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Singular perturbation problems in the calculus of variations

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Singular Perturbation Problems in the Calculus of Variations (*).

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1. – Introduction.

In this paper we study the following singular perturbation problem in the Calculus of Variations; given an integral functional of the form

$$F(u) = \int_{\Omega} f(x, u, Du, D^2u, \ldots, D^m u)\, dx\,;$$

determine the asymptotic behaviour (as $\varepsilon \to 0^+$) of the infima of the functionals

$$F_\varepsilon(u) = \int_{\Omega} f(x, u, \varepsilon Du, \varepsilon^2 D^2 u, \ldots, \varepsilon^m D^m u)\, dx$$

(here $D^k u$ denotes the vector $(D^k u)_{|\alpha| = k}$ of all k-th order partial derivatives of u).

By means of the Γ-convergence theory we prove that, under suitable assumptions on the integrand f, there exists a convex integrand $\psi\colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that for every $\varphi \in L^q(\Omega)$

$$\lim_{\varepsilon \to 0^+} \inf \left\{ F_\varepsilon(u) + \int_{\Omega} \varphi u\, dx\colon u \in W^{m,r}(\Omega) \cap L^p(\Omega) \right\}$$

$$= \lim_{\varepsilon \to 0^+} \inf \left\{ F_\varepsilon(u) + \int_{\Omega} \varphi u\, dx\colon u \in W_0^{m,r}(\Omega) \cap L^p(\Omega) \right\}$$

$$= \min \left\{ \int_{\Omega} [\psi(x, u) + \varphi u]\, dx\colon u \in L^p(\Omega) \right\},$$

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where the exponents r and p are related to the behaviour of the integrand f and $1/p + 1/q = 1$. Moreover a formula for the function ψ is given.

There is an intimate relationship between this kind of problems and some singular perturbation problems in Optimal Control Theory. Consider for example a control problem with a cost functional of the form

$$J(u, v) = \int_\Omega [N|v(x)|^2 + |u(x) - b(x)|^p [dx$$

and with a singularity perturbed state equation of the form

$$(E_\varepsilon) \begin{cases} \varepsilon^2 \Delta u + g(u) = v \\ u \in H_0^1(\Omega) . \end{cases}$$

($N > 0$, $b \in L^p(\Omega)$, and $g: \mathbb{R} \to \mathbb{R}$ are given; u and v are respectively the state variable and the control variable). Problems of this kind have been studied by J. L. Lions in his courses at the Collège de France in 1981-82 and 1982-83, and by A. Bensoussan [2], A. Haraux and F. Murat [11], [12], and V. Komornik [13]. By substituting $v = \varepsilon^2 \Delta u + g(u)$ in the cost functional, the study of the asymptotic behaviour (as $\varepsilon \to 0^+$) of

$$\inf \{J(u, v) \colon (u, v) \text{ is a solution of } (E_\varepsilon)\}$$

is reduced to the study of

$$\inf \left\{ \int_\Omega [N|\varepsilon^2 \Delta u + g(u)|^2 + |u - b(x)|^p] \, dx \colon u \in H^2_{\mathrm{loc}}(\Omega) \cap H_0^1(\Omega) \right\},$$

which is the problem considered in Section 5.

Some of the results proved in this paper were announced without proof in [4].

We wish to thank Prof. E. De Giorgi for many helpful discussions on this subject.

2. – Γ-convergence.

In this section we collect some known results of Γ-convergence theory that are used in the sequel. For a general exposition of this subject we refer to [6] and [7].

Let Λ, X be two topological spaces (we consider Λ as a space of parameters, in general $\Lambda = \overline{\mathbb{N}} = \mathbb{N} \cup \{+\infty\}$ or $\Lambda = \mathbb{R}$); let $\Lambda_0 \subseteq \Lambda$ and $X_0 \subseteq X$

with X_0 dense in X; for every $\lambda \in \Lambda_0$ let F_λ be a function from X_0 into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$; let $\lambda_0 \in \Lambda$, $x \in X$ with $\lambda_0 \in \overline{\Lambda}_0$; following [8] we define

$$\Gamma(\Lambda^-, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \sup_{U \in \mathfrak{J}(x)} \liminf_{\substack{\lambda \to \lambda_0 \\ \lambda \in \Lambda_0}} \inf_{y \in U \cap X_0} F_\lambda(y), \tag{2.1}$$

$$\Gamma(\Lambda^+, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \sup_{U \in \mathfrak{J}(x)} \limsup_{\substack{\lambda \to \lambda_0 \\ \lambda \in \Lambda_0}} \inf_{y \in U \cap X_0} F_\lambda(y), \tag{2.2}$$

where $\mathfrak{J}(x)$ denotes the family of all neighbourhoods of x in the space X. When the Γ-limits (2.1) and (2.2) coincide, their common value is indicated by

$$\Gamma(\Lambda, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y).$$

The main properties of Γ-limits are given by the following propositions, proved in [3] and [9].

PROPOSITION 2.1. *For every $x \in X$ define*

$$F^-(x) = \Gamma(\Lambda^-, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y)$$

$$F^+(x) = \Gamma(\Lambda^+, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y).$$

The functions $F^-: X \to \overline{\mathbb{R}}$ and $F^+: X \to \overline{\mathbb{R}}$ are lower semicontinuous on X.

PROPOSITION 2.2. *Suppose that X has a countable base for the open sets. For every sequence (F_h) of functions from X_0 into $\overline{\mathbb{R}}$, there exists a subsequence (F_{h_k}) and a function $F: X \to \overline{\mathbb{R}}$ such that*

$$F(x) = \Gamma(\overline{\mathbb{N}}, X^-) \lim_{\substack{k \to \infty \\ y \to x}} F_{h_k}(y)$$

for every $x \in X$.

