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SINGULAR PERTURBATIONS FOR DIFFERENCE EQUATIONS

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ABSTRACT. This paper discusses singular perturbations for second-order linear difference equations with a small parameter. It is found that there exhibits boundary layer behavior for the two-point boundary-value problem as well as for the final-value problem, but not for the initial-value problem. In contrast to problems for differential equations, a boundary layer exists only at the right end point. By virtue of a stretching transformation, a formal procedure is developed for treating such problems, and the justification of this procedure is established through a discrete maximum principle.

1. **Introduction.** We consider here the singular perturbation for the second-order linear difference equations. More precisely we consider the boundary value problem (P_ϵ) defined by

$$(1.1)\quad \epsilon Y_{k+2} + a_k Y_{k+1} + b_k Y_k = 0 \quad (k = 0, 1, 2, \cdots, N-2),$$

and

$$(1.2)\qquad Y_0 = \alpha, \qquad Y_N = \beta.$$

Here $\epsilon > 0$ is a small parameter; a_k and b_k are non-zero discrete functions which are assumed to be bounded; α and β are given constants. We shall study the asymptotic behavior of the solution when ϵ approaches zero. By analogy with the ordinary differential equations [2], the problem (P_ϵ) is said to be *singular* in the sense that the *degenerate problem* (or the *reduced problem*) (P_0),

$$(1.3)\qquad a_k Y_{k+1} + b_k Y_k = 0 \qquad (k = 0, 1, 2, \cdots, N-2)$$

together with (1.2) has no solution.

Our goal here is to develop a procedure for treating such problems. In particular, we are interested in the boundary layer behavior as well as the feasibility of applying the method of inner and outer expansions to singular perturbation problems for the difference equations. We will see that in contrast to the differential equation problem,

$$(1.4)\qquad \begin{aligned} &\epsilon y'' + ay' + by = 0 \qquad (x_0 < x < x_N) \\ &y(x_0) = \alpha \quad \text{and} \quad y(x_N) = \beta \end{aligned}$$

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there is a "boundary layer" only at the right end point, regardless of the sign of the coefficient a_k in (1.1) (this discrepancy will be explained in § 5). In fact, the initial value problem for the difference equation (1.1) together with $Y_0 = \alpha$ and $Y_1 = \beta$ is unstable with no boundary layer, while the final value problem, (1.1), $Y_N = \beta$ and $Y_{N-1} = \alpha$, is quite stable with a boundary layer on the right. For both problem (P_ϵ) and final value problem there are maximum principles available which we use to establish our boundary layer results.

In order to gain some insight of the singular behavior of the solution of (P_ϵ), we first consider the special case of (1.1) where a_k and b_k are non-zero constants. In this case the problem (P_ϵ) may be solved explicitly, and hence one can easily obtain the asymptotic development of the solution. Based on this simple illustrative example, we then establish a general procedure for treating the case of non-constant coefficients. It is our hope that the present investigation may shed some light on the development of asymptotic analysis as well as on the numerical treatment for the nonlinear difference equations, whose solutions in general cannot be found explicitly.

2. **A Simple Example.** We begin with the simple problem:

$$\begin{aligned} &\epsilon Y_{k+2} + aY_{k+1} + bY_k = 0 \qquad (k = 0, 1, 2, \cdots, N-2) \\ &\qquad\qquad Y_0 = \alpha \quad \text{and} \quad Y_N = \beta, \end{aligned} \tag{2.1}$$

where $a \neq 0$, $b \neq 0$ are constants. The solution in this case can be obtained explicitly as

$$Y_k = \{(\alpha m_2{}^N - \beta)m_1{}^k + (\beta - \alpha m_1{}^N)m_2{}^k\}(m_2{}^N - m_1{}^N)^{-1},$$

where m_1 and m_2 are the roots of the characteristic equation $\epsilon m^2 + am + b = 0$. To obtain the asymptotic expansion of Y_k, one has to examine the behavior of the m_i's for small ϵ:

$$m_1 = \frac{-a}{2\epsilon}\{1 + \sqrt{1 - 4\epsilon b/a^2}\} = -\frac{1}{\epsilon}\left\{a - \frac{\epsilon b}{a} - \frac{\epsilon^2 b^2}{2a^3} + O(\epsilon^3)\right\},$$

$$m_2 = \frac{-a}{2\epsilon}\{1 - \sqrt{1 - 4\epsilon b/a^2}\} = -\left\{\frac{b}{a} + \frac{b^2}{2a^3}\epsilon + O(\epsilon^2)\right\}.$$

