varepsilon(z,t) - H^\varepsilon(z,t) \tag{7.6}$$
$$\Psi^\varepsilon(z_i^\varepsilon(t),t) = 0$$

Next we give a bound for U_1^ε.

Lemma 7.3 *There exist constants* $C_1(R_0)$ *and* $C_2(R_0)$ *such that* U_1^ε *satisfies*

$$\|U_1^\varepsilon(.,t)\|_{H^{1,\infty}(z_-^\varepsilon, z_+^\varepsilon)} \le C_1(R_0)(1+|\lambda^\varepsilon(t)|), \quad \text{for all} \quad t \in [\bar{t}_1, \bar{t}_2] \tag{7.7}$$

and

$$\sup_{t\in[\bar{t}_1,\bar{t}_2]} \|U_1^\varepsilon(.,t)\|_{H^{1,\infty}(z_-^\varepsilon, z_+^\varepsilon)} \le C_2(R_0)\varepsilon^{-1/2}. \tag{7.8}$$

Proof. Applying [[4], Lemma A.3] we deduce that for all $t \in [\bar{t}_1, \bar{t}_2]$

$$\|U_{1i}^\varepsilon(.,t)\|_{H^{1,\infty}(z_-^\varepsilon, z_+^\varepsilon)} \le 12(\|V^\varepsilon(.,t)\|_{L^\infty(z_-^\varepsilon, z_+^\varepsilon)} + |\lambda^\varepsilon(t)|) \quad \text{for all} \quad i \in [1,m]. \tag{7.9}$$

Using (4.9) we deduce that $\|U_{1i}^\varepsilon(.,t)\|_{H^{1,\infty}(z_-,z_+)} \le K_1(R_0)(1+|\lambda^\varepsilon(t)|)$, for all $i \in [1,m]$. Finally using the definition of U_1^ε and the fact that E_i^ε and $(E_i^\varepsilon)_z$ are uniformly bounded in ε we deduce that

$$\|U_1^\varepsilon(.,t)\|_{H^{1,\infty}(z_-,z_+)} \le K_2(R_0)(1+|\lambda^\varepsilon(t)|), \quad \text{for all} \quad t \in [\bar{t}_1, \bar{t}_2].$$

This coincides with (7.7). Moreover using the bound on λ^ε given in (3.11) we deduce (7.8). This completes the proof of Lemma 7.3. Next we state a lemma, which will be useful to prove Theorem 7.1. □

Lemma 7.4

$$\lim_{\varepsilon\to 0}(z^\varepsilon_{i+1}(t) - z^\varepsilon_i(t)) = -\infty, \quad \textit{for almost} \;\; t \in (\bar t_1, \bar t_2) \tag{7.10}$$

for all $i \in [1, m(\overline{r}(\overline{t}))-1]$. *Moreover there exists* $K(\delta) > 0$ *such that* $K(\delta) \to 0$ *as* $\delta \downarrow 0$ *satisfying*

$$[\Psi^\varepsilon\Theta^\varepsilon]_- \le K(\delta) \;\; \textit{for all} \;\; (z,t) \in [-\varepsilon^{-3/5} + z^\varepsilon_{m(\overline{r}(\overline{t}))}, \varepsilon^{-3/5}] \times [\bar t_1, \bar t_2]. \tag{7.11}$$

Proof. (7.10) is a direct consequence of (6.13). Next we prove (7.11). By definitions of Ψ^ε and Θ^ε we first note that

$$\begin{aligned}[\Psi^\varepsilon\Theta^\varepsilon(z,t)]_- \le{}& [((U^\varepsilon - U^\varepsilon_0)U^\varepsilon_0)(z,t)]_- + \varepsilon|(U^\varepsilon - U^\varepsilon_0)(z,t)||U^\varepsilon_1(z,t)| \\ &+ \varepsilon|U^\varepsilon_1(z,t)||U^\varepsilon_0(z,t)| + \varepsilon^2|U^\varepsilon_1(z,t)|^2\end{aligned}$$

for all $(z,t) \in [-\varepsilon^{-3/5} + z^\varepsilon_m, \varepsilon^{-3/5}] \times [\bar t_1, \bar t_2]$. Using (4.10), (7.8) and the fact that $|U^\varepsilon_0| \le 1$ we deduce that

$$[\Psi^\varepsilon\Theta^\varepsilon(z,t)]_- \le [((U^\varepsilon - U^\varepsilon_0)U^\varepsilon_0)(z,t)]_- + C(R_0,t)\varepsilon^{1/2} \tag{7.12}$$

for all $(z,t) \in [-\varepsilon^{-3/5} + z^\varepsilon_m, \varepsilon^{-3/5}] \times [\bar t_1, \bar t_2]$. Next we prove that

$$[((U^\varepsilon - U^\varepsilon_0)U^\varepsilon_0)(z,t)]_- \le K(\delta), \quad \text{for all} \;\; (z,t) \in [z^\varepsilon_{i+1}, z^\varepsilon_i] \times [\bar t_1, \bar t_2]. \tag{7.13}$$

We denote by τ_i the sign of $U^\varepsilon(.,t) - a^\varepsilon$ on the interval $(z^\varepsilon_{i+1}, z^\varepsilon_i)$ and we note that $\tau_i = (-1)^i$.

We first consider the case that i is even. We have in this case that

$$\begin{aligned}U^\varepsilon_0(z,t) &= E^\varepsilon_{i+1}\tanh(z - z^\varepsilon_{i+1} + \mu^\varepsilon) + E^\varepsilon_i \tanh(z^\varepsilon_i - z + \mu^\varepsilon) \\ &\ge E^\varepsilon_{i+1}\tanh(\mu^\varepsilon) + E^\varepsilon_i\tanh(\mu^\varepsilon) \ge a^\varepsilon \ge 0,\end{aligned} \tag{7.14}$$

for all $z \in [z^\varepsilon_{i+1}, z^\varepsilon_i]$. This with the fact that $(ab)_- = (a)_- b$ for $b > 0$ implies that $[(U^\varepsilon - U^\varepsilon_0)U^\varepsilon_0]_- = [U^\varepsilon - U^\varepsilon_0]_-[U^\varepsilon_0]$, on $[z^\varepsilon_{i+1}, z^\varepsilon_i] \times [\bar t_1, \bar t_2]$. Therefore using the fact that $\tau_i > 0$ and Theorem 6.1 we deduce that

$$[((U^\varepsilon - U^\varepsilon_0)U^\varepsilon_0)(z,t)]_- \le K(\delta), \quad \text{for all} \;\; (z,t)[z^\varepsilon_{i+1}, z^\varepsilon_i] \times [\bar t_1, \bar t_2]. \tag{7.15}$$

This implies (7.13).

In the case that i is odd we have that

$$U_0^\varepsilon(z,t) = E_{i+1}^\varepsilon \tanh(-z + z_{i+1}^\varepsilon + \mu^\varepsilon) + E_i^\varepsilon \tanh(-z_i^\varepsilon + z + \mu^\varepsilon), \tag{7.16}$$

so that $-U_0^\varepsilon(z,t) \geq 0$ for all $(z,t) \in [z_{i+1}^\varepsilon + \mu^\varepsilon, z_i^\varepsilon - \mu^\varepsilon] \times [\bar{t}_1, \bar{t}_2]$. The inequality (7.13) then follows in a similar way. In the same way one can deduce from the results (ii) and (iii) of Theorem 6.1 that

$$[((U^\varepsilon - U_0^\varepsilon)U_0^\varepsilon)(z,t)]_- \leq K(\delta) \tag{7.17}$$

for all $z \in [-\varepsilon^{-3/5} + z_m^\varepsilon, z_m^\varepsilon] \cup [z_1^\varepsilon, z_1^\varepsilon + \varepsilon^{-3/5}]$ and $t \in [\bar{t}_1, \bar{t}_2]$. Finally (7.12), (7.13) and (7.17) imply (7.11). This completes the proof of Lemma 7.4.

□

Next we give a bound for G^ε. More precisely we prove the following result

Lemma 7.5 *There exists a constant $C_3(R_0)$ such that G^ε satisfies*

$$\int_{\bar{t}_1}^{\bar{t}_2} \int_{z_-}^{z_+} |G^\varepsilon(z,t)|^2 dz\, dt \leq C(R_0)\varepsilon^2 \tag{7.18}$$

Proof. By definition of G^ε (7.4) we have

$$\int_{\bar{t}_1}^{\bar{t}_2} \int_{z_-}^{z_+} |G^\varepsilon(z,t)|^2 dz \leq 2\varepsilon^4 \int_{\bar{t}_1}^{\bar{t}_2} \int_{z_-}^{z_+} |u_t^\varepsilon(\varepsilon z + r_j^\varepsilon, t)|^2 dz\, dt$$
$$+ 2(N-1)^2 \varepsilon^2 \int_{\bar{t}_1}^{\bar{t}_2} \int_{z_-}^{z_+} \frac{|U_z^\varepsilon(z,t)|^2}{|\varepsilon z + r_j^\varepsilon|^2} dz,$$

After performing the change of variable $r = \varepsilon z + r_j^\varepsilon$ we obtain

$$\int_{\bar{t}_1}^{\bar{t}_2} \int_{z_-}^{z_+} |G^\varepsilon(z,t)|^2 dz \leq 2\varepsilon^3 \int_{\bar{t}_1}^{\bar{t}_2} \int_{R_0}^{1} |u_t^\varepsilon(r,t)|^2 dr\, dt$$
$$+ 2(N-1)^2 \varepsilon^3 \int_{\bar{t}_1}^{\bar{t}_2} \int_{R_0}^{1} \frac{|u_r^\varepsilon|^2}{r^2} dr\, dt.$$

Finally using (3.7) we deduce that $\int_{\bar{t}_1}^{\bar{t}_2} \int_{z_-}^{z_+} |G^\varepsilon(z,t)|^2 dz\, dt \leq C_2(R_0)\varepsilon^2$.

