Technical Notes and Correspondence

Singular LQ Control, Optimal PD Controller and Inadmissible Initial Conditions

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***Abstract*—We consider a classical control problem: the infinite horizon *singular* LQ problem, i.e., some inputs are unpenalized in the quadratic performance index. In this case, it is known that the slow dynamics is constrained to be in a proper subspace of the state-space, with the optimal input for the slow dynamics implementable by feedback. In this technical note we show that both the fast dynamics and the slow dynamics can be implemented by a feedback controller. Moreover, we show that the feedback controller cannot be a static feedback controller but can be PD, i.e., $\texttt{proportional} + \texttt{differentiate exactly once}$, in the state. We show that the closed loop system is a singular descriptor state space system and we also characterize the conditions on the system/performance index for existence of *inadmissible* initial conditions, i.e., initial conditions that cause impulsive solutions. There are no inadmissible initial conditions in the controlled system if and only if in the strictly proper transfer matrix from the unpenalized inputs to the penalized states, there exists at least one maximal minor of relative degree equal to the number of unpenalized inputs. In addition to the above, we prove solvability of the infinite horizon singular LQ problem under milder assumptions than in the literature.**

***Index Terms*—Cheap control, impulsive solutions, PD controller, singular LQ problem, singular system, transmission zeros, zeros at infinity.**

I. Introduction

Singular linear quadratic problem has been extensively studied for the past few decades. For example see [5], [10], [12] and [4]. The need to consider distributional inputs instead of smooth functions, and hence the concern about an ill-defined quadratic form, has been well addressed in the literature. Notable amongst the several techniques used are the cases of [5], [12] and [4], where only the regular part of the input is shown to be implementable by feedback. In this technical note we obtain the optimal controller explicitly as a dynamic state feedback PD controller. Further, we investigate conditions on the system and the performance index under which the controlled system has inadmissible initial conditions, i.e., initial conditions for which the states contain an impulse or its derivatives. We prove that there do not exist inadmissible initial conditions if and only if the strictly proper transfer matrix from the unpenalized inputs to the penalized states has a full column rank first moment about $s=\infty$. It follows that under this condition the optimal control input may contain impulses δ but never δ'. While it is well-known that restriction of the slow[1] dynamics to a proper subspace of the state space [9] can be achieved using PD control, and while it is also known that the optimal input for the slow dynamics in the singular LQ problem can be obtained by feedback, our technical note combines these two: a PD feedback controller for both fast and slow dynamics. In the context of the structure at infinity and impulsive solutions, related work can be found in [7], [8] and [1].

The technical note is organized as follows. Section II has a formulation of the problem, assumptions applicable in this technical note and the main results of this technical note. Section III has some behavioral preliminaries, the regular LQ optimal control problem and a result proving that, in the singular case too, the performance index with cross-terms can be dealt with by modification using a preliminary state feedback to eliminate the cross-terms. Section IV deals with the design of an optimal controller for the singular LQ optimal control problem. In Section V we prove the main result of this technical note, Theorem 2.5: this section contains other results essential for this proof. Section VI summarizes the conclusions in this technical note.

II. Problem Formulation and Main Results

Consider the following continuous, linear, time-invariant, multi-input state-space system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$. Over these system trajectories, the LQ problem defines the performance index[2] as

$$J(x_0; u) = \int_0^\infty (x^T Qx + u^T Ru)dt \tag{2}$$

where $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $R \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ and x_0 the initial condition of the states, $x(0^-)$.

A. Assumptions and Their Justifications

This subsection lists the conditions on the problem data that we assume in this technical note. Also justification of these assumptions are discussed below.

Assumptions 2.1: For the system $\dot{x} = Ax + Bu$ and the matrices Q and R in $J(x_0; u)$ we assume[3] the following:

A1) (A, B) is controllable and (Q, A) is observable.

A2) $Q \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $R \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ are symmetric and positive semidefinite and have the following structure:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{3}$$

where the sizes of I in Q and R depend on their respective ranks. Assume R has rank $m - \nu$.

A3) $\begin{bmatrix} B \\ R \end{bmatrix}$ is full column rank.

A4) Partition $B = [B_1 \quad B_2]$ with $B_2 \in \mathbb{R}^{\mathtt{n}\times\nu}$ and define $T(s) := Q(sI - A)^{-1}B_2 \in \mathbb{R}^{\mathtt{n}\times\nu}(s)$, i.e., $T(s)$ is the transfer

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[1]In the situation where there is a jump/impulsive part in the states only at $t = 0$, "slow dynamics" refers to the evolution on the time scale from 0^+ to ∞: see [2] for example.

[2]In the case of singular R too, cross-terms in the performance index can be eliminated using a preliminary state feedback: Section III, Lemma 3.1 deduces rank conditions on the modified Q and R.

[3]In Remark 2.2 below, we summarize which of our assumptions are milder than in the literature and also which other of our assumptions can be relaxed.

matrix from the unpenalized inputs to the penalized states. Assume the rational transfer matrix $T(s)$ is left-invertible[4]. Further, assume $T(s)$ has no transmission[5] zeros on the imaginary axis.

