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Singular Limits For The Compressible Euler Equation In An Exterior Domain

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INTRODUCTION

The equations in fluid dynamics are clasified into 4 categories according to the viscousity and the compressibility. The following diagram represents the relations between them:

$$\begin{array}{ccc} \text{Compressible viscous fluids} & \longrightarrow & \text{Compressible ideal fluids} \\ \downarrow & & \downarrow \\ \text{Incompressible viscous fluids} & \dashrightarrow & \text{Incompressible ideal fluids.} \end{array}$$

It is generally believed that to pass from viscous fluids to ideal fluids, one lets the viscous coefficients tend to 0, and to pass from compressible fluids to incompressible fluids, we have only to let the Mach number tend to 0, where the Mach number = (the mean flow speed)/(the mean sound speed). The justification of this diagram proposes attracting problems in the theory of non-linear equations. In this note, we consider the equation of compressible ideal fluids (the Euler equation).

Let Ω be a domain in $\mathbb{R}^3$ exterior to a bounded obstacle with smooth boundary S. We assume that Ω is arcwise connected, but nothing is assumed on the shape of the boundary. Suppose that Ω is occupied by an ideal gas. Let P be its pressure and V the velocity. Then the Euler equation is written as

$$(1) \qquad \begin{cases} \partial_t P + (V\cdot\nabla)P + \gamma P\nabla\cdot V = 0, \\ \partial_t V + (V\cdot\nabla)V + P^{-1/\gamma}\nabla P = 0, \end{cases}$$

with the boundary condition $\langle V,n\rangle = 0$, n being the unit normal to S. Here γ is a constant > 1, for the air $\gamma = 1,4$. We consider the case of low speed. It means that we replace V by $\lambda^{-1}V$ and t by λt, λ being a large constant. Then we have

$$(2)\qquad \begin{cases} \partial_t P + (V\cdot\nabla)P + \gamma P\nabla\cdot V = 0, \\ \partial_t V + (V\cdot\nabla)V + \lambda^2 P^{-1/\gamma}\nabla P = 0. \end{cases}$$

A simple consideration shows that λ^{-1} is propotional to the Mach number. First we explain the result roughly.

Assume that the initial data behaves like

$$(3)\qquad \begin{cases} P^\lambda(0) = P_0 + O(\lambda^{-1}) \quad (P_0 \text{ being a positive constant}), \\ V^\lambda(0) \to V_0^\infty, \end{cases}$$

as $\lambda \to \infty$. Then there exists a time interval $[0,T]$ independent of λ, in which the solution $P^\lambda(t)$, $V^\lambda(t)$ of the above equation exists, and for $0 < t \leq T$, $P^\lambda(t) \to P_0$ and $V^\lambda(t) \to V^\infty(t)$ as $\lambda \to \infty$. Moreover, $V^\infty(t)$ satisfies the incompressible Euler equation

$$(4)\qquad \begin{cases} \partial_t V^\infty + P_S(V^\infty\cdot\nabla)V^\infty = 0, \qquad 0 \leq t \leq T, \\ V^\infty(0) = P_S V_0^\infty, \end{cases}$$

where P_S is the projection onto the solenoidal fields.

A small history should be explained before going into the details. Ebin [1] considered this problem in a bounded domain using Laglangean coordinates. Klainerman-Majda [2] treated it in $\mathbf{R}^n$ or under the periodic boundary condition with the assumption that $P^\lambda(0) = P_0 + O(\lambda^{-2})$, $\operatorname{div} V^\lambda(0) = 0$. These results have been extended by Agemi [3] and Schochet [4] to the interior boundary value problem. One can see a good explanation in Majda's book [5]. Recently, Asano [6] and Ukai [7] studied this problem in $\mathbf{R}^3$ without assuming that div $V^\lambda(0) = 0$, whence they found the initial layer of the solution. The aim of this note is to extend their results to an exterior domain.

MAIN RESULTS

Now we go into the details. It is convenient to change the dependent variable P into the form $Q = \frac{\gamma}{\gamma-1} P^{1-1/\gamma}$. Then we have

$$\begin{cases} \partial_t Q + (V\cdot\nabla)Q + (\gamma-1)Q\nabla\cdot V = 0, \\ \partial_t V + (V\cdot\nabla)V + \lambda^2\nabla Q = 0. \end{cases}$$

For the sake of simplicity, we set $\gamma = 2$. Since we shall assume that the initial data has an asymptotic form : Const. + $O(\lambda^{-1})$, we set without loss of generality $Q = 1 + p/\lambda$ and write v instead of V. Then we have

$$(5) \qquad \begin{cases} \partial_t p + (v\cdot\nabla)p + p\nabla\cdot v + \lambda\nabla\cdot v = 0, \\ \partial_t v + (v\cdot\nabla)v \qquad\qquad + \lambda\nabla p = 0, \end{cases}$$

with the boundary condition

$$(6) \qquad \langle v,n\rangle = 0 \quad \text{on} \quad S.$$

Let $H^m(\Omega)$ be the usual Sobolev space of order m, $W^{n,1}(\Omega)$ the Sobolev space of order n with L^1-derivatives. The following assumptions are imposed on the initial data p_0^λ and v_0^λ.