□

Next we give a bound on H^ε. More precisely we prove the following result.

Lemma 7.6 *There exists a constant $C_4(R_0)$ such that H^ε satisfies*

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{(-\varepsilon^{-3/5}+z^\varepsilon_m,z^\varepsilon_m)\cup(z^\varepsilon_1,\varepsilon^{-3/5})}|H^\varepsilon(z,t)|^2dz\,dt \le C_4(R_0)\varepsilon^{7/5}. \quad (7.19)$$

Moreover we have that

$$\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{z^\varepsilon_m}^{z^\varepsilon_1}|H^\varepsilon(z,t)|^2dz\,dt = 0. \quad (7.20)$$

Proof. Using the definition of the partition of unity $\{E^\varepsilon_i\}$ we have $E^\varepsilon_i = 0$ for all $i \in [1, m-1]$ and $E^\varepsilon_m = 1$ on $[z_-, z^\varepsilon_m]$. This together with (7.5) implies that

$$H^\varepsilon(z,t) = \varepsilon^2\left(6U^\varepsilon_{0m}(U^\varepsilon_{1m})^2 + 2\varepsilon(U^\varepsilon_{1m})^3\right)(z,t),$$
$$\text{for all } \ (z,t)\in[z^\varepsilon_-, z^\varepsilon_m]\times[\bar{t}_1,\bar{t}_2] \quad (7.21)$$

Moreover we have that $E^\varepsilon_i = 0$ for all $i \in [2,m]$ and $E^\varepsilon_1 = 1$ on $[z^\varepsilon_1, z^\varepsilon_+]\times[\bar{t}_1,\bar{t}_2]$, which implies that

$$H^\varepsilon(z,t) = \varepsilon^2\left(6U^\varepsilon_{01}(U^\varepsilon_{11})^2 + 2\varepsilon(U^\varepsilon_{11})^3\right)(z,t),$$
$$\text{for all } \ (z,t)\in[z^\varepsilon_1, z^\varepsilon_+]\times[\bar{t}_1,\bar{t}_2]. \quad (7.22)$$

Moreover we have

$$|U^\varepsilon_{0i}|_{L^\infty((z_-,z_+)\times(\bar{t}_1,\bar{t}_2))} \le 1, \quad \text{for all } \ i\in[1,m] \quad (7.23)$$

In view of (7.21), (7.22) and (7.23) we deduce that

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{(-\varepsilon^{-3/5}+z^\varepsilon_m,z^\varepsilon_m)\cup(z^\varepsilon_1,\varepsilon^{-3/5})}|H^\varepsilon(z,t)|^2dz\,dt$$
$$\le C\varepsilon^4\Big[\varepsilon^2|U^\varepsilon_1|^6_{L^\infty((z^\varepsilon_-,z^\varepsilon_+)\times(\bar{t}_1,\bar{t}_2))} + \varepsilon|U^\varepsilon_1|^5_{L^\infty((z^\varepsilon_-,z^\varepsilon_+)\times(\bar{t}_1,\bar{t}_2))}$$
$$+ |U^\varepsilon_1|^4_{L^\infty((z_-,z_+)\times(\bar{t}_1,\bar{t}_2))}\Big]\varepsilon^{-3/5}|\bar{t}_2-\bar{t}_1|.$$

Applying Lemma 7.3 we deduce that

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{(-\varepsilon^{-3/5}+z^\varepsilon_m,z^\varepsilon_m)\cup(z^\varepsilon_1,\varepsilon^{-3/5})}|H^\varepsilon(z,t)|^2dz\,dt \le C_3(R_0)\varepsilon^{7/5}.$$

This completes the proof of (7.19). Next we prove (7.20). We set

$$H_1^\varepsilon(z) = \varepsilon^2 \sum_{i=1}^{i=m} E_i^\varepsilon(6U_{0i}^\varepsilon(U_{1i}^\varepsilon)^2 + 2\varepsilon(U_{1i}^\varepsilon)^3)$$

$$H_2^\varepsilon(z) = \sum_{i=1}^{i=m-1} (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)\big[(\zeta_i^\varepsilon)_{zz} + 2E_i^\varepsilon E_{i+1}^\varepsilon\big((1+E_i^\varepsilon)(\Theta_i^\varepsilon)^2 + (E_i^\varepsilon - 2)(\Theta_{i+1}^\varepsilon)^2 + (1-2E_i^\varepsilon)\Theta_i^\varepsilon\Theta_{i+1}^\varepsilon\big)\big]$$

$$H_3^\varepsilon(z) = \sum_{i=1}^{i=m-1} (\Theta_{i+1}^\varepsilon - \Theta_i^\varepsilon)_z(\zeta_i^\varepsilon)_z,$$

so that

$$H^\varepsilon(z,t) = H_1^\varepsilon(z,t) + H_2^\varepsilon(z,t) + H_3^\varepsilon(z,t). \tag{7.24}$$

We deduce that

$$\int_{z_m^\varepsilon}^0 |H^\varepsilon(z,t)|^2 dz \le C\Big[\int_{z_m^\varepsilon}^0 |H_1^\varepsilon(z,t)|^2 dz + \int_{z_m^\varepsilon}^0 |H_2^\varepsilon(z,t)|^2 dz + \int_{z_m^\varepsilon}^0 |H_3^\varepsilon(z,t)|^2 dz\Big] \tag{7.25}$$

Next we prove that

$$\int_{\bar t_1}^{\bar t_2}\int_{z_m^\varepsilon}^0 |H_1^\varepsilon(z,t)|^2 dz\, dt \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{7.26}$$

Using (7.7) (7.23) and the fact that $|z_m^\varepsilon|$ is bounded by $\frac{1}{\varepsilon}$ we obtain that for all $i \in [1,m]$

$$\begin{aligned}
&\int_{\bar t_1}^{\bar t_2}\int_{z_m^\varepsilon}^0 |H_1^\varepsilon(z,t)|^2 dz\, dt \\
&\le C\varepsilon^4\Big[\int_{\bar t_1}^{\bar t_2}\int_{z_m^\varepsilon}^0 (E_i^\varepsilon)^2\left((U_{0i}^\varepsilon)^2(U_{1i}^\varepsilon)^4 + \varepsilon^2(U_{1i}^\varepsilon)^6 dz\, dt\right)\Big] \\
&\le C\varepsilon^3\Big[\Big(1 + \int_{\bar t_1}^{\bar t_2} |\lambda^\varepsilon(t)|^2 dt\Big)\|U_1^\varepsilon\|^2_{L^\infty((z_-^\varepsilon, z_+^\varepsilon)\times(\bar t_1,\bar t_2))} \\
&\qquad + \varepsilon^2\Big(1 + \int_{\bar t_1}^{\bar t_2} |\lambda^\varepsilon(t)|^2 dt\Big)\|U_1^\varepsilon\|^4_{L^\infty((z_-^\varepsilon, z_+^\varepsilon)\times(\bar t_1,\bar t_2))}\Big]
\end{aligned}$$

Applying Lemma 4.4 and (7.8) we obtain

$$\varepsilon^4\left[\int_{\bar{t}_1}^{\bar{t}_2}\int_{z_m^\varepsilon}^0 (E_i^\varepsilon)^2\left((U_{0i}^\varepsilon)^2(U_{1i}^\varepsilon)^4+\varepsilon^2(U_{1i}^\varepsilon)^6 dz\,dt\right)\right]\le K(R_0)\varepsilon^2,$$

which implies (7.26). Moreover as it is done in [[4], Lemma 6.6] one can check that. Next we prove that

$$\int_{z_m^\varepsilon}^0 |H_2^\varepsilon(z,t)|^2 dz\to 0 \;\text{ for a.e. }\; t\in(\bar{t}_1,\bar{t}_2), \tag{7.27}$$

and

$$\int_{z_m^\varepsilon}^0 |H_3^\varepsilon(z,t)|^2 dz\to 0 \;\text{ for a.e. }\; t\in(\bar{t}_1,\bar{t}_2), \tag{7.28}$$

as $\varepsilon\downarrow 0$. Moreover since H_2^ε and H_3^ε are uniformly bounded we deduce from the dominated convergence theorem and from (7.25), (7.26), (7.27) and (7.28) that $\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{z_m^\varepsilon}^0 |H^\varepsilon(z,t)|^2 dz\,dt=0$. This completes the proof of Lemma 7.6. □

We are now in a position to prove Theorem 7.1.

