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Singular limit of the incompressible ideal magneto-fluid motion with respect to the Alfvén number

Shun'ichi GOTO
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0. Introduction.

We discuss the singular limit of the incompressible ideal magneto-fluid motion with respect to the Alfvén number in the three dimensional torus $\boldsymbol{T}^3$ (i.e., the periodic motion).

In the fluid dynamics there appear many systems of non-linear differential equations involving parameters such as the Mach number and the Alfvén number etc. One problem on the singular limit is to determine the limiting system which has a completely different property comparing with the original system, as such a parameter tends to some value.

When the system is hyperbolic, this problem has been studied in C. Browning—H.-O. Kreiss [2], S. Klainerman—A. Majda [4], A. Majda [5] and S. Schochet [6]. In particular, Browning and Kreiss studied the Alfvén limit for the compressible magneto-fluid motion as an example of their theorems. However, to show this, they needed more assumptions on the initial data than those in other papers above.

The purpose of this paper is to determine the limiting system for the incompressible magneto-fluid motion under the natural assumptions on the initial data. The limiting system becomes essentially the system of the two dimensional motion (see (1.6)).

We state main results in Section 1. In Section 2, we show the uniform estimates with respect to the Alfvén number, which are obtained by the energy method. The convergence of the solutions is generally proved in Section 3. Especially, Lemma 3.3 is employed to determine the limiting system. The proof of our theorem is finally completed in Section 4.

1. The statement of results.

We consider the system of the incompressible ideal magneto-fluid motion involving a large parameter α.

(1.1.a) $(\partial_t+(v^\alpha, \nabla))v^\alpha+\nabla p^\alpha+\alpha^2 H^\alpha\times \mathrm{rot}\, H^\alpha=0$

(1.1.b) $(\partial_t+(v^\alpha, \nabla))H^\alpha-(H^\alpha, \nabla)v^\alpha=0 \qquad \text{in } [0, T^\alpha]\times \boldsymbol{T}^3$

(1.1.c) $\operatorname{div} v^{\alpha}=\operatorname{div} H^{\alpha}=0$

(1.1.d) $v^{\alpha}(0)=v_0^{\alpha},\ H^{\alpha}(0)=H_0^{\alpha}$ on $\boldsymbol{T}^3$.

Here the fluid velocity $v^{\alpha}=v^{\alpha}(t,x)={}^t(v_1^{\alpha}, v_2^{\alpha}, v_3^{\alpha})$, the magnetic field $H^{\alpha}=H^{\alpha}(t,x)={}^t(H_1^{\alpha}, H_2^{\alpha}, H_3^{\alpha})$ and the pressure $p^{\alpha}=p^{\alpha}(t,x)$ are unknowns depending on α. The reciprocal of α is the Alfvên number which is in proportion to $|v_m|/|H_m|$, where $|v_m|$, $|H_m|$ are typical mean values of the velocity and the magnetic field.

Let $s\geq 3$ be an integer and assume that the initial data (1.1.d) satisfy

(1.2.a) $H_0^{\alpha}=\bar{H}+\alpha^{-1}K_0^{\alpha},\ (v_0^{\alpha}, K_0^{\alpha})\in H_{\sigma}^{s}(\boldsymbol{T}^3)$,

where $\bar{H}$ is a non zero constant vector, and there exist vector fields $(v_0^{\infty}, K_0^{\infty})\in H_{\sigma}^{s}(\boldsymbol{T}^3)$ and a constant $\Delta_0>0$ such that

(1.2.b) $(v_0^{\alpha}, K_0^{\alpha})\to(v_0^{\infty}, K_0^{\infty})$ in $H^{s}(\boldsymbol{T}^3)$, as $\alpha\to\infty$,

(1.2.c) $\Delta_0^{-1}\leq\alpha\|(\bar{H},\nabla)v_0^{\alpha}\|_{s-1}+\alpha\|(\bar{H},\nabla)K_0^{\alpha}\|_{s-1}\leq\Delta_0$.

Throughout this paper, $H^{r}(\boldsymbol{T}^3)$ denotes the Sobolev space of the L^2-type with inner product $(\cdot,\cdot)_r$ and norm $\|\cdot\|_r$ and $H_{\sigma}^{r}(\boldsymbol{T}^3)$ denotes the solenoidal subspace of $H^{r}(\boldsymbol{T}^3)$. The function space $H^{r}(\boldsymbol{T}^3)$ is identified with a space of functions in $H^{r}((-\pi,\pi)^3)$ with periodic boundary conditions and an element of $H^{r}(\boldsymbol{T}^3)$ has a Fourier development $f(x)=\sum_{(n)}f_n e^{in\cdot x}$ such that $\sum_{(n)}(1+|n|^2)^r|f_n|^2<\infty$.

We note that (1.2.b) and (1.2.c) imply that $(\bar{H},\nabla)v_0^{\infty}=(\bar{H},\nabla)K_0^{\infty}=0$, which are the compatibility condition of the limiting system, and there exists a constant $\Delta_1>0$ such that

(1.3) $\|v_0^{\alpha}\|_s+\|K_0^{\alpha}\|_s\leq\Delta_1$.