PROPOSITION 2.3. *If $G: X \to \mathbb{R}$ is lower semicontinuous at the point $x \in X$, then*

$$\Gamma(\Lambda^-, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} [G + F_\lambda](y) \geqslant G(x) + \Gamma(\Lambda^-, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y)$$

$$\Gamma(\Lambda^+, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} [G + F_\lambda](y) \geqslant G(x) + \Gamma(\Lambda^+, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y);$$

if in addition G is continuous at the point x, then the above inequalities are equalities.

PROPOSITION 2.4. *Suppose that there exists $F\colon X \to \overline{\mathbb{R}}$ such that*

$$F(x) = \Gamma(\Lambda, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y)$$

for every $x \in X$. Assume further that the functions F_λ are equicoercive on X, i.e. for every $s \in \mathbb{R}$ there exists a compact subset K_s of X (independent of λ) such that $\{x \in X_0\colon F_\lambda(x) \leqslant s\} \subseteq K_s$ for every $\lambda \in \Lambda_0$.

Then we have

$$\min_X F = \lim_{\lambda \to \lambda_0} \left[\inf_{X_0} F_\lambda\right].$$

Moreover, if $(x_\lambda)_{\lambda \in \Lambda_0}$ is a family of elements of X_0 such that $\lim_{\lambda \to \lambda_0} \lambda_\lambda = x$ and $\lim_{\lambda \to \lambda_0} \left[F_\lambda(x_\lambda) - \inf_{X_0} F_\lambda\right] = 0$, then x is a minimum point of F in X.

Let $S_0(\lambda_0)$ be the set of all sequences in Λ_0 converging to λ_0 in Λ, and let $S(x)$ be the set of all sequences in X_0 converging to x; we define (the subscript seq stands for sequential)

$$\Gamma_{\text{seq}}(\Lambda^-, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \inf_{(\lambda_h) \in S_0(\lambda_0)} \inf_{(x_h) \in S(x)} \liminf_{h \to \infty} F_{\lambda_h}(x_h) \tag{2.3}$$

$$\Gamma_{\text{seq}}(\Lambda^+, X^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \sup_{(\lambda_h) \in S_0(\lambda_0)} \inf_{(x_h) \in S(x)} \limsup_{h \to \infty} F_{\lambda_h}(x_h). \tag{2.4}$$

REMARK 2.5. If the spaces Λ and X satisfy the first axiom of countability it is possible to prove (see [3]) that the Γ_{seq}-limits (2.3) and (2.4) coincide respectively with the Γ-limits (2.1) and (2.2).

REMARK 2.6. It is not difficult to see that in the case $\Lambda = \overline{\mathbb{N}}$, $\Lambda_0 = \mathbb{N}$, $\lambda_0 = \infty$, the Γ_{seq}-limits (2.3) and (2.4) of a sequence $(F_h)_{h \in \mathbb{N}}$ of functions reduce respectively to

$$\inf_{(x_h) \in S(x)} \liminf_{h \to \infty} F_h(x_h) \quad \text{and} \quad \inf_{(x_h) \in S(x)} \limsup_{\lambda \to \infty} F_h(x_h).$$

Suppose that X is a reflexive separable Banach space with dual X'. Let (x'_h) be a sequence dense in the unit ball of X'; we introduce the metric δ

on X defined by

$$\delta(x, y) = \sum_{h=1}^{\infty} 2^{-h} |\langle x'_h, x - y\rangle| .$$

It is known that the metric space (X, δ) is separable.

Let us denote by w the weak topology of X.

We shall use the following proposition proved in [1].

PROPOSITION 2.7. *Assume that X is a reflexive Banach space, that λ_0 has a countable neighbourhood base in Λ, and that there exist two constants $c_1, c_2 \in \mathbb{R}$, with $c_2 > 0$, such that*

$$F_\lambda(x) \geqslant c_1 + c_2 \|x\|$$

for every $\lambda \in \Lambda_0$, $x \in X_0$.

Then for every $x \in X$

$$\Gamma_{\text{seq}}(\Lambda^-, w^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \Gamma(\Lambda^-, w^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \Gamma(\Lambda^-, \delta^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y)$$

$$\Gamma_{\text{seq}}(\Lambda^+, w^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \Gamma(\Lambda^+, w^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = \Gamma(\Lambda^+, \delta^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) .$$

Using Proposition 2.3 and some general properties of Γ-limits (see [3], [8]) it is easy to obtain the following proposition.

PROPOSITION 2.8. *Under the hypotheses of Proposition 2.7, for every $x \in X$, $s \in \mathbb{R}$ the following conditions are equivalent:*

i) $\Gamma(\Lambda, w^-) \lim_{\substack{\lambda \to \lambda_0 \\ y \to x}} F_\lambda(y) = s$

ii) *for every sequence (λ_h) in Λ_0 converging to λ_0 in Λ there exists a subsequence (λ_{h_k}) such that*

$$\Gamma(\overline{\mathbf{N}}, w^-) \lim_{\substack{k \to \infty \\ y \to x}} F_{h_k}(y) = s .$$

3. – Statement of the result.

Let Ω be a bounded open subset of $\mathbb{R}^n$, let $m \geqslant 1$ be an integer, and let p, r be two real numbers with $p > 1$, $1 \leqslant r \leqslant p$.

