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SINGULAR LIMITS IN LIOUVILLE-TYPE EQUATIONS

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Abstract. We consider the boundary value problem $\Delta u + \varepsilon^2 k(x) e^u = 0$ in a bounded, smooth domain Ω in $\mathbb{R}^2$ with homogeneous Dirichlet boundary conditions. Here $\varepsilon > 0$, $k(x)$ is a non-negative, not identically zero function. We find conditions under which there exists a solution u_ε which blows up at exactly m points as $\varepsilon \to 0$ and satisfies $\varepsilon^2 \int_\Omega k e^{u_\varepsilon} \to 8m\pi$. In particular, we find that if $k \in C^2(\bar\Omega)$, $\inf_\Omega k > 0$ and Ω is not simply connected then such a solution exists for any given $m \geq 1$

1. Introduction and statement of main results

Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary and $\varepsilon > 0$. This paper is concerned with analysis of solutions to the boundary value problem

$$\begin{cases} \Delta u + \varepsilon^2 k(x) e^u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $k(x)$ is a non-negative, not identically zero function of class $C^2(\bar\Omega)$. Sometimes called *Liouville equation* after [24], this problem and qualitatively similar ones have attracted great attention over the last decades. In a two-dimensional domain or a compact manifold this type of equation arises in a broad range of applications, in particular in astrophysics and combustion theory, see [10, 19, 26] and references, the prescribed Gaussian curvature problem, [21, 12, 13], mean field limit of vortices in Euler flows, [8, 14], and vortices in the relativistic Maxwell-Chern-Simons-Higgs theory [6, 9, 28, 23].

In the 20th. century, mathematical treatment of this problem traces back at least to [7, 19, 20]. It is a standard fact that Problem (1.1) does not admit any solutions for large ε, as testing against a first eigenfunction of the Laplacian readily shows, while for small ε a solution close to zero exists, which represents a strict local minimizer of the energy functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \varepsilon^2 \int_\Omega k(x) e^u. \tag{1.2}$$

Moreover, Trudinger-Moser embedding yields necessary compacness to apply in this range of ε the mountain pass lemma thus getting a second solution, which clearly becomes unbounded as $\varepsilon \downarrow 0$. This second, "large" solution of (1.1) was found in simply connected domains in [32] when $k \equiv 1$, see also [11] for earlier work on existence. While subcritical in the sense of Trudinger-Moser embedding, this problem exhibits loss of compactness as $\varepsilon \to 0$, similar to that present in equations at the critical exponent in higher dimensions. For instance in the Brezis-Nirenberg problem in dimension $N \geq 4$, [5],

$$\begin{cases} \Delta u + \varepsilon^2 u + u^{\frac{N+2}{N-2}} = 0, \quad \text{in } \Omega, \\ \\ u > 0 , \quad u = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

in which the Mountain-pass solution ceases to exist by blowing-up as $\varepsilon \downarrow 0$. The behavior of blowing-up families of solutions to problem (1.1) when $\inf_\Omega k > 0$ has become understood after the works [4, 22, 27, 30]. It is known that if u_ε is an unbounded family of solutions for which $\varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon}$ remains uniformly bounded, then necessarily

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon} = 8m\pi, \tag{1.4}$$

for some integer $m \geq 1$. Moreover there are m-tuples of distinct points of Ω, $(x_1^\varepsilon, \dots, x_m^\varepsilon)$, separated at uniformly positive distance from each other and from $\partial\Omega$ as $\varepsilon \to 0$ for which u_ε remains uniformly bounded on $\Omega \setminus \cup_{j=1}^m B_\delta(x_i^\varepsilon)$ and

$$\sup_{B_\delta(x_i^\varepsilon)} u_\varepsilon \to +\infty, \tag{1.5}$$

for any $\delta > 0$.

An obvious question is the reciprocal, namely existence of solutions of Problem (1.1) with the property (1.4). In this paper we prove that such a family indeed exists if Ω is not simply connected.

Theorem 1. *Assume that Ω is not simply connected and that $\inf_\Omega k > 0$. Then given any $m \geq 1$ there exists a family of solutions u_ε to (1.1) with*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon} = 8m\pi \ .$$

In case of existence, location of blowing-up points is well-understood: it is established in [27, 30] that the m-tuple $(x_1^\varepsilon, \dots, x_m^\varepsilon)$ in (1.5) converges, up to subsequences, to a critical point of the functional

$$\varphi_m(y_1, \dots, y_m) = -\sum_{j=1}^m [2 \log k(y_j) + H(y_j, y_j)] - \sum_{i \neq j} G(y_i, y_j), \tag{1.6}$$

where $G(x, y)$ is the Green's function of the problem

$$-\Delta_x G = 8\pi \delta_y(x), \quad x \in \Omega,$$

$$G(x, y) = 0, \quad x \in \partial\Omega,$$

and H its regular part defined as

$$H(x, y) = G(x, y) - 4 \log \frac{1}{|x - y|}.$$

The proofs in [27, 30] are actually for the case $k \equiv 1$ but, as pointed out in [25], they apply to the general case. Obvious question is the reciprocal, namely presence of multiple-bubbling solutions with concentration at a critical point of φ_m.

Baraket and Pacard [2] established that for $k \equiv 1$ and any *nondegenerate critical point* of φ_m, a family of solutions u_ε concentrating at this point as $\varepsilon \to 0$ does exist. See also [33] for an extension of their technique in the radial case for $m = 1$. As remarked in [2], their construction, based on a very precise approximation of the actual solution and an application of Banach fixed point theorem, uses nondegeneracy in essential way. This assumption, however, is hard to check in practice and in general not true, an annulus being an obvious example. Another construction of these solutions, for the related *mean field* version of Problem (1.1) in a compact two-dimensional Riemannian manifold was carried out by Chen and Lin as a major step in their program for computation of degrees in [15]. Their construction shares elements with that of [2] but the functional-analytic setting is closer to that of [1, 29] where bubbling for problems at the critical exponent was analyzed. This construction also seems to rely in essential way on the assumption that the corresponding analogue of φ_m has only non-degenerate critical points.

In this paper we present a construction of blowing-up families of solutions of (1.1) which lifts the nondegeneracy assumption of [2], and it is in particular enough for the proof of Theorem 1. More precisely, we consider the role of *non-trivial critical values* of φ_m in existence of solutions of (1.1). Let Ω^m denote the cross product of m copies of Ω. We also denote

$$\tilde{\Omega} = \{x \in \Omega \ / \ k(x) > 0 \ \}, \tag{1.7}$$

set we always assume non-empty. An observation we make is that in any compact subset of $\tilde{\Omega}^m$, we may define, without ambiguity,

$$\varphi_m(x_1, \dots , x_m) = -\infty \quad \text{if } x_i = x_j \text{ for some } i \neq j.$$

Let $\mathcal{D}$ be an open set in Ω^m compactly contained in $\tilde{\Omega}^m$ with smooth boundary. We recall that φ_m *links in $\mathcal{D}$ at critical level $\mathcal{C}$ relative to B and B_0* if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: Let us set Γ to be the class of all maps $\Phi \in C(B, \mathcal{D})$ with the property that there exists a function $\Psi \in C([0, 1] \times B, \mathcal{D})$ such that:

$$\Psi(0, \cdot) = \mathrm{Id}_B, \quad \Psi(1, \cdot) = \Phi, \quad \Psi(t, \cdot)|_{B_0} = \mathrm{Id}_{B_0} \text{ for all } t \in [0, 1].$$

We assume

$$\sup_{y\in B_0} \varphi_m(y) \ < \mathcal{C} \ \equiv \ \inf_{\Phi\in\Gamma} \sup_{y\in B} \varphi_m(\Phi(y)) \,, \tag{1.8}$$

and for all $y \in \partial\mathcal{D}$ such that $\varphi_m(y) = \mathcal{C}$, there exists a vector τ_y tangent to $\partial\mathcal{D}$ at y such that

$$\nabla\varphi_m(y) \cdot \tau_y \neq 0 \,. \tag{1.9}$$

Under these conditions a critical point $\bar{y} \in \mathcal{D}$ of φ_m with $\varphi_m(\bar{y}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_m shows. Condition (1.8) is a general way of describing a change of topology in the level sets $\{\varphi_m \le c\}$ in $\mathcal{D}$ taking place at $c = \mathcal{C}$, while (1.9) prevents intersection of the level set $\mathcal{C}$ with the boundary. It is easy to check that the above conditions hold if

$$\inf_{x\in\mathcal{D}} \varphi_m(x) < \inf_{x\in\partial\mathcal{D}} \varphi_m(x), \quad \text{or} \quad \sup_{x\in\mathcal{D}} \varphi_m(x) > \sup_{x\in\partial\mathcal{D}} \varphi_m(x)\,,$$

namely the case of (possibly degenerate) local minimum or maximum points of φ_m. The level $\mathcal{C}$ may be taken in these cases respectively as that of the minimum and the maximum of φ_m in $\mathcal{D}$. These hold also if φ_m is C^1-close to a function with a non-degenerate critical point in $\mathcal{D}$. We call $\mathcal{C}$ a non-trivial critical level of φ_m in $\mathcal{D}$.

In the next result we assume $k \ge 0$, $k \not\equiv 0$ and $k \in C(\bar{\Omega}) \cap C^2(\tilde{\Omega})$ where $\tilde{\Omega}$ is given by (1.7).

Theorem 2. *Let $m \ge 1$ and assume that there is an open set $\mathcal{D}$ compactly contained in $\tilde{\Omega}^m$ where φ_m has a non-trivial critical level $\mathcal{C}$. Then, there exists a solution u_ε, with*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon} = 8m\pi\ .$$

Moreover, there is an m-tuple $(x_1^\varepsilon, \dots, x_m^\varepsilon) \in \mathcal{D}$, such that as $\varepsilon \to 0$

$$\nabla\varphi_m(x_1^\varepsilon, \dots, x_m^\varepsilon) \to 0, \quad \varphi_m(x_1^\varepsilon, \dots, x_m^\varepsilon) \to \mathcal{C},$$

for which u_ε remains uniformly bounded on $\Omega \setminus \cup_{j=1}^m B_\delta(x_i^\varepsilon)$, and

$$\sup_{B_\delta(x_i^\varepsilon)} u_\varepsilon \to +\infty,$$

for any $\delta > 0$.

We will see that if Ω is not simply connected, such a set $\mathcal{D}$ actually exists for any $m \ge 1$, thus yielding the result of Theorem 1. For $m = 1$, a multiplicity result is also available, see Remark 7.1. If Ω has d holes, then there exist at least $d + 1$ solutions u_ε, with

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon} = 8\pi.$$

Theorem 2 is of course applicable to situations in which $\inf_\Omega k = 0$. As an application in this direction we consider the following problem involving a *singular source*,

$$\begin{cases} \Delta u + \varepsilon^2 e^u - 4\pi\alpha\,\delta_P = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where δ_P denotes Dirac mass supported at P. Replacing u by $-\frac{\alpha}{2}G(x,P)+u$, Problem (1.10) is then equivalent to (1.1) with $k(x) = e^{-\frac{\alpha}{2}G(x,P)}$, so that

k is positive everywhere except at $x = P$ and $k(x) \sim |x - P|^{2\alpha}$. We have the validity of the following result, analogue of Theorem 1, in which the assumption of non-simply connectedness becomes replaced by the presence of a source with sufficiently large weight.

Theorem 3. *Assume that $\alpha > 0$ and that $1 \leq m < 1+\alpha$. Then there exists a family of solutions u_ε to Problem (1.10) with*

$$\lim_{\varepsilon \to 0} \varepsilon^2 \int_\Omega e^{u_\varepsilon} = 8m\pi \ .$$

The solutions found in the above result have concentration at points different from the locations of the source. The problem of finding solutions with additional concentration around the source is of different nature. In case they exist, they provide an extra contribution $8\pi(1+\alpha)$ to the above limit, see [3, 31]. We do not treat this case in this paper, but we believe the functional-analytic setting used in the proof Theorem 2 may render existence results for this type of concentration phenomena.

The proof of Theorem 2 relies on the construction of an approximate solution, different from those in [2, 15], which turns out to be precise enough, not only with its local maxima near a critical point of φ_m but everywhere in its domain. Then we carry out a finite dimensional variational reduction for which the main ingredient is an analysis, of independent interest, of bounded invertibility up to translations of the linearized operator in suitable L^∞-weighted spaces. This functional analytic setting yields in fairly smooth way the reduced variational problem to be that of a functional C^1-close to φ_m on every compact subset of its domain. L^∞-weighted spaces have been used in [17, 18] to detect bubbling from above the critical exponent in higher dimensional problems improving the method in [1, 29], both in lifting criticality (or subcriticality) required there, and non-degeneracy of critical points of the analogue of φ_m in that context. The local notion of nontrivial critical value in (1.8)-(1.9) was introduced in [16] in the analysis of concentration phenomena in nonlinear Schrödinger equations.