It is known that, under the assumption (1.2.a), for fixed α, there exists a local in time ($T^{\alpha}>0$ depend on α) unique classical solution of (1.1.a)—(1.1.d) (for example, see [1],[3]). The solution belongs to the following function space

$$(v^{\alpha}, H^{\alpha}-\bar{H})\in C([0,T^{\alpha}];H^{s}(\boldsymbol{T}^3))\cap C^1([0,T^{\alpha}];H^{s-1}(\boldsymbol{T}^3)),$$
$$\nabla p^{\alpha}\in C([0,T^{\alpha}];H^{s-1}(\boldsymbol{T}^3)).$$

Here, for a Banach space X and a constant $T>0$, $C^k([0,T];X)$ denotes a set of all k-times continuously differentiable functions on a time interval $[0,T]$ with values in X, and this set becomes a Banach space with norm

$$|||f|||_{X,T}=\sup_{t\in[0,T]}\sum_{j=0}^{k}\|\partial_t^j f(t)\|_X.$$

Setting $K^{\alpha}=\alpha(H^{\alpha}-\bar{H})$, we can write (1.1.a)—(1.1.d) in the form

$$\begin{aligned}
&(1.4.a) \quad (\partial_t+(v^{\alpha},\nabla))v^{\alpha}+K^{\alpha}\times \mathrm{rot}\, K^{\alpha}+\nabla(p^{\alpha}+\alpha\bar{H}\cdot K^{\alpha})-\alpha(\bar{H},\nabla)K^{\alpha}=0\\
&(1.4.b) \quad (\partial_t+(v^{\alpha},\nabla))K^{\alpha}-(K^{\alpha},\nabla)v^{\alpha}-\alpha(\bar{H},\nabla)v^{\alpha}=0 \qquad \text{in } [0, T^{\alpha}]\times \boldsymbol{T}^3\\
&(1.4.c) \quad \mathrm{div}\, v^{\alpha}=\mathrm{div}\, K^{\alpha}=0\\
&(1.4.d) \quad v^{\alpha}(0)=v_0^{\alpha},\ K^{\alpha}(0)=K_0^{\alpha} \qquad \text{on } \boldsymbol{T}^3.
\end{aligned}$$

The aim of this paper is to prove the following

THEOREM 1.1. *Assume that* (1.2.*a*)—(1.2.*c*) *hold. Then there exist a constant* $T_*>0$, *independent of* α, *and vecter fields*

$$(v^{\infty}, K^{\infty})\in C([0, T_*]\,;H^s(\boldsymbol{T}^3))\cap C^1([0, T_*]\,;H^{s-1}(\boldsymbol{T}^3))$$

such that

$$(v^{\alpha}, K^{\alpha})\rightarrow(v^{\infty}, K^{\infty}) \text{ weak* in } L^{\infty}([0, T_*]\,;H^s(\boldsymbol{T}^3)), \text{ as } \alpha\rightarrow\infty,$$

and (v^{∞}, K^{∞}) *is a unique solution of the following system*

$$\begin{aligned}
&(1.5.a) \quad (\partial_t+(v^{\infty},\nabla))v^{\infty}+K^{\infty}\times \mathrm{rot}\, K^{\infty}+\nabla q^{\infty}=0\\
&(1.5.b) \quad (\partial_t+(v^{\infty},\nabla))K^{\infty}-(K^{\infty},\nabla)v^{\infty}=0\\
&(1.5.c) \quad \mathrm{div}\, v^{\infty}=\mathrm{div}\, K^{\infty}=0 \qquad in\ [0, T_*]\times \boldsymbol{T}^3\\
&(1.5.d) \quad (\bar{H},\nabla)v^{\infty}=(\bar{H},\nabla)K^{\infty}=0\\
&(1.5.e) \quad v^{\infty}(0)=v_0^{\infty},\ K^{\infty}(0)=K_0^{\infty} \qquad on\ \boldsymbol{T}^3.
\end{aligned}$$

Here ∇q^{∞} *is uniquely determined by*

$$\nabla(p^{\alpha}+\alpha\bar{H}\cdot K^{\alpha})\rightarrow\nabla q^{\infty} \text{ weak* in } L^{\infty}([0, T_*]\,;H^{s-1}(\boldsymbol{T}^3)).$$

REMARKS. (1) It follows that $\nabla q^{\infty}\in C([0, T_*]\,;H^{s-1}(\boldsymbol{T}^3))$, $q^{\infty}\in L^{\infty}([0, T_*]\,;L^2(\boldsymbol{T}^3))$ and $(\bar{H},\nabla)q^{\infty}=0$.

(2) The motion described by (1.5) is essentially the two dimensional motion in the plane which is orthogonal to $\bar{H}$. In fact, let $\bar{H}={}^t(0,0,1)$, $U^{\infty}={}^t(v_1^{\infty}, v_2^{\infty})$ and $B^{\infty}={}^t(K_1^{\infty}, K_2^{\infty})$, where $v^{\infty}={}^t(v_1^{\infty}, v_2^{\infty}, v_3^{\infty})$ and $K^{\infty}={}^t(K_1^{\infty}, K_2^{\infty}, K_3^{\infty})$, we can write (1.5.a)—(1.5.e) in the two dimensional system of the incompressible ideal magneto-fluid motion

$$\begin{aligned}
&(1.6.a) \quad (\partial_t+(U^{\infty},\nabla))U^{\infty}+\nabla(q^{\infty}+\frac{1}{2}\cdot(K_3^{\infty})^2)+B^{\infty}\times \mathrm{rot}\, B^{\infty}=0\\
&(1.6.b) \quad (\partial_t+(U^{\infty},\nabla))B^{\infty}-(B^{\infty},\nabla)U^{\infty}=0 \qquad \text{in } [0, T_*]\times \boldsymbol{T}^2\\
&(1.6.c) \quad \mathrm{div}\, U^{\infty}=\mathrm{div}\, B^{\infty}=0\\
&(1.6.d) \quad U^{\infty}(0)=U_0^{\infty},\ B^{\infty}(0)=B_0^{\infty} \qquad \text{on } \boldsymbol{T}^2,
\end{aligned}$$

where $U_0^{\infty}={}^t(v_{01}^{\infty}, v_{02}^{\infty})$ and $B_0^{\infty}={}^t(K_{01}^{\infty}, K_{02}^{\infty})$, and two linear equations

(1.7.a) $(\partial_t+(U^\infty,\nabla))v_3^\infty-(B^\infty,\nabla)K_3^\infty=0$
(1.7.b) $(\partial_t+(U^\infty,\nabla))K_3^\infty-(B^\infty,\nabla)v_3^\infty=0$ in $[0, T_*]\times \boldsymbol{T}^2$
(1.7.c) $v_3^\infty(0)=v_{03}^\infty$, $K_3^\infty(0)=K_{03}^\infty$ on $\boldsymbol{T}^2$.