We indicate by $d = d(n, m)$ the number of multi-indices $\alpha \in \mathbf{N}^n$ such that $1 \leqslant |\alpha| \leqslant m$, by $\mathcal{A}(\mathbb{R}^n)$ the family of all bounded open subsets of $\mathbb{R}^n$, and by $\mathcal{A} = \mathcal{A}(\Omega)$ the family of all open subsets of Ω.

For every $k = 1, 2, \ldots, m$ and every $u \in W^{m,r}_{\mathrm{loc}}(A)$, with $A \in \mathcal{A}(\mathbb{R}^n)$, we denote by $D^k u$ the vector $(D^\alpha u)_{|\alpha|=k}$ of all k-th order partial derivatives of u.

The integrands we shall consider are Borel functions $f: \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0, +\infty[$ which satisfy the following properties:

(3.1) *there exist* $c \geq 1$ *and* $a \in L^1(\Omega)$ *such that*

$$-a(x) + |s|^p \leq f(x, s, z) \leq a(x) + c[|s|^p + |z|^r]$$

for every $x \in \Omega$, $s \in \mathbb{R}$, $z \in \mathbb{R}^d$;

(3.2) *there exist* $a \in L^1(\Omega)$, *an increasing continuous function* $\sigma: [0, +\infty[\to [0, +\infty[$ *with* $\sigma(0) = 0$, *and a Borel function* $\omega: \Omega \times \mathbb{R}^n \to [0, +\infty[$ *with*

$$\lim_{y \to 0} \int_\Omega \omega(x, y)\, dx = \int_\Omega \omega(x, 0)\, dx = 0\,,$$

such that

$$|f(y, t, w) - f(x, s, z)| \leq \omega(x, y - x) + \sigma(|y - x| + |t - s| + |w - z|)(a(x) + f(x, s, z))$$

for every $x \in \Omega$, $s \in \mathbb{R}$, $z \in \mathbb{R}^d$;

(3.3) *there exists* $a \in L^1(\Omega)$, *a Borel function* $\gamma: \mathbb{R} \times \mathbb{R}^d \to [0, +\infty[$, *and a function* $\lambda: \mathcal{A}(\mathbb{R}^n) \times \mathcal{A}(\mathbb{R}^n) \to [0, +\infty[$ *such that*

(i) *for every* $x \in \Omega$, $s \in \mathbb{R}$, $z \in \mathbb{R}^d$

$$\gamma(s, z) \leq f(x, s, z) + |s|^p + a(x)$$

(ii) *for every pair* $A, A' \in \mathcal{A}(\mathbb{R}^n)$ *with* $A \subset\subset A'$ *and for every* $u \in W^{m,r}(A')$

$$\int_A \sum_{|\alpha| \leq m} |D^\alpha u|^r dx \leq \lambda(A, A') \int_{A'} \gamma(u, Du, D^2 u, \ldots, D^m u)\, dx$$

(iii) *for every pair* $A, A' \in \mathcal{A}(\mathbb{R}^n)$ *with* $A \subset\subset A'$

$$\limsup_{t \to +\infty} \lambda(tA, tA') < +\infty\,.$$

For every $\varepsilon > 0$ we consider the functional $F_\varepsilon(u, A)$ defined for every $A \in \mathcal{A}$ and for every $u \in W^{m,r}_{\mathrm{loc}}(A)$ by

$$F_\varepsilon(u, A) = \int_A f(x, u, \varepsilon Du, \varepsilon^2 D^2 u, \ldots, \varepsilon^m D^m u)\, dx\,. \tag{3.4}$$

It is possible to verify (see section 6) that hypotheses (3.1), (3.2), (3.3) are fulfilled, for example, by the functionals

$$F_\varepsilon(u, A) = \int_A \left[\left(\varepsilon|Du| + P_k(u) + a(x)\right)^2 + |u - b(x)|^{2k}\right] dx\,,$$

$$F_\varepsilon(u, A) = \int_A \left[|\varepsilon^2 \Delta u + P_k(u) + a(x)|^2 + |u - b(x)|^{2k}\right] dx\,,$$

$$F_\varepsilon(u, A) = \int_A \left[\varphi(x, u, \varepsilon Du, \varepsilon^2 D^2 u)|\varepsilon^2 \Delta u + P_k(u) + a(x)|^2 + |u - b(x)|^{2k}\right] dx\,,$$

where $k \geqslant 1$ is an integer, P_k is a polynomial of degree less than or equal to k, $a \in L^2(\Omega)$, $b \in L^{2k}(\Omega)$, and $\varphi\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is uniformly continuous and satisfies $0 < \inf \varphi \leqslant \sup \varphi < +\infty$.

Other examples of functionals verifying hypotheses (3.1), (3.2), (3.3) can be found in Section 5.

Define now for every $A \in \mathcal{A}$, $u \in L^p(A)$

$$T(u, A) = \begin{cases} 0 & \text{if } u \in W_0^{m,r}(A) \\ +\infty & \text{otherwise}. \end{cases} \tag{3.5}$$

Let us denote by $w - L^p(A)$ the weak topology of $L^p(A)$. The main result we prove in this paper is the following.