The rest of this paper will be devoted to the Proofs of Theorems 1 and 2. In Sections §2 to §6, the hypotheses of Theorem 2 will always be assumed.

2. A first approximation of the solution

In this section we will provide an ansatz for solutions of problem (1.1). The "basic cells" for the construction of an approximate solution of problem (1.1) are the radially symmetric solutions of the problem

$$\begin{cases} \Delta u + e^u = 0, & \text{in } \mathbb{R}^2, \\ u(x) \to -\infty, & \text{as } |x| \to \infty, \end{cases} \tag{2.1}$$

which are given by the one-parameter family of functions

$$\omega_\mu(r) = \log \frac{8\mu^2}{(\mu^2 + r^2)^2},$$

where μ is any positive number.

Let m be a positive integer and choose m distinct points in $\tilde{\Omega}$, say $\xi_1, \dots, \xi_m$ with $k(\xi_j) > 0$. Let μ_j, $j = 1, \dots, m$ be positive numbers. We observe that the function

$$u_j(x) = \log \frac{8\mu_j^2}{(\mu_j^2 \varepsilon^2 + |x - \xi_j|^2)^2 k(\xi_j)} = \omega_{\mu_j}(\frac{|x - \xi_j|}{\varepsilon}) + 4 \log \frac{1}{\varepsilon} - \log k(\xi_j),$$

satisfies in entire $\mathbb{R}^2$

$$\Delta u_j + k(\xi_j) \varepsilon^2 e^{u_j} = 0.$$

We would like to take $\sum_{j=1}^m u_j$ as a first approximation to a solution of the equation. We need to modify it in order to satisfy zero Dirichlet boundary conditions. We consider $H_j(x)$ solution of

$$\begin{cases} -\Delta H_j(x) = 0, \quad \text{in } \Omega, \\ \\ H_j(x) = -\omega_{\mu_j}(\frac{|x-\xi_j|}{\varepsilon}) - 4 \log \frac{1}{\varepsilon} + \log k(\xi_j), \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We consider as initial approximation $U = \sum_{i=1}^m (u_j + H_j)$, which by definition satisfies the boundary conditions. This approximation is less accurate near ξ_j than u_j alone unless $H_j(\xi_j) + \sum_{i=1, i \neq j}^m [H_i(\xi_j) + u_i(\xi_j)] \sim 0$ as $\varepsilon \to 0$. We can achieve this by further adjusting the numbers μ_j. As we will justify below, the good choice of these numbers is

$$\log 8\mu_j^2 = \log k(\xi_j) + H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j), \tag{2.3}$$

where G and H are Green's function and its regular part as defined in the introduction. Thus we consider the first approximation

$$U = \sum_{i=1}^m (u_i + H_i) = \sum_{i=1}^m (\omega_i(\frac{|x - \xi_i|}{\varepsilon}) - \log k(\xi_i) \varepsilon^4 + H_i), \tag{2.4}$$

where $\omega_i = \omega_{\mu_i}$ and with the numbers μ_j defined in (2.3). Let us analyze the asymptotic behavior of H_j as $\varepsilon \to 0$. We observe that for $x \in \partial\Omega$,

$$H_j(x) = -2 \log \frac{1}{\mu_j^2 \varepsilon^2 + |x - \xi_j|^2} - \log \frac{8\mu_j^2}{k(\xi_j)}$$

from where it follows that

$$H_j(x) = H(x, \xi_j) - \log \frac{8\mu_j^2}{k(\xi_j)} + O(\mu_j^2 \varepsilon^2), \tag{2.5}$$

uniformly in C^2-sense for x on compact subsets of Ω. Observe also that, away from each ξ_j

$$w_j = \log \frac{8\mu_j^2}{k(\xi_j)} + 4\log\frac{1}{|x-\xi_j|} + O(\mu_j^2\varepsilon^2),$$

and hence

$$w_j(x) + H_j(x) = G(x,\xi_j) + O(\varepsilon^2), \tag{2.6}$$

where the term $O(\,)$ is uniform in C^2-sense on compact subsets of $\bar\Omega \setminus \{\xi_j\}$.

A useful observation is that u satisfies equation (1.1) if and only if

$$v(y) = u(\varepsilon y) - 4\log\frac{1}{\varepsilon}$$

satisfies

$$\begin{cases} \Delta v + \, k(\varepsilon y)\, e^v = 0, & \text{in } \Omega_\varepsilon, \\ \\ u > 0, \quad \text{in } \Omega_\varepsilon, \quad v = -4\log\frac{1}{\varepsilon}, & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.7}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. We also write $\xi_i' = \varepsilon^{-1}\xi_i$ and define the initial approximation in expanded variables as $V(y) = U(\varepsilon y) - 4\log\frac{1}{\varepsilon}$. We want to measure how well V solves the above problem. Let us fix a small number $\delta > 0$ and observe that $k(\varepsilon y)e^{V(y)} = \varepsilon^4 k(x)e^{U(x)}$ with $x = \varepsilon y$, hence we see that

$$k(\varepsilon y)e^{V(y)} = O(\varepsilon^4) \quad \text{if } |y-\xi_j'| > \frac{\delta}{\varepsilon} \text{ for all } j = 1,\dots,m. \tag{2.8}$$

Similarly, $\Delta V(y) = \varepsilon^2\Delta U(x)$ and (2.6) implies

$$\Delta V(y) = O(\varepsilon^4) \quad \text{if } |y-\xi_j'| > \frac{\delta}{\varepsilon} \text{ for all } j = 1,\dots,m. \tag{2.9}$$

On the other hand, assume that for certain j, $|y-\xi_j'| < \delta$. Then setting $y = \xi_j' + z$ we get

$$k(\varepsilon y)e^{V(y)} = k(\xi_j + \varepsilon z)\frac{8\mu_j^2}{k(\xi_j)(\mu_j^2 + |z|^2)^2} \times$$

$$\exp\left(H_j(\xi_j+\varepsilon z) + \sum_{l\neq j} \log[\frac{8\mu_l^2}{k(\xi_l)(\mu_l^2\varepsilon^2 + |\xi_l - \xi_j + \varepsilon z|^2)^2}] + H_l(\xi_j + \varepsilon z)\right).$$

Now, by definition

$$\log\frac{1}{|\xi_l - \xi_j|^4} + H(\xi_l,\xi_j) = G(\xi_l,\xi_j).$$

Taking into account this relation, the asymptotic expansion (2.5) and the definition of the numbers μ_l in (2.3) we get then that

$$k(\varepsilon y)e^{V(y)} = \frac{8\mu_j^2}{(\mu_j^2 + |y-\xi_j'|^2)^2}[1 + O(\varepsilon z) + O(\varepsilon^2)], \quad |y - \xi_j'| < \frac{\delta}{\varepsilon}. \tag{2.10}$$

We also have in this region

$$\Delta V(y) = \Delta w_{\mu_j}(|y-\xi'_j|) + O(\varepsilon^4) = -\frac{8\mu_j^2}{(\mu_j^2+|y-\xi'_j|^2)^2} + O(\varepsilon^4). \tag{2.11}$$

In summary, combining (2.8)-(2.11) we have established the following fact: if we set

$$R = \Delta V(y) + k(\varepsilon y)e^{V(y)}, \tag{2.12}$$

then

$$|R(y)| \le C\varepsilon \sum_{j=1}^m \frac{1}{1+|y-\xi'_j|^3}. \tag{2.13}$$

Let us stay in these expanded variables. In the rest of this paper we will look for a solution v of Problem (2.7) of the form $v = V + \phi$, where V is defined as above. Let us set

$$W(y) = k(\varepsilon y)e^{V(y)}.$$

In terms of ϕ, Problem (2.7) becomes

$$\begin{cases} L(\phi) := \Delta\phi + W\phi = -[R+N(\phi)], \quad \text{in } \Omega_\varepsilon, \\ \phi = 0, \quad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.14}$$

where

$$N(\phi) = W[e^\phi - 1 - \phi]. \tag{2.15}$$

A main step in solving Problem (2.14) for small ϕ under a suitable choice of the points ξ_j is that of a solvability theory for the linear operator L. In developing this theory we will take into account the invariance, under translations and dilations, of the problem $\Delta w + e^w = 0$ in $\mathbb{R}^2$. We shall devote the next section to prove bounded invertibility of the operator L in this sense using L^∞-norms naturally attached to the setting of Problem (2.14).

3. Analysis of the linearized operator

In this section we will develop a solvability theory for the linearized operator under suitable orthogonality conditions. Thus we set

$$L(\phi) = \Delta\phi + W(y)\phi,$$

for functions ϕ defined on Ω_ε, where

$$W(y) = \sum_{j=1}^m \frac{8\mu_j^2}{(\mu_j^2+|y-\xi'_j|^2)^2}\,[1+\theta_\varepsilon(y)],$$

and θ_ε has the property that for some constant C independent of ε,

$$|\theta_\varepsilon(y)| \le C\varepsilon \sum_{j=1}^m [|y-\xi'_j|+1]\ .$$

If we center the system of coordinates at, say ξ_j' by setting $z = y - \xi_j'$, then the operator formally approaches the linear operator in $\mathbb{R}^2$,

$$L_j(\phi) = \Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi,$$

namely, equation $\Delta v + e^v = 0$ linearized around the radial solution $v_j(z) = \log \frac{8\mu_j^2}{(\mu_j^2+|z|^2)^2}$. An important fact to develop the desired solvability theory is the non-degeneracy of v_j modulo the natural invariance of the equations under translations and dilations, $\zeta \mapsto v_j(z-\zeta)$ and $s \mapsto v_j(sz) - 2\log s$. Thus we set,

$$z_{ij}(z) = \tfrac{\partial}{\partial \zeta_i} v_j(z+\zeta)\,|_{\zeta=0}, \quad i = 1, 2,$$

$$z_{0j}(z) = \tfrac{\partial}{\partial s}[v_j(sz) + 2\log s]\,|_{s=1}.$$

It turns out that the only bounded solutions of $L_j(\phi) = 0$ in $\mathbb{R}^2$ are precisely the linear combinations of the z_{ij}, $i = 0, 1, 2$, see [2] for a proof. Let us denote also $Z_{ij}(y) := z_{ij}(y - \xi_j')$.

Additionally, let us consider a large but fixed number $R_0 > 0$ and a non-negative function $\chi(\rho)$ with $\chi(\rho) = 1$ if $\rho < R_0$ and $\chi(\rho) = 0$ if $\rho > R_0 + 1$. We denote

$$\chi_j(y) = \chi(|y - \xi_j'|).$$

Given h of class $C^{0,\alpha}(\Omega_\varepsilon)$, we consider the linear problem of finding a function ϕ and scalars c_{ij} $i = 1, 2$, $j = 1, \dots, m$ such that

$$L(\phi) \;=\; h + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\chi_j Z_{ij}, \qquad \text{in } \Omega_\varepsilon, \tag{3.1}$$

$$\phi = 0, \qquad \text{on } \partial\Omega_\varepsilon, \tag{3.2}$$

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0, \qquad \text{for all } i = 1, 2,\ j = 1, \dots, m. \tag{3.3}$$

Our main result for this problem states its bounded solvability, uniform in small ε and points ξ_j uniformly separated from each other and from the boundary. Thus we consider the norms

$$\|\psi\|_\infty = \sup_{y\in\Omega_\varepsilon} |\psi(y)|, \quad \|\psi\|_* = \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^{m} (1 + |y - \xi_j'|)^{-3} + \varepsilon^2\right)^{-1} |\psi(y)|.$$

Proposition 3.1. *Let $\delta > 0$ be fixed. There exist positive numbers ε_0 and C, such that for any points ξ_j, $j = 1, \dots, m$ in Ω, with*

$$\operatorname{dist}(\xi_j, \partial\Omega) \geq \delta, \quad |\xi_l - \xi_j| \geq \delta \text{ for } l \neq j\ , \tag{3.4}$$

there is a unique solution to problem (3.1)–(3.3) for all $\varepsilon < \varepsilon_0$. Moreover

$$\|\phi\|_\infty \leq C(\log\frac{1}{\varepsilon})\|h\|_*. \tag{3.5}$$

We observe that the orthogonality conditions in the problem above are only taken with respect to the elements of the approximate kernel due to translations.