(3) When $(\alpha(\bar{H},\nabla)v_0^\alpha,\ \alpha(\bar{H},\nabla)K_0^\alpha)$ converges to zero faster than (1.2.c), we can easily find the limiting system (1.5).

2. Uniform estimates.

In this section, we show uniform estimates in α of the solutions to (1.4.a)—(1.4.d), which will be stated in Proposition 2.2 and Corollary 2.3. To this end we assume that the solutions $(v^\alpha, K^\alpha, p^\alpha)$ are sufficiently smooth.

LEMMA 2.1. *There exists a constant* $\Delta_2>0$, *independent of* α, *such that*

$$\|\partial_t v^\alpha(0)\|_{s-1}+\|\partial_t K^\alpha(0)\|_{s-1}\le\Delta_2.$$

PROOF. First, we estimate $\partial_t K^\alpha$ by H^{s-1}-norm at $t=0$. Since $H^r(\boldsymbol{T}^3)$ forms a Banach algebra for any $r>3/2$, it follows from (1.2.c), (1.3) and (1.4.b) that

$$\begin{aligned}\|\partial_t K^\alpha(0)\|_{s-1}&\le\|(v_0^\alpha,\nabla)K_0^\alpha-(K_0^\alpha,\nabla)v_0^\alpha\|_{s-1}+\alpha\|(\bar{H},\nabla)v_0^\alpha\|_{s-1}\\&\le C(\Delta_0^2+\Delta_1),\end{aligned}$$

where C is a positive constant depending on s.

Next, in order to estimate $\partial_t v^\alpha$, let P_σ be the orthogonal projection on $L^2(\boldsymbol{T}^3)$ to $L^2_\sigma(\boldsymbol{T}^3)$. Applying P_σ to (1.4.a), we have

$$\partial_t v^\alpha=-P_\sigma[(v^\alpha,\nabla)v^\alpha+K^\alpha\times\mathrm{rot}\,K^\alpha]+\alpha(\bar{H},\nabla)K^\alpha.$$

Since P_σ is a bounded operator on $H^r(\boldsymbol{T}^3)$ for any $r\ge0$, it follows that

$$\begin{aligned}\|\partial_t v^\alpha(0)\|_{s-1}&\le\|(v_0^\alpha,\nabla)v_0^\alpha+K_0^\alpha\times\mathrm{rot}\,K_0^\alpha\|_{s-1}+\alpha\|(\bar{H},\nabla)K_0^\alpha\|_{s-1}\\&\le C(\Delta_0^2+\Delta_1),\end{aligned}$$

where C is the same as above one. Now, putting $\Delta_2=2C(\Delta_0^2+\Delta_1)$, we have proved the lemma. □

PROPOSITION 2.2. *There exist constants* $T_*>0$ *and* $\Delta_3>0$ *which are independent of* α *such that, for any* $t\in[0, T_*]$,

$$\|v^\alpha(t)\|_s+\|\partial_t v^\alpha(t)\|_{s-1}+\|K^\alpha(t)\|_s+\|\partial_t K^\alpha(t)\|_{s-1}\le\Delta_3.$$

PROOF. For simplicity, we ignore the superscripts α of $(v^\alpha, K^\alpha, p^\alpha)$ and put $v^{i,\beta}=\partial_t^i D^\beta v$, etc., where $i=0,1$ and $i+|\beta|\le s$.

Applying $\partial_t^i D^\beta$ to (1.4.a) and (1.4.b), we have

(2.1.a) $$(\partial_t+(v,\nabla))v^{i,\beta}+K\times\mathrm{rot}\,K^{i,\beta}+\nabla(p^{i,\beta}+\alpha\bar{H}\cdot K^{i,\beta}) -\alpha(\bar{H},\nabla)K^{i,\beta}=F^{i,\beta},$$

(2.1.b) $$F^{i,\beta}=(v,\nabla)v^{i,\beta}-\partial_t^i D^\beta((v,\nabla)v)+K\times\mathrm{rot}\,K^{i,\beta}-\partial_t^i D^\beta(K\times\mathrm{rot}\,K),$$

(2.2.a) $$(\partial_t+(v,\nabla))K^{i,\beta}-(K,\nabla)v^{i,\beta}-\alpha(\bar{H},\nabla)v^{i,\beta}=G^{i,\beta},$$

(2.2.b) $$G^{i,\beta}=(v,\nabla)K^{i,\beta}-\partial_t^i D^\beta((v,\nabla)K)-(K,\nabla)v^{i,\beta}+\partial_t^i D^\beta((K,\nabla)v).$$

Using the integration by parts, we know that (2.1.a) and (2.2.a) imply

$$\frac{d}{dt}(v^{i,\beta},v^{i,\beta})_0=-2(K\times\mathrm{rot}\,K^{i,\beta},v^{i,\beta})_0+2\alpha((\bar{H},\nabla)K^{i,\beta},v^{i,\beta})_0 +2(F^{i,\beta},v^{i,\beta})_0,$$

$$\frac{d}{dt}(K^{i,\beta},K^{i,\beta})_0=2((K,\nabla)v^{i,\beta},K^{i,\beta})_0+2\alpha((\bar{H},\nabla)v^{i,\beta},K^{i,\beta})_0 +2(G^{i,\beta},K^{i,\beta})_0,$$

where $(\cdot,\cdot)_0$ stands for the inner product in $H^0=L^2$. Since $(K\times\mathrm{rot}\,K^{i,\beta}, v^{i,\beta})_0=((K,\nabla)v^{i,\beta}-(v^{i,\beta},\nabla)K,K^{i,\beta})_0$, it follows that