Assumption A1 helps in the theorem statements and proofs and Remark 2.3 below describes the situation for stabilizability and detectability. Assumption A2 is without loss of generality: this structure of Q and R can be obtained using suitable choice of bases in the input and state spaces. Assumption A3 rules out inputs that are both unpenalized ($\ker R$) and redundant ($\ker B$); of course, Assumption A3 follows if B has full column rank. A3, together with left-invertibility assumed in A4, ensures that the closed loop system is autonomous[6]. The assumption of absence of imaginary axis transmission zeros of $T(s)$ is elaborated in the following remark.

Remark 2.2: Note that [12] assumes that there are no imaginary axis transmission zeros in the transfer matrix from *all* inputs to the penalized states. It is reasonable that imaginary axis transmission zeros in the transfer matrix from *only unpenalized* inputs to the penalized states should be ruled out: this can be seen by the familiar method of studying cheap control using ρ for those diagonal entries in R corresponding to unpenalized inputs, and letting $\rho \to 0$. Consider the transfer matrix $T(s)$ from the unpenalized inputs to the penalized states and let $\rho \to 0$, but $\rho \neq 0$. It is known that standard root-locus arguments help to conclude that as $\rho \to 0$, the closed loop poles converge to the zeros of $T(s)$ if the zeros are in the LHP and to the reflection about the imaginary axis for those zeros in the RHP. For imaginary axis zeros of $T(s)$, the optimal input fails to be stabilizing in the limit $\rho \to 0$. Example 5.3 demonstrates both the necessity of Assumption A4 and also how imaginary axis transmission zeros from *penalized* inputs do not contribute to closed loop poles. Further, slight modification in that example shows how transmission zeros in the open *right* half complex plane do not hinder existence of an optimal stabilizing input.

Remark 2.3: For the regular LQ problem, it is known that (A, B)-stabilizability, instead of controllability, is necessary and sufficient for solvability. While stabilizability is clearly necessary, further investigation is required to prove sufficiency for the results in this technical note. Our proofs use the notion of an 'image representation' (see Section III below), which exists for just controllable systems. For uncontrollable systems (A, B), the image representation considers the controllable part, say (A_c, B_c), and the procedure adopted in our technical note yields a PD feedback controller that is optimal for the controllable part. The question arises whether solving for the feedback controller using just (A_c, B_c) and using this controller with the original system results in the same closed loop pole locations as that by solving using (A, B). This is known to be true for the regular LQ case and remains to be shown for the singular LQ case. (For the regular LQ case, the closed loop *state transition matrices* in these two approaches can be different due to the uncontrollable subspace not being A-invariant in general, see [6, Example 4.1]).

The assumption of detectability of (Q, A) is enough to ensure a stabilizing optimal input. In the absence of imaginary axis (Q, A)-unobservable poles, but presence of right half plane (Q, A)-unobservable poles, the procedure described in this technical note yields the solution to the LQ problem *with internal stability*. For the regular case, using Hamiltonian matrix arguments for example, it is well-known that imaginary axis (Q, A)-unobservable poles end up as closed loop poles. This is inevitable in the singular case too, see proof of Lemma 4.4-2 below.

The space of inputs over which $J(x_0; u)$ is to be minimized is central to the singular LQ problem. The set of all infinitely differentiable functions with domain $\mathbb{R}$ and co-domain $\mathbb{R}^{\mathtt{w}}$ is represented by $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and its subspace of compactly supported functions by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. The distributions on the test functions $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ form the set $\mathfrak{D}'(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Consider $\mathfrak{D}'_0 := \{u \in \mathfrak{D}'(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) | u = \sum_{i=0}^{N} a_i \delta^{(i)}, a_i \in \mathbb{R}^{\mathtt{m}}\}$, where δ is the Dirac delta distribution and δ^i denotes its ith distributional derivative. The set $\mathfrak{D}'_0$ consists of all distributions supported at zero. We define the following two input spaces:

1- Regular inputs: $u \in \mathfrak{L}_2^{\text{loc}}(\mathbb{R}^+, \mathbb{R}^{\mathtt{m}})$, i.e., u satisfies $\int_0^T \|u\|^2 dt < \infty$ for all $T \in \mathbb{R}^+$.

2- Distributional inputs:

$$\mathcal{U}_{\text{dist}} := \left\{ u \in \mathfrak{D}'(\mathbb{R}, \mathbb{R}^{\mathtt{m}}) \mid u = u^{\mathtt{imp}} + u^{\mathtt{reg}}, \right.$$
$$\left. u^{\mathtt{reg}} \in \mathfrak{L}_2^{\text{loc}}(\mathbb{R}^+, \mathbb{R}^{\mathtt{m}}) \ \& \ u^{\mathtt{imp}} \in \mathfrak{D}'_0 \right\}.$$

It is known (see [5], [12], for example) that considering u from the subset $\mathcal{U}_{\text{dist}}$ of all distributions is enough for the singular LQ problem. As mentioned in the introduction, allowing inputs, and hence states, to be impulsive causes $J(x_0; u)$ to possibly be ill-defined[7]. Exactly like in [12], set $J(x_0; u) = \infty$ whenever any impulses or its-derivatives in either x or u result in an ill-defined $x^T Q x + u^T R u$. Define the optimum value for the performance index as

$$J^*(x_0) := \inf_{u \in \mathcal{U}_{\text{dist}}} J(x_0; u)$$

with x satisfying the system (1). The problems that we address in this technical note are as follows:

Problem 2.4: For the singular LQ optimal control problem

(a) Find conditions on A, B, Q, R such that $J^*(x_0)$ can be *achieved* by some input u^* for each $x_0 \in \mathbb{R}^{\mathtt{n}}$.
(b) Find conditions when there exists a *feedback* controller that implements u^*. If one exists, can it be static in state?
(c) For the closed loop system, find conditions on A, B, Q and R under which there exist inadmissible initial conditions.