From these expansions, it is clear that one should write the solution of (2.1) in the form:

$$\begin{aligned} Y_k = &\{(\alpha - \beta m_1{}^{-N})m_2{}^k + (\beta - \alpha m_2{}^N)m_1{}^{-(N-k)}\} \\ &\{1 - [m_2/m_1]^N\}^{-1}. \end{aligned} \tag{2.2}$$

Note that from the definition of m_1 and m_2, $m_1{}^{-1}$ and $m_2 m_1{}^{-1}$ are both of $O(\epsilon)$. Hence for small ϵ, we have

$$Y_k \sim (\beta - \alpha m_2^N) m_1^{-(N-k)} + \alpha m_2^k = \alpha \left[-\frac{b}{a} \right]^k + O(\epsilon) \quad \text{for} \quad k \neq N. \tag{2.3}$$

The degenerate equation for (2.1) reads

$$a z_{k+1} + b z_k = 0$$

which has the solution in the form:

$$z_k = c(-b/a)^k$$

for arbitrary constant c. From (2.3) it indicates that if one chooses $c = \alpha$, that is, if z_k is the solution of the reduced boundary value problem:

$$a z_{k+1} + b z_k = 0 \quad \text{and} \quad z_0 = Y_0, \tag{2.4}$$

then it is clear that $Y_k \to z_k$ as $\epsilon \to 0^+$. But the convergence is *non-uniform* at $k = N$. This non-uniform convergence is due to the fact that the term $m_1^{-(N-k)}$ in (2.3) is small only for $k \neq N$. By analogy with singular perturbation problems for differential equations [2], we may refer to the point $k = N$ as the *boundary-layer point.* Our goal now is to develop a procedure for obtaining a correction term say w_k, so that $z_k + w_k$ can be used to approximate the exact solution Y_k at the boundary-layer point as well as other points.

Remark. It is interesting to point out that the boundary-layer point is always at $k = N$ whether $a > 0$ or $a < 0$, while in the case of differential equation problem (1.4), the boundary layer occurring in the neighborhood of different boundary points depends generally upon the sign of a. However, this is not really surprising, as we will see the explanations in § 5.

Before we discuss how to obtain the correction term, let us for the moment study the initial value problem:

$$\begin{gathered} \epsilon Y_{k+2} + a Y_{k+1} + b Y_k = 0 \qquad (k = 0, 1, \cdots) \\ Y_0 = \alpha \quad \text{and} \quad Y_1 = \beta. \end{gathered} \tag{2.5}$$

The solution is obviously

$$Y_k = \{(\beta - \alpha m_2) m_1^{k-1} + (\alpha - \beta m_1^{-1}) m_2^k\} \{1 - m_2/m_1\}^{-1}. \tag{2.6}$$

Using our expansions of m_1 and m_2 for small ϵ, (2.6) becomes

$$\begin{aligned} Y_k \sim \{(\beta + \alpha b/a)(-a/\epsilon)^{k-1} \\ + (\alpha + \beta\epsilon/a)(-b/a)^k\} \{1 - \epsilon b/a^2\}^{-1}. \end{aligned} \tag{2.7}$$

For $k > 1$ this solution grows without bound as $\epsilon \to 0^+$ (unless α, β, a, b are just right). On the other hand, suppose we consider the "final value" problem,

$$\text{(2.8)} \qquad \begin{gathered} \epsilon Y_{k+2} + aY_{k+1} + bY_k = 0 \qquad (k = N-2, N-3, \cdots) \\ Y_N = \beta \quad \text{and} \quad Y_{N-1} = \alpha. \end{gathered}$$

Now the solution is

$$\text{(2.9)} \qquad \begin{aligned} Y_k = {} & \{(\beta - \alpha m_2)m_1{}^{k-N} \\ & + (\alpha - \beta m_1{}^{-1})m_2{}^{k-N+1}\}\{1 - m_2/m_1\}^{-1} \\ & \cong \{(\beta + \alpha b/a)(-a/\epsilon)^{k-N} - (\epsilon b/a^2)(a\alpha/\epsilon \\ & + \beta)(-b/a)^{k-N}\}\{1 - \epsilon b/a^2\}^{-1}. \end{aligned}$$