(2.3) $$\frac{1}{2}\cdot\frac{d}{dt}\{(v^{i,\beta},v^{i,\beta})_0+(K^{i,\beta},K^{i,\beta})_0\} =((v^{i,\beta},\nabla)K,K^{i,\beta})_0+(F^{i,\beta},v^{i,\beta})_0+(G^{i,\beta},K^{i,\beta})_0.$$

To estimate (2.1.b) and (2.2.b) by the L^2-norm, we use the Gagliard—Nirenberg inequality: for any i, r with $0\le i\le r$,

$$|D^i f|_{L^{2r/i}}\le C_r|f|_{L^\infty}^{1-i/r}\cdot\|D^r f\|_0^{i/r},$$

and the Sobolev inequality: for any $r>3/2$,

$$|f|_{L^\infty}\le C_r\|f\|_r,$$

where C_r are positive constants depending on r. Then we can prove

(2.4) $$\|F^{i,\beta}\|_0+\|G^{i,\beta}\|_0\le C\|(v(t),K(t))\|_E^2,$$

where $\|(v(t),K(t))\|_E=1+\|v(t)\|_s+\|\partial_t v(t)\|_{s-1}+\|K(t)\|_s+\|\partial_t K(t)\|_{s-1}$ and C is a positive constant depending on s.

For example,

$$(v,\nabla)v^{i,\beta}-\partial_t^i D^\beta((v,\nabla)v)=\sum_{1\le j+|\gamma|,\,j\le i,\,|\gamma|\le|\beta|}C_{i,j,\beta,\gamma}v^{j,\gamma}\cdot\nabla v^{i-j,\beta-\gamma},$$

where $C_{i,j,\beta,\gamma}=\dfrac{i!}{(i-j)!j!}\cdot\dfrac{\beta!}{(\beta-\gamma)!\gamma!}$, and each terms of the right hand side are estimated by

$$\begin{aligned}\|v^{j,\gamma}\cdot\nabla v^{i-j,\beta-\gamma}\|_0 &\leq |v^{j,\gamma}|_{L^{2p}}|\nabla v^{j-j,\beta-\gamma}|_{L^{2p}} \text{ (for } 1/p+1/q=1)\\ &\leq C_p|v^{j,0}|_{L^\infty}^{1-1/p}\|D^{p|\gamma|}v^{j,0}\|_0^{1/p}\\ &\quad\times C_q|v^{i-j,0}|_{L^\infty}^{1-1/q}\|D^{q(|\beta|-|\gamma|+1)}v^{i-j,0}\|_0^{1/q}\\ &\leq C_{p,s}\|\partial_t^j v\|_{s-1}^{1-1/p}\|\partial_t^j v\|_{p|\gamma|}^{1/p}\\ &\quad\times C_{q,s}\|\partial_t^{i-j}v\|_{s-1}^{1-1/q}\|\partial_t^{i-j}v\|_{q(|\beta|-|\gamma|+1)}^{1/q}.\end{aligned}$$

Setting $p=(s-j)/|\gamma|$, we find $p|\gamma|+j\leq s$ and $q(|\beta|-|\gamma|+1)+i-j\leq s$. Hence, we have

$$\|(v,\nabla)v^{i,\beta}-\partial_t^i D^\beta((v,\nabla)v)\|_0\leq C(\|v\|_s^2+\|\partial_t v\|_{s-1}^2).$$

Now, we have from (2.3) and (2.4)

$$(2.5.\text{a})\qquad \frac{d}{dt}\|(v^\alpha(t),K^\alpha(t))\|_E^2\leq C\|(v^\alpha(t),K^\alpha(t))\|_E^3.$$

On the other hand, by (1.3) and Lemma 2.1, we get

$$(2.5.\text{b})\qquad \|(v^\alpha(0),K^\alpha(0))\|_E\leq 1+\Delta_1+\Delta_2\equiv\Delta_4.$$

Solving (2.5.a)–(2.5.b), we find

$$\|(v^\alpha(t),K^\alpha(t))\|_E\leq 2\Delta_4/(2-C\Delta_4 t).$$

Hence, choosing constants T_* and Δ_3 which satisfy

$$(2.6)\qquad 0<T_*<2(C\Delta_4)^{-1},\ \Delta_3=2\Delta_4/(2-C\Delta_4 T_*),$$

we have proved the proposition. □

Note that $\nabla(p^\alpha+\alpha\bar{H}\cdot K^\alpha)$ and $(\bar{H},\nabla)K^\alpha$ are orthogonal in L^2, then the following result follows easily from Proposition 2.2 and the equations (1.4.a) and (1.4.b).

COROLLARY 2.3. *There exists a constant $\Delta_5>0$, independent of α, such that, for any $t\in[0,T_*]$,*

$$\alpha\|(\bar{H},\nabla)v^\alpha\|_{s-1}+\alpha\|(\bar{H},\nabla)K^\alpha\|_{s-1}+\|\nabla(p^\alpha+\alpha\bar{H}\cdot K^\alpha)\|_{s-1}\leq\Delta_5.$$

3. The convergence of functions.

In this section, we discuss in general the convergence of the sequences of functions having the uniform estimate such as Proposition 2.2 or Corollary 2.3.

The following lemma can be proved similar to [5], but we show it for completeness.