THEOREM 3.1. *Let $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0, +\infty[$ be a Borel function satisfying hypotheses* (3.1), (3.2), (3.3), *and let F_ε be the functionals defined by* (3.4). *Then there exists a Borel function $\psi\colon \Omega \times \mathbb{R} \to [0, +\infty[$ such that*

(i) *for every $A \in \mathcal{A}$, $u \in L^p(A)$, $w_0 \in W^{m,r}(A) \cap L^p(A)$*

$$\int_A \psi(x, u)\,dx = \Gamma(\mathbb{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \to 0^+ \\ v \to u}} F_\varepsilon(v, A)$$

$$= \Gamma(\mathbb{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \to 0^+ \\ v \to u}} [F_\varepsilon(v, A) + T(v - w_0, A)]\,;$$

(ii) *for every $x \in \Omega$ the function $s \to \psi(x, s)$ is convex on $\mathbb{R}$;*

(iii) *for every $(x, s) \in \Omega \times \mathbb{R}$*

$$f^-(x, s, 0) \leqslant \psi(x, s) \leqslant f^+(x, s, 0)$$

where $f^+(x, s, z)$ is the greatest function convex in s which is less than or equal to $f(x, s, z)$ and $f^-(x, s, z)$ is the greatest function convex in (s, z) which is less than or equal to $f(x, s, z)$.

Moreover the following representation formulae for ψ hold for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$\begin{aligned}\psi(x,s) &= \lim_{\varepsilon\to 0^+} \inf\Big\{F_\varepsilon(x,u)\colon u \in W^{m,r}(Y)\cap L^p(Y), \int_Y u\,dy = s\Big\}\\ &= \lim_{\varepsilon\to 0^+} \inf\Big\{F_\varepsilon(x,u)\colon u - s \in W_0^{m,r}(Y)\cap L^p(Y), \int_Y u\,dy = s\Big\}\\ &= \inf\Big\{F_\varepsilon(x,u)\colon \varepsilon>0,\ u - s \in W_0^{m,r}(Y)\cap L^p(Y), \int_Y u\,dy = s\Big\}\\ &= \inf\Big\{F_\varepsilon(x,u)\colon \varepsilon>0,\ u \in W_\#^{m,r}(Y)\cap L^p(Y), \int_Y u\,dy = s\Big\},\end{aligned}$$

where Y denotes the unit cube $]0,1[^n$, $W_\#^{m,r}(Y)$ denotes the space of all Y-periodic functions of $W_{\mathrm{loc}}^{m,r}(\mathbb{R}^n)$, and

$$F_\varepsilon(x,u) = \int_Y f(x, u(y), \varepsilon Du(y), \varepsilon^2 D^2u(y), \ldots, \varepsilon^m D^m u(y))\,dy\,.$$

COROLLARY 3.2. *Let $w_0 \in W^{m,r}(\Omega)\cap L^p(\Omega)$, let $W(w_0) = \{u \in L^p(\Omega)\colon u - w_0 \in W_0^{m,r}(\Omega)\}$, and let V be a set such that $W(w_0) \subseteq V \subseteq W_{\mathrm{loc}}^{m,r}(\Omega)\cap L^p(\Omega)$. Then we have*

$$\lim_{\varepsilon\to 0^+} \inf\Big\{F_\varepsilon(u,\Omega)\,dx + \int_\Omega gu\,dx\colon u \in V\Big\} = \min\Big\{\int_\Omega \psi(x,u)\,dx + \int_\Omega gu\,dx\colon u \in L^p(\Omega)\Big\} \tag{3.6}$$

for every $g \in L^q(\Omega)$ $(1/p + 1/q = 1)$.

PROOF. It follows from Theorem 3.1, Proposition 2.3 and Proposition 2.4 that

$$\begin{aligned}&\min\Big\{\int_\Omega \psi(x,u)\,dx + \int_\Omega gu\,dx\colon u \in L^p(\Omega)\Big\}\\ &\quad= \lim_{\varepsilon\to 0^+} \inf\Big\{F_\varepsilon(u,\Omega) + \int_\Omega gu\,dx\colon u \in W_{\mathrm{loc}}^{m,r}(\Omega)\cap L^p(\Omega)\Big\}\\ &\quad= \lim_{\varepsilon\to 0^+} \inf\Big\{F_\varepsilon(u,\Omega) + \int_\Omega gu\,dx + T(u-w_0,\Omega)\colon u \in W_{\mathrm{loc}}^{m,r}(\Omega)\cap L^p(\Omega)\Big\}\\ &\quad= \lim_{\varepsilon\to 0^+} \inf\Big\{F_\varepsilon(u,\Omega) + \int_\Omega gu\,dx\colon u \in W(w_0)\Big\}.\end{aligned}$$

Since $W(w_0) \subseteq V \subseteq W_{\mathrm{loc}}^{m,r}(\Omega)\cap L^p(\Omega)$ we obtain (3.6). ∎

4. – Proof of the result.

In this section we prove Theorem 3.1.

The function f and the functionals F_ε are supposed to satisfy the hypotheses of the theorem. In what follows we shall write briefly $f(x, u, \varepsilon^k D^k u)$ instead of $f(x, u, \varepsilon Du, \varepsilon^2 D^2 u, \ldots, \varepsilon^m D^m u)$. Let (ε_h) be a sequence in $]0, +\infty[$ converging to 0. For every $A \in \mathcal{A}$, $u \in L^p(A)$ set

$$F^+(u, A) = \Gamma(\overline{\mathbf{N}}^+, w - L^p(A)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_{\varepsilon_h}(u, A) .$$

LEMMA 4.1. *For every $A \in \mathcal{A}$, $u \in L^p(A)$ we have*

$$F^+(u, A) \leq \int_A f(x, u, 0)\, dx .$$

PROOF. Let $A \in \mathcal{A}$, $u \in L^p(A)$. Let ϱ be a non-negative function in $C_0^\infty(\mathbf{R}^n)$ such that $\int \varrho\, dx = 1$, let $\theta = 1/(n + m + 1)$, let $\varrho_h(x) = \varepsilon_h^{-n\theta} \varrho(\varepsilon_h^{-\theta} x)$, and let $u_h = \varrho_h * u$. We have

$$F_{\varepsilon_h}(u_h, A) = \int_A f(x, \varrho_h * u, \varepsilon_h^k D^k \varrho_h * u)\, dx .$$