This exhibits a nice boundary layer behavior as $k \to N$, and is well behaved for all other k as $\epsilon \to 0^+$. The reduced problem in this case is then defined by

$$\text{(2.10)} \qquad az_{k+1} + bz_k = 0 \quad \text{and} \quad z_{N-1} = Y_{N-1}.$$

Thus our difference equation in (2.1) has a backwards stability, but not a forwards stability in some sense like a parabolic equation. However, as we shall prove, there is an elliptic type of maximum principle available by means of which we can show that the boundary value-broundary layer approach is valid.

3. **Formal Procedure.** Now let us continue our discussion on the correction term for the problem (2.1). Examining the difference between the solution Y_k in (2.3) and the reduced solution z_k in (2.4), one expects that a reasonable correction term w_k should be deduced from the term

$$\begin{aligned} (\beta - \alpha m_2{}^N)m_1{}^{-(N-k)} & \sim [\beta - \alpha(-b/a)^N](-a/\epsilon)^{-(N-k)} \\ & = \epsilon^{(N-k)}(Y_N - z_N)(-a)^{-(N-k)}. \end{aligned}$$

Let us set $\tilde{w}_k = (Y_N - z_N)(-a)^{-(N-k)}$. Then one observes that $\tilde{w}_k$ is the solution of the problem

$$\text{(3.1)} \qquad \begin{gathered} \tilde{w}_{k+2} + a\tilde{w}_{k+1} = 0 \qquad (k = N-2, N-1, \cdots, 0) \\ \tilde{w}_N = Y_N - z_N. \end{gathered}$$

A simple comparison of (2.1) and (3.1) shows that if we introduce $\tilde{Y}_k$,

$$\text{(3.2)} \qquad Y_k = \epsilon^{N-k}\tilde{Y}_k,$$

as the *stretched variable,* then (2.1) can be rewritten in the form:

$$\tilde{Y}_{k+2} + a\tilde{Y}_{k+1} + \epsilon b\tilde{Y}_k = 0$$
$$\text{(3.3)} \qquad \tilde{Y}_0 = \epsilon^{-N}\alpha, \qquad \tilde{Y}_N = \beta = Y_N.$$

Note that the difference equation in (3.3) assumes a similar form as the "boundary layer equation" for the differential equation in (1.4). It is easy to see that $\tilde{w}_k$ is the solution of the corresponding reduced problem of (3.3) with β replaced by $\beta - z_N$; namely the problem defined by (3.1). Thus we arrive at the following formal procedure for constructing the solution of (2.1). That is, one can approximate the solution of (2.1) by

$$Y_k \cong z_k + \epsilon^{(N-k)}\tilde{w}_k.$$

More precisely we have proved the following theorem:

THEOREM 1. *Let Y_k be the solution of the boundary value problem* (2.1). *Then Y_k can be represented in the form*

$$\text{(3.4)} \qquad Y_k = z_k + \epsilon^{N-k}\tilde{w}_k + 0(\epsilon) \quad as \quad \epsilon \to 0^+,$$

uniformly for $k = 0, 1, 2, \cdots, N$, where z_k and $\tilde{w}_k$ are solutions of BVP's defined by (2.4) *and* (3.1) *respectively.*

REMARK. The procedure for the construction of outer and inner expansions should now be straightforward.

4. **Main Theorem.** We return now to the problem (P_ϵ). Clearly we wish to establish a similar result (3.4) for (P_ϵ). In this case z_k and $\tilde{w}_k$ are the unique solutions to the boundary value problems:

$$\text{(4.1)} \qquad a_k z_{k+1} + b_k z_k = 0 \qquad (k = 0, 1, \cdots, N-1); \quad z_0 = \alpha.$$

and

$$\tilde{w}_{k+2} + a_k\tilde{w}_{k+1} = 0 \qquad (k = N-2, N-3, \cdots, 0);$$
$$\text{(4.2)} \qquad \tilde{w}_N = (\beta - z_N).$$