LEMMA 3.1. *Let $\{U^\alpha(t,x)\}$ be the sequesnce of functions satisfying the following assumptions :*

(3.1.a) $U^{\alpha}\in C([0, T_*]; H^s(\boldsymbol{T}^3))\cap C^1([0, T_*]; H^{s-1}(\boldsymbol{T}^3))$

and there exists a constant $\Delta_6>0$, *independent of* α, *such that*

(3.1.b) $\|U^{\alpha}(t)\|_s+\|\partial_t U^{\alpha}(t)\|_{s-1}\leq\Delta_6$ *for any* $t\in[0, T_*]$.

Then, by passing to a subsequence, there exists a function $U^{\infty}\in C([0, T_*]\times \boldsymbol{T}^3)$ *such that, as* $\alpha\to\infty$,

(3.2.a) $U^{\alpha}\to U^{\infty}$ *weak* in* $L^{\infty}([0, T_*]; H^s(\boldsymbol{T}^3))$,
(3.2.b) $U^{\alpha}\to U^{\infty}$ *in* $C([0, T_*]; H^{s-\varepsilon}(\boldsymbol{T}^3))$ *for any* $\varepsilon>0$,

and furthermore,

(3.2.c) $U^{\infty}\in C_w([0, T_*]; H^s(\boldsymbol{T}^3))\cap Lip([0, T_*]; H^{s-1}(\boldsymbol{T}^3))$,
(3.2.d) $\partial_t U^{\alpha}\to\partial_t U^{\infty}$ *weak* in* $L^{\infty}([0, T_*]; H^{s-1}(\boldsymbol{T}^3))$, *as* $\alpha\to\infty$.

PROOF. The first notice is that, by (3.1.a) and (3.1.b), $\{U^{\alpha}(t, x)\}$ is uniformly bounded and equi-continuous with respect to α. That is, for any (t, x), $(s, y)\in[0, T_*]\times \boldsymbol{T}^3$ and any α,

$$|U^{\alpha}(t, x)|\leq C\Delta_6,\ |U^{\alpha}(t, x)-U^{\alpha}(s, y)|\leq C\Delta_6(|t-s|+|x-y|),$$

where C is a positive constant depending on s.

By the Ascoli-Arzela theorem and passing to a subsequence, there exists $U^{\infty}\in C([0, T_*]\times \boldsymbol{T}^3)$ such that

$$\sup_{(t,x)\in[0, T_*]\times \boldsymbol{T}^3}|U^{\alpha}(t, x)-U^{\infty}(t, x)|\to 0, \text{ as } \alpha\to\infty.$$

Since $\boldsymbol{T}^3$ is a compact manifold, it follows that

$$\begin{aligned}\sup_{t\in[0, T_*]}&\left(\int_{\boldsymbol{T}^3}|U^{\alpha}(t, x)-U^{\infty}(t, x)|^2dx\right)^{1/2}\\ &\leq C\sup_{(t,x)\in[0, T_*]\times \boldsymbol{T}^3}|U^{\alpha}(t, x)-U^{\infty}(t, x)|,\end{aligned}$$

where $C=(2\pi)^{3/2}$. Hence, we have

(3.3) $U^{\alpha}\to U^{\infty}$ in $C([0, T_*]; L^2(\boldsymbol{T}^3))$, as $\alpha\to\infty$.

On the other hand, by (3.1.b) and passing to a subsequence, we have

(3.4) $U^{\alpha}\to U^{\infty}$ weak* in $L^{\infty}([0, T_*]; H^s(\boldsymbol{T}^3))$, as $\alpha\to\infty$,

because this topology is stronger than that of (3.3). By the resonance theorem, we know that (3.1.b) and (3.4) imply

(3.5) $|||U^{\infty}|||_{s,T_*}\leq\underline{\lim}_{\alpha\to\infty}|||U^{\alpha}|||_{s,T_*}\leq\Delta_6.$

Using the Interpolation inequality: for any r, r' with $0 \leq r' \leq r$,

$$\|f\|_{r'} \leq C_r \|f\|_0^{1-r'/r} \|f\|_r^{r'/r},$$

we have from (3.3) and (3.5)

(3.6) $\quad U^\alpha \to U^\infty$ in $C([0, T_*]; H^{s-\varepsilon}(\boldsymbol{T}^3))$ for any $\varepsilon > 0$, as $\alpha \to \infty$.

Next, we show two regularities (3.2.c) of U^∞. Let $V^\alpha = U^\alpha - U^\infty$. We note that, for any $\varphi \in H^s(\boldsymbol{T}^3)$, there exist $\varphi_k \in C^\infty(\boldsymbol{T}^3)$ such that $\|\varphi - \varphi_k\|_s \to 0$, as $k \to \infty$. For each φ_k,

$$\begin{aligned}(V^\alpha(t), \varphi_k)_s &= (V^\alpha(t), \varphi_k)_{s-1} + (D^s V^\alpha(t), D^s \varphi_k)_0 \\ &= (V^\alpha(t), \varphi_k)_{s-1} - (D^{s-1} V^\alpha(t), D^{s+1} \varphi_k)_0.\end{aligned}$$

The right hand side of above equality converges uniformly on $[0, T_*]$ to zero, as $\alpha \to \infty$. Now, $(V^\alpha(t), \varphi)_s = (V^\alpha(t), \varphi - \varphi_k)_s + (V^\alpha(t), \varphi_k)_s$. The first term of right hand side is estimated by

$$|(V^\alpha(t), \varphi - \varphi_k)_s| \leq |||V^\alpha|||_{s, T_*} \|\varphi - \varphi_k\|_s \leq 2\Delta_6 \|\varphi - \varphi_k\|_s.$$

Therefore, $(V^\alpha(t), \varphi)_s$ converges uniformly on $[0, T_*]$ to zero. By $(U^\alpha(\cdot), \varphi)_s \in C([0, T_*])$, we have $(U^\infty(\cdot), \varphi)_s \in C([0, T_*])$. This means

$$U^\infty \in C_w([0, T_*]; H^s(\boldsymbol{T}^3)).$$

On the other hand, for any $t, s \in [0, T_*]$, we have

$$\|U^\alpha(t) - U^\alpha(s)\|_{s-1} \leq |||\partial_t U^\alpha|||_{s-1, T_*} |t-s| \leq \Delta_6 |t-s|.$$

By (3.6) we get $\|U^\infty(t) - U^\infty(s)\|_{s-1} \leq \Delta_6 |t-s|$. This means

(3.7) $\quad U^\infty \in Lip([0, T_*]; H^{s-1}(\boldsymbol{T}^3))$.