Proof of Theorem 7.1. Multiplying (7.6) by $|\xi^\varepsilon|^2\Psi^\varepsilon$ and integrating the result on $I\times(\bar{t}_1,\bar{t}_2)$, where $I=(-\infty,z_m^\varepsilon)\cup(0,+\infty)$ or $(z_m^\varepsilon,0)$ we obtain that

$$\begin{aligned}
&\int_{\bar{t}_1}^{\bar{t}_2}\int_I \left(-\Psi_{zz}^\varepsilon - f'(\Theta^\varepsilon)\Psi^\varepsilon\right)\Psi^\varepsilon|\xi^\varepsilon|^2\\
&\quad = -6\int_{\bar{t}_1}^{\bar{t}_2}\int_I \Theta^\varepsilon(\Psi^\varepsilon)^3|\xi^\varepsilon|^2 - 2\int_{\bar{t}_1}^{\bar{t}_2}\int_I (\Psi^\varepsilon)^4|\xi^\varepsilon|^2\\
&\qquad + \int_{\bar{t}_1}^{\bar{t}_2}\int_I G^\varepsilon\Psi^\varepsilon|\xi^\varepsilon|^2 - \int_{\bar{t}_1}^{\bar{t}_2}\int_I H^\varepsilon\Psi^\varepsilon|\xi^\varepsilon|^2
\end{aligned}$$

Using Lemma A.4 in Appendix of [4] we deduce that

$$\begin{aligned}
&S_1\int_{\bar{t}_1}^{\bar{t}_2}\int_I |\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2 + S_2\int_{\bar{t}_1}^{\bar{t}_2}\int_I |\Psi^\varepsilon|^2|\xi^\varepsilon|^2\\
&\quad \le \frac{1}{S_1}\int_{\bar{t}_1}^{\bar{t}_2}\int_I |\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2 + \frac{1}{S_2}\left(\int_{\bar{t}_1}^{\bar{t}_2}\int_I |G^\varepsilon|^2|\xi^\varepsilon|^2 + \int_{\bar{t}_1}^{\bar{t}_2}\int_I |H^\varepsilon|^2|\xi^\varepsilon|^2\right)\\
&\qquad + 6([\Theta^\varepsilon\Psi^\varepsilon]_-)_{L^\infty((-\varepsilon^{-3/5}+z_m^\varepsilon,\varepsilon^{-3/5})\times(\bar{t}_1,\bar{t}_2))}\int_{\bar{t}_1}^{\bar{t}_2}\int_I |\Psi^\varepsilon|^2|\xi^\varepsilon|^2
\end{aligned}$$

This implies that

$$\begin{aligned} S_1 \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi_z^\varepsilon|^2 |\xi^\varepsilon|^2 \\ + \Big(\frac{3}{4} S_2 - 6([\Theta^\varepsilon \Psi^\varepsilon]_-)_{L^\infty((-\varepsilon^{-3/5}+z_m^\varepsilon, \varepsilon^{-3/5})\times(\bar{t}_1,\bar{t}_2))}\Big) \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi^\varepsilon|^2 |\xi^\varepsilon|^2 \\ \le \frac{2}{S_1} \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi^\varepsilon|^2 |\xi_z^\varepsilon|^2 + \frac{1}{2S_2} \Big(\int_{\bar{t}_1}^{\bar{t}_2} \int_I |G^\varepsilon|^2 |\xi^\varepsilon|^2 + \int_{\bar{t}_1}^{\bar{t}_2} \int_I |H^\varepsilon|^2 |\xi^\varepsilon|^2 \Big) \end{aligned} \tag{7.29}$$

By (7.11) we may choose δ small enough so that

$$([\Theta^\varepsilon \Psi^\varepsilon]_-)_{L^\infty([z_-,z_+]\times[\bar{t}_1,\bar{t}_2])} < \frac{1}{24} S_2$$

Substituting this into (7.29) we obtain

$$\begin{aligned} S_1 \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi_z^\varepsilon|^2 |\xi^\varepsilon|^2 + \frac{S_2}{2} \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi^\varepsilon|^2 |\xi^\varepsilon|^2 \\ \le \frac{2}{S_1} \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi^\varepsilon|^2 |\xi_z^\varepsilon|^2 + \frac{2}{S_2} \Big(\int_{\bar{t}_1}^{\bar{t}_2} \int_I |G^\varepsilon|^2 |\xi^\varepsilon|^2 + \int_{\bar{t}_1}^{\bar{t}_2} \int_I |H^\varepsilon|^2 |\xi^\varepsilon|^2 \Big). \end{aligned} \tag{7.30}$$

We first consider the case that $I = [z_m^\varepsilon, 0]$. Using (7.30) lemmas 7.5 and 7.6 and the fact that in this case $\xi^\varepsilon = 1$ we deduce that

$$\lim_{\varepsilon \to 0} \int_{\bar{t}_1}^{\bar{t}_2} \int_{(z_m^\varepsilon, 0)} (|\Psi_z^\varepsilon|^2 + |\Psi^\varepsilon|^2) \, dz \, dt = 0.$$

Next we consider the case that $I = (-\infty, z_m^\varepsilon]$ or $[0, +\infty)$. Using (7.30) and the Lemmas 7.5 and 7.6 we deduce that

$$\begin{aligned} S_1 \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi_z^\varepsilon|^2 |\xi^\varepsilon|^2 + \frac{S_2}{2} \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi^\varepsilon|^2 |\xi^\varepsilon|^2 \\ \le \frac{2}{S_1} \int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi^\varepsilon|^2 |\xi_z^\varepsilon|^2 + K(R_0)(\varepsilon^2 + \varepsilon^{6/5}) \end{aligned} \tag{7.31}$$

Next we estimate $\int_{\bar{t}_1}^{\bar{t}_2} \int_I |\Psi^\varepsilon|^2 |\xi_z^\varepsilon|^2$. Using the fact that $|\xi_z^\varepsilon| \le C\varepsilon^{3/5}$ we deduce that

$$\int_{\bar t_1}^{\bar t_2}\int_I |\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2 \le C\varepsilon^{6/5}\Big[\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-3/5}+z_m^\varepsilon}^{-\varepsilon^{-1/2}+z_m^\varepsilon}|\Psi^\varepsilon|^2+\int_{\bar t_1}^{\bar t_2}\int_{\varepsilon^{-1/2}}^{\varepsilon^{-3/5}}|\Psi^\varepsilon|^2\Big] \tag{7.32}$$

As it is done by Henry in [4] one can deduce from (7.32) the estimate

$$\int_I |\Psi^\varepsilon|^2|\xi_z^\varepsilon|^2 \le C_3(R_0)\varepsilon^{6/5}. \tag{7.33}$$

Finally using (7.31) and integrating (7.33) on $[\bar t_1, \bar t_2]$ we deduce that

$$\int_{\bar t_1}^{\bar t_2}\int_I |\Psi_z^\varepsilon|^2|\xi^\varepsilon|^2+\int_{\bar t_1}^{\bar t_2}\int_I |\Psi^\varepsilon|^2|\xi^\varepsilon|^2 \le C(R_0)\varepsilon^{6/5}$$

This completes the proof of Theorem 7.1. □

As in [4] one can then deduce from Theorem 7.1 the following result.

Corollary 7.7 *Let* $J := [-\varepsilon^{-1/2}+z^\varepsilon_{m(\bar r(\bar t))}, z^\varepsilon_{m(\bar r(\bar t))}]\cup[0,\varepsilon^{-1/2}]$; *we have that*

$$\int_{\bar t_1}^{\bar t_2}\|\Psi^\varepsilon(.,t)\|^2_{H^{1,\infty}(J)}dt \le C(R_0)\varepsilon^{6/5}, \tag{7.34}$$

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\|\Psi^\varepsilon(.,t)\|^2_{H^{1,\infty}(0,z^\varepsilon_{m(\bar r(\bar t))})}dt = 0, \tag{7.35}$$

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\|U^\varepsilon(.,t)-\nu(r)U_0^\varepsilon(.,t)\|^2_{H^1([-\varepsilon^{-1/2}+z^\varepsilon_{m(\bar r(\bar t))},\varepsilon^{-1/2}])}dt = 0, \tag{7.36}$$

$$\int_{\bar t_1}^{\bar t_2}\|U^\varepsilon(.,t)-\nu(r)U_0^\varepsilon(.,t)\|^2_{H^{1,\infty}(J)}dt \le C(R_0)\varepsilon^{6/5}, \tag{7.37}$$

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\|U^\varepsilon(.,t)-\nu(r)U_0^\varepsilon(.,t)\|^2_{H^{1,\infty}(0,z^\varepsilon_{m(\bar r(\bar t))})}dt = 0. \tag{7.38}$$

8. The limit as $\varepsilon \downarrow 0$

Let $\bar r(\bar t)$ be a jump of $u(.,\bar t)$ and $R_0\in(0,1)$ such that $\bar r(\bar t) > R_0$. We refer to the introduction of Section 7 for the properties of $\bar r(\bar t)$. Moreover we set

$$\mu(r_j(s),s) = \begin{cases} \nu(r_j(s)) & \text{if } r_j(s) \text{ is a jump} \\ 0 & \text{if } r_j(s) \text{ is not a jump.} \end{cases}$$

In this Section we derive the motion equation of a jump $\overline{r}$. To that purpose we first obtain the equation of the limit function r_j.

8.1. Evolution equation for the limit r_l

Lemma 8.1 *For each $j \in [1,\ldots,m(\overline{r}(\overline{t}))]$ we have*

$$\partial_t r_j(t) = -\frac{N-1}{r_j(t)} + \frac{3}{2}\mu(r_j(t),t)[\lambda_0(t) + v(r_j(t),t)], \tag{8.1}$$

a.e. in $[\overline{t}_1, \overline{t}_2]$.

Proof. For the sake of simplicity we only prove Lemma 8.1 in the case that the function $r \longrightarrow (u^\varepsilon - a^\varepsilon)(r,t)$ only has one zero for all $t \in [\overline{t}_1, \overline{t}_2]$. This amounts to suppose that $m = 1$ so that $j = 1$ and $z^\varepsilon_m = z^\varepsilon_1 = 0$. We refer to [5] for the proofs in the general case where the function $u^\varepsilon - a^\varepsilon$ may have m zeros close to each other. Moreover we also suppose that $\nu = 1$. Note that the function U^ε_0, which is given by

$$U^\varepsilon_0(z) = \tanh(z + \mu^\varepsilon) \;\; \text{where} \;\; \mu^\varepsilon = \tanh^{-1}(a^\varepsilon), \tag{8.2}$$

does not depend on time. Moreover we have that

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} (U^\varepsilon_0)_z dz = 2, \tag{8.3}$$

and

$$\lim_{\varepsilon \to 0} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} ((U^\varepsilon_0)_z)^2 dz = \frac{4}{3}. \tag{8.4}$$

We write (4.1) in the variable $z := \frac{r - r^\varepsilon_1(t)}{\varepsilon}$, this gives that

$$\varepsilon U^\varepsilon_t - [\partial_t r^\varepsilon_1(t)] U^\varepsilon_z - \frac{1}{\varepsilon} U^\varepsilon_{zz} - \frac{N-1}{\varepsilon z + r^\varepsilon_1} U^\varepsilon_z - \frac{1}{\varepsilon} f(U^\varepsilon) + V^\varepsilon + \lambda^\varepsilon = 0 \tag{8.5}$$

Multiplying (8.5) by $\zeta(t) U^\varepsilon_z(z,t)$ where ζ is a smooth test function with compact support in $(\overline{t}_1, \overline{t}_2)$ and integrating the result on $(-\varepsilon^{-1/2}, \varepsilon^{-1/2}) \times$

$(\bar{t}_1, \bar{t}_2)$ we obtain

$$\varepsilon \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta U_t^\varepsilon U_z^\varepsilon - \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta \left[\partial_t r_1^\varepsilon - \frac{N-1}{\varepsilon z + r_1^\varepsilon} r_1^\varepsilon\right] (U_z^\varepsilon)^2$$
$$+ \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta V^\varepsilon U_z^\varepsilon - \frac{1}{\varepsilon} \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta \left[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\right]_z$$
$$+ \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta U_z^\varepsilon \lambda^\varepsilon = 0 \tag{8.6}$$

In order to pass to the limit in (8.6), we first prove the following lemma.