In order to show that

$$\text{(4.3)} \qquad Y_k = z_k + \epsilon^{N-k}\tilde{w}_k + O(\epsilon),$$

one needs some a priori estimates for the solution of the problem (P_ϵ). Before we pursue the idea, let us first consider the error term R_k defined by

$$\text{(4.4)} \qquad R_k \equiv Y_k - (z_k + \epsilon^{N-k}\tilde{w}_k).$$

A simple computation then shows that R_k satisfies the following problem:

$$\begin{aligned} &\epsilon R_{k+2} + a_k R_{k+1} + b_k R_k = f_k(\epsilon) \ (k = 0, 1, 2, \cdots, N-2) \\ &R_0 = -\epsilon^N \tilde{w}_0, \quad R_N = 0, \end{aligned} \tag{4.5}$$

where $f_k(\epsilon) \equiv -\{\epsilon z_{k+2} + \epsilon^{N-k} b_k \tilde{w}_k\}$. Clearly from the definition of z_k and $\tilde{w}_k$ it follows that $f_k(\epsilon) = O(\epsilon)$ uniformly for $k = 0, 1, 2, \cdots, N-2$. Hence a suitable a priori estimate of R_k in terms of $f_k(\epsilon)$, R_0 and R_N is desirable in order to establish the result (4.3) in this case. For this purpose, we need the following lemma, a sort of *maximum principle* (in [1], a discrete maximum principle is available for the positive type difference operator):

LEMMA 1. *Let U_k be the solution of the following problem:*

$$\begin{aligned} &\epsilon U_{k+2} + a_k U_{k+1} + b_k U_k = f_k \qquad (k = 0, 1, 2, \cdots, N-2) \\ &U_0 \text{ and } U_N \text{ are prescribed,} \end{aligned} \tag{4.6}$$

where the coefficients a_k, b_k are bounded non-zero discrete functions. Then for ϵ sufficiently small there exists a constant $\gamma > 0$ such that

$$\|U\|_N \leqq \gamma \{|U_0| + \epsilon |U_N| + \|f\|_{N-2}\}, \tag{4.7}$$

where the notation $\|g\|_n \equiv \max_{k=0,1,2,\cdots,n} \{|g_k|\}$ for any discrete function g.

The proof of the lemma, an induction argument, is tedious but straightforward. We omit the proof in order to save space.

From Lemma 1, one can easily show that $R_k = O(\epsilon)$ uniformly for $k = 0, 1, 2, \cdots, N$. We thus prove the main theorem:

THEOREM 2. *Under the assumptions of Lemma 1 on a_k and b_k, the solution Y_k of (P_ϵ) has the representation form:*

$$Y_k = z_k + \epsilon^{N-k} \tilde{w}_k + O(\epsilon) \quad \text{as} \quad \epsilon \to 0^+ \tag{4.8}$$

uniformly for $k = 0, 1, 2, \cdots, N$. The functions z_k and $\tilde{w}_k$ are defined by (4.1) *and* (4.2) *respectively.*

Similar theorems are available for the final value problem, the details of which are left to the reader.

5. **Concluding Remarks.** The results in this paper may shed some light on the difference between various finite difference schemes. That the boundary layer for (1.1) is always at the *right* end point has a simple explanation. The fact is that the difference equation (1.1) is *not always* a discrete analog of the differential equation (1.4). If the centered difference approximation is used for the second derivative and the backward difference approximation is used for the first derivative in (1.4), then indeed a difference equation of the form (1.1) can

be obtained (assuming that the mesh size is much bigger than ϵ). If instead, the forward difference approximation is used for the first derivative in (1.4), one will obtain a difference equation of the form:

$$b_k Y_{k+2} + a_k Y_{k+1} + \epsilon Y_k = 0 \tag{5.1}$$

with suitable discrete functions a_k and b_k. Then it can easily be deduced from our results that the difference equation (5.1) always has a boundary layer at the *left* end point. In [1], it has been observed that (1.1) is a good approximation to (1.1) for small ϵ when a in (1.4) is *negative,* and that (5.1) is the correct approximation to (1.4) when a is *positive.* In this sense it shows that the results for the difference equations (1.1) and (5.1) are indeed consistent with those for the differential equation (1.4).

Finally it would be worthwhile to note that although one may also approximate both the first and the second derivatives in (1.4) by using centered difference, yet the corresponding difference equation obtained does not fall into the general type of equations considered here; moreover, the corresponding difference equation oscillates rapidly for small ϵ, and hence, it is not a good approximation to (1.4).

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