(A-1) $\qquad p_0^\lambda,\ v_0^\lambda \in C_0^\infty(\Omega)$,

(A-2) $\qquad \{(p_0^\lambda, v_0^\lambda);\lambda>0\}$ is a bounded set in $H^{N+1}(\Omega)\cap W^{7,1}(\Omega)$, (A-3) $\quad P_S v_0^\infty \to v_0^\infty$ in $H^N(\Omega)$ as $\lambda \to \infty$,

where $N \geq 8$ and P_S is the projection onto the solenoidal fields defined below. The assumption (A-1) is stronger than really needed. We have only to assume the compatibility condition up to some finite order.

Our first result is concerned with the interval of existence, independent of λ, of the solution of (5), (6) and its uniform estimate.

<u>THEOREM</u> A (Uniform Estimates). There exist constants $T > 0$ and $\Lambda > 0$ such that for any $\lambda > \Lambda$, the solution $p^\lambda(t)$, $v^\lambda(t)$ of the compressible Euler equation (5), (6) with the initial data $p^\lambda(0) = p_0^\lambda$, $v^\lambda(0) = v_0^\lambda$ exists uniquely in the interval $I = [0,T]$. Moreover, we have the following uniform estimate

$$\sup_{\lambda>\Lambda,\, t\in I} \left(\|p^\lambda(t)\|_{H^N(\Omega)} + \|v^\lambda(t)\|_{H^N(\Omega)} \right) < \infty .$$

The following theorem is the main theme of this note.

<u>THEOREM</u> B (Incompressible Limit). For $0 < t \leq T$, $p^\lambda(t) \to 0$ in $L^2_{loc}(\bar{\Omega})$ and $v^\lambda(t) \to v^\infty(t)$ in $L^2_{loc}(\bar{\Omega})$ as $\lambda \to \infty$. Furthermore $v^\infty(t)$ is the classical solution to the incompressible Euler equation

$$(7) \qquad \begin{cases} \partial_t v^\infty + P_S(v^\infty \cdot \nabla) v^\infty = 0 \quad \text{in} \quad \Omega,\ t \in I, \\ v^\infty(0) = v_0^\infty = P_S v_0^\infty . \end{cases}$$

<u>Sketch of the proof of Theorem A</u>

Let L be the linearized operator of acoustics:

$$L = -i \begin{pmatrix} 0 & \nabla \\ {}^t\nabla & 0 \end{pmatrix}, \quad \nabla = (\partial_1, \partial_2, \partial_3),$$

with the boundary condition (6). Introducing the notations $f^\lambda = (p^\lambda, v^\lambda)$ and

$$A(f^\lambda) = \begin{pmatrix} v^\lambda\cdot\nabla & p^\lambda\nabla \\ 0 & v^\lambda\cdot\nabla \end{pmatrix},$$

one can rewrite (5) as

(8) $$\partial_t f^\lambda + A(f^\lambda)f^\lambda + i\lambda L f^\lambda = 0.$$

Let Γ_0 and Γ be the orthogonal projections onto $N(L)$ = the null space of L, and its orthogonal complement, respectively. The important estimates on which we are based are the following coerciveness estimates:

(9) If $f \in D(L) \cap N(L)^\perp$,

$$\|f\|_{H^{m+1}} \leq C_m(\|f\|_{L^2} + \|Lf\|_{H^m}), \quad m = 0, 1, 2, \ldots.$$

(10) If $f \in D(L^m) \cap N(L)^\perp$,

$$\|f\|_{H^m} \leq C_m(\|f\|_{L^2} + \|L^m f\|_{L^2}), \quad m = 0, 1, 2, \ldots.$$

The structure of $N(L)$ is closely related with the Helmholtz decomposition of $L^2(\Omega)^3$. Let $H_D(\Omega)$ be the completion of $C_0^\infty(\bar{\Omega})$ with respect to the Dirichlet norm $\|\phi\|_D = (\int_\Omega |\nabla\phi|^2 dx)^{1/2}$. We define