□

Lemma 8.2

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta \left[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\right]_z dz\, dt = 0$$

Proof. We note that

$$\frac{1}{\varepsilon} \left| \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t) \left[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\right]_z \right|$$
$$\le \frac{1}{\varepsilon} \|\zeta\|_\infty \int_{\bar{t}_1}^{\bar{t}_2} \left\{ \frac{1}{2} \Big((U_z^\varepsilon)^2(\varepsilon^{-1/2}, t) + (U_z^\varepsilon)^2(-\varepsilon^{-1/2}, t) \Big) \right.$$
$$\left. + F\big(U^\varepsilon(\varepsilon^{-1/2}, t)\big) + F(U^\varepsilon(-\varepsilon^{-1/2}, t)) \right\} dt \tag{8.7}$$

Moreover we have that

$$|U_z^\varepsilon|^2 \le (|U_z^\varepsilon - (U_0^\varepsilon)_z| + |(U_0^\varepsilon)_z|)^2 \le 2\left[|U_z^\varepsilon - (U_0^\varepsilon)_z|^2 + |(U_0^\varepsilon)_z|^2\right]. \tag{8.8}$$

Using the definition of U_0^ε we obtain

$$|(U_0^\varepsilon)_z(\varepsilon^{-1/2})| = 1 - \tanh^2(\varepsilon^{-1/2} + \mu^\varepsilon) \tag{8.9}$$

and

$$|(U_0^\varepsilon)_z(-\varepsilon^{-1/2})| = 1 - \tanh^2(-\varepsilon^{-1/2} + \mu^\varepsilon), \tag{8.10}$$

so that applying (8.8) at the points $-\varepsilon^{-1/2}$ and $\varepsilon^{-1/2}$ and also using the fact that the points $-\varepsilon^{-1/2}$ and $\varepsilon^{-1/2}$ are in the interval J defined in the

Corollary 7.7 we deduce that

$$
\begin{aligned}
&|U_z^\varepsilon(\varepsilon^{-1/2},t)|^2 + |U_z^\varepsilon(-\varepsilon^{-1/2},t)|^2 \\
&\quad \le 4\|[U_z^\varepsilon - (U_0^\varepsilon)_z](.,t)\|^2_{L^\infty(J)} + 2(1-\tanh^2(\varepsilon^{-1/2}+\mu^\varepsilon))^2 \\
&\qquad + 2(1-\tanh^2(-\varepsilon^{-1/2}+\mu^\varepsilon))^2, \qquad (8.11)
\end{aligned}
$$

for all $t \in [\bar{t}_1, \bar{t}_2]$. Furthermore we have that

$$F(U_0^\varepsilon(\varepsilon^{-1/2})) = \frac{1}{2}[1-\tanh^2(\varepsilon^{-1/2}+\mu^\varepsilon)]^2$$

and

$$F(U_0^\varepsilon(-\varepsilon^{-1/2})) = \frac{1}{2}[1-\tanh^2(-\varepsilon^{-1/2}+\mu^\varepsilon)]^2.$$

This implies that

$$
\begin{aligned}
F(U^\varepsilon(\varepsilon^{-1/2},t)) &\le |(F(U^\varepsilon) - F(U_0^\varepsilon))(\varepsilon^{-1/2},t)| \\
&\quad + \frac{1}{2}[1-\tanh^2(\varepsilon^{-1/2}+\mu^\varepsilon)]^2 \qquad (8.12)
\end{aligned}
$$

and

$$
\begin{aligned}
F(U^\varepsilon(-\varepsilon^{-1/2},t)) &\le |(F(U^\varepsilon) - F(U_0^\varepsilon))(-\varepsilon^{-1/2},t)| \\
&\quad + \frac{1}{2}[1-\tanh^2(-\varepsilon^{-1/2}+\mu^\varepsilon)]^2 \qquad (8.13)
\end{aligned}
$$

for all $t \in [\bar{t}_1, \bar{t}_2]$. Substituting (8.11), (8.12), and (8.13) into (8.7) we obtain

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left|\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta\left[\frac{1}{2}(U_z^\varepsilon)^2 - F(U^\varepsilon)\right]_z dz\,dt\right| \\
&\le C(T)\|\zeta\|_\infty\Big(\frac{2}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2} \|U_z^\varepsilon - (U_0^\varepsilon)_z\|^2_{L^\infty(J)} dt \\
&\qquad + \frac{1}{\varepsilon}[1-\tanh^2(\varepsilon^{-1/2}+\mu^\varepsilon)]^2 + \frac{1}{\varepsilon}[1-\tanh^2(-\varepsilon^{-1/2}+\mu^\varepsilon)]^2 \\
&\qquad + \frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2} \Big|(F(U^\varepsilon) - F(U_0^\varepsilon))(\varepsilon^{-1/2},t))\Big|\,dt \\
&\qquad\quad + \frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2} \Big|(F(U^\varepsilon) - F(U_0^\varepsilon))(-\varepsilon^{-1/2},t)\Big|\,dt\Big) \qquad (8.14)
\end{aligned}
$$

Moreover there exists $\tilde{U}$ between $U_0^\varepsilon(\varepsilon^{-1/2})$ and $U^\varepsilon(\varepsilon^{-1/2},t)$ such that

$$\begin{aligned}&(F(U^\varepsilon)-F(U_0^\varepsilon))(\varepsilon^{-1/2},t)\\&=(U^\varepsilon-U_0^\varepsilon)(\varepsilon^{-1/2},t)F'(U_0^\varepsilon(\varepsilon^{-1/2}))+\frac{(U^\varepsilon-U_0^\varepsilon)^2(\varepsilon^{-1/2},t)}{2}F'(\tilde{U}).\end{aligned}$$

From (4.10) and from the fact that $|U_0^\varepsilon|\le 1$ we deduce that $F'(x)$ is bounded for all $\tilde{U}$ between $U_0^\varepsilon(\varepsilon^{-1/2})$ and $U^\varepsilon(\varepsilon^{-1/2},t)$ and moreover since $|F'(U_0^\varepsilon)|\le 2[1-\tanh^2(\varepsilon^{-1/2}+\mu^\varepsilon)]$ we obtain that

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}|(F(U^\varepsilon)-F(U_0^\varepsilon))(\varepsilon^{-1/2},t)|dt\\&\le C\Bigg(\frac{1}{\varepsilon}[1-\tanh^2(\varepsilon^{-1/2}+\mu^\varepsilon)]\bigg(\int_{\bar{t}_1}^{\bar{t}_2}|U_0^\varepsilon(\varepsilon^{-1/2})-U^\varepsilon(\varepsilon^{-1/2},t)|^2dt\bigg)^{1/2}\\&\qquad+\frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}|U_0^\varepsilon(\varepsilon^{-1/2})-U^\varepsilon(\varepsilon^{-1/2},t)|^2dt\Bigg).\end{aligned}$$

Using (7.37) we deduce that

$$\frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}|(F(U^\varepsilon)-F(U_0^\varepsilon))(\varepsilon^{-1/2},t)|dt\to 0,\quad\text{as}\quad \varepsilon\downarrow 0.$$

Similarly one can prove that

$$\frac{1}{\varepsilon}\int_{\bar{t}_1}^{\bar{t}_2}|(F(U^\varepsilon)-F(U_0^\varepsilon))(-\varepsilon^{-1/2},t)|dt\to 0,\quad\text{as}\quad \varepsilon\downarrow 0.$$

Coming back to (8.14) and using again (7.37) we deduce the result of Lemma 8.2. □

Lemma 8.3

$$\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta\left[\frac{N-1}{\varepsilon z+r_1^\varepsilon}+\partial_t r_1^\varepsilon\right](U_z^\varepsilon)^2dz\,dt=\frac{4}{3}\int_{\bar{t}_1}^{\bar{t}_2}\zeta\left[\frac{N-1}{r_1}+\partial_t r_1\right]dt$$

Proof. We first prove that

$$\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta\frac{1}{\varepsilon z+r_1^\varepsilon}(U_z^\varepsilon)^2dz\,dt=\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta\frac{1}{\varepsilon z+r_1^\varepsilon}(U_0^\varepsilon)_z^2dz\,dt \tag{8.15}$$

and that

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta\partial_t r_1^\varepsilon (U_z^\varepsilon)^2 dz\,dt = \lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta\partial_t r_1^\varepsilon (U_0^\varepsilon)_z^2 dz\,dt. \tag{8.16}$$