B. Main Results

Problem 2.4(a) has been studied extensively in the literature; for example in [5] and [12], it is discussed that $J^*(x_0)$ can be achieved. In addition to some assumptions being milder in Assumption 2.1 as elaborated in Remark 2.2 above, Theorem 2.5, the main result of this technical note, infers the existence of a *PD feedback controller* and characterizes conditions for nonexistence of inadmissible initial conditions for the closed loop system.

Theorem 2.5: Consider the singular LQ optimal control problem satisfying Assumption 2.1. The following statements hold.

1) The optimal control u^* that achieves $J^*(x_0) = J(x_0; u^*)$ can be implemented as a feedback controller.
2) The feedback controller cannot be static state feedback. Further, there exists a feedback controller that is PD in the state: proportional and first-derivative terms in the state. More precisely, there

[4]It is assumed in [5] that the transfer matrix from *all* inputs to the penalized states is left-invertible. We prove solvability of the singular LQ optimal control under milder assumptions. See also Footnote 6.

[5]Among the many variants of transmission-zero definitions, we define the transmission zeros of a rational matrix $T(s)$ as the roots of the product of all the numerator polynomials in the Smith-McMillan form of $T(s)$. When $T(s)$ is a left-invertible rational matrix, the transmission zeros are precisely those complex numbers λ such that the complex-valued signal $u(t) = ae^{\lambda t}$, for some nonzero complex vector a, and zero initial conditions, results in the output $y(t) \equiv 0$.

[6]Closed loop system autonomy is guaranteed by A3-A4 which ensure that the Euler Lagrange equations $Z(\mathrm{d}/\mathrm{d}t)\ell = 0$ (defined below within Lemma 4.4) are independent and equal the number of inputs. This is linked with optimal input's uniqueness in a distributional sense: see [5, page 373].

[7]Since we are seeking the infimum of a non-negative quantity, and under (A, B) controllability/stabilizability assumptions, the unboundedness of the interval of integration is not the concern of ill-definedness of $J(x_0; u)$.

exist $F_p, F_d \in \mathbb{R}^{m \times n}$ such that $u = F_p x + F_d \dot{x}$ is an optimal feedback controller.
3) The closed loop system is a singular descriptor system of the form $E\dot{x} = A_F x$ with $\operatorname{rank} E = n - \dim(\ker R)$.
4) There are no inadmissible initial conditions in the closed loop system if and only if $\dim(QB \ker R) = \dim(\ker R)$.

Section V contains the proof of the various statements in the above theorem. Remark 2.6 elaborates on the system theoretic significance of statement 4 above.

Remark 2.6: Consider the strictly proper transfer matrix $T(s)$ from Assumption 2.1-A4. We have $\lim_{s\to\infty} sT(s) = QB_2$, i.e., QB_2 contains the relative degree one terms in $T(s)$, equivalently QB_2 is the first moment about $s = \infty$ of the transfer matrix T from unpenalized inputs to penalized states. Thus the condition $\dim(QB \ker R) = \dim(\ker R)$ is equivalent to $T(s)$ containing a maximal minor with relative degree ν, the number of unpenalized inputs.

III. Preliminaries: The Behavioral Approach and Regular LQ Problem

In this section we review some behavioral preliminaries and the regular LQ problem from the behavioral perspective.

A. Behavioral Approach

A behavior is a set of allowed trajectories in the variables of interest: w. In this technical note, w is (x, u). More precisely, a linear differential behavior $\mathfrak{B}$ is defined as

$$\mathfrak{B} := \left\{ w \in \mathfrak{L}_2^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt w}) \mid P\left(\frac{d}{dt}\right) w = 0 \right\},$$

where $P \in \mathbb{R}^{\bullet \times {\tt w}}[s]$. The differential equations are assumed to be satisfied in the *distributional sense*, i.e., weak sense. In [13] it has been shown that behavioral controllability of system (1) is equivalent to state space controllability of the pair (A, B): the matrix $P(s) = [sI - A \ \ -B]$ has full row rank for s equal to every complex number λ. A polynomial matrix $P(s)$ satisfying such a rank property is called *left prime*. An important characterizing property of controllable behaviors is the existence of an *image representation*, i.e., there exists a matrix $M(s) \in \mathbb{R}^{{\tt w} \times \bullet}[s]$ such that

$$\mathfrak{B} = \left\{ w \mid w = M(\frac{d}{dt})\ell \text{ for some } \ell \in \mathfrak{L}_2^{\rm loc}(\mathbb{R}, \mathbb{R}^{\bullet}) \right\} \qquad (4)$$

where ℓ is called *latent variable*. If ℓ is observable[8] from w, then the image representation is called observable. An image representation can be assumed to be observable without loss of generality. Since we have an observable image representation there also exists a matrix $F(s)$, such that $F(s)M(s) = I$. This matrix $F(s)$ is called a *left inverse* of the matrix $M(s)$.