It is easy to see that $(\varrho_h * u)_h$ converges to u in $L^p(A)$ and that $(\varepsilon_h^k D^k \varrho_h * u)_h$ converges to 0 in $L^p(A)$ (hence in $L^r(A)$) for $k = 1, 2, \ldots, m$. Since $f(x, s, z)$ is continuous in (s, z), inequalities (3.1) ensure that

$$\int_A f(x, u, 0)\, dx = \lim_{h\to\infty} \int_A f(x, \varrho_h * u, \varepsilon_h^k D^k \varrho_h * u)\, dx .$$

By Remark 2.6 and Proposition 2.7 we have

$$F^+(u, A) \leq \limsup_{h\to\infty} F_{\varepsilon_h}(u_h, A) = \int_A f(x, u, 0)\, dx$$

and the lemma is proved. ■

LEMMA 4.2. *Let $A, B, C \in \mathcal{A}$ with $C \subset\subset A \cup B$. For every $u \in L^p(A \cup B)$ we have*

$$F^+(u, C) \leq F^+(u, A) + F^+(u, B) .$$

PROOF. Let $K = \overline{C} - B$ and let A_0, B_0 be two open sets, with meas $(\partial A_0) = \text{meas}(\partial B_0) = 0$, such that $K \subseteq A_0 \subset\subset B_0 \subset\subset A$. Fix an integer ν and a family $(A_i)_{1\leqslant i \leqslant \nu}$ of open sets, with meas$(\partial A_i) = 0$, such that $A_0 \subset\subset A_1 \subset\subset \ldots \subset\subset A_\nu \subset\subset B_0$. Define $S_i = C \cap (A_i - \overline{A}_{i-1})$ and $S = C \cap (B_0 - A_0)$. For every $i = 1, 2, \ldots, \nu$ there exists $\varphi_i \in C_0^\infty(A_i)$ such that $0 \leqslant \varphi_i \leqslant 1$ and $\varphi_i = 1$ on A_{i-1}.

In what follows the letter c will denote various positive constants (independent of h, i, ν), whose value can change from one line to the next.

Fix $u \in L^p(A \cup B)$ and $\eta > 0$; there exists a sequence (u_h) in $W^{m,r}_{\text{loc}}(A) \cap L^p(A)$, converging to u weakly in $L^p(A)$ and a sequence (v_h) in $W^{m,r}_{\text{loc}}(B) \cap L^p(B)$ converging to u weakly in $L^p(B)$ such that

$$F^+(u, A) + \eta \geqslant \limsup_{h\to\infty} F_{\varepsilon_h}(u_h, A) \quad \text{and} \quad F^+(u, B) + \eta \geqslant \limsup_{h\to\infty} F_{\varepsilon_h}(v_h, B).$$

For every $i = 1, 2, \ldots, \nu$ and for every $h \in \mathbb{N}$ set

$$w_{i,h} = \varphi_i u_h + (1 - \varphi_i) v_h.$$

Using (3.1) we obtain

$$\begin{aligned} F_{\varepsilon_h}(w_{i,h}, C) &\leqslant F_{\varepsilon_h}(u_h, C \cap A_{i-1}) + F_{\varepsilon_h}(v_h, C - \overline{A}_i) \\ &+ c\int_{S_i}\Big[[a(x) + |w_{i,h}|^p + \sum_{k=1}^m |\varepsilon_h^k D^k w_{i,h}|^r\Big]dx \\ &\leqslant F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + c\int_{S_i}\Big\{a(x) + |u_h|^p + |v_h|^p + \sum_{k=1}^m\big[|\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^r\big] \\ &+ c_\nu \sum_{k=1}^m \varepsilon_h^{kr} \sum_{j=0}^{k-1}\big[|D^j u_h|^r + |D^j v_h|^r\big]\Big\}dx, \end{aligned}$$

where c_ν depends on $\sup|D^\alpha \varphi_i|$ for $i = 1, 2, \ldots, \nu$ and $|\alpha| \leqslant m$. Since the strips S_i are pairwise disjoint, for every $h \in \mathbb{N}$ there exists an index $i_h \in \{1, 2, \ldots, \nu\}$ such that

$$\int_{S_{i_h}} \{\ldots\} dx \leqslant \frac{1}{\nu}\int_S \{\ldots\} dx.$$

Define $w_h = w_{i_h,h}$. Then

$$\begin{aligned} F_{\varepsilon_h}(w_h, C) \leqslant F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + \frac{c}{\nu}\int_S\Big\{a(x) + |u_h|^p + |v_h|^p \\ + \sum_{k=1}^m\big[|\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^r\big] + c_\nu \sum_{k=1}^m \varepsilon_h^{kr} \sum_{j=0}^{k=1}\big[|D^j u_h|^r + |D^j v_h|^r\big]\Big\}dx. \end{aligned}$$

Let $E = A \cap B$. Since $S \subset\subset E$, there exists $S' \in \mathcal{A}$ such that $S \subset\subset S' \subset\subset E$. Since (u_h) and (v_h) are bounded in $L^p(S')$, using inequalities as

$$\int_S |D^k w|^r dx \leqslant \sigma \int_{S'} |D^m w|^r dx + c_\sigma \int_{S'} |w|^r dx$$