Since $\varepsilon z + r_1^\varepsilon = r \geq R_0$ we have

$$\begin{aligned}
&\left|\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta\frac{1}{\varepsilon z + r_1^\varepsilon}\Big[(U_z^\varepsilon)^2 - ((U_0^\varepsilon)_z)^2\Big]dz\,dt\right| \\
&\quad + \left|\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta\partial_t r_1^\varepsilon\Big[(U_z^\varepsilon)^2 - ((U_0^\varepsilon)_z)^2\Big]\right| \\
&\leq \|\zeta\|_\infty \int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \left(\frac{1}{\varepsilon z + r_1^\varepsilon} + |\partial_t r_1^\varepsilon|\right)\left|(U_z^\varepsilon)^2 - (U_0^\varepsilon)_z^2\right| dz\,dt \\
&\leq \|\zeta\|_\infty \int_{\bar t_1}^{\bar t_2}\left\{\left(\frac{1}{R_0} + |\partial_t r_1^\varepsilon|\right)\left[\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} |U_z^\varepsilon - (U_0^\varepsilon)_z|^2\right]^{1/2}\right. \\
&\qquad\qquad \left.\left[\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} |U_z^\varepsilon + (U_0^\varepsilon)_z|^2\right]^{1/2}\right\} dt
\end{aligned} \tag{8.17}$$

It follows from (4.7) and (8.4) that

$$\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} |U_z^\varepsilon + (U_0^\varepsilon)_z|^2 dz \leq \frac{\varepsilon}{R_0^{N-1}}\int_{R_0}^1 |u_r^\varepsilon|^2 r^{N-1} dr + \frac{4}{3} \leq C(R_0),$$

which we substitute into (8.17) to obtain

$$\begin{aligned}
&\left|\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \left[\frac{1}{\varepsilon z + r_1^\varepsilon} + \partial_t r_1^\varepsilon\right]\left[(U_z^\varepsilon)^2 - ((U_0^\varepsilon)_z)^2\right]\right| \\
&\leq C_2(R_0)\left(\int_{\bar t_1}^{\bar t_2}\left[\frac{1}{R_0} + |\partial_t r_1^\varepsilon|\right]^2\right)^{1/2}\left(\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} |U_z^\varepsilon - (U_0^\varepsilon)_z|^2\right)^{1/2}.
\end{aligned}$$

Using (7.36) and the fact that $\partial_t r_1^\varepsilon$ is bounded in $L^2(\bar t_1, \bar t_2)$ (cf. Theorem 5.9) we obtain (8.15) and (8.16). Moreover we deduce from (8.4), (8.16) and the the fact that $\partial_t r_1^\varepsilon$ tends to ∂r_1 weakly in $L^2(\bar t_1, \bar t_2)$ that

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta\partial_t r_1^\varepsilon (U_z^\varepsilon)^2 dz\,dt = \frac{4}{3}\int_{\bar t_1}^{\bar t_2} \zeta\partial_t r_1 dt \tag{8.18}$$

Next we prove that

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta(t)\frac{1}{\varepsilon z+r_1^\varepsilon(t)}((U_0^\varepsilon)_z)^2dz\,dt=\frac{4}{3}\int_{\bar t_1}^{\bar t_2}\zeta(t)\frac{1}{r_1(t)}\,dt. \tag{8.19}$$

We have

$$\begin{aligned}
&\left|\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta\left(\frac{1}{\varepsilon z+r_1^\varepsilon(t)}-\frac{1}{r_1(t)}\right)((U_0^\varepsilon)_z)^2\right|\\
&\le\frac{\|\zeta\|_\infty}{R_0^2}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\left(|r_1(t)-r_1^\varepsilon(t)|+\varepsilon^{-1/2}\right)((U_0^\varepsilon)_z)^2\\
&\le\frac{\|\zeta\|_\infty}{R_0^2}\left(\sup_{[\bar t_1,\bar t_2]}|r_1(t)-r_1^\varepsilon(t)|+\varepsilon^{1/2}\right)(\bar t_2-\bar t_1)\int_{-\varepsilon^{-1/2}}^{\varepsilon^{1/2}}((U_0^\varepsilon)_z)^2dz
\end{aligned} \tag{8.20}$$

Using (8.4) and the fact that r_1^ε tends to r_1 uniformly on $[\bar t_1,\bar t_2]$ (cf. Theorem 5.9) we deduce (8.19). Finally Lemma 8.3 follows from (8.15), (8.18) and (8.19). □

Lemma 8.4

$$\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta(t)\lambda^\varepsilon(t)U_z^\varepsilon(z,t)dz\,dt=2\int_{\bar t_1}^{\bar t_2}\zeta(t)\lambda_0(t)dt$$

Proof. We first prove that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta(t)\lambda^\varepsilon(t)U_z^\varepsilon\,dz\,dt\\
&\qquad=\lim_{\varepsilon\to 0}\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta(t)\lambda^\varepsilon(t)(U_0^\varepsilon)_z\,dz\,dt.
\end{aligned} \tag{8.21}$$

We have that

$$\begin{aligned}
&\left|\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta(t)\lambda^\varepsilon(t)[U_z^\varepsilon-(U_0^\varepsilon)_z]\,dz\,dt\right|\\
&=\left|\int_{\bar t_1}^{\bar t_2}\left\{\zeta(t)\lambda^\varepsilon(t)\left[(U^\varepsilon-U_0^\varepsilon)(\varepsilon^{-1/2},t)-(U^\varepsilon-U_0^\varepsilon)(-\varepsilon^{-1/2},t)\right]\right\}dt\right|\\
&\le 2\|\zeta\|_\infty\left(\int_{\bar t_1}^{\bar t_2}|\lambda^\varepsilon(t)|^2dt\right)^{1/2}\left(\int_{\bar t_1}^{\bar t_2}\|U^\varepsilon-U_0^\varepsilon\|_{L^\infty(J)}^2dt\right)^{1/2},
\end{aligned}$$

which in view of Lemma 4.4 and (7.37) implies (8.21). Moreover using the definition of U_0^ε given in (8.2) we have

$$\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)\lambda^\varepsilon(t)(U_0^\varepsilon)_z\,dz\,dt$$
$$= \Big[\tanh(\varepsilon^{-1/2}+\mu^\varepsilon) - \tanh(-\varepsilon^{-1/2}+\mu^\varepsilon)\Big]\int_{\bar{t}_1}^{\bar{t}_2} \zeta\lambda^\varepsilon\,dt$$

Letting ε tends to zero and using Corollary 4.5 and (8.21) we deduce Lemma 8.4. □

Lemma 8.5

$$\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)V^\varepsilon(z,t)U_z^\varepsilon(z,t)dz\,dt = 2\int_{\bar{t}_1}^{\bar{t}_2}\zeta(t)v(r_1(t),t)dt$$

Proof. We prove below that

$$\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)V^\varepsilon(z,t)U_z^\varepsilon(z,t)dz\,dt$$
$$= \lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)V^\varepsilon(z,t)(U_0^\varepsilon)_z(z,t)dz\,dt. \tag{8.22}$$

Integrating by parts the integral $\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} V^\varepsilon(U_z^\varepsilon-(U_0^\varepsilon)_z)dz$ and also using the bound of v^ε, (4.9), we obtain

$$\left|\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} V^\varepsilon(U_z^\varepsilon-(U_0^\varepsilon)_z)dz\right|$$
$$= \left|\Big[V^\varepsilon(U^\varepsilon-U_0^\varepsilon)\Big]_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} - \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} V_z^\varepsilon(U^\varepsilon-U_0^\varepsilon)dz\right|$$
$$\leq C(R_0,T)\|U^\varepsilon-U_0^\varepsilon\|_{L^\infty(J)}$$
$$+\left(\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}(V_z^\varepsilon)^2dz\right)^{1/2}\left(\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}(U^\varepsilon-U_0^\varepsilon)^2dz\right)^{1/2} \tag{8.23}$$

Moreover the energy estimate (4.7) implies that

$$\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}(V_z^\varepsilon)^2dz \leq \varepsilon\int_{R_0}^{1}(v_r^\varepsilon)^2dr \leq \varepsilon C(R_0), \tag{8.24}$$

which we substitute in (8.22) and (8.23) to obtain

$$\left|\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)V^\varepsilon(z,t)(U_z^\varepsilon-(U_0^\varepsilon)_z)(z,t)dz\,dt\right|$$
$$\leq \|\zeta\|_\infty C(R_0,T)$$
$$\left[\left(\int_{\bar{t}_1}^{\bar{t}_2}\|U^\varepsilon-U_0^\varepsilon\|^2_{L^\infty(J)}dt\right)^{1/2}+\varepsilon^{1/2}\left(\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}|U^\varepsilon-U_0^\varepsilon|^2dz\,dt\right)^{1/2}\right].$$

Using (7.36) and (7.37) we deduce (8.22). Next we prove that

$$\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)V^\varepsilon(z,t)(U_0^\varepsilon)_z(z,t)dz\,dt = 2\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\zeta(t)v^\varepsilon(r_1^\varepsilon,t)dt. \tag{8.25}$$

We first note that $v^\varepsilon(r_1^\varepsilon,t)=V^\varepsilon(0,t)$. We have that

$$\left|\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)V^\varepsilon(z,t)(U_0^\varepsilon)_z dz\,dt - 2\int_{\bar{t}_1}^{\bar{t}_2}\zeta(t)v^\varepsilon(r_1^\varepsilon,t)dt\right|$$
$$=\left|\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta\left[V^\varepsilon(z,t)-V^\varepsilon(0,t)\right](U_0^\varepsilon)_z dz\,dt\right.$$
$$\left.+\int_{\bar{t}_1}^{\bar{t}_2}\zeta v^\varepsilon(r_1^\varepsilon,t)\left[\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}(U_0^\varepsilon)_z dz-2\right]dt\right|$$
$$\leq \|\zeta\|_\infty\left(\int_{\bar{t}_1}^{\bar{t}_2}\left\{\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\left(\int_0^z|V_z^\varepsilon|^2dz\right)^{1/2}|z|^{1/2}|(U_0^\varepsilon)_z|\,dz\right\}dt\right.$$
$$\left.+\|v^\varepsilon\|_{L^\infty([R_0,1]\times[0,T])}(\bar{t}_2-\bar{t}_1)\left|\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}(U_0^\varepsilon)_z dz-2\right|\right).$$