The proof of this result consists of some steps. The first step is to prove uniform a priori estimates for the problem (3.1)–(3.3) when ϕ satisfies additionally orthogonality under dilations. Specifically, we consider the problem

$$L(\phi) = h, \qquad \text{in } \Omega_\varepsilon, \tag{3.6}$$

$$\phi = 0, \qquad \text{on } \partial\Omega_\varepsilon, \tag{3.7}$$

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij} \phi = 0, \qquad \text{for } i = 0,1,2,\ j = 1,\dots,m, \tag{3.8}$$

and prove the following estimate.

Lemma 3.1. *Let $\delta > 0$ be fixed. There exist positive numbers ε_0 and C, such that for any points ξ_j, $j = 1,\dots,m$ in Ω, which satisfy relations (3.4), and any solution ϕ to (3.6)-(3.8), one has*

$$\|\phi\|_\infty \le C\|h\|_*, \tag{3.9}$$

for all $\varepsilon < \varepsilon_0$.

Proof. We will carry out the proof of the a priori estimate (3.9) by contradiction. We assume then the existence of sequences $\varepsilon_n \to 0$, points $\xi_j^n \in \Omega$ which satisfy relations (3.4), functions h_n with $\|h_n\|_* \to 0$, ϕ_n with $\|\phi_n\|_\infty = 1$,

$$L(\phi_n) = h_n, \qquad \text{in } \Omega_\varepsilon, \tag{3.10}$$

$$\phi_n = 0, \qquad \text{on } \partial\Omega_\varepsilon, \tag{3.11}$$

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij} \phi_n = 0, \qquad \text{for all } i = 0,1,2,\ j = 1,\dots,m. \tag{3.12}$$

A key step in the proof is the fact that the operator L satisfies maximum principle in Ω_ε outside large balls centered at the points ξ_j'. Consider the function $z_0(r) = \frac{r^2-1}{1+r^2}$, radial solution in $\mathbb{R}^2$ of

$$\Delta z_0 + \frac{8}{(1+r^2)^2} z_0 = 0.$$

Define a comparison function in Ω_ε,

$$Z(y) = \sum_{j=1}^m z_0(a|y-\xi_j'|), \quad y \in \Omega_\varepsilon$$

Let us observe that

$$-\Delta Z = \sum_{j=1}^m \frac{8a^2(a^2|y-\xi_j'|^2 - 1)}{(1+a^2|y-\xi_j'|^2)^3}$$

So that for $|y-\xi_j'|^2 > 100a^{-2}$ for all j,

$$-\Delta Z \geq 2\sum_{j=1}^{m} \frac{a^2}{(1+a^2|y-\xi_j'|^2)^2} \geq \sum_{j=1}^{m} \frac{a^{-2}}{|y-\xi_j'|^4}.$$

On the other hand, in the same region,

$$WZ \leq C\sum_{j=1}^{m} \frac{1}{|y-\xi_j'|^4}.$$

Hence if a is taken small and fixed, and $R>0$ is chosen sufficiently large depending on this a, then we have that $L(Z)<0$ in $\tilde{\Omega}_\varepsilon := \Omega_\varepsilon \setminus \cup_{j=1}^m B(\xi_j', R)$. Since $Z>0$ in this region we then conclude that L satisfies Maximum principle, namely if $L(\psi)\leq 0$ in $\tilde{\Omega}_\varepsilon$ and $\psi \geq 0$ on $\partial\tilde{\Omega}_\varepsilon$ then $\psi\geq 0$ in $\tilde{\Omega}_\varepsilon$. Let us fix such a number $R>0$ which we may take larger whenever it is needed. Now, let us consider the "inner norm"

$$\|\phi\|_i = \sup_{\cup_{j=1}^m B(\xi_j',R)} |\phi|.$$

We make the following claim: there is a constant $C>0$ such that if $L(\phi)=h$ in Ω_ε then

$$\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*]. \tag{3.13}$$

We will establish this with the use of suitable barriers.

Let M be a large number such that for all j, $\Omega_\varepsilon \subset B(\xi_j', \frac{M}{\varepsilon})$. Consider now the solution of the problem

$$-\Delta\psi_j = \tfrac{2}{|y-\xi_j'|^3} + 2\varepsilon^2, \quad R<|y-\xi_j'|<\tfrac{M}{\varepsilon},$$

$$\psi_j(y) = 0 \text{ for } |y-\xi_j'| = R, \quad |y-\xi_j'| = \tfrac{M}{\varepsilon}.$$

A direct computation shows that

$$\psi(r) = \frac{1}{R} - \frac{1}{r} - \varepsilon^2(r-R) - \left[\frac{1}{R} - \frac{1}{r} - \varepsilon^2\left(\frac{M}{\varepsilon} - R\right)\right] \frac{\log\frac{r}{R}}{\log\frac{M}{\varepsilon R}},$$

hence these functions have a uniform bound independent of ε as long as $1<R<\frac{1}{2\varepsilon}$. On the other hand, let us consider the function $Z(y)$ defined above, and let us set

$$\tilde{\phi}(y) = 2\|\phi\|_i Z(y) + \|h\|_* \sum_{j=1}^{m} \psi_j(y).$$

Then, it is easily checked that, choosing R larger if necessary, $L(\tilde{\phi}) \leq h$, $\tilde{\phi} \geq \phi$ on $\partial\tilde{\Omega}_\varepsilon$. Hence $\phi \leq \tilde{\phi}$ on $\tilde{\Omega}_\varepsilon$. Similarly, $\phi \geq -\tilde{\phi}$ on $\tilde{\Omega}_\varepsilon$ and the claim follows.

Let us now go back to the contradiction argument. The above claim shows that since $\|\phi_n\|_\infty = 1$, then for some $\kappa>0$, $\|\phi_n\|_i \geq \kappa$. Let us set $\hat{\phi}_n(z) = \phi_n(\xi_j^n + z)$ where the index j is such that $\sup_{|y-\xi_j^n|<R} |\phi_n| \geq \kappa$. With no loss of generality we assume that this index j is the same for all n.

Elliptic estimates readily imply that $\hat\phi_n$ converges uniformly over compacts to a bounded solution $\hat\phi \neq 0$ of the problem in $\mathbb{R}^2$

$$\Delta\phi + \frac{8\mu_j^2}{(\mu_j^2 + |z|^2)^2}\phi = 0.$$

This implies that $\hat\phi$ is a linear combination of the functions z_{ij}, $i = 0, 1, 2$. However, our assumed orthogonality conditions on ϕ_n pass to the limit and yield $\int \chi(|z|) z_{ij}\hat\phi = 0$ and hence necessarily $\hat\phi \equiv 0$, a contradiction from which the result of the lemma follows. □

We want to establish next an a priori estimate for problem (3.6)-(3.8) with the orthogonality conditions $\int \chi_j \phi Z_{0j} = 0$ dropped, namely the problem

$$L(\phi) = h, \qquad \text{in } \Omega_\varepsilon, \tag{3.14}$$

$$\phi = 0, \qquad \text{on } \partial\Omega_\varepsilon, \tag{3.15}$$

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0, \qquad \text{for } i = 1, 2,\ j = 1, \dots, m. \tag{3.16}$$

Lemma 3.2. *Let $\delta > 0$ be fixed. There exist positive numbers ε_0 and C, such that for any points ξ_j, $j = 1, \dots, m$ in Ω which satisfy (3.4), and any solution ϕ to problem (3.14)-(3.16), one has*

$$\|\phi\|_\infty \le C(\log\frac{1}{\varepsilon})\|h\|_*, \tag{3.17}$$

for all $\varepsilon < \varepsilon_0$.

Proof. Let $R > R_0 + 1$ be a large and fixed number, and let $\hat z_{0j}$ be the solution of the problem

$$\Delta\hat z_{0j} + \tfrac{8\mu_j^2}{(\mu_j^2 + |y - \xi_j'|^2)^2}\hat z_{0j} = 0,$$

$$\hat z_{0j}(y) = z_{0j}(R) \text{ for } |y - \xi_j'| = R, \quad \hat z_{0j}(y) = 0 \text{ for } |y - \xi_j'| = \tfrac{\delta}{3\varepsilon}.$$

A direct computation shows that this function is explicitly given by

$$\hat z_{0j}(y) = z_{0j}(r)\left[1 - \frac{\int_R^r \frac{ds}{s z_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0j}^2(s)}}\right], \quad r = |y - \xi_j'|.$$

Next we consider smooth cut-off functions $\eta_1(r)$ and $\eta_2(r)$ with the following properties: $\eta_1(r) = 1$ for $r < R$, $\eta_1(r) = 0$ for $r > R + 1$, $|\eta_1'(r)| \le 2$. $\eta_2(r) = 1$ for $r < \frac{\delta}{4\varepsilon}$, $\eta_2(r) = 0$ for $r > \frac{\delta}{3\varepsilon}$, $|\eta_2'(r)| \le C\varepsilon$, $|\eta''{}_2(r)| \le C\varepsilon^2$. Then we set

$$\eta_{1j}(y) = \eta_1(|y - \xi_j'|), \quad \eta_{2j}(y) = \eta_2(|y - \xi_j'|). \tag{3.18}$$

and define a test function

$$\tilde z_{0j} = \eta_{1j} Z_{0j} + (1 - \eta_{1j})\eta_{2j}\hat z_{0j}, \quad Z_{0j}(y) = z_{0j}(|y - \xi_j'|).$$

Intuitively, $\tilde z_{0j}$ resembles the eigenfunction of the operator L associated to the invariance of L under dilations when L is considered in the whole $\mathbb{R}^2$.

Let ϕ be a solution to (3.14)-(3.16). We will modify ϕ so that the orthogonality conditions with respect to Z_{0j}'s are satisfied. We set

$$\tilde\phi = \phi + \sum_{j=1}^{m} d_j \tilde z_{0j}$$

where the numbers d_j are defined as

$$d_j \int_{\Omega_\varepsilon} \chi_j |Z_{0j}|^2 + \int_{\Omega_\varepsilon} \chi_j Z_{0j}\phi = 0.$$

Then

$$L(\tilde\phi) = h + \sum_{j=1}^{m} d_j L(\tilde z_{0j}), \tag{3.19}$$

and $\int_{\Omega_\varepsilon} \chi_j Z_{0i}\tilde\phi = 0$ for all i and all j. The previous lemma thus allows us to estimate

$$\|\tilde\phi\|_\infty \le C[\|h\|_* + \sum_{j=1}^{m} |d_j| \|L(\tilde z_{0j})\|_*]. \tag{3.20}$$

Testing equation (3.19) against $\tilde z_{0l}$ we find

$$\langle \tilde\phi, L(\tilde z_{0l})\rangle = \langle h, \tilde z_{0l}\rangle + d_l \langle L(\tilde z_{0l}), \tilde z_{0l}\rangle.$$

where $\langle f, g\rangle = \int_{\Omega_\varepsilon} fg$. This relation in combination with (3.20) gives us that

$$d_l \langle L(\tilde z_{0l}), \tilde z_{0l}\rangle \le C\|h\|_*[1 + \|L(\tilde z_{0l})\|_*] + C\sum_{j=1}^{m} |d_j| \|L(\tilde z_{0j})\|_*^2. \tag{3.21}$$

We will measure next the size of $\|L(\tilde z_{0j})\|_*$. We have

$$\begin{aligned} L(\tilde z_{0j}) \;&=\; 2\nabla\eta_{1j}\nabla(Z_{0j} - \hat z_{0j}) + \Delta\eta_{1j}(Z_{0j} - \hat z_{0j}) \\ &\quad + 2\nabla\eta_{2j}\nabla\hat z_{0j} + \Delta\eta_{2j}\hat z_{0j} + O(\varepsilon^4). \end{aligned}$$

Let us observe first that, for $r \in (R, R+1)$, $r = |y - \xi'_j|$, we have

$$\hat z_{0j} - Z_{0j} = -z_{0j}(r) \frac{\int_R^r \frac{ds}{s z_{0j}^2(s)}}{\int_R^{\frac{\delta}{3\varepsilon}} \frac{ds}{s z_{0j}^2(s)}},$$

so that

$$|\hat z_{0j} - Z_{0j}| \le \frac{C}{\log\frac{1}{\varepsilon}},$$

in this region. Similarly,

$$|\hat z'_{0j} - Z'_{0j}| \le \frac{C}{\log\frac{1}{\varepsilon}}.$$

On the other hand for $r \in (\frac{\delta}{4\varepsilon}, \frac{\delta}{3\varepsilon})$,

$$\hat z_{0j}(r) \le \frac{C}{\log\frac{1}{\varepsilon}},$$

and

$$|\hat z_{0j}'(r)| \le \frac{C\varepsilon}{\log\frac{1}{\varepsilon}}.$$

We observe then that from the definition of the *-norm,

$$\|L(\tilde z_{0j})\|_* \le \frac{C}{\log\frac{1}{\varepsilon}}, \tag{3.22}$$

where the number C depends in principle of the chosen large constant R. Now we want to measure the size of $\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle$. We decompose

$$\begin{aligned}\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle &= \int_{R<r<R+1} L(\tilde z_{0l})\tilde z_{0l} + \int_{\frac{\delta}{4\varepsilon}<r<\frac{\delta}{3\varepsilon}} L(\tilde z_{0l})\tilde z_{0l} + O(\varepsilon)\\ &= I + II + O(\varepsilon).\end{aligned}$$