(which hold for $1 \leqslant k \leqslant m$ and for every $\sigma > 0$) we get

$$(4.1) \qquad F_{\varepsilon_h}(w_h, C) \leqslant F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B) + \frac{c}{\nu}(1 + \varepsilon_h c_{\nu,\sigma}) + \frac{c}{\nu}(1 + \sigma c_\nu) \int_{S'} \sum_{k=1}^{m} \left[|\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^r \right] dx .$$

Define now $U_h(x) = u_h(\varepsilon_h x)$ and $V_h(x) = v_h(\varepsilon_h x)$; then, using (3.3), we get

$$(4.2) \qquad \int_{S'} \sum_{k=1}^{m} \left[|\varepsilon_h^k D^k u_h|^r + |\varepsilon_h^k D^k v_h|^k \right] dx$$

$$= \varepsilon_h^n \int_{\varepsilon_h^{-1} S'} \sum_{k=1}^{m} \left[|D^k U_h|^r + |D^k V_h|^r \right] dx \leqslant \lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} E) \varepsilon_h^n \int_{\varepsilon_h^{-1} E} [\gamma(U_h, D^k U_h) + \gamma(V_h, D^k V_h)] dx$$

$$= \lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} E) \int_E [\gamma(u_h, \varepsilon_h^k D^k u_h) + \gamma(v_h, \varepsilon_h^k D^k v_h)] dx$$

$$\leqslant \lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} E)[c + F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(v_h, B)] .$$

Since the sequences $(w_{i,k})$ converge to u weakly in $L^p(C)$, it is easy to see that the sequence (w_h) converges to u weakly in $L^p(C)$. Therefore, passing to the limit in (4.1) as $h \to \infty$, and using (4.2) and (3.3) (iii) we get

$$F^+(u, C) \leqslant F^+(u, A) + F^+(u, B) + 2\eta + \frac{c}{\nu} + \frac{c}{\nu}(1 + \sigma c_\nu) M[c + F^+(u, A) + F^+(u, B) + 2\eta] ,$$

where $M = \limsup_{t \to +\infty} \lambda(tS', tE)$. Passing to the limit first as $\sigma \to 0$, then as $\nu \to +\infty$, and finally as $\eta \to 0$, we obtain

$$F^+(u, C) \leqslant F^+(u, A) + F^+(u, B) . \qquad \blacksquare$$

REMARK 4.3. In the same way we can prove that for every $A, B \in \mathcal{A}$, with $B \subset\subset A$, and for every compact subset K of B

$$F^+(u, A) \leq F^+(u, B) + F^+(u, A - K)$$

for every $u \in L^p(A)$. This fact, combined with Lemma 4.1 and inequalities (3.1), implies that

$$F^+(u, A) = \sup \{F^+(u, B): B \in \mathcal{A}, B \subset\subset A\}.$$

LEMMA 4.4. *There exist a subsequence* (ε_{h_k}) *of* (ε_h) *and a functional* F *such that*

$$F(u, A) = \Gamma(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{k \to \infty \\ v \to u}} F_{\varepsilon_{h_k}}(v, A) \tag{4.3}$$

for every $A \in \mathcal{A}$ *and for every* $u \in L^p(A)$. *Moreover for every* $u \in L^p(\Omega)$ *the set function* $A \to F(u, A)$ *is the trace on* $\mathcal{A}$ *of a regular Borel measure defined on* Ω.

PROOF. *Let* $\mathcal{U}$ be a countable base for the open subsets of Ω, closed under finite unions; note that for every $A, B \in \mathcal{A}$ with $A \subset\subset B$, there exists $U \in \mathcal{U}$ such that $A \subset\subset U \subset\subset B$. By the compactness of Γ-convergence (see Propositions 2.2 and 2.7) there exists a subsequence of (ε_h) (which we still denote by (ε_h)) such that for every $B \in \mathcal{U}$, $u \in L^p(B)$ there exists the Γ-limit

$$G(u, B) = \Gamma(\overline{\mathbf{N}}, w - L^p(B)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_{\varepsilon_h}(v, B).$$

For every $A \in \mathcal{A}$, $u \in L^p(A)$ we set

$$F(u, A) = \sup \{G(u, B): B \in \mathcal{U}, B \subset\subset A\}.$$

It is easy to see that for every $u \in L^p(\Omega)$ the set function $A \to G(u, A)$ is superadditive on $\mathcal{U}$, so $A \to F(u, A)$ is superadditive on $\mathcal{A}$. It follows from Lemma 4.2 that $A \to F(u, A)$ is subadditive. So $A \to F(u, A)$ is increasing, superadditive, subadditive, and inner regular. By a result of measure theory (see [10] Proposition 5.5 and Theorem 5.6) this implies that $A \to F(u, A)$ is the trace on $\mathcal{A}$ of a regular Borel measure defined on Ω. It remains to prove (4.3). Let

$$F^-(u, A) = \Gamma(\overline{\mathbf{N}}^-, w - L^p(A)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_{\varepsilon_h}(v, A)$$

and

$$F^{+}(u, A) = \Gamma(\overline{\mathbf{N}}^{+}, w - L^{p}(A)^{-}) \lim_{\substack{h\to\infty \\ v\to u}} F_{\varepsilon_h}(v, A).$$

By Remark 4.3 we have

$$\begin{aligned} F^{+}(u, A) &= \sup\{F^{+}(u, B)\colon B \in \mathcal{A}, B \subset\subset A\} \\ &= \sup\{G(u, B)\colon B \in \mathcal{A}, B \subset\subset A\} = F(u, A) \leqslant F^{-}(u, A) \leqslant F^{+}(u, A), \end{aligned}$$

which proves (4.3). ∎

LEMMA 4.5. *Let F be the functional introduced in Lemma 4.4. There exists a Borel function $\psi\colon \Omega\times\mathbf{R} \to [0, +\infty[$ such that*