In view of (8.24) and (4.9) this gives that

$$\left|\int_{\bar{t}_1}^{\bar{t}_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)V^\varepsilon(z,t)(U_0^\varepsilon)_z dz\,dt - 2\int_{\bar{t}_1}^{\bar{t}_2}\zeta(t)v^\varepsilon(r_1^\varepsilon,t)dt\right|$$
$$\leq \|\zeta\|_\infty C(R_0,T)\left(\varepsilon^{1/4}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}|(U_0^\varepsilon)_z|dz+\left|\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}(U_0^\varepsilon)_z dz-2\right|\right).$$

Together with (8.3) this implies (8.25). We now prove that

$$2\lim_{\varepsilon\to 0}\int_{\bar{t}_1}^{\bar{t}_2}\zeta(t)v^\varepsilon(r_1^\varepsilon,t)dt = 2\int_{\bar{t}_1}^{\bar{t}_2}\zeta(t)v(r_1,t)dt. \tag{8.26}$$

We have that

$$[v^\varepsilon - v](r_1,t) = -\int_{r_1}^{r_1^\varepsilon} v_r^\varepsilon ds + v^\varepsilon(r_1^\varepsilon,t) - v(r_1,t). \tag{8.27}$$

Moreover we note that

$$\begin{aligned}\int_{r_1(t)}^{r_1(t)+h}&\left[\int_r^{r_1(t)} [v^\varepsilon - v]_r(s,t)\,ds\right] dr \\ &= \int_{r_1(t)}^{r_1(t)+h} \left([v^\varepsilon - v](r_1(t),t) - [v^\varepsilon - v](r,t)\right) dr \\ &= h[v^\varepsilon - v](r_1(t),t) - \int_{r_1(t)}^{r_1(t)+h} [v^\varepsilon - v](r,t)dr, \end{aligned} \tag{8.28}$$

for all $h > 0$. Substituting (8.27) into (8.28) we obtain that

$$\begin{aligned} v^\varepsilon&(r_1^\varepsilon(t),t) - v(r_1(t),t) \\ &= \int_{r_1(t)}^{r_1^\varepsilon(t)} v_r^\varepsilon(s,t)ds + \frac{1}{h}\int_{r_1(t)}^{r_1(t)+h}\int_r^{r_1(t)} (v^\varepsilon - v)_r ds\,dr \\ &\quad + \frac{1}{h}\int_{r_1(t)}^{r_1(t)+h} [v^\varepsilon - v](r,t)dr \end{aligned} \tag{8.29}$$

Multiplying (8.29) by ζ and integrating the result on $[\bar{t}_1, \bar{t}_2]$ we deduce that

$$\left|\int_{\bar{t}_1}^{\bar{t}_2} \zeta(t)\,[v^\varepsilon(r_1^\varepsilon(t),t) - v(r_1(t),t)]\,dt\right| \le H^\varepsilon + 2J^\varepsilon + K^\varepsilon, \tag{8.30}$$

where we have set

$$H^\varepsilon := \int_{\bar{t}_1}^{\bar{t}_2} |\zeta(t)||r_1^\varepsilon(t) - r_1(t)|^{1/2}\left(\int_{r_1(t)}^{r_1^\varepsilon(t)} |v_r^\varepsilon|^2 dr\right)^{1/2} dt, \tag{8.31}$$

and

$$J^\varepsilon := \frac{1}{h}\int_{\bar{t}_1}^{\bar{t}_2} |\zeta(t)| \int_{r_1(t)}^{r_1(t)+h} |r - r_1(t)|^{1/2}\left(\int_{r_1(t)}^{r} (|v_r^\varepsilon|^2 + |v_r|^2)dr\right)^{1/2} dr\,dt \tag{8.32}$$

and

$$K^\varepsilon := \frac{1}{h}\left|\int_{\bar{t}_1}^{\bar{t}_2}\int_{r_1(t)}^{r_1(t)+h} \zeta(t)[v^\varepsilon - v)](r,t)dr\,dt\right|. \tag{8.33}$$

Using (4.7) and the fact that r_1^ε tends to r_1 uniformly on $[\bar{t}_1, \bar{t}_2]$ we deduce that

$$\lim_{\varepsilon\to 0} H^\varepsilon = 0. \tag{8.34}$$

Next we estimate J^ε. For all $1 > h > 0$ and $r \in [r_1, r_1 + h]$ we have

$$J^\varepsilon \le \frac{1}{h}\left[\int_{\bar{t}_1}^{\bar{t}_2} |\zeta|^2 \left(\int_{r_1(t)}^{r_1(t)+h} |r - r_1(t)|^{1/2} dr\right)^2 dt\right]^{1/2} \left[\int_{\bar{t}_1}^{\bar{t}_2}\int_{R_0}^{1} (|v_r^\varepsilon|^2 + |v_r|^2) dr\, dt\right]^{1/2} \tag{8.35}$$

Since $\int_{r_1(t)}^{r_1(t)+h} |r - r_1(t)|^{1/2} dr \le h^{3/2}$, also using (4.7) and the fact that $v \in L^2(0, T, H^1(\Omega))$ we deduce from (8.35) that

$$J^\varepsilon \le C(R_0, T) h^{1/2} \|\zeta\|_\infty. \tag{8.36}$$

Using the fact that v^ε tends to v weakly in $L^2(0, T, L^2(\Omega))$ we deduce that

$$\lim_{\varepsilon\to 0} K^\varepsilon = 0. \tag{8.37}$$

Letting ε tends to zero into (8.30) and using (8.34), (8.36) and (8.37) we obtain

$$\overline{\lim_{\varepsilon\to 0}} \left|\int_{\bar{t}_1}^{\bar{t}_2} \zeta(t)\,[v^\varepsilon(r_1^\varepsilon(t), t) - v(r_1(t), t)]\, dt\right| \le C(R_0, T)\|\zeta\|_\infty h^{1/2},$$

for all $h > 0$. So that letting h tends to zero we obtain (8.26). Finally (8.22), (8.25), and (8.26) imply the result of Lemma 8.5. □

Lemma 8.6

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t)(U_t^\varepsilon U_z^\varepsilon)(z, t) dz\, dt = 0.$$

Proof. We first prove that

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t) U_t^\varepsilon U_z^\varepsilon dz\, dt = \lim_{\varepsilon\to 0} \varepsilon \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta(t) U_t^\varepsilon (U_0^\varepsilon)_z dz\, dt. \tag{8.38}$$

Indeed

$$\varepsilon\left|\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta(t)U_t^\varepsilon[U_z^\varepsilon-(U_0^\varepsilon)_z]dz\,dt\right|$$
$$\le\varepsilon\|\zeta\|_\infty\left(\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}|U_t^\varepsilon|^2\right)^{1/2}\left(\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}|U_z^\varepsilon-(U_0^\varepsilon)_z|^2\right)^{1/2}. \tag{8.39}$$

Moreover we have that

$$\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}|U_t^\varepsilon|^2dz\,dt$$
$$\le 2\int_{\bar t_1}^{\bar t_2}\int_{z_-}^{z_+}\left|U_t^\varepsilon-\frac{1}{\varepsilon}\partial_t r_1^\varepsilon U_z^\varepsilon\right|^2+\left|\frac{1}{\varepsilon}\partial_t r_1^\varepsilon U_z^\varepsilon\right|^2dz\,dt$$
$$\le\frac{2}{\varepsilon}\int_{\bar t_1}^{\bar t_2}\int_{R_0}^1(|u_t^\varepsilon|^2+|\partial_t r_1^\varepsilon||u_r^\varepsilon|^2)dr\,dt$$
$$\le\frac{C(R_0)}{\varepsilon}\left[\left(\int_{\bar t_1}^{\bar t_2}\int_0^1|u_t^\varepsilon|^2r^{N-1}dr\,dt\right)\right.$$
$$\left.+\left(\int_{\bar t_1}^{\bar t_2}|\partial_t r_1^\varepsilon|^2\right)^{1/2}\left(\int_{\bar t_1}^{\bar t_2}\int_0^1|u_r^\varepsilon|^2r^{N-1}dr\,dt\right)^{1/2}\right]$$

Using (3.7) and the fact that $\int_{\bar t_1}^{\bar t_2}|\partial_t r_1^\varepsilon|^2$ is bounded (cf. Lemma 5.1) we deduce that

$$\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}|U_t^\varepsilon|^2(z,t)dz\,dt\le\frac{C_3(R_0)}{\varepsilon^2}. \tag{8.40}$$

Substituting (8.40) into (8.39) and also using (7.36) we obtain (8.38). Next we check that

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\bar t_1}^{\bar t_2}\int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}\zeta(t)(U_t^\varepsilon(U_0^\varepsilon)_z)(z,t)dz\,dt=0 \tag{8.41}$$

Integrating by parts the integral $\int_{\bar t_1}^{\bar t_2}\zeta(t)U_t^\varepsilon(U_0^\varepsilon)_z\,dt$ and using the fact that $[(U_0^\varepsilon)_z]_t=0$ we obtain that

$$\int_{\bar t_1}^{\bar t_2}\zeta U_t^\varepsilon(U_0^\varepsilon)_z dt=\left[\zeta U^\varepsilon U_0^\varepsilon\right]_{\bar t_1}^{\bar t_2}-\int_{\bar t_1}^{\bar t_2}\zeta_t U^\varepsilon(U_0^\varepsilon)_z\,dt,$$

so that

$$\left|\varepsilon \int_{\bar{t}_1}^{\bar{t}_2} \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} \zeta U_t^\varepsilon (U_0^\varepsilon)_z dz\, dt\right|$$
$$\leq \varepsilon^{1/2} 2\|\zeta\|_\infty \|u^\varepsilon\|_{L^\infty([R_0,1]\times[0,T])}$$
$$+ \varepsilon(\bar{t}_2 - \bar{t}_1)\|u^\varepsilon\|_{L^\infty([R_0,1]\times[0,T])}\|\zeta_t\|_\infty \int_{-\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} |(U_0^\varepsilon)_z| dz \qquad (8.42)$$

Using (4.10) and (8.3) we deduce (8.41). Finally (8.38) and (8.41) imply the result of Lemma 8.6. □

We now return to the proof of Lemma 8.1.