Quadratic Differential forms (QDFs) are quadratic functionals in the system variables and a finite number of their derivatives[9]. The QDF Q_Φ induced by $\Phi(\zeta, \eta) \in \mathbb{R}^{{\tt w} \times {\tt w}}[\zeta, \eta]$ is a map $Q_\Phi : \mathfrak{L}_2^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt w}) \to \mathfrak{L}_2^{\rm loc}(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{i,k} \left(\frac{d^i w}{dt^i}\right)^T \Phi_{ik} \left(\frac{d^k w}{dt^k}\right).$$

[8]For a behavior $\mathfrak{B} \subseteq \mathfrak{L}_2^{\rm loc}(\mathbb{R}, \mathbb{R}^{{\tt w}+\ell})$ with variables w and ℓ, the variable ℓ is said to be observable from w in $\mathfrak{B}$ if whenever $(w, \ell_1), (w, \ell_2) \in \mathfrak{B}$ then $\ell_1 = \ell_2$.

[9]A concern when dealing with $\mathfrak{L}_2^{\rm loc}$ trajectories instead of $\mathfrak{C}^\infty$ trajectories is that the quadratic differential form may not be well-defined: this is central to this technical note since the quadratic performance index $J(x_0; u)$ deals with distributional inputs/states. See text before Problem 2.4.

We often require the one variable polynomial matrix $Z(s) := \Phi(-s, s)$. See [13] for a detailed treatment of QDFs.

The next lemma states that for the singular LQ problem, the performance index involving the cross terms can be brought to the form in (2) by a preliminary static state feedback.

Lemma 3.1: Consider the LQ problem: $\dot{x} = Ax + Bu$ with $J(x(0); u) = \int_0^\infty (x^T Q x + 2x^T S u + u^T R u) dt$, where $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$, $S \in \mathbb{R}^{n \times m}$, with Q and R symmetric. Assume $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is positive semi-definite and has rank p. Then, the following hold.

1) There exists $F \in \mathbb{R}^{m \times n}$ such that $S + F^T R = 0$.
2) The preliminary static state feedback law $u = Fx + v$ results in the following system and performance index: $\dot{z} = (A + BF)z + Bv$; $z(0) = x(0)$ and $\tilde{J}(z(0); v) = \int_0^\infty (z^T \tilde{Q} z + v^T R v) dt$ with $\tilde{Q} := Q - F^T R F$.
3) $\operatorname{rank} \tilde{Q} = p - \operatorname{rank} R$.

Statement 1 above crucially utilizes the positive semi-definiteness assumption, after which the proof is exactly the same as that for the *regular* case with cross terms. For shortage of space, the proof is referred to [6, Lemma 2.2.1]. See also Corollary 2.2.2 there.

B. Regular LQ Problem and Solution

In this subsection we review the regular LQ problem. The performance index defined in (2) is re-written as a QDF induced by $\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$, with x and u as the system variables

$$J(x_0; u) = \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt.$$

This QDF is written in terms of a latent variable using the image representation matrix $M \in \mathbb{R}^{n \times m}[s]$ and we have a QDF induced by $\Phi \in \mathbb{R}^{m \times m}[\zeta, \eta]$ as follows. $J(x_0 : u) = \int_0^\infty Q_\Phi(\ell)\, dt$ where $\Phi(\zeta, \eta) = M(\zeta)^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} M(\eta)$. For a polynomial matrix $M(s)$, $M(s)^\sim := M(-s)^T$. We use this in the following proposition. [11, Proposition 1] characterizes the stationary and stable trajectories of the performance functional in (2) and gives the optimal trajectories for the regular LQ control problem. See also [3] for use of behavioral approach for solving the LQ problem.

Proposition 3.2: ([11, Proposition 1]) Consider the system with behavior $\mathfrak{B}$ described by $\mathfrak{B} = \{w \in \mathfrak{L}_2^{\rm loc} | \ w = M({\rm d}/{\rm d}t)\ell$ for some $\ell \in \mathfrak{L}_2^{\rm loc}(\mathbb{R}, \mathbb{R}^{\tt w})\}$. Let $Z(s) := M(s)^\sim \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} M(s)$, $R > 0$ and $Q > 0$. Then there exists a square polynomial matrix H, with H *Hurwitz*[10] such that $Z(s) = H(s)^\sim H(s)$. The set of optimal trajectories for the performance functional J in (2) denoted by $\mathfrak{B}^*$ is given by $\mathfrak{B}^* = \{w^* = M({\rm d}/{\rm d}t)\ell^* | H({\rm d}/{\rm d}t)\ell^* = 0\}$.

IV. Singular LQ Control: Preliminary Results/Proofs

The main focus of this section is to construct an optimal controller for the singular LQ problem. A careful analysis of the degree of the Hurwitz factor $H(s)$ from the above proposition is crucial for the results of this technical note.

Definition 4.1: Consider a polynomial matrix $T(s) \in \mathbb{R}^{n \times m}[s]$ with rank r and $r = \min(n, m)$. The slow-McMillan-degree of $T(s)$ denoted by ${\rm Mc}_{\rm slow}\, T(s)$ is defined as the maximum among the degrees of all $r \times r$ minors of $T(s)$. The *degree* of a polynomial vector $t(s)$ is defined as the highest degree among the entries in that vector. Let $c_1, \ldots, c_m$ denote the degrees of the column vectors of $T(s)$. We say the matrix $T(s)$ is *column reduced* if ${\rm Mc}_{\rm slow}\,(T(s)) = \sum_{i=1}^m c_i$.

[10]A square nonsingular polynomial matrix $H(s)$ is called Hurwitz if each of the roots of $\det H$ has real part negative.