(i) *for every $A \in \mathcal{A}$, $u \in L^{p}(A)$*

$$F(u, A) = \int_A \psi(x, u)\, dx,$$

(ii) *for every $x \in \Omega$ the function $s \to \psi(x, s)$ is convex on $\mathbf{R}$,*

(iii) *for every $x \in \Omega$, $s \in \mathbf{R}$*

$$-a(x) + |s|^{p} \leqslant \psi(x, s) \leqslant f^{+}(x, s, 0).$$

PROOF. Let us denote by $\mathcal{B} = \mathcal{B}(\Omega)$ the class of all Borel subsets of Ω. For every $u \in L^{p}(\Omega)$ we denote by $\Phi(u,\cdot)$ the measure on $\mathcal{B}$ which extends $F(u,\cdot)$; it is easy to see that for every $B \in \mathcal{B}$

$$\Phi(u, B) = \inf\{F(u, A)\colon A \in \mathcal{A}, A \supseteq B\}.$$

First of all we prove that the functional Φ is local on $\mathcal{B}$, that is: if $u = v$ a.e. on a Borel set B, then $\Phi(u, B) = \Phi(v, B)$. Let $u, v \in L^{p}(\Omega)$ and let $B \in \mathcal{B}$ with $u = v$ a.e. on B; without loss of generality we may suppose that $u = v$ everywhere on B and $u \leqslant v$ everywhere on Ω. By Lusin's theorem, for every $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{A}$, with meas $(A_\varepsilon) < \varepsilon$, such that the restrictions $u|_{\Omega - A_\varepsilon}$ and $v|_{\Omega - A_\varepsilon}$ are continuous. Then the set $B_\varepsilon = A_\varepsilon \cup \{x \in \Omega\colon v(x) < u(x) + \varepsilon\}$ is open; moreover $B_\varepsilon \supseteq B$. Define now

$$u_\varepsilon(x) = \begin{cases} v(x) & \text{if } x \in B_\varepsilon \\ u(x) + \varepsilon & \text{if } x \in \Omega - B_\varepsilon; \end{cases}$$

it is easy to see that (u_ε) converges to u strongly in $L^p(\Omega)$ as $\varepsilon \searrow 0$. For every $\eta > 0$ there exist an open set A and a compact set K such that $K \subseteq B \subseteq A \subseteq \Omega$, $F(u, A) < \Phi(v, B) + \eta$ and $\int_{A-K} [a(x) + c|u|^p]\, dx < \eta$.

Since $F(\cdot, A)$ is lower semicontinuous with respect to the weak topology of $L^p(A)$ (see Proposition 2.1) and F is local on $\mathcal{A}$, using Lemma 4.1 and inequalities (3.1) we obtain

$$\Phi(u, B) \leqslant F(u, A) \leqslant \liminf_{\varepsilon \to 0^+} F(u_\varepsilon, A) \leqslant \liminf_{\varepsilon \to 0^+} [F(v, A \cap B_\varepsilon) + F(u_\varepsilon, A - K)]$$

$$\leqslant F(v, A) + \liminf_{\varepsilon \to 0^+} \int_{A-K} [a(x) + c|u_\varepsilon|^p]\, dx \leqslant \Phi(v, B) + 2\eta\,.$$

Since $\eta < 0$ was arbitrary, we get

$$\Phi(u, B) \leqslant \Phi(v, B)\,.$$

The opposite inequality can be proved in a similar way.

So the functional $\Phi\colon L^p(\Omega) \times \mathcal{B} \to [0, +\infty[$ is local on B, for every $u \in L^p(\Omega)$ the set function $\Phi(u, \cdot)$ is a measure, and the function $\Phi(\cdot, \Omega)$ is lower semicontinuous in the weak topology of $L^p(\Omega)$. This implies (see [5]) that there exists a non-negative Borel function $\psi(x, s)$, convex in s, such that

$$\Phi(u, B) = \int_B \psi(x, u)\, dx$$

for every $u \in L^p(\Omega)$, $B \in \mathcal{B}$. Since $\Phi(u, A) = F(u, A)$ for every $A \in \mathcal{A}$, we obtain (i) and (ii). Finally, (iii) follows from inequalities (3.1) and from Lemma 4.1. ■

LEMMA 4.6. *For every $A \in \mathcal{A}$ and for every $u \in W^{m,r}(A) \cap L^p(A)$ we have*

$$F^+(u, A) \geqslant \Gamma(\overline{\mathbb{N}}^+, w - L^p(A)^-) \lim_{\substack{h \to \infty \\ v \to u}} [F_{\varepsilon_h}(v, A) + T(v - u, A)]$$

where T is the functional defined by (3.5).