Finally, we know from (3.7) that there exist $\partial_t U^\infty(\cdot)$ having finite values in H^{s-1}-norm, on $[0, T_*]$ almost everywhere. On the other hand, by (3.1.b) and passing to a subsequence, there exists a function $W(t, x)$ such that

$$\partial_t U^\alpha \to W \text{ weak* in } L^\infty([0, T_*]; H^{s-1}(\boldsymbol{T}^3)), \text{ as } \alpha \to \infty.$$

Since W is equal to $\partial_t U^\infty$ in distribution sense, the proof is completed. □

By using the Sobolev inequality, the following convergence follows easily from Lemma 3.1.

COROLLARY 3.2. *Let $\{U^\alpha(t)\}$ be the same sequence of functions as Lemma 3.1, then*

$$U^\alpha \cdot D^1 U^\alpha \to U^\infty \cdot D^1 U^\infty \text{ weak* in } L^\infty([0, T_*]; H^{s-1}(\boldsymbol{T}^3)), \text{ as } \alpha \to \infty.$$

Next, we consider the convergence of functions having the estimate such as Corollary 2. 3.

LEMMA 3. 3. *Let $\{V^{\alpha}(t, x)\}$ be the sequence of functions satisfying the following assumptions :*

(3.8.a) $V^{\alpha}\in C([0, T_{*}]; H^{s}(\boldsymbol{T}^{3}))$

and there exists a constant $\Delta_{7}>0$, independent of α, such that

(3.8.b) $\|(\bar{H}, \nabla) V^{\alpha}(t)\|_{s-1}\leq\Delta_{7}$ *for any* $t\in[0, T_{*}]$.

Then, by passing to a subsequence, there exists a function $V^{\infty}(t, x)$ such that, as $\alpha\rightarrow\infty$,

$$\tilde{V}^{\alpha}\rightarrow V^{\infty} \text{ weak* in } L^{\infty}([0, T_{*}]; L^{2}(\boldsymbol{T}^{3})),$$
$$(\bar{H}, \nabla)\tilde{V}^{\alpha}=(\bar{H}, \nabla) V^{\alpha}\rightarrow(\bar{H}, \nabla) V^{\infty} \text{ weak* in } L^{\infty}([0, T_{*}]; H^{s-1}(\boldsymbol{T}^{3})),$$

where $\tilde{V}^{\alpha}(t, x)=V^{\alpha}(t, x)-V^{\alpha}(t, x-(\bar{H}, x)\bar{H}/|\bar{H}|^{2})$.

PROOF. We can assume $\bar{H}={}^{t}(0,0,1)$ without loss of generality. By the definition of $\tilde{V}^{\alpha}$, we have

$$\tilde{V}^{\alpha}(t,x)=V^{\alpha}(t,x)-V^{\alpha}(t,x_{1},x_{2},0),\ (\bar{H},\nabla)\tilde{V}^{\alpha}=(\bar{H},\nabla)V^{\alpha}.$$

Since $\tilde{V}^{\alpha}(t, x_{1}, x_{2}, 0)=0$, it follows that, for any $x_{3}\in(-\pi, \pi)$,

$$\tilde{V}^{\alpha}(t,x)=\int_{0}^{x_{3}}\partial_{3}\tilde{V}^{\alpha}(t,x_{1},x_{2},\xi)d\xi.$$

Using the Schwarz inequality, we get

$$|\tilde{V}^{\alpha}(t,x)|^{2}\leq\pi\int_{-\pi}^{\pi}|\partial_{3}\tilde{V}^{\alpha}(t,x_{1},x_{2},\xi)|^{2}d\xi.$$

By integrating both sides of above inequality over $\boldsymbol{T}^{3}$, we have from (3.8. b) that

(3.9) $\|\tilde{V}^{\alpha}(t)\|_{0}\leq C\|(\bar{H},\nabla)V^{\alpha}(t)\|_{0}\leq C\Delta_{7}$,

where C is a positive constant.

By (3.8.b) and passing to a subsequence, there exists a function $W(t,x)$ such that

$$(\bar{H},\nabla)V^{\alpha}\rightarrow W \text{ weak* in } L^{\infty}([0, T_{*}]; H^{s-1}(\boldsymbol{T}^{3})), \text{ as } \alpha\rightarrow\infty.$$

On the other hand, by (3.9) and passing to a subsequence, there exists a function $V^{\infty}(t,x)$ such that

$$\tilde{V}^{\alpha}\rightarrow V^{\infty} \text{ weak* in } L^{\infty}([0, T_{*}]; L^{2}(\boldsymbol{T}^{3})), \text{ as } \alpha\rightarrow\infty.$$

Now, return to the proof of the lemma, we know that $W=(\bar{H},\nabla)V^{\infty}$ in distribution sense and this completes the proof. □