(11) $$G(\Omega) = \{\nabla\phi;\ \phi \in H_D(\Omega)\},$$

(12) $$S(\Omega) = G(\Omega)^\perp \quad \text{in} \quad L^2(\Omega)^3.$$

Let P_G, P_S be the orthogonal projections onto $G(\Omega)$ and $S(\Omega)$, respectively. Then, for $f = (p,v)$,

(13) $$\Gamma_0 f = (0, P_S v), \quad \Gamma f = (p, P_G v).$$

Now, to prove Theorem A, we consider the linearized equation

(14) $$\partial_t f + A(g)f + i\lambda L f = F, \quad \langle v,n\rangle = 0 \ \text{on} \ S.$$

The treatment of this equation is rather difficult, since the boundary is characteristic and the boundary matrix is not of constant rank near the boundary. We decompose the solution of (14) into two parts: $f = \Gamma_0 f + \Gamma f$. The part $\Gamma_0 f$ satisfies the linearized incompressible Euler equation whose treatment has already been given by, e.g., Agemi [8]. To estimate the part Γf, we utilize the coerciveness estimate (9).

Let us introduce the following norm

$$\| f(t) \|_{X^m} = \sum_{k=0}^{m-1} \| (\tfrac{1}{\lambda}\partial_t)^k f(t) \|_{H^{m-k}(\Omega)}, \quad m \geq 1.$$

Let $\gamma = \sup_{t \in I} \| g(t) \|_{X^N}$. Then we have the following energy estimate

$$\| f(t) \|_{X^m} \leq C\, e^{tC(\gamma)} \Big(\| f(0) \|_{X^m} + \int_0^t \| F(s) \|_{X^m} ds + \frac{1}{\lambda} \| F(t) \|_{X^m} \Big), \tag{15}$$

$1 \leq m \leq N$. Once (15) is established, Theorem A follows by the usual method of iteration.

Sketch of the proof of Theorem B

We rewrite (5) into the integral equation

$$f^\lambda(t) = e^{-it\lambda L} f^\lambda(0) - \int_0^t e^{-i(t-s)\lambda L} A(f^\lambda(s)) f^\lambda(s) ds. \tag{16}$$

The crucial fact is the following decay lemma.

LEMMA C.

$$\| \Gamma e^{-itL}(L+i)^{-3} f \|_{L^\infty(\Omega)} \leq C(t) (\| f \|_{W^{7,1}(\Omega)} + \| f \|_{H^7(\Omega)}),$$

where $C(t) \to 0$ as $t \to \infty$.

We recall that in order that the solution of a mixed problem is regular, the initial data must satisfy the compatibility condition. For the equation $\partial_t f + iLf = 0$, it means that the initial data must belong to the domain of some power of L. This is why we inserted the resolvent in the above lemma. We are not mentioning the rate of decay, hence no assumption is necessary on the shape of the boundary.

Using the above lemma, it is rather easy to see that

$$\Gamma f^{\lambda}(t) \to 0 \quad \text{in} \quad L^2_{loc}(\bar{\Omega}) \quad \text{as} \quad \lambda \to \infty,\ t > 0.$$

The part $\Gamma_0 f^{\lambda}(t)$ satisfies

$$\Gamma_0 f^{\lambda}(t) = \Gamma_0 f_0^{\lambda} - \int_0^t \Gamma_0 A(f^{\lambda}(s)) f^{\lambda}(s)\,ds.$$

Letting λ tend to infinity, we can obtain formally

$$\Gamma_0 f^{\infty}(t) = f^{\infty}(t) = \Gamma_0 f_0^{\infty} - \int_0^t \Gamma_0 A(f^{\infty}(s)) f^{\infty}(s)\,ds. \tag{17}$$

In view of (13), we see that (17) is nothing but the incompressible Euler equation (7).

The main tool for proving Lemma C is the spectral theory for symmetric hyperbolic systems. In particular, we make use of the limiting absorption principle due to Mochizuki [9], and the micro-local estimates for the resolvent developed for the study of Schrödinger operators by Isozaki-Kitada [10]. The complete proof is given in the paper [11].

PROBLEMS.

(P.1) The first problem is the asymptotic expansion of the solution in λ. Symbolically, we expect that

Compressible Euler = Incompressible Euler

$$+ \frac{1}{\lambda}(\text{Linear Acoustics}) + O(\lambda^{-2}).$$

Majda's book [3] and Asano's work [6] will be a good guide for this problem.

(P.2) Our method does not work well for the 2-dimensional case. The study of the above problem in 2-dimension will be an interesting mathematicl problem.

(P.3) It is known that, by letting the mean free path tend to 0, one can obtain the Euler equation from the Boltzman equation. This has been proved by Nishida [12] and Asano-Ukai [13] for the whole space. It will also be an interesting problem to extend their results to the boundary value problems with the aid of the spectral theory.

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