Generally $\mathrm{Mc}_{\mathrm{slow}}(T(s)) \leqslant \sum_{i=1}^{m} c_i$. The next lemma (proof in [6, Lemma 3.1.4]) quantifies $\mathrm{Mc}_{\mathrm{slow}}$ of submatrices under column reduced assumption for the state space case.

Lemma 4.2: Consider a controllable pair (A, B) with $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$ and define $P(s) = [\, sI - A \quad -B \,]$. Let $M(s) \in \mathbb{R}^{(\mathtt{m}+\mathtt{n})\times\mathtt{m}}[s]$ be a column reduced, maximal right annihilator of $P(s)$. Let $M(s)$ be partitioned as $\begin{bmatrix} M_x \\ M_u \end{bmatrix}$ with $M_x \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[s]$ and $M_u \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[s]$. Then,

1) $\mathrm{Mc}_{\mathrm{slow}}(M_x) = n - m$ and $\mathrm{Mc}_{\mathrm{slow}}(M_u) = n$.
2) Both M_x and M_u are column reduced.

The following lemma plays a key role in relating the fall in the determinantal degree of a Hurwitz factor with relative degree one assumptions on the transfer matrix. See [6, Lemma 3.1.5] for the proof.

Lemma 4.3: Consider a polynomial matrix $\begin{bmatrix} N \\ M \end{bmatrix}$, $N \in \mathbb{R}^{\ell\times\mathtt{m}}[s]$, $M \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[s]$ with $\deg\det(M) = n$, $n \geqslant m$ and NM^{-1} strictly proper. Pick k satisfying $0 \leqslant k \leqslant m-1$ and let S be the first $m-k$ rows of M. Construct the matrix $U = \begin{bmatrix} N \\ S \end{bmatrix}$. Define $D := \lim_{s\to\infty} sNM^{-1}$ and partition D into $[\, D_1 \quad D_2 \,]$ with $D_2 \in \mathbb{R}^{\ell\times\mathtt{k}}$. Then $\deg\det U^{\sim}U = 2(n-k)$ if and only if $\mathrm{rank}\ D_2 = k$.

Proposition 3.2 characterizes the optimal trajectories for the regular LQ optimal control problem. We extend this result to the singular case in the following key lemma.

Lemma 4.4: Consider the system $\dot{x} = Ax + Bu$, $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, whose behavior $\mathfrak{B}$ with variables $w = (x, u) \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{(\mathtt{m}+\mathtt{n})})$ is $\mathfrak{B} = \{w \mid w = M(\mathrm{d}/\mathrm{d}t)\ell$ for some $\ell \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})\}$, where M is a column reduced, maximal right annihilator[11] of $[\, sI - A \quad -B \,]$. Let $Z(s) := M(s)^{\sim} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} M(s)$. Suppose the conditions in Assumption 2.1 are satisfied. Then

1) $\det Z(s) \not\equiv 0$
2) There exists a Hurwitz factor $H(s)$ such that $H(-s)^T H(s) = Z(s)$. The set of optimal trajectories for the performance functional J in the singular LQ problem denoted by $\mathfrak{B}^*$ is given by

$$\mathfrak{B}^* = \left\{ w^* = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell^* \mid H(\frac{\mathrm{d}}{\mathrm{d}t})\ell^* = 0 \right\}. \tag{5}$$

3) The following are equivalent.
   a) $\deg\det H(s) = n - \dim(\ker R)$
   b) $\dim(QB\ker R) = \dim(\ker R)$.

Proof:

1) Consider $Z(s)$ as follows with M partitioned in accordance with the sizes of Q and R:

$$Z(s) = [\, M_x^{\sim} \quad M_u^{\sim} \,] \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} M_x \\ M_u \end{bmatrix}. \tag{6}$$

From the structure of Q and R in Assumption A2, we get $Z(s)$ as follows:

$$Z(s) = M_1^{\sim} M_1 \text{ where } M_1 := \begin{bmatrix} M_{x_1} \\ M_{u_1} \end{bmatrix}. \tag{7}$$

M_{x_1} and M_{u_1} are the first few rows of M_x and M_u depending on the rank of Q and R respectively. If $\det Z(s) \not\equiv 0$, then M_1 has full column rank as a polynomial matrix. Hence $M_1 M_u^{-1}$ (M_u is nonsingular, refer Lemma 4.2) also has full column rank. We have

$$M_1 M_u^{-1} = \left[\frac{M_{x_1} M_u^{-1}}{I_{\mathtt{m}-\nu} \ \mid\ 0} \right]. \tag{8}$$

This implies that the last ν columns of $M_{x1} M_u^{-1}$ has full column rank, i.e the transfer matrix from the inputs not penalized by R to the states penalized by Q has full column rank. Assumption 2.1-A4 ensures this. Hence we have $\det Z(s) \not\equiv 0$.