We have that

$$|II| \le C\int |\nabla\eta_{2l}||\nabla\hat z_{0l}||\hat z_{0l}| + C\int|\Delta\eta_{2l}||\hat z_{0l}|^2 + O(\varepsilon^2),$$

hence from the above obtained estimates,

$$|II| \le \frac{C}{(\log\frac{1}{\varepsilon})^2}.$$

Let us estimate now I. We have

$$I = 2\int\nabla\eta_1\nabla(Z_{0j}-\hat z_{0j})\hat z_{0j} + \int\Delta\eta_1(Z_{0j}-\hat z_{0j})\hat z_{0j} + O(\varepsilon).$$

Thus integrating by parts we find

$$I = \int\nabla\eta_1\nabla(Z_{0j}-\hat z_{0j})\hat z_{0j} - \int\nabla\eta_1(Z_{0j}-\hat z_{0j})\nabla\hat z_{0j} + O(\varepsilon).$$

Now, we observe that in the considered region, $r \in (R, R+1)$ with $r = |y-\xi_j'|$, $|\hat z_{0j} - Z_{0j}| \le \frac{C}{\log\frac{1}{\varepsilon}}$, while $|\hat z_{0j}'| \sim \frac{1}{R^3} + \frac{1}{R}\frac{1}{\log\frac{1}{\varepsilon}}$. In conclusion (R is large but independent of ε) we find

$$\left|\int\nabla\eta_1(\hat z_{0j}-Z_{0j})\nabla\hat z_{0j}\right| \le \frac{D}{R^3}\frac{1}{\log\frac{1}{\varepsilon}},$$

where D may be chosen independent of R. Now,

$$\begin{aligned}\int\nabla\eta_1\nabla(Z_{0j}-\hat z_{0j})\hat z_{0j} &= \int_R^{R+1}\eta_1'(z_{0j}-\hat z_{0j})'\hat z_{0j}\, r dr\\ &= \frac{1}{\int_R^{\frac{\delta}{3\varepsilon}}\frac{ds}{sz_{0j}^2}}\int_R^{R+1}\eta_1'\left[1+\frac{(\mu_j r)^2 z_{0j}\int_R^r\frac{ds}{sz_{0j}^2}}{1+(\mu_j r)^2}\right]dr + O(\frac{1}{\log\frac{1}{\varepsilon}})\\ &= -\frac{E}{\log\frac{1}{\varepsilon}}[1+O(\frac{1}{\log\frac{1}{\varepsilon}})],\end{aligned}$$

where E is a positive constant independent on ε. Thus we conclude, choosing R large enough, that $I \sim -\frac{E}{\log\frac{1}{\varepsilon}}$ Combining this and the estimate for II we find

$$\langle L(\tilde z_{0l}), \tilde z_{0l}\rangle \le -\frac{E}{\log\frac{1}{\varepsilon}}[1+O(\frac{1}{\log\frac{1}{\varepsilon}})]. \tag{3.23}$$

Combining relations (3.23) with (3.21) and (3.22) we finally get that

$$|d_j| \le C(\log\frac{1}{\varepsilon})\|h\|_*,$$

for all $j = 1, \dots, m$. We thus conclude from estimate (3.20) that

$$\|\phi\|_\infty \le C(\log\frac{1}{\varepsilon})\|h\|_*.$$

The proof is complete. □

We are now ready for the proof of our main result of this section.

Proof of Proposition 3.1 We begin by establishing the validity of the a priori estimate (3.5). The previous lemma yields

$$\|\phi\|_\infty \le C(\log\frac{1}{\varepsilon})[\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|], \tag{3.24}$$

hence it suffices to estimate the values of the constants $|c_{ij}|$. Let us consider the cut-off function η_{2j} introduced in (3.18). We test equation (3.1) against $Z_{ij}\eta_{2j}$ to find

$$\langle L(\phi), \eta_{2j}Z_{ij}\rangle = \langle h, \eta_{2j}Z_{ij}\rangle + c_{ij}\int_{\Omega_\varepsilon}\chi_j|Z_{ij}|^2 \,. \tag{3.25}$$

Now,

$$\langle L(\phi), \eta_{2j}Z_{ij}\rangle = \langle \phi, L(\eta_{2j}Z_{ij})\rangle.$$

We have

$$L(\eta_{2j}Z_{ij}) = \Delta\eta_{2j}Z_{ij} + 2\nabla\eta_{2j}\nabla Z_{ij} + \varepsilon O((1+r)^{-3}),$$

with $r = |y-\xi_j'|$. Since $\Delta\eta_{2j} = O(\varepsilon^2)$, $\nabla\eta_{2j} = O(\varepsilon)$, and besides $Z_{ij} = O(r^{-1})$, $\nabla Z_{ij} = O(r^{-2})$, we find

$$L(\eta_{2j}Z_{ij}) = O(\varepsilon^3) + \varepsilon O((1+r)^{-3}).$$

Thus

$$|\langle \phi, L(\eta_{2j}Z_{ij})\rangle| \le C\varepsilon\|\phi\|_\infty.$$

Combining this estimate with (3.25) and (3.24) we obtain

$$|c_{ij}| \le C[\|h\|_* + \varepsilon\log\frac{1}{\varepsilon}\sum_{l,k}|c_{lk}|],$$

which implies $|c_{ij}| \le C\|h\|_*$. It follows finally from (3.24) that $\|\phi\|_\infty \le C(\log\frac{1}{\varepsilon})\|h\|_*$ and the a priori estimate has been thus proven. It only remains to prove the solvability assertion. To this purpose we consider the space

$$H = \Big\{ \phi \in H_0^1(\Omega_\varepsilon) \ : \ \int_{\Omega_\varepsilon} \chi_j Z_{ij}\, \phi \, = \, 0 \quad \text{ for } \ i = 1, 2, \, j = 1, \dots , m \Big\},$$

endowed with the usual inner product $[\phi, \psi] = \int_{\Omega_\varepsilon} \nabla\phi\nabla\psi$. Problem (3.1)-(3.3) expressed in weak form is equivalent to that of finding a $\phi \in H$, such that

$$[\phi, \psi] = \int_{\Omega_\varepsilon} [-W\phi + h]\, \psi \, dx, \qquad \text{for all } \ \psi \in H.$$

With the aid of Riesz's representation theorem, this equation gets rewritten in H in the operator form $\phi = K(\phi) + \tilde{h}$, for certain $\tilde{h} \in H$, where K is a compact operator in H. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation $\phi = K(\phi)$ has only the zero solution in H. This last equation is equivalent to (3.1)-(3.3) with $h \equiv 0$. Thus existence of a unique solution follows from the a priori estimate (3.5). This finishes the proof. $\square$

The result of Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (3.1)-(3.3) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$, into L^∞, with norm bounded uniformly in ε.

It is important for later purposes to understand the differentiability of the operator T with respect to the variables ξ_i'. Fix $h \in \mathcal{C}_*$ and let $\phi = T(h)$. Let us recall that ϕ satisfies the equation

$$L(\phi) = h + \sum_{i,\,j} c_{ij}\, Z_{ij}\chi_j\,,$$

and the vanishing and orthogonality conditions, for some (uniquely determined) constants c_{ij}. We want to compute derivatives of ϕ with respect to the parameters ξ_{kl}'. Formally $Z = \partial_{\xi_{kl}'}\phi$ should satisfy

$$L(Z) = -\partial_{\xi_{kl}'}(W)\,\phi + \sum_{i=1}^{2} c_{il}\,\partial_{\xi_{kl}'}(Z_{il}\chi_l) \, + \sum_{i,\,j} d_{ij}\, Z_{ij}\chi_j\,,$$

where (still formally) $d_{ij} = \partial_{\xi_{kl}'}(c_{ij})$. The orthogonality conditions now become

$$\int_{\Omega_\varepsilon} Z_{ij}\chi_j Z = 0, \qquad \text{if } j \neq l$$
$$\int_{\Omega_\varepsilon} Z_{il}\chi_l Z = -\int \partial_{\xi_{kl}'}(Z_{il}\chi_l)\phi, \quad i = 1, 2.$$

We will recast Z as follows. Let us consider η_{2l}, a smooth cut-off function as in (3.18) for j replaced by l. We consider the constants b_{il} defined as

$$b_{il}\int_{\Omega_\varepsilon} \chi_l|Z_{il}|^2 \equiv \int_{\Omega_\varepsilon} \phi\ \partial_{\xi_{kl}'}(\chi_l Z_{il}),$$

and the function

$$f \equiv -\sum_{i=1}^{2}\Big[b_{il}\, L(\eta_{2l}\, Z_{il}) - c_{il}\, \partial_{\xi'_{kl}}(\chi_l Z_{il})\Big] + \partial_{\xi'_{kl}}(W)\,\phi.$$

Then the function Z above can be uniquely expressed as

$$Z = T(f) + \sum_{i=1}^{2} b_{il}\,\eta_{2l}\, Z_{il}.$$

This computation is not just formal. Arguing directly by definition it shows that indeed $\partial_{\xi'_{kl}}\phi = Z$. Moreover, using Proposition 3.1 we find that $\|f\|_* \le C(\log\frac{1}{\varepsilon})\|h\|_*$, hence

$$\|\partial_{\xi'_{kl}} T(h)\|_\infty \le C\,(\log\frac{1}{\varepsilon})^2\|h\|_* \quad \text{for all } k=1,2, l=1,\dots,m. \tag{3.26}$$

This estimate is of crucial importance in the arguments to come.

4. The nonlinear problem

In what follows we keep the notation introduced in the previous sections. We recall that our goal is to solve Problem (2.14). Rather than doing so directly, we shall solve first the intermediate problem

$$L(\phi) = -[R + N(\phi)] + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}\chi_j Z_{ij}, \quad \text{in } \Omega_\varepsilon, \tag{4.1}$$

$$\phi = 0, \quad \text{on } \partial\Omega_\varepsilon, \tag{4.2}$$

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0, \quad \text{for all } i=1,2,\ j=1,\dots,m, \tag{4.3}$$

using the theory developed in the previous section. We assume that the conditions in Proposition 3.1 hold. We have the following result.

Lemma 4.1. *Under the assumptions of Proposition 3.1 there exist positive numbers C and ε_0, such that Problem (4.1)-(4.3) has a unique solution ϕ which satisfies*

$$\|\phi\|_\infty \le C\,\varepsilon|\log\varepsilon|.$$

Proof. In terms of the operator T defined in Proposition 3.1, Problem (4.1)-(4.3) becomes

$$\phi = T(-(N(\phi) + R)) \equiv A(\phi)\,. \tag{4.4}$$

For a given number $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma \equiv \{\phi \in C(\bar\Omega_\varepsilon)\ :\ ||\phi||_\infty \le \gamma\,\varepsilon|\log\varepsilon|\}.$$

From Proposition 3.1, we get

$$\|A(\phi)\|_\infty \le C|\log\varepsilon|\,\Big[\|N(\phi)\|_* + \|R\|_*\Big]\,.$$

Estimate (2.13) implies that $\|R\| \le C\varepsilon$. Also, the definition of N in (2.15) immediately yields $\|N(\phi)\|_* \le C\,\|\phi\|_\infty^2$. It is also immediate that N satisfies, for $\phi_1, \phi_2 \in \mathcal{F}_\gamma$,

$$\|N(\phi_1) - N(\phi_2)\|_* \le C\gamma\varepsilon|\log\varepsilon|\,\|\phi_1 - \phi_2\|_*,$$

where C is independent of γ. Hence we get

$$\begin{aligned} \|A(\phi)\|_\infty &\le C|\log\varepsilon|\varepsilon\left[\gamma^2\,\varepsilon|\log\varepsilon|^2 + 1\right], \\ \|A(\phi_1) - A(\phi_2)\|_\infty &\le C\,\gamma\,\varepsilon|\log\varepsilon|^2\,\|\phi_1 - \phi_2\|_*\,. \end{aligned}$$

It follows that for all sufficiently small ε we get that A is a contraction mapping of $\mathcal{F}_\gamma$, and therefore a unique fixed point of A exists in this region. This concludes the proof. □