LEMMA 3.4. *Let $\{V^{a}(t,x)\}$ be the sequence of functions satisfying the following assumptions: $V^{a}\in C([0,T_{*}];H^{s}(\boldsymbol{T}^{3}))$ and there exists a constant $\Delta_{8}>0$, independent of a, such that*

$$\|\nabla V^{a}(t)\|_{s-1}\leq\Delta_{8}\ \textit{for any}\ t\in[0,T_{*}].$$

Then, by passing to a subsequence, there exists a function $V^{\infty}(t,x)$ such that, as $a\to\infty$,

$$V^{a}-\bar{V}^{a}\to V^{\infty}\ \textit{weak*}\ \textit{in}\ L^{\infty}([0,T_{*}];L^{2}(\boldsymbol{T}^{3})),$$
$$\nabla V^{a}\to\nabla V^{\infty}\ \textit{weak*}\ \textit{in}\ L^{\infty}([0,T_{*}];H^{s-1}(\boldsymbol{T}^{3})),$$

where $\bar{V}^{a}(t)=(\textit{meas}\ \boldsymbol{T}^{3})^{-1}\int_{\boldsymbol{T}^{3}}V^{a}(t,x)dx.$

PROOF. When G is a bounded domain, the following Poincaré inequality holds:

$$\|f\|_{L^{2}(G)}^{2}\leq C\left(\|\nabla f\|_{L^{2}(G)}^{2}+\left|\int_{G}f(x)dx\right|^{2}\right)\ \text{for any}\ f\in H^{1}(G),$$

where a constant C depends on G. Setting $f=V^{a}-\bar{V}^{a}$ and using $\nabla\bar{V}^{a}=0$, we have $\|V^{a}-\bar{V}^{a}\|_{0}\leq C\|\nabla V^{a}\|_{0}$. Hence, Lemma 3.4 is proved similar to Lemma 3.3. □

4. The proof of Theorem.

By the results of Section 2 and 3, it is proved that there exist a constant T_{*} determined in (2.6) and vector fields

(4.1.a) $(v^{\infty},K^{\infty})\in C_{w}([0,T_{*}];H^{s}(\boldsymbol{T}^{3}))\cap Lip([0,T_{*}];H^{s-1}(\boldsymbol{T}^{3}))$,
(4.1.b) $(q^{\infty},u^{\infty},L^{\infty})\in L^{\infty}([0,T_{*}];L^{2}(\boldsymbol{T}^{3}))$

such that, as $a\to\infty$,

(4.2) $(v^{a},K^{a})\to(v^{\infty},K^{\infty})$ weak* in $L^{\infty}([0,T_{*}];H^{s}(\boldsymbol{T}^{3}))$

and each terms of (1.4.a) and (1.4.b) converge weakly in $L^{\infty}([0,T_{*}];H^{s-1}(\boldsymbol{T}^{3}))$ to suitable terms, that is,

(4.3.a) $(\partial_{t}v^{a},\partial_{t}K^{a})\to(\partial_{t}v^{\infty},\partial_{t}K^{\infty})$,
(4.3.b) $((v^{a},\nabla)v^{a},K^{a}\times\mathrm{rot}\,K^{a},(v^{a},\nabla)K^{a},(K^{a},\nabla)v^{a})$
$\to((v^{\infty},\nabla)v^{\infty},K^{\infty}\times\mathrm{rot}\,K^{\infty},(v^{\infty},\nabla)K^{\infty},(K^{\infty},\nabla)v^{\infty})$,
(4.3.c) $(a(\bar{H},\nabla)v^{a},a(\bar{H},\nabla)K^{a})\to((\bar{H},\nabla)u^{\infty},(\bar{H},\nabla)L^{\infty})$,
(4.3.d) $\nabla(p^{\infty}+a\bar{H}\cdot K^{a})\to\nabla q^{\infty}$.

In fact, (4.1.a), (4.2), (4.3.a) and (4.3.b) follow easily from Lemma 3.1 and Corollary 3.2. Setting $V^a=av^a$ or aK^a in Lemma 3.3 and $V^a=p^a+a\bar{H}\cdot K^a$ in Lemma 3.4, we obtain (4.1.b), (4.3.c) and (4.3.d).

Now, it follows from (4.3.a)—(4.3.d) that $(v^\infty, K^\infty, q^\infty, u^\infty, L^\infty)$ satisfy the equations

(4.4.a) $(\partial_t+(v^\infty, \nabla))v^\infty+K^\infty\times\text{rot}\, K^\infty+\nabla q^\infty-(\bar{H}, \nabla)L^\infty=0,$

(4.4.b) $(\partial_t+(v^\infty, \nabla))K^\infty-(K^\infty, \nabla)v^\infty-(\bar{H}, \nabla)u^\infty=0.$

Because $(\bar{H}, \nabla)v^a$ and $(\bar{H}, \nabla)K^a$ converge in $L^\infty([0, T_*]; H^{s-1}(\boldsymbol{T}^3))$ to zero by Corollary 2.3, (4.2) implies that

$$(\bar{H}, \nabla)v^\infty=(\bar{H}, \nabla)K^\infty=0.$$

By (1.4.c) and (4.2) we have (1.5.c) and

(4.4.c) $\text{div}\, u^\infty=\text{div}\, L^\infty=0.$

The initial data (1.5.e) follow from (1.2.b), (4.1.a) and (4.2).

Next, we show the regularity of the solution to (4.4.a)—(4.4.c) and (1.5.c)—(1.5.e). To this end we prove the following a priori estimate.

PROPOSITION 4.1. *For any* $t, t_0\in[0, T_*]$,

$$\|v^\infty(t)\|_s+\|K^\infty(t)\|_s\le\{\|v^\infty(t_0)\|_s+\|K^\infty(t_0)\|_s\}\times\exp\left[C(|\nabla v^\infty|_{L^\infty}+|\nabla K^\infty|_{L^\infty})|t-t_0|\right],$$

where C is a positive constant depending on s.