2) Since the image representation (4) is observable, the variable ℓ is observable from (x, u). From the structure of Q, (Q, A) observable implies $(x_1, u) = 0 \Rightarrow x_2 = 0$. Hence in fact ℓ is observable from (x_1, u). Utilize this to rule out imaginary axis roots of $Z(s)$ as follows. By symmetry and positive semidefiniteness of Q and R, we have $Z(-s) = Z(s)^T$, i.e $Z(s)$ is para-Hermitian, and further, $Z(i\omega) \geqslant 0$ for each $\omega \in \mathbb{R}$. A Hurwitz factorization exists if, in fact, $Z(i\omega)$ is positive definite for each $\omega \in \mathbb{R}$. Assume to the contrary, i.e., there exist $\omega_0 \in \mathbb{R}$ and $v \in \mathbb{R}^m \backslash \{0\}$ such that $Z(i\omega_0)v = 0$. Therefore from (7), $M_1(i\omega_0)v = 0$, i.e., $\bar{x}_1 := M_{x_1}(i\omega_0)v = 0$ and $\bar{u}_1 := M_{u_1}(i\omega)v = 0$. The observability of ℓ from (x_1, u) implies $\bar{u}_2 := M_{u_2}(i\omega_0)v \neq 0$. Existence of a transmission zero at $i\omega_0$ in the transfer matrix from u_2 to x_1 can be concluded (see Footnote 5) by using $\bar{u}_2 \neq 0$ to define the nonzero input $u(t) := \bar{u}_2 e^{i\omega_0}$ which results in solutions $x_1(t)$ and $x_2(t)$ identically zero. This contradicts Assumption 2.1 A4 and therefore $Z(i\omega)$ is positive definite for each $\omega \in \mathbb{R}$. Hence $Z(s)$ admits a Hurwitz factorization, say $H(s)$. Finally, the infimum[12] for the integral of a supply rate is attained by the set $\mathfrak{B}^*$ as defined in (5): this follows from [13, Theorem 5.7, Equation (A.20)]].

3) This proof is straightforward by utilizing Lemma 4.3 above, and hence is referred to [6, Lemma 3.1.6]. □

The following remark is about the relation between the degree of controlled system and $H(s)$.

Remark 4.5: It is easy to prove (see [6, Lemma 3.1.8]) that the optimal controller given by $H(d/dt)\ell = 0$ in Lemma 4.4, when attached to the system $P(\mathrm{d}/\mathrm{d}t)w = 0$ with $P(s) := [sI–A \quad -B]$, results in the dimension of the slow subspace of the controlled system to be precisely $\deg(\det H)$. This can be shown by using an observable image representation $w = M(\mathrm{d}/\mathrm{d}t)\ell$ of the system, then a left-inverse $F(s)$ of $M(s)$, and noting that the polynomial matrix $[P^T F^T]$ is unimodular.

A. Optimal Control Law

Lemma 4.4 gave the optimal trajectories for the singular LQ optimal control problem. In this section we give an explicit expression for the optimal control law in terms of the state and the input. We begin with the following proposition.

Proposition 4.6: ([13, Section 2]) Consider a controllable system in the input/state representation, $\dot{x} = Ax + Bu$. Let an observable image representation for the above system be $\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} M_x \\ M_u \end{bmatrix} \ell$. Assume $v = L(\mathrm{d}/\mathrm{d}t)\ell$. Then, LM_u^{-1} is proper if and only if there exists constant matrices V_x and V_u such that $v = V_x x + V_u u$. Further, LM_u^{-1} is strictly proper if and only if V_u is zero, while LM_u^{-1} is biproper if and only if V_u is invertible.

We use this proposition for the following theorem which says that the optimal controller for the singular LQ problem can be written as a static equation in terms of the states and input.

Theorem 4.7: Consider the singular LQ optimal control Problem 2.4 and suppose conditions in Assumptions 2.1 are satisfied. Let $H(\mathrm{d}/\mathrm{d}t)\ell = 0$, with $H(s) \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[s]$, $\ell \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ denote an optimal controller for this problem as in Lemma 4.4. Then the following statements are true.

1) For every $H(s)$ there exist constant matrices $V_x \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$ and $V_u \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}$ such that $H(\mathrm{d}/\mathrm{d}t)\ell = V_x x + V_u u$ and $\ker V_u = \ker R$.
2) There exists an optimal controller $Vx + Ru = 0$, $V \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}$.

[11]For the purpose of this technical note, given a full row rank matrix $P(s) \in \mathbb{R}^{\mathtt{n}\times(\mathtt{m}+\mathtt{n})}[s]$, we define a maximal right annihilator, $M(s) \in \mathbb{R}^{(\mathtt{m}+\mathtt{n})\times\mathtt{m}}[s]$ as one such that $PM = 0$ and $M(\lambda)$ is full column rank for all $\lambda \in \mathbb{C}$.

[12]This is over the open interval $(0, \infty)$, i.e the regular slow subspace.

Proof: It can be easily verified that HM_u^{-1} is proper. Using Proposition 4.6 we write the controller as $H(\mathrm{d}/\mathrm{d}t)\ell = V_x x + V_u u$. Since x and u depend on the latent variable ℓ, we get $H(\mathrm{d}/\mathrm{d}t)\ell = V_x M_x(\mathrm{d}/\mathrm{d}t)\ell + V_u M_u(\mathrm{d}/\mathrm{d}t)\ell$. Noting that the above equation is true for every $\ell \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ we have the polynomial identity $H(s) - V_x M_x(s) - V_u M_u(s) = 0$. Post-multiplying by $M_u(s)^{-1}$ we get $H(s)M_u(s)^{-1} = V_x M_x(s) M_u(s)^{-1} + V_u$. Hence $\lim_{s\to\infty} HM_u(s)^{-1} = V_u$ as $M_x(s)M_u(s)^{-1}$ is strictly proper. From the expression of $Z(s)$ along with Hurwitz factorization we can verify that $\lim_{s\to\infty} (HM_u^{-1})^{\sim} HM_u^{-1} = R$. Therefore $V_u^T V_u = R$. This proves $\ker V_u = \ker R$.