Since R depends continuously (in the *-norm) on the m-tuple

$$\xi' = (\xi'_1, \dots, \xi'_m),$$

the fixed point characterization obviously yields so for the map $\xi' \mapsto \phi$. We shall next analyze the differentiability of this map. Assume for instance that the partial derivative $\partial_{\xi'_{kl}}\phi$ exists. Then, formally,

$$-\partial_{\xi'_{kl}}N(\phi) = \partial_{\xi'_{kl}}W(e^\phi - \phi - 1) + W[e^\phi - 1]\partial_{\xi'_{kl}}\phi\,.$$

It is readily found that $\|\partial_{\xi'_{kl}}W\|_*$ is uniformly bounded. Hence we conclude

$$\|\partial_{\xi'_{kl}}N(\phi)\|_* \le C\left[\|\phi\|_\infty + \|\partial_{\xi'_{kl}}\phi\|_\infty\right]\|\phi\|_\infty \le C\left[\varepsilon|\log\varepsilon| + \|\partial_{\xi'_{kl}}\phi\|_\infty\right]\varepsilon|\log\varepsilon|\,.$$

Also observe that we have

$$\partial_{\xi'_{kl}}\phi = (\partial_{\xi'_{kl}}T)\Big(-(N(\phi) + R)\Big) + T\Big(-\partial_{\xi'_{kl}}\Big[N(\phi) + R\Big]\Big)$$

so that, using (3.26),

$$\|\partial_{\xi'_{kl}}\phi\|_\infty \le C\,|\log\varepsilon|\Big[|\log\varepsilon|\|(N(\phi) + R)\|_* + \|\partial_{\xi'_{kl}}N(\phi)\|_* + \|\partial_{\xi'_{kl}}R\|_*)\Big].$$

Since it is also easily checked that $\|\partial_{\xi'_{kl}}R\| \le C\varepsilon$, we conclude from the above computation that

$$\|\partial_{\xi'_{kl}}\phi\|_\infty \le C\,\varepsilon|\log\varepsilon|^2, \quad \text{for all } k = 1, 2, l = 1, \dots, m.$$

The above computation can be made rigorous by using the implicit function theorem and the fixed point representation (4.4) which guarantees C^1 regularity in ξ'. Thus we have the validity of the following:

Lemma 4.2. *Consider the map $\xi' \mapsto \phi$ into the space $C(\bar\Omega_\varepsilon)$. Under the assumptions of Proposition 3.1 and Lemma 4.1 the derivative $D_{\xi'}\phi$ exists and defines a continuous function of ξ'. Besides, there is a constant $C > 0$, such that*

$$\|D_{\xi'}\phi\|_* \le C\,\varepsilon|\log\varepsilon|^2.$$

After Problem (4.1)-(4.3) has been solved, we will find solutions to the full problem (2.14) (or equivalently (1.1)) if we manage to adjust the m-tuple ξ' in such a way that $c_{ij}(\xi') = 0$ for all i, j. A nice feature of this system of equations is that it turns out to be equivalent to finding critical points of a functional of ξ which is close, in appropriate sense, to the energy of the first approximation V. We make this precise in the next sections.

5. Variational reduction

As we have said, after Problem (4.1)-(4.3) has been solved, we find a solution to Problem (2.14) and hence to the original problem if ξ' is such that

$$c_{ij}(\xi') = 0 \quad \text{for all } i, j. \tag{5.1}$$

This problem is indeed variational: it is equivalent to finding critical points of a function of $\xi = \varepsilon\xi'$. To see that let us consider the energy functional J_ε associated to Problem (1.1), namely

$$J_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \varepsilon^2 \int_\Omega k(x)e^u\,dx. \tag{5.2}$$

We define

$$F(\xi) \equiv J_\varepsilon(U(\xi) + \tilde{\phi}(\xi)), \tag{5.3}$$

where U is the function defined in (2.4) and $\tilde{\phi} = \tilde{\phi}(\xi) = \tilde{\phi}(x,\xi)$ is the function defined on Ω from the relation $\tilde{\phi}(x,\xi) = \phi(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$, with ϕ the solution of Problem (4.1)–(4.3) given by Proposition 3.1. Critical points of F correspond to solutions of (5.1) for small ε, as the following result states.

Lemma 5.1. *Under the assumptions of Proposition 3.1, the functional $F(\xi)$ is of class C^1. Moreover, for all $\varepsilon > 0$ sufficiently small, if $D_\xi F(\xi) = 0$ then ξ satisfies System (5.1).*

Proof. Define

$$I_\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla v|^2\,dy - \int_{\Omega_\varepsilon} k(\varepsilon y)e^v\,dy. \tag{5.4}$$

Let us differentiate the function $F(\xi)$ with respect to ξ. Since $J_\varepsilon(U + \tilde{\phi}) = I_\varepsilon(V + \phi)$, we can differentiate directly $I_\varepsilon(V + \phi)$ under the integral sign, so that

$$\begin{aligned}
\partial_{\xi_{kl}} F(\xi) &= \varepsilon^{-1} DI_\varepsilon(V+\phi)\Big[\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\Big] \\
&= \varepsilon^{-1}\sum_{i=1}^{2}\sum_{j=1}^{m}\int_{\Omega_\varepsilon} c_{ij}\,\chi_j Z_{ij}\left[\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right].
\end{aligned}$$

From the results of the previous section, this expression defines a continuous function of ξ', and hence of ξ. Let us assume that $D_\xi F(\xi) = 0$. Then

$$\sum_{i=1}^{2}\sum_{j=1}^{m}\int_{\Omega_\varepsilon} c_{ij}\,\chi_j Z_{ij}\left[\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi\right] = 0, \quad k = 1,2, \quad l = 1,\dots,m.$$

We recall that we proved $\|D_{\xi'}\phi\|_\infty \le C\varepsilon\,|\log\varepsilon|^2$, thus we directly check that as $\varepsilon \to 0$, we have $\partial_{\xi'_{kl}} V + \partial_{\xi'_{kl}}\phi = -[Z_{kl} + o(1)]$ with $o(1)$ small in terms of the L^∞ norm, as $\varepsilon \to 0$.

We get that $D_\xi F(\xi) = 0$ implies the validity of a system of equations of the form

$$\sum_{i=1}^{m}\sum_{j=1}^{2} c_{ij}\int_{\Omega_\varepsilon} \chi_j Z_{ij}\,[Z_{kl} + o(1)] = 0, \quad k = 1,2, \quad l = 1,\dots,m,$$

with $o(1)$ small in the sense of the L^∞ norm as $\varepsilon \to 0$. The above system is diagonal dominant and we thus get $c_{ij} = 0$ for all i,j. This concludes the proof of the lemma. □

In order to solve for critical points of the function F, a key step is its expected closeness to the function $J_\varepsilon(U)$, which we will analyze in the next section.

Lemma 5.2. *The following expansion holds*

$$F(\xi) = J_\varepsilon(U) + \theta_\varepsilon(\xi)\,,$$

where

$$|\theta_\varepsilon| + |\nabla\theta_\varepsilon| \to 0,$$

uniformly on points satisfying the constraints in Proposition 3.1.

Proof. Since $I_\varepsilon(V) = J_\varepsilon(U)$, $I_\varepsilon(V+\phi) = J_\varepsilon(U+\tilde\phi)$, it is enough to show that $\tilde\theta_\varepsilon(\xi') = \theta_\varepsilon(\varepsilon\xi')$ satisfies

$$|\tilde\theta| + \varepsilon^{-1}|\nabla_{\xi'}\tilde\theta_\varepsilon| = o(1).$$

Taking into account $DI_\varepsilon(V+\phi)[\phi] = 0$, a Taylor expansion gives

$$I_\varepsilon(V+\phi) - I_\varepsilon(V) \tag{5.5}$$
$$= \int_0^1 D^2 I_\varepsilon(V + t\phi)[\phi]^2\,(1-t)\,dt \tag{5.6}$$
$$= \int_0^1 \left(\int_{\Omega_\varepsilon} [N(\phi) + R]\,\phi + \int_{\Omega_\varepsilon} k(\varepsilon y)e^V[1 - e^{t\phi}]\,\phi^2\right)(1-t)\,dt\,.$$

Since $\|\phi\|_\infty \le C\varepsilon|\log\varepsilon|$, we get

$$I_\varepsilon(V+\phi) - I_\varepsilon(V) = \tilde\theta_\varepsilon = O(\varepsilon^2|\log\varepsilon|^3)\,.$$

Let us differentiate with respect to ξ'. We use the representation (5.6) and differentiate directly under the integral sign, thus obtaining, for each $k = 1,2$, $l = 1,\dots,m$,

$$\partial_{\xi'_{kl}}[I_\varepsilon(V+\phi) - I_\varepsilon(V)]$$
$$= \int_0^1\!\left(\int_{\Omega_\varepsilon}\partial_{\xi'_{kl}}[(N(\phi)+R)\,\phi] + \int_{\Omega_\varepsilon}\partial_{\xi'_{kl}}[k(\varepsilon y)e^V[1 - e^{t\phi}]\,\phi^2]\right)(1-t)\,dt\,.$$

Using the fact that $\|\partial_{\xi'}\phi\|_* \le C\,\varepsilon|\log\varepsilon|^2$ and the computations in the proof of Lemma 4.2 we get

$$\partial_{\xi'_{kl}}[I_\varepsilon(V+\phi)-I_\varepsilon(V)] = \partial_{\xi'_{kl}}\tilde\theta_\varepsilon = O(\varepsilon^2|\log\varepsilon|^4)\,.$$

The continuity in ξ of all these expressions is inherited from that of ϕ and its derivatives in ξ in the L^∞ norm. The proof is complete. $\square$

6. Asymptotics of energy of approximate solution

The purpose of this section is to give an asymptotic estimate of $J_\varepsilon(U)$ where U is the approximate solution defined in (2.4) and J_ε is the energy functional (5.2) associated to Problem (1.1).

We have the following result.

Lemma 6.1. *Let $\delta > 0$ be a fixed small number and U be the function defined in (2.4). With the choice (2.3) for the parameters μ_j, the following expansion holds*

$$J_\varepsilon(U) = -16m\pi + 8m\pi\log 8 - 16m\pi\log\varepsilon + 4\pi\varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi) \tag{6.1}$$

where the function φ_m is defined by

$$\varphi_m(\xi_1,\dots,\xi_m) = -\sum_{j=1}^m[2\log k(\xi_j) + H(\xi_j,\xi_j)] - \sum_{i\neq j} G(\xi_i,\xi_j). \tag{6.2}$$

Here G and H are the Green function for the Laplacian on Ω with Dirichlet boundary condition and its regular part, as defined in section 1. In (6.1), Θ_ε is a smooth function of $\xi = (\xi_1,\dots,\xi_m)$, bounded together with its derivatives, as $\varepsilon \to 0$ uniformly on points $\xi_1,\dots,\xi_m \in \Omega$ that satisfy $\operatorname{dist}(\xi_i,\partial\Omega) > \delta$ *and $|\xi_i - \xi_j| > \delta$.*

Remark 6.1. *In the sequel, by $\theta_\varepsilon, \Theta_\varepsilon$ we will denote generic functions of ξ that are bounded, together with its derivatives, in the region* $\operatorname{dist}(\xi_i,\partial\Omega) > \delta$ *and $|\xi_i - \xi_j| > \delta$.*

Proof. We will first evaluate the quadratic part of the energy evaluated at U, that is

$$\frac12\int_\Omega |\nabla U|^2\,dx = \frac12\left\{\sum_{j=1}^m\int_\Omega |\nabla U_j|^2\,dx + \sum_{j\neq i}\int_\Omega \nabla U_j\nabla U_i\,dx\right\} \tag{6.3}$$

Let j be fixed. Using $U_j(x) = u_j(x) + H_j(x)$, we write

$$\frac12\int_\Omega|\nabla U_j|^2\,dx = \frac12\int_\Omega|\nabla u_j|^2\,dx + \int_\Omega \nabla u_j\nabla H_j\,dx + \frac12\int_\Omega|\nabla H_j|^2\,dx. \tag{6.4}$$

Since H_j is harmonic in Ω and U_j is zero on the boundary $\partial\Omega$, we first get

$$\int_\Omega \nabla u_j\nabla H_j\,dx + \frac12\int_\Omega|\nabla H_j|^2\,dx = \int_{\partial\Omega} u_j\frac{\partial H_j}{\partial\nu}d\sigma + \frac12\int_{\partial\Omega} H_j\frac{\partial H_j}{\partial\nu}d\sigma$$

$$= -\frac{1}{2}\int_{\partial\Omega} H_j \frac{\partial H_j}{\partial \nu} d\sigma \tag{6.5}$$

where ν denotes the unitary outer normal of $\partial\Omega$.