PROOF. Let the solution to (4.4.a)—(4.4.3) and (1.5.c)—(1.5.e) be sufficiently smooth, which is justified by approximating the initial data by smooth data.

Using the Gagliard—Nirenberg inequality, we can prove similar to the proof of Proposition 2.2 that

$$\frac{d}{dt}\{\|v^\infty(t)\|_s+\|K^\infty(t)\|_s\}\le C_s\{|\nabla v^\infty|_{L^\infty}+|\nabla K^\infty|_{L^\infty}\}\{\|v^\infty(t)\|_s+\|K^\infty(t)\|_s\},$$

where C_s is a positive constant depending on s. By the Gronwall's inequality, we have proved the proposition. □

By Proposition 4.1, we have

$$\overline{\lim_{t\to t_0}}\{\|v^\infty(t)\|_s+\|K^\infty(t)\|_s\}\le\|v^\infty(t_0)\|_s+\|K^\infty(t_0)\|_s.$$

On the other hand, since $(v^\infty, K^\infty)\in C_w([0, T_*]; H^s(\boldsymbol{T}^3))$, it follows from the resonance theorem that

$$\|v^\infty(t_0)\|_s+\|K^\infty(t_0)\|_s\leq\varliminf_{t\to t_0}\{\|v^\infty(t)\|_s+\|K^\infty(t)\|_s\}.$$

Hence, we have

$$(v^\infty, K^\infty)\in C([0, T_*]\,;H^s(\boldsymbol{T}^3)).$$

Let new projection define as $P_S:L^2(\boldsymbol{T}^3)\to S^\perp$ where $S^\perp$ is orthogonal complement of $S\equiv\{(\bar{H},\nabla)f\,;f\in H^1(\boldsymbol{T}^3)\}$ in L^2. Applying P_σ to (4.4.a) and next applying P_S, we have

$$\partial_t v^\infty=-P_SP_\sigma[(v^\infty,\nabla)v^\infty+K^\infty\times\mathrm{rot}\,K^\infty].$$

Since $(v^\infty,\nabla)v^\infty+K^\infty\times\mathrm{rot}\,K^\infty\in C([0, T_*]\,;H^{s-1}(\boldsymbol{T}^3))$ and P_S, P_σ are bounded operators on $H^r(\boldsymbol{T}^3)$ for any $r\geq0$, it follows that

$$\partial_t v^\infty\in C([0, T_*]\,;H^{s-1}(\boldsymbol{T}^3)).$$

Similarly, it is proved that

$$(\partial_t K^\infty, \nabla q^\infty, (\bar{H},\nabla)L^\infty, (\bar{H},\nabla)u^\infty)\in C([0, T_*]\,;H^{s-1}(\boldsymbol{T}^3)).$$

The next lemma shows that $(\bar{H},\nabla)L^\infty=(\bar{H},\nabla)u^\infty=0$ and $(\bar{H},\nabla)q^\infty=0$ in (4.4.a)—(4.4.b). For simplicity, we put $\bar{H}=(0, 0, 1)$.

LEMMA 4.2. *Let $f\in L^2(\boldsymbol{T}^3)$ and $\partial_3^2 f\in L^2(\boldsymbol{T}^3)$. If $\partial_3^2 f=0$, then f is independent of x_3.*

PROOF. Any function $f\in L^2(\boldsymbol{T}^3)$ has a Fourier development $f(x)=\sum_{(n)}f_n e^{in\cdot x}$. Since the right hand side is a convergent series in the L^2-sence, we have

$$\partial_3^2 f(x)=-\sum_{(n)}(n_3)^2 f_n e^{in\cdot x}$$

in distribution sense. By the assumptions, the right hand side is belong to $L^2(\boldsymbol{T}^3)$ and is equal to zero. Note that $\{e^{in\cdot x}\}$ is complete in $L^2(\boldsymbol{T}^3)$, then we have $(n_3)^2 f_n=0$ for any n. This means that f is dndependent of x_3. □

Applying $(\bar{H},\nabla)$ to (4.4.a) and (4.4.b), we have from (1.5.d)

(4.9.a) $\nabla(\bar{H}\cdot\nabla q^\infty)-(\bar{H},\nabla)^2L^\infty=0,$
(4.9.b) $(\bar{H},\nabla)^2u^\infty=0.$

Applying *div* to (4.9.a), we get $\Delta(\bar{H}\cdot\nabla q^\infty)=0$ by (4.4.c). We can prove that $(\bar{H},\nabla)q^\infty$ is equal to a constant, similar to Lemma 4.2. Now, we have $(\bar{H},\nabla)^2L^\infty=(\bar{H},\nabla)^2u^\infty=0$ and $(\bar{H},\nabla)^2q^\infty=0$. By Lemma 4.2, we have

$$(\bar{H}, \nabla)L^{\infty}=(\bar{H}, \nabla)u^{\infty}=0, \ (\bar{H}, \nabla)q^{\infty}=0.$$

Finally, we prove the uniqueness of the limiting solution. Let $(\tilde{v}^{\infty}, \tilde{K}^{\infty})$ be a solution to (1.5.a)—(1.5.e). The following inequality follows easily from the same arguement in Proposition 4. 1,

$$\|(v^{\infty}-\tilde{v}^{\infty})(t)\|_0+\|(K^{\infty}-\tilde{K}^{\infty})(t)\|_0\leq\|(v^{\infty}-\tilde{v}^{\infty})(0)\|_0+\|(K^{\infty}-\tilde{K}^{\infty})(0)\|_0,$$

which implies the uniqueness of the solution. Therefore, we have proved our theorem. □

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Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan

Present Address
Department of Applied Science
Faculty of Engineering 36
Kyushu University
Kukuoka 812, Japan