2) We showed that an optimal controller is given by $V_x x + V_u u = 0$ with $R = V_u^T V_u$. Premultiply V_x by a suitable nonsingular constant matrix to be able to choose $V_u = R$, and define the modified V_x as V. □

V. Proof of Main Results

In Theorem 4.7 of previous section the optimal controller was not in feedback form. We proceed to show that the optimal input can be implemented as a feedback from the states: statement 1 of of our main result; Theorem 2.5.

Proof of Statement 1 of Theorem 2.5: We show that the optimal controller can be implemented as a feedback from states. From Theorem 4.7, an optimal controller is given by $Vx + Ru = 0$. The equations of the controlled system are

$$\begin{bmatrix} sI - A & -B \\ V & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0; \text{ where } V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}. \tag{9}$$

Here $V_1 \in \mathbb{R}^{(\mathtt{m}-\nu)\times\mathtt{n}}$, $V_2 \in \mathbb{R}^{\nu\times\mathtt{n}}$ and $B = [\,B_1 \quad B_2\,]$ where $B_2 \in \mathbb{R}^{\mathtt{n}\times\nu}$. Assumptions 2.1-A2 and A3 implies that B_2 is full column rank. Hence there exists a nonsingular submatrix of B_2, say $B_\nu \in \mathbb{R}^{\nu\times\nu}$. In (9) above, add the ν rows which form the submatrix B_ν to the last ν rows and we get

$$\left[\begin{array}{c|c|c} sI - A & -B_1 & -B_2 \\ \hline V_1 & I_{m-\nu} & 0 \\ \hline W(s) & -C_1 & -B_\nu \end{array}\right] \begin{bmatrix} x \\ u \end{bmatrix} = 0.$$

Here $C_1 \in \mathbb{R}^{\nu\times(\mathtt{m}-\nu)}$ is a submatrix of B_1 and $W(s) \in \mathbb{R}^{\nu\times\mathtt{n}}[s]$ is a polynomial matrix with degree of each entry at most one, i.e., $W(s) = W_0 + W_1 s$ for some constant matrices W_0 and W_1. The control law can be rewritten as

$$u = \begin{bmatrix} -I & 0 \\ C_2 & B_\nu^{-1} \end{bmatrix} \left\{ \begin{bmatrix} V_1 \\ W_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ W_1 \end{bmatrix} \dot{x} \right\} \tag{10}$$

with $C_2 := B_\nu^{-1} C_1$. Hence we have the control law as a feedback law, from the states of the system. This proves statement 1 of Theorem 2.5. □

Proof of Statement 2 of Theorem 2.5: We first show that the optimal input if implemented as a feedback from states cannot be a static controller. Assume the optimal controller can be implemented as a static state feedback controller. Then the controller is of the form $V_x x + V_u u = 0$ with V_u invertible. Hence from Proposition 4.6 we have HM_u^{-1} is biproper: this contradicts $\deg \det H(s) < n$ which we showed in Lemma 4.4. Hence V_u is not invertible and the optimal input cannot be implemented using a *static* state feedback law. Equation (10) gives the optimal input in feedback form, expanding which gives

$$u = F_p x + F_d \dot{x} \tag{11}$$

where $F_p = \begin{bmatrix} -I & 0 \\ C_2 & B_\nu^{-1} \end{bmatrix} \begin{bmatrix} V_1 \\ W_0 \end{bmatrix}$ and $F_d = \begin{bmatrix} 0 \\ B_\nu^{-1} W_1 \end{bmatrix}$. Hence we conclude that if the optimal controller is implemented as a state feedback controller, then there exists a PD controller. This proves statement 2 of Theorem 2.5. □

The following algorithm summarizes the key steps involved.

Algorithm 5.1: Input: A, B, Q, R satisfying Assumption 2.1 and output: optimal PD controller.

1) Define $P(s) := [\,sI - A \quad -B\,]$ and find an observable image representation.
2) Construct the matrix $Z(s)$ and obtain a Hurwitz factorization $H(s)$ to define the optimal controller, $H(\mathrm{d}/\mathrm{d}t)\ell = 0$.
3) Rewrite the controller equation in terms of x and u using the left-inverse $F(s)$ (Remark 4.5) and use elementary row operations to obtain the controller as a *static* law.
4) Finally perform operations described before (10) and (11) to get the controller as a PD controller.

A. Closed Loop Singular Descriptor System

We have seen that the optimal controller can be implemented as a PD feedback controller. We proceed to show that the closed loop system is a singular descriptor system.

Proof of Statement 3 of Theorem 2.5: Substitute the feedback law of the optimal control in the system equation and simplify: this gives the closed loop as $E\dot{x} = A_F x$, where E is $(I - BF_d)$ and A_F is $(A + BF_p)$. Use a permutation matrix $P \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that PB ensures that the nonsingular submatrix B_ν (see proof of statement 1 of Theorem 2.5) occurs in the first ν rows of B. Therefore the matrix PEP^T evaluates to $\begin{bmatrix} 0_\nu & 0 \\ * & I_{\mathtt{n}-\nu} \end{bmatrix}$ and hence rank of E is $n - \nu$. This proves statement 3 of Theorem 2.5. □

B. Inadmissible Initial Conditions

We obtained the controlled system to be an autonomous singular descriptor system, thus possibly causing the state response to have impulses. The following proposition gives conditions to rule out inadmissible initial conditions.