We will now evaluate $\int_\Omega |\nabla u_j|^2\, dx$. Observe first that

$$\nabla u_j(x) = \nabla \omega_j(\frac{|x-\xi_j|}{\varepsilon}) = -\frac{4(x-\xi_j)}{\mu_j^2\varepsilon^2 + |x-\xi_j|^2}.$$

Let now $\tilde\delta > 0$ be small and fixed, independent of ε. We will split the previous integral into two pieces, namely

$$\int_\Omega |\nabla u_j|^2\, dx = \int_{B(\xi_j,\tilde\delta)} |\nabla\tilde\omega_j|^2\, dx + \int_{\Omega\setminus B(\xi_j,\tilde\delta)} |\nabla\tilde\omega_j|^2\, dx, \tag{6.6}$$

with $\tilde\omega_j(x) = \omega_j(\frac{|x-\xi_j|}{\varepsilon})$. Now, a direct computation yields

$$\int_{B(\xi_j,\tilde\delta)} |\nabla\tilde\omega_j|^2\, dx = 16\int_{B(\xi_j,\tilde\delta)} \frac{|x-\xi_j|^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}\, dx$$

$$= 16\int_{B(0,\frac{\tilde\delta}{\mu_j\varepsilon})} \frac{|y|^2}{(1+|y|^2)^2}\, dy \qquad (y = \frac{x-\xi_j}{\varepsilon\mu_j})$$

$$= 32\pi\int_0^{\frac{\tilde\delta}{\varepsilon\mu_j}} \frac{r^3}{(1+r^2)^2}\, dr = 16\pi\left[\int_0^{\frac{\tilde\delta}{\varepsilon\mu_j}} \frac{2r}{(1+r^2)} - \int_0^{\frac{\tilde\delta}{\varepsilon\mu_j}} \frac{2r}{(1+r^2)^2}\right]$$

$$= 16\pi\left[-2\log\varepsilon\mu_j - 1 + \log[(\varepsilon\mu_j)^2+\tilde\delta^2] + \frac{(\varepsilon\mu_j)^2}{(\varepsilon\mu_j)^2+\tilde\delta^2}\right] \tag{6.7}$$

On the other hand,

$$\int_{\Omega\setminus B(\xi_j,\tilde\delta)} |\nabla\tilde\omega_j|^2\, dx = 16\int_{\Omega\setminus B(\xi_j,\tilde\delta)} \frac{|x-\xi_j|^2}{(\mu_j^2\varepsilon^2+|x-\xi_j|^2)^2}\, dx$$

$$= 16\int_{\Omega\setminus B(\xi_j,\tilde\delta)} \frac{1}{|x-\xi_j|^2}\, dx + (\varepsilon\mu_j)^2\Theta_{\tilde\delta}(\xi_j),$$

where $\Theta_{\tilde\delta}(\xi_j)$ is a function dependent on $\tilde\delta$ which has the explicit form $\Theta_{\tilde\delta}(\xi_j) = \int_{\Omega\setminus B(\xi_j,\tilde\delta)} \frac{1}{|x-\xi_j|^6}\, dx + o(\varepsilon\mu_j)$, where $o(\varepsilon\mu_j)$ is uniform in the region $\operatorname{dist}(\xi_j,\partial\Omega) > \delta$.

Since $\Gamma(x,y) = 4\log\frac{1}{|x-y|}$, we have

$$16\int_{\Omega\setminus B(\xi_j,\tilde\delta)} \frac{1}{|x-\xi_j|^2}\, dx = \int_{\Omega\setminus B(\xi_j,\tilde\delta)} |\nabla\Gamma(x,\xi_j)|^2\, dx$$

$$= \int_{\partial\Omega} \Gamma\frac{\partial\Gamma}{\partial\nu} d\sigma - \int_{\partial B(\xi_j,\tilde\delta)} \Gamma\frac{\partial\Gamma}{\partial\nu} d\sigma$$

$$= -\int_{\partial\Omega} H(x,\xi_j)\frac{\partial\Gamma}{\partial\nu} d\sigma - 32\pi\log\frac{1}{\tilde\delta},$$

where we use the fact that $H = -\Gamma$ on $\partial\Omega$. The last integral is a direct computation.

So we have

$$\int_{\Omega\setminus B(\xi_j,\tilde\delta)} |\nabla\tilde\omega_j|^2\,dx = -\int_{\partial\Omega} H(x,\xi_j)\frac{\partial\Gamma}{\partial\nu}d\sigma - 32\pi\log\frac{1}{\tilde\delta} + (\varepsilon\mu_j)^2\Theta_{\tilde\delta}$$

$$= -\int_{\Omega} H(x,\xi_j)\Delta\Gamma(x,\xi_j)\,dx +$$

$$\int_{\partial\Omega} H(x,\xi_j)\frac{\partial H}{\partial\nu}(x,\xi_j)\,d\sigma - 32\pi\log\frac{1}{\tilde\delta} + (\varepsilon\mu_j)^2\Theta_{\tilde\delta}$$

$$= 8\pi H(\xi_j,\xi_j) + \int_{\partial\Omega} H(x,\xi_j)\frac{\partial H}{\partial\nu}(x,\xi_j)\,d\sigma - 32\pi\log\frac{1}{\tilde\delta} + (\varepsilon\mu_j)^2\Theta_{\tilde\delta}. \tag{6.8}$$

Noticing that the integral on the left hand side in (6.6) is independent from $\tilde\delta$, (6.7) and (6.8) imply that

$$\frac12\int_\Omega |\nabla u_j|^2\,dx = -8\pi + 16\pi\log\frac{1}{\varepsilon\mu_j} + 4\pi H(\xi_j,\xi_j)$$

$$+\frac12\int_{\partial\Omega} H(x,\xi_j)\frac{\partial H}{\partial\nu}(x,\xi_j)\,d\sigma + (\varepsilon\mu_j)^2\Theta_\varepsilon(\xi_j), \tag{6.9}$$

with the term Θ_ε bounded in the region $\operatorname{dist}(\xi_j,\partial\Omega) > \delta$ and independent from $\tilde\delta$.

A direct application of (2.5) yields

$$\int_{\partial\Omega} H(x,\xi_j)\frac{\partial H}{\partial\nu}(x,\xi_j)\,d\sigma - \int_{\partial\Omega} H_j(x)\frac{\partial H_j}{\partial\nu}(x)\,d\sigma = O((\varepsilon\mu_j)^2).$$

From (6.5) and (6.9) we thus conclude that, for $j = 1,\dots,m$,

$$\frac12\int_\Omega |\nabla U_j|^2\,dx = -8\pi + 16\pi\log\frac{1}{\varepsilon\mu_j} + 4\pi H(\xi_j,\xi_j) + \varepsilon^2\Theta_\varepsilon. \tag{6.10}$$

We next deal with the mixed term in (6.3). Fix $i \neq j$.

Notice that $\Delta U_i = \Delta u_i + \Delta H_i = \varepsilon^{-2}\Delta\omega_i = -\varepsilon^{-2}e^{\omega_i}$. Moreover $U_i = 0$ on $\partial\Omega$. Hence we can write

$$\int_\Omega \nabla U_i\nabla U_j\,dx = \varepsilon^{-2}\int_\Omega e^{\tilde\omega_i}U_j\,dx$$

$$= \int_\Omega \frac{8\mu_i^2\varepsilon^2}{((\varepsilon\mu_i)^2+|x-\xi_i|^2)^2}\left[\omega_j(\frac{|x-\xi_j|}{\varepsilon}) + \log\frac{1}{k(\xi_j)\varepsilon^4} + H_j(x)\right]dx \;=$$

$$\int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)} \frac{8}{(1+|y|^2)^2}\left[\omega_j(\frac{|\varepsilon\mu_i y+\xi_i-\xi_j|}{\varepsilon}) + \log\frac{1}{k(\xi_j)\varepsilon^4}\right]dy$$

$$+\int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)} \frac{8}{(1+|y|^2)^2}H_j(\xi_i+\varepsilon\mu_i y)\,dy$$

$$= \int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)} \frac{8}{(1+|y|^2)^2}\left[\log\frac{1}{(\varepsilon^2\mu_j^2+|\varepsilon\mu_i+\xi_i-\xi_j|^2)^2} + \log\frac{8\mu_j^2}{k(\xi_j)}\right]dy$$

$$+\int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)}\frac{8}{(1+|y|^2)^2}H_j(\xi_i+\varepsilon\mu_i y)\,dy$$

$$=\int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)}\frac{8}{(1+|y|^2)^2}[\log\frac{1}{(\varepsilon^2\mu_j^2+|\varepsilon\mu_i y+\xi_i-\xi_j|^2)^2}-4\log\frac{1}{|\xi_i-\xi_j|}]\,dy$$

$$+\int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)}\frac{8}{(1+|y|^2)^2}[H_j(\xi_i+\varepsilon\mu_i y)-H_j(\xi_i)]\,dy$$

$$+\int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)}\frac{8}{(1+|y|^2)^2}[H_j(\xi_i)-H(\xi_j,\xi_i)+\log\frac{8\mu_j^2}{k(\xi_j)}]\,dy$$

$$+\int_{\frac{1}{\varepsilon\mu_i}(\Omega-\xi_i)}\frac{8}{(1+|y|^2)^2}[H(\xi_i,\xi_j)+4\log\frac{1}{|\xi_i-\xi_j|}]\,dy$$

$$=8\pi G(\xi_i,\xi_j)+O(\varepsilon^2\log\frac{1}{\varepsilon})+O(\varepsilon^2),\tag{6.11}$$

where $O(\cdot)$ terms have uniform bounds in ξ the region considered.

Summing up all the previous information contained in (6.10) and (6.11) we finally get the estimate for (6.3), namely

$$\frac{1}{2}\int_\Omega|\nabla U|^2\,dx=-8m\pi+\sum_{j=1}^m 16\pi\log\frac{1}{\varepsilon\mu_j}$$
$$+4\pi\left(\sum_{j=1}^k H(\xi_j,\xi_j)+\sum_{i\neq j}G(\xi_i,\xi_j)\right)+\varepsilon^2\log\frac{1}{\varepsilon}\Theta_\varepsilon.\tag{6.12}$$

Let us now evaluate the second term in the summation in (5.2). We have

$$\varepsilon^2\int_\Omega k(x)e^U\,dx=\varepsilon^2\left[\sum_{j=1}^m\int_{B(\xi_j,\tilde\delta)}k(x)e^U\,dx\right]+A_\varepsilon.\tag{6.13}$$

First observe that

$$A_\varepsilon=\varepsilon^2\Theta_\varepsilon(\xi)\tag{6.14}$$

with Θ_ε a uniformly bounded function as $\varepsilon\to 0$. Now,

$$\varepsilon^2\int_{B(\xi_j,\tilde\delta)}k(x)e^U\,dx=\varepsilon^2\int_{B(\xi_j,\tilde\delta)}k(x)e^{U_j}e^{\sum_{i\neq j}U_i}\,dx$$

$$=\varepsilon^2\int_{B(\xi_j,\tilde\delta)}\frac{k(x)8\mu_j^2e^{H_j}}{k(\xi_j)(\varepsilon^2\mu_j^2+|x-\xi_j|^2)^2}e^{\sum_{i\neq j}(\log\frac{8\mu_i^2}{\varepsilon^4k(\xi_i)(\mu_i^2+\frac{|x-\xi_i|^2}{\varepsilon^2})^2}+H_i(x))}\,dx$$

(using (2.5))

$$=\frac{1}{\varepsilon^2\mu_j^4}\int_{B(\xi_j,\tilde\delta)}\frac{k(x)e^{H(x,\xi_j)+O(\varepsilon^2\mu_j^2)}}{(1+(\frac{|x-\xi_j|}{\varepsilon\mu_j})^2)^2}e^{\sum_{i\neq j}[\log\frac{1}{(\varepsilon^2\mu_i^2+|x-\xi_i|^2)^2}+H(x,\xi_i)+O(\varepsilon^2\mu_i^2)]}\,dx$$

$(x - \xi_j = \varepsilon\mu_j y)$

$$= \frac{1}{\mu_j^2} \int_{B(0,\frac{\tilde{\delta}}{\varepsilon\mu_j})} \frac{k(\xi_j + \varepsilon\mu_j y) \times e^{H(\xi_j+\varepsilon\mu_j y,\xi_j)}}{(1+|y|^2)^2}$$

$$\times e^{\sum_{i\neq j}[\log \frac{1}{(\varepsilon^2\mu_i^2+|\varepsilon\mu_j y+\xi_j-\xi_i|^2)^2}+H(\xi_j+\varepsilon\mu_j y,\xi_i)]} dy + O(\varepsilon^2)$$

$$= \pi \frac{k(\xi_j)}{\mu_j^2} e^{H(\xi_j,\xi_j)+\sum_{i\neq j} G(\xi_i,\xi_j)} + \varepsilon\Theta_\varepsilon(\xi).$$

From (6.13), (6.14) and the choice (2.3) for the μ_i's, we get

$$\varepsilon^2 \int_\Omega k(x)e^U\,dx = 8m\pi + \varepsilon\Theta_\varepsilon(\xi). \tag{6.15}$$

Using again the expression for the μ_i's by (2.3), together with formulas (6.12) and (6.15), we can write the whole asymptotic expansion of the energy (5.2) evaluated at the U, namely

$$J_\varepsilon(U) = -16m\pi + 8m\pi\log 8 - 16m\pi\log\varepsilon + 4\pi\varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi) \tag{6.16}$$

where the function φ_m is given by (6.2). The C^1-closeness is a direct consequence of the fact that $\Theta_\varepsilon(\xi)$ is bounded together with its derivatives in the considered region. □

7. Proofs of theorems

In this section we carry out the proofs of our main results.