Letting ε tends to zero into (8.6) and using Lemmas 8.2–8.6 we obtain that

$$\partial_t r_1(t) = -\frac{N-1}{r_1(t)} + \frac{3}{2}[\lambda_0(t) + v(r_1(t), t)], \text{ a.e. in } (\bar{t}_1, \bar{t}_2),$$

which coincides with (8.1).

8.2. Evolution equation of the jumps

The geometrical context

From now on we suppose that the number $\mathcal{N}_0$ of jumps of u is finite and constant in time on a time interval $[t_1, t_2]$. Let $\{t \to \overline{r}_i(t)\}$ be the jumps of $u(., t)$ for $t \in [t_1, t_2]$. We choose R_0 such that

$$R_0 < \overline{r}_{\mathcal{N}_0}(t) < \cdots < \overline{r}_1(t) < 1,$$

for all $t \in [t_1, t_2]$. Let $\bar{t} \in [t_1, t_2] \cap A_{R_0}$ (cf. (5.1) for the definition of A_{R_0}). In view of Theorem 5.9 there exist, for each jump $\overline{r}_l(\bar{t})$, an interval $[\bar{t}_1, \bar{t}_2]$ such that $\bar{t} \in [\bar{t}_1, \bar{t}_2] \subset [t_1, t_2] \cap A_{R_0}$ and functions $\{t \to r_{i,l}(t)\}$ and $\{t \to r_{i,l}^\varepsilon(t)\}$, defined on $[\bar{t}_1, \bar{t}_2]$, where $i \in [1, \ldots, m_l(\overline{r}_l(\bar{t}))]$ such that $r_{1,l}^\varepsilon(t) > \cdots > r_{m(\overline{r}_l(\bar{t})),l}^\varepsilon(t)$ for all $t \in [\bar{t}_1, \bar{t}_2]$. Moreover $r_{i,l}^\varepsilon$ tends to $r_{i,l}$ uniformly on $[\bar{t}_1, \bar{t}_2]$ and $r_{i,l}(\bar{t}) = \overline{r}_l(\bar{t})$. In order to simplify the notation we replace $m(\overline{r}_l(\bar{t}))$ in this Section. In particular, we have

$$1 > r_{1,1} > \cdots r_{m_1,1} > \cdots > r_{1,l+1} > \cdots > r_{m_{l+1},l+1} > r_{1,l} > r_{2,l} > \cdots$$
$$\cdots > r_{m_l,l} > \cdots > r_{1,\mathcal{N}_0} > \cdots > r_{m_{\mathcal{N}_0},\mathcal{N}_0} \cdots > R_0.$$

This implies that there exists a box $[\overline{r}_l(\bar{t}) - h, \overline{r}_l(\bar{t}) + h] \times [\tilde{t}_1, \tilde{t}_2]$ such that $\bar{t} \in [\tilde{t}_1, \tilde{t}_2] \subset [\bar{t}_1, \bar{t}_2]$ and such that the solutions of the equation $u^\varepsilon(r, t) = a^\varepsilon$

are the pairs $(r^\varepsilon_{i,l}(t), t)$ where $t \in [\tilde{t}_1, \tilde{t}_2]$ and $i \in [1, \ldots, m_l]$. We set

$$\mathcal{U}^- := \left\{(r,t),\ \overline{r}_l(\overline{t}) - \frac{h}{2} < r < r_{m_l,l}(t) \ \text{ and } \ t \in [\tilde{t}_1, \tilde{t}_2]\right\},$$

$$\mathcal{U}^+ := \left\{(r,t),\ r_{1,l}(t) < r < \overline{r}_l(\overline{t}) + \frac{h}{2} \ \text{ and } \ t \in [\tilde{t}_1, \tilde{t}_2]\right\}.$$

We suppose the case that $\nu(\overline{r}_l(\overline{t})) = 1$. Since $u^\varepsilon - a^\varepsilon$ is of constant sign on $\mathcal{U}^-$ and on $\mathcal{U}^+$, we have that $u = 1$ a.e. in $\mathcal{U}^+$ and $u = -1$ a.e. in $\mathcal{U}^-$. Thus

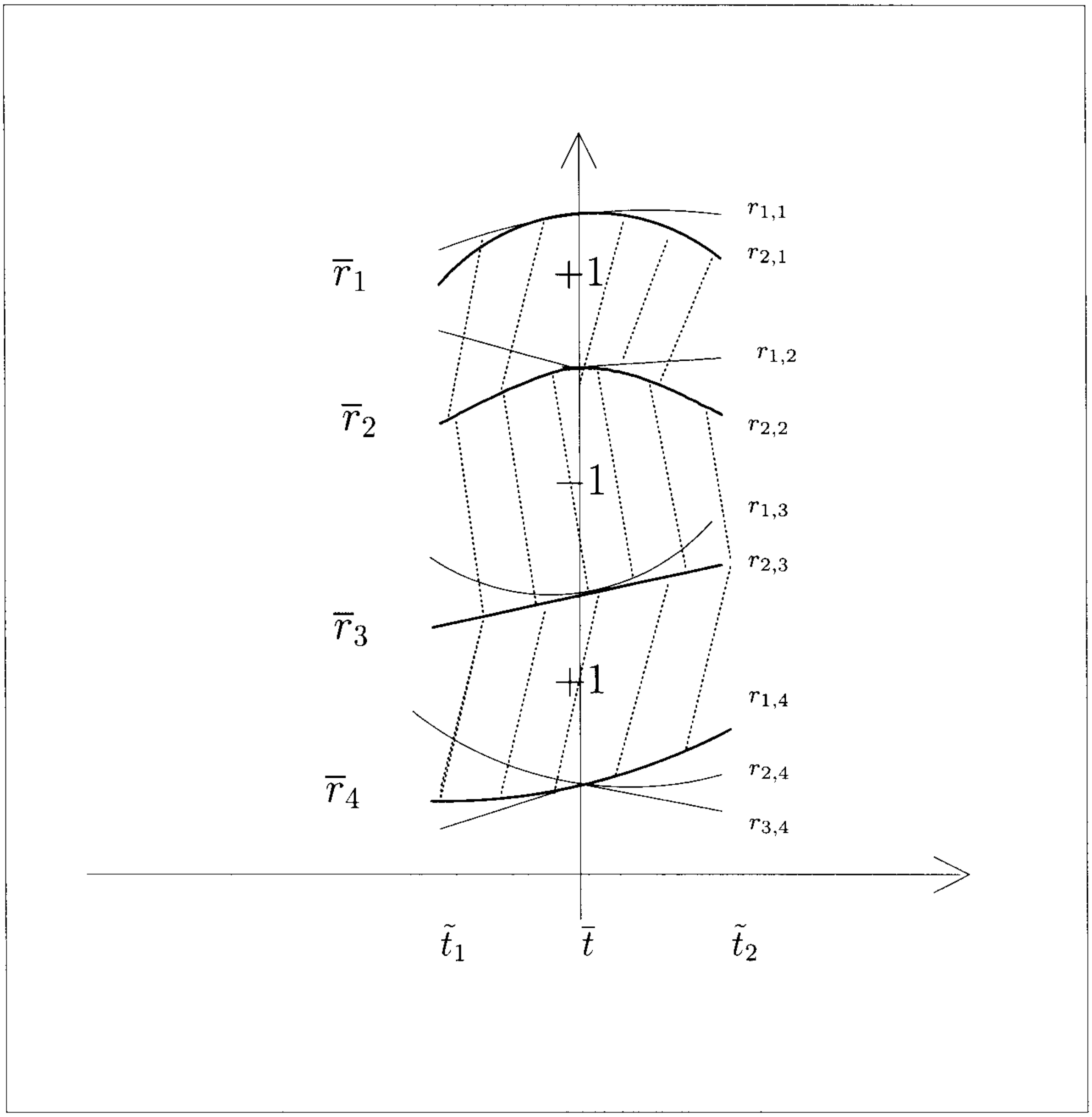

Fig. 3. A possible configuration of the limits

we deduce that there exists a jump of $u(.,t)$ in $[\overline{r}_l(\overline{t})-h,\overline{r}_l(\overline{t})+h]$ for all $t\in[\tilde{t}_1,\tilde{t}_2]$. Using the fact that the number of jumps is finite we choose h small enough such that for all $l\in[1,\ldots,\mathcal{N}_0]$ there exists a jump of $u(.,t)$ in $[\overline{r}_l(\overline{t})-h,\overline{r}_l(\overline{t})+h]$ for all $t\in[\tilde{t}_1,\tilde{t}_2]$. Therefore in each $\mathcal{N}_0$ box defined by $[\overline{r}_l(\overline{t})-h,\overline{r}_l(\overline{t})+h]\times[\tilde{t}_1,\tilde{t}_2]$ there is only one jump of $u(.,t)$. Thus for each jump $\overline{r}_l$ we obtain that

$$\nu(\overline{r}_l(t))=\nu(\overline{r}_l(\overline{t}))\quad\text{for all }\ t\in[\tilde{t}_1,\tilde{t}_2],\ \ \text{which we rewrite }\ \nu(\overline{r}_l). \tag{8.43}$$

Moreover we also have shown that for all $t\in[\tilde{t}_1,\tilde{t}_2]$ there exists $i:=i(t)\in[1,\ldots,m_l]$ such that

$$\overline{r}_l(t)=r_{i,l}(t) \tag{8.44}$$

Lemma 8.7 *Let $\overline{t}\in[t_1,t_2]\cap A_{R_0}$ there exist an interval $[\tilde{t}_1,\tilde{t}_2]\subset[t_1,t_2]\cap A_{R_0}$ such that for all $l\in[1,\ldots,\mathcal{N}_0]$ the function $\overline{r}_l$ is Hölder continuous of exponent $1/2$ on $[\tilde{t}_1,\tilde{t}_2]$ and moreover*

$$\partial_t\overline{r}_l(t)=-\frac{N-1}{\overline{r}_l(t)}+\nu(\overline{r}_l)\frac{3}{2}\big(\lambda_0(t)+v(\overline{r}_l(t),t)\big), \tag{8.45}$$

for a.e. $t\in[\tilde{t}_1,\tilde{t}_2]$.