Proposition 5.2: ([7]) Consider the singular state space system $E\dot{x} = A_F x$ with $\det\,(sE - A_F) \not\equiv 0$. The response of the system for arbitrary initial conditions has no impulses if and only if $\deg \det\,(sE - A_F) = \operatorname{rank} E$.

Proof of Statement 4 of Theorem 2.5: In Remark 4.5 we noted that the degree of the controlled system equals the determinantal degree of $H(s)$, i.e., $\deg \det\,(sE - A_F) = \deg \det H(s)$. Therefore from Proposition 5.2 there are no impulsive solutions if and only if $\operatorname{rank} E = \deg \det H(s)$. From statement 3 of Theorem 2.5 the rank of E is $n - \dim(\ker R)$. Using Lemma 4.4 we conclude that there are no impulsive solutions for arbitrary initial conditions if and only if $\dim(QB \ker R) = \dim(\ker R)$. This completes the proof of statement 4 of Theorem 2.5. □

C. Examples

We consider an example bringing out where assumptions in Assumption 2.1 are milder than in the literature and also where they are essential.

Example 5.3: Consider the state space system $\dot{x} = Ax + Bu$ and the resulting transfer function $G(s)$ from u to x

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\; B = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},\; G(s) = \begin{bmatrix} \frac{s-2}{s^2-2s+2} & \frac{-(s-2)}{s^2-2s+2} \\ \frac{s}{s^2-2s+2} & \frac{-s}{s^2-2s+2} \end{bmatrix}.$$

Case 1: Consider penalty defined by $x^TQx + u^TRu := x_1^2 + u_1^2$. The transfer function from u_2 to x_1 does not have a transmission zero on the imaginary axis, but in RHP: at 2. Using Algorithm 5.1 above, we solve this control problem to obtain the closed loop pole at -2: a reflection about the imaginary axis of the open loop RHP zero. This example does not meet the stronger assumptions in the literature as the transfer matrix from u to x_1 is not left-invertible.

We briefly compare the solution to this problem using the limiting approach. For $\epsilon = 10^{-7}$, with $x^TQx + u^TR_\epsilon u := x_1^2 + u_1^2 + \epsilon u_2^2$, we obtain closed loop poles at -2 and approximately $-\sqrt{10^7}$, with the static feedback law $u = \begin{bmatrix} 3.16\times10^{-4} & -6.33\times10^{-4} \\ -3161.3 & 6327.6 \end{bmatrix} x$. The ARE solution is close to being singular but with bounded entries, see also [6, Chapter 5].

Case 2: For the same system, redefine the penalty as $x^TQx + u^TRu := x_2^2 + u_1^2$. The transfer function from u_2 to x_2 has a transmission zero on the imaginary axis, and does not satisfy Assumption 2.1-A4. A limiting approach, i.e., of considering penalty as $x_2^2 + u_1^2 + \rho u_2^2$, reveals that the closed loop pole approaches 0 as $\rho \to 0$. This means u_ρ^* fails to be stabilizing in the limit $\rho \to 0$.

VI. Conclusion

We showed that an optimal input which minimizes the cost function $J(x_0; u)$ could be implemented as a feedback from the states of the system. Further, we showed that the feedback controller cannot be a static state feedback law, but there exists a PD control law. With the obtained PD feedback controller, the closed loop system is singular descriptor system, and hence the slow dynamics is now restricted to a lower dimensional subspace of the state space. For the states in the closed loop system to be impulse-free for arbitrary initial conditions, we formulated and proved a necessary and sufficient condition: the strictly proper transfer matrix from the unpenalized inputs to the penalized states satisfies the property that there exists at least one maximal minor such that the relative degree of this minor is equal to the number of unpenalized inputs. The assumptions in this technical note, like left-invertibility and absence of transmission zeros on the imaginary axis, were milder than those in the literature. We considered an example to illustrate the need for these assumptions.

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Plug-and-Play Decentralized Model Predictive Control for Linear Systems

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***Abstract*—In this technical note, we consider a linear system structured into physically coupled subsystems and propose a decentralized control scheme capable to guarantee asymptotic stability and satisfaction of constraints on system inputs and states. The design procedure is totally decentralized, since the synthesis of a local controller uses only information on a subsystem and its neighbors, i.e. subsystems coupled to it. We show how to automatize the design of local controllers so that it can be carried out in parallel by smart actuators equipped with computational resources and capable to exchange information with neighboring subsystems. In particular, local controllers exploit tube-based Model Predictive Control (MPC) in order to guarantee robustness with respect to physical coupling among subsystems. Finally, an application of the proposed control design procedure to frequency control in power networks is presented.**

***Index Terms*—Decentralized control, decentralized synthesis, large-scale systems, model predictive control, plug-and-play control, robust control.**

I. Introduction

Decentralized regulators have been studied since the 70's as a viable solution to the control of large-scale system composed by physically coupled subsystems [1]. The problem of guaranteeing stability and suitable performance levels for decentralized control system has been addressed in the 70's and 80's mainly for unconstrained systems [1], [2]. Distributed (also known as overlapping decentralized) control, where controllers can exchange pieces of information through a communication network, has also been studied (see, e.g., [3], and references therein).

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