7.1. **Proof of Theorem 2.** Let us consider the set $\mathcal{D}$ as in the statement of the theorem, $\mathcal{C}$ the associated critical value and $\xi \in \mathcal{D}$. According to Lemma 5.1, we have a solution of Problem (1.1) if we adjust ξ so that it is a critical point of $F(\xi)$ defined by (5.3). This is equivalent to finding a critical point of

$$\tilde{F}(\xi) = F(\xi) + 16m\pi\log\varepsilon.$$

On the other hand, from Lemmas 5.2 and 6.1, we have that for $\xi \in \mathcal{D}$, such that its components satisfy $|\xi_i - \xi_j| \geq \delta$,

$$\alpha\tilde{F}(\xi) + \beta = \varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi)$$

where Θ_ε and $\nabla_\xi\Theta_\varepsilon$ are uniformly bounded in the considered region as $\varepsilon \to 0$, and $\alpha \neq 0$ and β are universal constants.

Let us observe that if $M > \mathcal{C}$, then assumptions (1.8), (1.9) still hold for the function $\min\{M, \varphi_m(\xi)\}$ as well as for $\min\{M, \varphi_m(\xi) + \varepsilon\Theta_\varepsilon(\xi)\}$. It follows that the function $\min\{M, \alpha\tilde{F}(\xi)+\beta\}$ satisfies for all ε small assumptions (1.8),(1.9) in $\mathcal{D}$ and therefore has a critical value $\mathcal{C}_\varepsilon < M$ which is close to $\mathcal{C}$ in this region. If $\xi_\varepsilon \in \mathcal{D}$ is a critical point at this level for $\alpha\tilde{F}(\xi) + \beta$, then since

$$\alpha\tilde{F}(\xi_\varepsilon) + \beta \leq \mathcal{C}_\varepsilon < M$$

we have that there exists a $\delta > 0$ such that $|\xi_{\varepsilon,j} - \xi_{\varepsilon,i}| > \delta$, $dist(\xi_{\varepsilon,j}, \partial\Omega) > 0$. This implies C^1-closeness of $\alpha\tilde{F}(\xi) + \beta$ and $\varphi_m(\xi)$ at this level, hence $\nabla\varphi_m(\xi_\varepsilon) \to 0$. The function $u_\varepsilon = U(\xi_\varepsilon) + \tilde{\phi}(\xi_\varepsilon)$ is therefore a solution as predicted by the theorem. □

7.2. **Proof of Theorem 1.** According to the result of Theorem 2, it is sufficient to establish that given $m \geq 1$, φ_m has a nontrivial critical value in some open set $\mathcal{D}$, compactly contained in Ω^m. Our choice of $\mathcal{D}$ is just given by

$$\mathcal{D} = \{y \in \Omega^m \ / \ \mathrm{dist}\,(y, \partial\Omega^m) > \delta\}$$

where δ is a small positive number yet to be chosen. We observe that in this set function $\sum_{j=1}^m H(y_j, y_j)$ is bounded and $\sum_{i\neq j} G(y_i, y_j)$ is bounded below. Consequently function $\varphi_m(y)$ is also bounded below in $\mathcal{D}$.

Let Ω_1 be a bounded nonempty component of $\mathbb{R}^2 \setminus \bar{\Omega}$, and consider a closed, smooth Jordan curve γ contained in Ω which encloses Ω_1. We let S to be the image of γ, $B_0 = \emptyset$ and $B = S \times \cdots \times S = S^m$.

Then define

$$\mathcal{C} = \inf_{\Phi\in\Gamma} \sup_{z\in B} \varphi_m(\Phi(z)), \tag{7.1}$$

where $\Phi \in \Gamma$ if and only if $\Phi(z) = \Psi(1, z)$ with $\Psi : [0, 1] \times B \to \mathcal{D}$ continuous and $\Psi(0, z) = z$.

Lemma 7.1. *There exists $K > 0$, independent of the small number δ used to define $\mathcal{D}$ such that $\mathcal{C} \geq -K$.*

Proof. We need to prove the existence of $K > 0$ independent of small δ such that if $\Phi \in \Gamma$, then there exists a $\bar{z} \in B$ with

$$\varphi_m(\Phi(\bar{z})) \geq -K. \tag{7.2}$$

Let us assume that $0 \in \Omega_1$ and write

$$\Phi(z) = (\Phi_1(z), \ldots, \Phi_m(z)).$$

Identifying the components of the above m-tuple with complex numbers, we shall establish the existence of $\bar{z} \in B$ such that

$$\frac{\Phi_j(\bar{z})}{|\Phi_j(\bar{z})|} = e^{\frac{2j\pi i}{m}} \quad \text{for all } j = 1, \ldots, m. \tag{7.3}$$

Clearly in such a situation, there is a number $\mu > 0$ depending only on m and Ω such that

$$|\Phi_j(\bar{z}) - \Phi_l(\bar{z})| \geq \mu.$$

This, and the definition of φ_m clearly yields the validity of estimate (7.2) for a number K only dependent of Ω. To prove (7.3), we consider an orientation-preserving homeomorphism $h : S^1 \to S$ and the map $f : T^m \to T^m$ defined as $f(\zeta) = (f_1(\zeta), \ldots, f_m(\zeta))$ with

$$T^m = \underbrace{S^1 \times \cdots \times S^1}_{m},$$

and

$$f_j(\zeta_1, \dots, \zeta_m) = \frac{\Phi_j(h(\zeta_1), \dots, h(\zeta_m))}{|\Phi_j(h(\zeta_1), \dots, h(\zeta_m))|}.$$

We define a homotopy $F : [0,1] \times T^m \to T^m$ by

$$F_j(t, \zeta) = \frac{\Psi_j(t, h(\zeta_1), \dots, h(\zeta_m))}{|\Psi_j(t, h(\zeta_1), \dots, h(\zeta_m))|}.$$

Notice that $F(1, \zeta) = f(\zeta)$ and

$$F(0, \zeta) = (\frac{h(\zeta_1)}{|h(\zeta_1)|}, \dots, \frac{h(\zeta_m)}{|h(\zeta_m)|}),$$

which is a homeomorphism of T^m. The existence of $\bar{z}$ such that relation (7.3) holds follows from establishing that f is onto, which we show next.

The torus T^m can be identified with the closed manifold embedded in $\mathbb{R}^{m+1}$ parameterized as

$$\zeta : (\theta_1, \dots, \theta_m) \in [0, 2\pi)^m \mapsto$$
$$(\rho_1 e^{i\theta_1}, 0_{m-1}) + (0_1, \rho_2 e^{i\theta_2}, 0_{m-2}) + \cdots + (0_{m-1}, \rho_m e^{i\theta_m}),$$

where $0 < \rho_m < \cdots < \rho_1$ and we have denoted $0_k = \underbrace{(0, \dots, 0)}_{k}$, $e^{i\theta_j} = (\cos\theta_j, \sin\theta_j)$. We consider as well the solid torus $\hat{T}^m$ parameterized as

$$(\theta_1, \dots, \theta_m, \rho) \in [0, 2\pi)^m \times [0, \rho_m] \mapsto$$
$$(\rho_1 e^{i\theta_1}, 0_{m-1}) + (0_1, \rho_2 e^{i\theta_2}, 0_{m-2}) + \cdots + (0_{m-1}, \rho e^{i\theta_m}).$$

Obviously $\partial\hat{T}^m = T^m$ in $\mathbb{R}^{m+1}$.

With slight abuse of notation, we consider the map $f : T^m \to T^m$, induced from the original f under the above identification, namely

$$f(\zeta) = (\rho_1 f_1(\zeta), 0_{m-1}) + (0_1, \rho_2 f_2(\zeta), 0_{m-2}) + \cdots + (0_{m-1}, \rho_m f_m(\zeta)).$$

f then can be extended continuously to the whole solid torus as $\tilde{f} : \hat{T}^m \to R^{m+1}$ defined simply as

$$f(\zeta, \rho) = (\rho_1 f_1(\zeta), 0_{m-1}) + (0_1, \rho_2 f_2(\zeta), 0_{m-2}) + \cdots + (0_{m-1}, \rho f_m(\zeta)).$$

$\tilde{f}$ is homotopic to a homeomorphism of $\hat{T}^m$, along a deformation which applies $\partial\hat{T}^m$ into itself. Thus if $P \in int(\hat{T}^m)$ then $\deg(\tilde{f}, \hat{T}^m, P) \neq 0$ and hence there exists $Q \in \hat{T}^m$ such that $\tilde{f}(Q) = P$. Thus if we fix angles $(\theta_1^*, \dots, \theta_m^*) \in [0, 2\pi)^m$ and $\rho^* \in (0, \rho_m)$ then there exist $\zeta^{**} \in T^m$ and $\rho^{**} \in (0, \rho_m)$ such that

$$(\rho_1 f_1(\zeta^{**}), 0_{m-1}) + (0_1, \rho_2 f_2(\zeta^{**}), 0_{m-2}) + \cdots + (0_{m-1}, \rho^{**} f_m(\zeta^{**})) =$$
$$(\rho_1 e^{i\theta_1^*}, 0_{m-1}) + (0_1, \rho_2 e^{i\theta_2^*}, 0_{m-2}) + \cdots + (0_{m-1}, \rho^* e^{i\theta_m^*}).$$

A direct computation shows then that $f_j(\zeta^{**}) = e^{i\theta_j^*}$ for all j and also $\rho^* = \rho^{**}$. It then follows that f is onto. This concludes the proof. $\square$

The second step we have to carry out to make Theorem 1 applicable is to establish the validity of assumption (1.9). To this end we need to establish a couple of preliminary facts on the half plane

$$\mathcal{H} = \{(x^1, x^2) \ : \ \ x^1 \geq 0\}.$$

Lemma 7.2. *Consider the function of k distinct points on $\mathcal{H}$*

$$\Psi_k(x_1, \dots , x_k) = -4 \sum_{i \neq j} \log |x_i - x_j|.$$

Let I_+ denote the set of indices i for which $x_i^1 > 0$ and I_0 that for which $x_i^1 = 0$. Then, either

$$\nabla_{x_i} \Psi_k(x_1, \dots , x_k) \neq 0, \quad \textit{for some } i \in I_+,$$

or

$$\frac{\partial}{\partial x_{i2}} \Psi_k(x_1, \dots , x_k) \neq 0, \quad \textit{for some } i \in I_0.$$

Proof. We have that

$$\frac{\partial}{\partial \lambda} \Psi_k(\lambda x_1, \dots , \lambda x_k)|_{\lambda=1} =$$

$$\sum_{i \in I_+} \nabla_{x_i} \Psi_k(x_1, \dots , x_k) \cdot x_i + \sum_{i \in I_0} \partial x_{i2} \Psi_k(x_1, \dots , x_k) x_{i2}.$$

On the other hand,

$$\frac{\partial}{\partial \lambda} \Psi_k(\lambda x_1, \dots , \lambda x_k)|_{\lambda=1} = -4 \frac{\partial}{\partial \lambda} [k(k-1) \log \lambda]|_{\lambda=1} \neq 0,$$

and the result follows. □

A second result we need concerns the analogue of the function φ_k, for the half-plane $\mathcal{H}$.