Proof. We first prove that the functions $\{\overline{r}_l\}_{l\in[1,\ldots,\mathcal{N}_0]}$ are Hölder continuous of exponent $1/2$ on $[\tilde{t}_1,\tilde{t}_2]$. Let $l\in[1,\ldots,\mathcal{N}_0]$. Using (8.44) and the fact that the functions $\{r_{i,l}\}$ are Hölder continuous of exponent $1/2$ we deduce that there exist $K>0$ such that

$$|\overline{r}_l(t)-\overline{r}_l(\overline{t})|=|r_{i,l}(t)-r_{i,l}(\overline{t})|\le K\sqrt{|t-\overline{t}|},\quad\text{for all }\ t\in[\tilde{t}_1,\tilde{t}_2].$$

Thus the function $\overline{r}_l$ is Hölder continuous of exponent $1/2$ locally on all the intervals $[\tilde{t}_1,\tilde{t}_2]$ with the same constant K. So that $\overline{r}_l$ is Holder continuous of exponent $1/2$ on $[\tilde{t}_1,\tilde{t}_2]$. Moreover setting

$$\mathcal{R}_i:=\left\{t\in[\tilde{t}_1,\tilde{t}_2],\ \overline{r}_l(t)=r_i(t)\right\},\quad\text{for all }\ i\in[1,\ldots,m_l],$$

we deduce from Theorem A.4 [7] that $\partial_t\overline{r}_l=\partial_t r_{i,l}$ a.e. in $\mathcal{R}_i$ and thus using Lemma 8.1, (8.44) and the fact that $\mu(m(\overline{r}_l(t),t))=\nu(\overline{r}_l)$ we deduce that

$$\partial_t\overline{r}_l(t)=-\frac{N-1}{\overline{r}_l(t)}+\nu(\overline{r}_l)\frac{3}{2}v(\overline{r}_l(t),t)+\nu(\overline{r}_l)\lambda_0(t)$$

a.e. in $\mathcal{R}_i$ for all $i \in [1, \ldots, m_l]$ and so a.e. in $[\tilde{t}_1, \tilde{t}_2] = \bigcup_{i\in[1,\ldots,m_l]} \mathcal{R}_i$. This gives (8.45).

Identification of λ_0 .

Lemma 8.8 *The function λ_0 is given by*

$$\lambda_0(t) = \frac{2}{3}\frac{(N-1)\sum_{i=1}^{i=\mathcal{N}_0} \nu(\overline{r}_i)\overline{r}_i^{N-2}(t) - \frac{3}{2}\sum_{i=1}^{i=\mathcal{N}_0} v(\overline{r}_i(t),t)\overline{r}_i^{N-1}(t)}{\sum_{i=1}^{i=\mathcal{N}_0} \overline{r}_i^{N-1}(t)}, \tag{8.46}$$

fot all $t \in [\tilde{t}_1, \tilde{t}_2]$.

Proof. We have that

$$\begin{aligned}
&\int_\Omega u(x,t)dx \\
&= -\nu(\overline{r}_{\mathcal{N}_0}) \int_0^{\overline{r}_{\mathcal{N}_0}(t)} r^{N-1}dr + \sum_{i=1}^{\mathcal{N}_0-1} \nu(\overline{r}_{i+1}) \int_{\overline{r}_{i+1}(t)}^{\overline{r}_i(t)} r^{N-1}dr \\
&\quad + \nu(\overline{r}_1) \int_{\overline{r}_1(t)}^1 r^{N-1}dr \\
&= \frac{1}{N}\Bigg(-\nu(\overline{r}_{\mathcal{N}_0})(\overline{r}_{\mathcal{N}_0(t)})^N + \sum_{i=1}^{\mathcal{N}_0-1} \nu(\overline{r}_{i+1})\left[(\overline{r}_i(t))^N - (\overline{r}_{i+1}(t))^N\right] \\
&\qquad\qquad + \nu(\overline{r}_1)[1-(\overline{r}_1(t))^N]\Bigg),
\end{aligned} \tag{8.47}$$

for all $t \in [\tilde{t}_1, \tilde{t}_2]$. By (8.43) we remark that $\nu(\overline{r}_i)$ does not depend on time. Thus differentiating (8.47) we obtain in view of (1.6) that

$$\begin{aligned}
&-\nu(\overline{r}_{\mathcal{N}_0})(\overline{r}_{\mathcal{N}_0}(t))^{N-1}\partial_t\overline{r}_{\mathcal{N}_0}(t) \\
&+ \sum_{i=1}^{\mathcal{N}_0-1} \nu(\overline{r}_{i+1})\left[(\overline{r}_i(t))^{N-1}\partial_t(\overline{r}_i(t)) - (\overline{r}_{i+1}(t))^{N-1}\partial_t(\overline{r}_{i+1}(t))\right] \\
&- \nu(\overline{r}_1)(\overline{r}_1(t))^{N-1}\partial_t(\overline{r}_1(t)) = 0.
\end{aligned}$$

This implies that

$$\sum_{i=1}^{\mathcal{N}_0} \nu(\overline{r}_i)(\overline{r}_i(t))^{N-1}\partial_t(\overline{r}_i(t)) = 0,$$

for all $t \in [\tilde{t}_1, \tilde{t}_2]$. This with (8.45) imply that

$$\sum_{i=1}^{\mathcal{N}_0} \nu(\overline{r}_i)\left[-\frac{N-1}{\overline{r}_i(t)} + \frac{3}{2}\nu(\overline{r}_i(t))(\lambda_0(t) + v(\overline{r}_i(t), t))\right]\overline{r}_i(t)^{N-1} = 0,$$

for a.e. $t \in [\tilde{t}_1, \tilde{t}_2]$, so that

$$\lambda_0(t) = \frac{2}{3}\frac{(N-1)\sum_{i=1}^{i=\mathcal{N}_0} \nu(\overline{r}_i)\overline{r}_i(t)^{N-2} - \frac{3}{2}\sum_{i=1}^{i=\mathcal{N}_0} v(\overline{r}_i(t), t)\overline{r}_i(t)^{N-1}}{\sum_{i=1}^{i=\mathcal{N}_0} \overline{r}_i(t)^{N-1}},$$

for a.e. $t \in [\overline{t}_1, \overline{t}_2]$. This completes the proof of Lemma 8.8. □

We now are in position to prove Theorem 1.1. □

Proof of Theorem 1.1. Let $\overline{t} \in [t_1, t_2] \cap A_{R_0}$. By Lemmas 8.7 and 8.8 there exist an interval $[\tilde{t}_1, \tilde{t}_2] \subset [t_1, t_2] \cap A_{R_0}$ such that for all $l \in [1, \ldots, \mathcal{N}_0]$ the function $\overline{r}_l$ is Hölder continuous of exponent $1/2$ on $[\tilde{t}_1, \tilde{t}_2]$ and equalities (8.45) and (8.46) hold. Substituting (8.46) into (8.45) we obtain the evolution equation for the jump; for all $l \in [1, \ldots, \mathcal{N}_0]$ we have that

$$\partial_t \overline{r}_l(t) = -\frac{N-1}{\overline{r}_l(t)} + \nu(\overline{r}_l)\frac{3}{2}v(\overline{r}_l(t), t)$$
$$+ \nu(\overline{r}_l)\frac{(N-1)\sum_{i=1}^{i=\mathcal{N}_0} \nu(\overline{r}_i)\overline{r}_i(t)^{N-2} - 3/2\sum_{i=1}^{i=\mathcal{N}_0} v(\overline{r}_i(t), t)\overline{r}_i(t)^{N-1}}{\sum_{i=1}^{i=\mathcal{N}_0} \overline{r}_i(t)^{N-1}}, \tag{8.48}$$

a.e. $[\tilde{t}_1, \tilde{t}_2]$. Finally we check that the function $\overline{r}_l$ is Lipschitz continuous on $[t_1, t_2]$. Lemma 8.8 implies that $t \to \lambda_0(t)$ is bounded and thus by (8.48) $\partial_t \overline{r}_l(t)$ is also bounded, which implies that $\overline{r}_l$ is Lipschitz continuous locally on all the intervals $[\tilde{t}_1, \tilde{t}_2]$ with the same Lipschitz constant. Therefore $\overline{r}_l$ is Lipschitz continuous on $[\tilde{t}_1, \tilde{t}_2]$. This completes the proof of Theorem 1.1.

□

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M. Henry
Analyse Numérique et EDP
Université de Provence
Marseille, France

D. Hilhorst
Laboratoire de Mathématiques
CNRS et Université de Paris-Sud
91405 Orsay Cedex, France

Y. Nishiura
Laboratory of Nonlinear Studies and Computation
Research Institute for Electronic Science
Hokkaido University
Sapporo 060-0810, Japan