Let $x = (x^1, x^2)$, $y = (y^1, y^2)$. Then regular part of Green's function in $\mathcal{H}$ is now given by

$$H(x, y) = -4 \log \frac{1}{|x - \bar{y}|}, \quad \bar{y} = (y^1, -y^2).$$

Then

$$G(x, y) = 4 \log \frac{1}{|x - y|} - 4 \log \frac{1}{|x - \bar{y}|}.$$

Hence the associated function $\bar{\varphi}_k$ is given by

$$\bar{\varphi}_k(x_1, \dots , x_k) = 4 \sum_{i=1}^{k} \log \frac{1}{|x_i - \bar{x}_i|} + 4 \sum_{i \neq j} \log \frac{|x_i - x_j|}{|x_i - \bar{x}_j|}.$$

With identical proof as the previous lemma we now get

Lemma 7.3. *For any k distinct points $x_i \in int(\mathcal{H})$ we have*

$$\nabla \bar{\varphi}_k(x_1, \dots , x_k) \neq 0.$$

We will recall here some straightforward to verify facts about the regular part of the Green function $H(x,y) = G(x,y) - 4\log\frac{1}{|x-y|}$. Let $y \in \Omega$ be a point close to $\partial\Omega$ and let $\bar{y}$ be its uniquely determined reflection with respect to $\partial\Omega$. Set

$$\psi(x,y) = H(x,y) + 4\log\frac{1}{|x-\bar{y}|}.$$

Then it can be shown that $\psi(x,y)$ is bounded in $\bar{\Omega}\times\bar{\Omega}$ and

$$|\nabla_x\psi(x,y)| + |\nabla_y\psi(x,y)|. \le C_1 \tag{7.4}$$

Using (7.4) one can derive the following estimates

$$|\nabla_x H(x,y)| + |\nabla_y H(x,y)| \le C_1 \min\{\frac{1}{|x-y|}, \frac{1}{\mathrm{dist}\,(y,\partial\Omega)}\} + C_2. \tag{7.5}$$

Now we are ready to prove the validity of assumption (1.9) which in this case reads as follows:

Lemma 7.4. *Given $K > 0$, there exists a $\delta > 0$ such that if $(\xi_1,\dots,\xi_m) \in \partial\mathcal{D}$, and $|\varphi_m(\xi_1,\dots,\xi_m)| \le K$, then there is a vector τ, tangent to $\partial\mathcal{D}$ such that*

$$\nabla\varphi_m(\xi_1,\dots,\xi_m)\cdot\tau \ne 0.$$

Proof. Let us assume the opposite, namely the existence of a sequence $\delta \to 0$ and of points $\xi = \xi^\delta$ for which $\xi \in \partial\mathcal{D}$ and such that

$$\nabla_{\xi_i}\varphi_m(\xi_1,\dots,\xi_m) = 0 \quad \text{if } \xi_i \in \Omega_\delta, \tag{7.6}$$

and

$$\nabla_{\xi_i}\varphi_m(\xi_1,\dots,\xi_m)\cdot\tau_i = 0 \quad \text{if } \xi_i \in \partial\Omega_\delta, \tag{7.7}$$

for any vector τ_i tangent to $\partial\Omega_\delta$ at ξ_i, where $\Omega_\delta = \{x \in \Omega : \mathrm{dist}\,(x,\partial\Omega) > \delta\}$.

From the assumption of the lemma it follows that there is a point $\xi_l \in \partial\Omega_\delta$, such that $H(\xi_l) \to -\infty$ as $\delta \to 0$. Since the value of φ_m remains uniformly bounded, necessarily we must have that at least two points ξ_i and ξ_j that are becoming close. Let $\delta_n = \frac{1}{n}$, $\xi^n = (\xi_1^n,\dots,\xi_m^n) \in \Omega_{\delta_n}$ be a sequence of points such that (7.6), (7.7) hold, and

$$\rho_n = \inf_{i\ne j}|\xi_j^n - \xi_i^n| \to 0, \quad \textit{as } n\to\infty.$$

Without loss of generality we can assume that $\rho_n = |\xi_1^n - \xi_2^n|$. We define

$$x_j^n = \frac{\xi_1^n - \xi_j^n}{\rho_n}. \tag{7.8}$$

Clearly there exists a k, $2 \le k \le m$ such that

$$\lim_{n\to\infty}|x_j^n| < \infty, \quad j = 1,\dots,k \quad \text{and} \lim_{n\to\infty}|x_j^n| = \infty, \quad j > k.$$

For $j \le k$ we set

$$\tilde{x}_j = \lim_{n\to\infty} x_j^n$$

We consider two cases:

(1) either
$$\frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} \to \infty;$$
(2) or there exists $c_0 < \infty$ such that for almost all n we have
$$\frac{\operatorname{dist}(\xi_1^n, \partial\Omega_{\delta_n})}{\rho_n} < c_0.$$

Case 1. It is easy to see that in this case we actually have
$$\frac{\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} \to \infty, \quad j = 1, \dots, k.$$
Furthermore points $\xi_1^n, \dots, \xi_k^n$ are all interior to Ω_{δ_n} hance (7.6) is satisfied for all partial derivatives $\partial_{\xi_{lj}}$, $j \le k$. Define
$$\tilde\varphi_m(x_1, \dots, x_m) = \varphi_m(\xi_1 + \rho_n x_1, \dots, \xi_1 + \rho_n x_m).$$
We have for all $l = 1, 2, j = 1, \dots, k$
$$\partial_{x_{lj}} \tilde\varphi_m(x) = \rho_n \partial_{\xi_{lj}} \varphi_m(\xi_1^n + x\rho_n).$$
Then at $\tilde x = (\tilde x_1, \dots, \tilde x_k, 0, \dots, 0)$ we have
$$\partial_{x_{lj}} \tilde\varphi_m(\tilde x) = 0.$$
On the other hand, using (7.5) and letting $\rho_n \to 0$, we get
$$\lim_{n\to\infty} \rho_n \partial_{\xi_{lj}} \varphi_m(\xi_1^n + x\rho_n) = -4 \sum_{i \ne j, i \le k} \partial_{x_{lj}} \log \frac{1}{|\tilde x_j - \tilde x_i|} = 0.$$
Since this last equality is true for any $j \le k, l = 1, 2$ we arrive at a contradiction with Lemma 7.2 which proves impossibility of the Case 1 above.

It remains to consider:

Case 2. In this case there exists a constant C such that
$$\frac{\operatorname{dist}(\xi_j^n, \partial\Omega_{\delta_n})}{\rho_n} \le C, \quad j = 1, \dots, k.$$
If there points ξ_j^n are all interior to Ω_{δ_n} then after scaling with ρ_n we argue as in Case 1 above to reach a contradiction with Lemma 7.3.

Therefore, if Case 2 is to hold, we assume that for certain $j = j^*$ we have
$$\operatorname{dist}(\xi_{j*}^n, \partial\Omega_{\delta_n}) = 0.$$
Assume first that there exists a constant C such that $\delta_n \le C\rho_n$. Consider the following sum (summation here is taken with respect to all $i \ne j$)
$$s_n = \sum_{i \ne j} G(\xi_j^n, \xi_i^n)$$
The leading part, as $n \to \infty$, of s_n comes just from the points that become close as $n \to 0$. We can isolate groups of those points according to the asymptotic form of their mutual distances. For example we can define:
$$\rho_n^1 = \inf_{i \ne j, i, j > k} |\xi_j^n - \xi_i^n|,$$

and consider those points whose mutual distances are $O(\rho_n^1)$, and so on. For each group of those points (also those with indices higher than k) the argument given above in the Case 1 applies. This means that not only those points become close to one another but also that their distance to the boundary $\partial\Omega_{\delta_n}$ is comparable with their mutual distance. Applying the asymptotic formula for the Green's function we see that

$$s_n = O(1), \quad \text{as } n \to \infty. \tag{7.9}$$

On the other hand we have

$$\sum_j H(\xi_j^n, \xi_j^n) \le H(\xi_{j^*}^n, \xi_{j^*}^n) + C \le -4\log\frac{1}{|\xi_{j^*}^n - \bar{\xi}_{j^*}^n|} + C.$$

Since $|\xi_{j^*}^n - \bar{\xi}_{j^*}^n| \le 2\delta_n$ (because $\xi_{j^*}^n \in \partial\Omega_{\delta_n}$) we have that

$$\sum_j H(\xi_j^n, \xi_j^n) \to -\infty, \quad as\ n \to \infty,$$

which together with (7.9) contradicts the fact that $\varphi_m(\xi^n)$ is bounded uniformly in n.

Finally assume that $\rho_n = o(\delta_n)$. In this case after scaling with ρ_n around $\xi_{j^*}^n$ and arguing similarly as in the Case 1 we get a contradiction with Lemma 7.2 since those points ξ_j^n that are on $\partial\Omega_{\delta_n}$, after passing to the limit, give rise to points that lie on the same straight line. Thus Case 2 cannot hold.

In summary we reached now a contradiction with the assumptions of the Lemma. The proof is complete. □

Remark 7.1. If Ω has d holes, namely d bounded components for its complement, then at least $d+1$ solutions u_ε with

$$\lim_{\varepsilon\to 0} \varepsilon^2 \int_\Omega k(x) e^{u_\varepsilon} = 8\pi$$

exist. We observe that $\varphi_1(\xi) = H(\xi)$. Since $H(\xi)$ approaches $+\infty$ as ξ approaches $\partial\Omega$, Ljusternik-Schnirelman theory yields that H has at least $cat(\Omega) = d+1$ critical points with critical levels characterized through $d+1$ min-max quantities. The same property is thus inherited for $F(\xi)$ and the fact is thus established.

7.3. **Proof of Theorem 3.** We want to apply Theorem 2 in this situation. Observe that the function φ_m now becomes

$$\varphi_m(y_1, \dots, y_m) = \sum_{j=1}^m [\alpha G(y_j, P) - H(y_j, y_j)] - \sum_{i\neq j} G(y_i, y_j),$$

so we want to investigate the existence of a nontrivial critical value for this function. We proceed similarly as in the proof of Theorem 1, except that now the domain $\mathcal{D}$ is chosen as $\mathcal{D} = \Omega_\delta^m$ where now

$$\Omega_\delta = \{ y \in \Omega \ / \ \text{dist}\,(y, \partial\Omega) > \delta,\ |y - P| > \delta\ \}$$

where δ is a small positive number. We consider the same min-max quantity $\mathcal{C}$ as in (7.1), except that now the curve γ is chosen to enclose the point P. We need to get that $\mathcal{C}$ is uniformly bounded below independently of δ. Assume $P = 0$. Arguing exactly as in Theorem 1, this fact follows if we find that for $y \in \mathcal{D}$ with the property

$$\frac{y_j}{|y_j|} = e^{\frac{2i\pi j}{m}}, \quad j = 1, \dots, m$$

we have that

$$A \equiv \sum_{j=1}^{m} \alpha G(y_j, 0) - \sum_{l \neq j} G(y_l, y_j) \geq -K$$

Then $|y_l - y_j| \geq C|y_l|$ for all $j \neq l$. Clearly we have

$$A = \alpha \sum_{l=1}^{m} \log \frac{1}{|y_l|} - \sum_{j \neq l} \log \frac{1}{|y_l - y_j|} + O(1)$$

where $O(1)$ is a quantity uniformly bounded independently of δ. We have, for a fixed l, that

$$\alpha \log \frac{1}{|y_l|} - \sum_{j \neq l} \log \frac{1}{|y_l - y_j|} \geq \alpha \log \frac{1}{|y_l|} - (m-1) \log \frac{1}{|y_l|} + O(1).$$

Since $\alpha > m - 1$ by assumption, the above quantity is uniformly bounded below, hence the value $\mathcal{C}$ is bounded below independently of δ, as desired.

To prove the assertion of tangential derivatives being non-zero over the boundary of $\mathcal{D}$ for uniformly bounded values of φ_m, provided that δ is small enough, we argue by contradiction in similar terms as those in Theorem 1. The situation we end up with now, with exactly same proof, is that all points ξ_i that are close to one another, say by $\rho(\delta) \to 0$, as $\delta \to 0$, must be at $O(\rho)$ distance from $\partial\Omega_\delta$. Scaling arguments as in the proof of Theorem 1 work as long as those points remain interior to Ω_δ or $\rho = o(\delta)$. Once this is excluded, we only need to consider the case $C\rho > \delta$. But this is impossible as well, since on the one hand for points ξ_i that are at $O(\rho)$ from $\partial\Omega$, $G(\xi_i, \xi_j)$ remains uniformly bounded, while for those close either to P (or to $\partial\Omega_\delta$), their contribution to the total value of φ_m is at least of unbounded of order $[\alpha - (m-1)] \log \frac{1}{\rho}$ (or $O(\log \frac{1}{\rho})$ due to the asymptotic behavior of H); in any case this is in contradiction with the fact that φ_m is uniformly bounded. Hence Theorem 2 becomes applicable to this situation and the proof is concluded. □

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