PROOF. Let $A \in \mathcal{A}$, $u \in W^{m,r}(A) \cap L^p(A)$, and $\eta > 0$. There exists a sequence (u_h) in $W^{m,r}_{\mathrm{loc}}(A) \cap L^p(A)$ converging to u weakly in $L^p(A)$ such that

$$F^+(u, A) + \eta \geqslant \limsup_{h \to \infty} F_{\varepsilon_h}(u_h, A)\,.$$

Let A_0, B_0 be two open sets with $A_0 \subset\subset B_0 \subset\subset A$ and $\mathrm{meas}(\partial A_0) = \mathrm{meas}(\partial B_0) = 0$. Fix an integer ν and, for $i = 1, 2, \ldots, \nu$, define A_i and φ_i as in Lemma 4.2. Set

$$w_{i,h} = \varphi_i u_h + (1 - \varphi_i) u ;$$

we have $T(w_{i,h} - u, A) = 0$. With the same argument used in the proof of Lemma 4.2 we get

$$F_{\varepsilon_h}(w_{i_h,h}, A) \leqslant F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(u, A - \bar{A}_0) + \frac{c}{\nu}(1 + \varepsilon_h c_{\nu,\sigma}) + \frac{c}{\nu}(1 + \sigma c_\nu)\lambda(\varepsilon_h^{-1} S', \varepsilon_h^{-1} A)[c + F_{\varepsilon_h}(u_h, A) + F_{\varepsilon_h}(u, A - \bar{A}_0)],$$

where $B_0 - \bar{A}_0 \subset\subset S' \subset\subset A$. Since $(w_{i_h,h})$ converges to u weakly in $L^p(A)$ we have

$$\inf\Big\{\limsup_{h\to\infty} [F_{\varepsilon_h}(v_h, A) + T(v_h - u, A)] \colon v_h \to u \text{ in } w - L^p(A)\Big\} \leqslant F^+(u, A) + \eta + \int_{A - \bar{A}_0} [a(x) + c|u|^p]\, dx + \frac{c}{\nu} + \frac{c}{\nu}(1 + \sigma c_\nu) M \Big\{c + F^+(u, A) + \eta + \int_{A - \bar{A}_0} [a(x) + c|u|^p]\, dx\Big\},$$

where $M = \limsup_{t\to+\infty} \lambda(tS', tA)$. Passing to the limit first as $\sigma \to 0$, next as $\nu \to +\infty$, then as $\eta \to 0$, and finally as $A_0 \uparrow A$, we get the thesis. ∎

LEMMA 4.7. *Assume that*

$$\int_A \psi(x, u)\, dx = \Gamma(\bar{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_{\varepsilon_h}(v, A)$$

for every $A \in \mathcal{A}$ *and for every* $u \in L^p(A)$. *Then*

$$\int_A \psi(x, u)\, dx = \Gamma(\bar{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{h\to\infty \\ v\to u}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)]$$

for every $A \in \mathcal{A}$, $u \in L^p(A)$, $w_0 \in W^{m,r}(A) \cap L^p(A)$.

PROOF. Let $A \in \mathcal{A}$, $u \in L^p(A)$, $w_0 \in W^{m,r}(A) \cap L^p(A)$. There exists a sequence (u_k) in $W^{m,r}(A) \cap L^p(A)$ converging to u strongly in $L^p(A)$ such that

$u_k - w_0 \in W_0^{m,r}(A)$. Using Lemma 4.6 we obtain for every $k \in \mathbf{N}$

$$\int_A \psi(x, u_k)\,dx \geq \Gamma(\overline{\mathbf{N}}^+, w - L^p(A)^-) \lim_{\substack{h\to\infty \\ v\to u_k}} [F_{\varepsilon_h}(v, A) + T(v - u_k, A)]$$
$$= \Gamma(\overline{\mathbf{N}}^+, w - L^p(A)) \lim_{\substack{h\to\infty \\ v\to u_k}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)]\,.$$

Since Γ-limits are lower semicontinuous (see Proposition 2.1) and $\int_A \psi(x, v)\,dx$ is continuous in $L^p(A)$ (see Lemma 4.5), passing to the limit as $k \to +\infty$ we obtain

$$\int_A \psi(x, u)\,dx \geq \Gamma(\overline{\mathbf{N}}^+, w - L^p(A)^-) \lim_{\substack{h\to\infty \\ v\to u}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)]$$
$$\geq \Gamma(\overline{\mathbf{N}}^-, w - L^p(A)^-) \lim_{\substack{h\to\infty \\ v\to u}} [F_{\varepsilon_h}(v, A) + T(v - w_0, A)] \geq \int_A \psi(x, u)\,dx. \quad \blacksquare$$

Let $Y =]0, 1[^n$ and let $W_\#^{m,r}(Y)$ be the space of all Y-periodic functions of $W_{\mathrm{loc}}^{m,r}(\mathbf{R}^n)$; for every $\varepsilon > 0$, $x \in \Omega$, $s \in \mathbf{R}$ we set

$$(4.4)\quad \begin{cases} W(s) = \left\{u \in W^{m,r}(Y) \cap L^p(Y) : \int_Y u(y)\,dy = s\right\} \\ W_0(s) = \left\{u \in W^{m,r}(Y) \cap L^p(Y) : \int_Y u(y)\,dy = s,\ u - s \in W_0^{m,r}(Y)\right\} \\ W_\#(s) = \left\{u \in W_\#^{m,r}(Y) \cap L^p(Y) : \int_Y u(y)\,dy = s\right\} \\ m^\varepsilon(x, s) = \inf\left\{\int_Y f(x, u(y), \varepsilon^k D^k u(y))\,dy : u \in W(s)\right\} \\ m_0^\varepsilon(x, s) = \inf\left\{\int_Y f(x, u(y), \varepsilon^k D^k u(y))\,dy : u \in W_0(s)\right\} \\ m_\#^\varepsilon(x, s) = \inf\left\{\int_Y f(x, u(y), \varepsilon^k D^k u(y))\,dy : u \in W_\#(s)\right\} \\ m_0(x, s) = \inf\{m_0^\varepsilon(x, s) : \varepsilon > 0\} \\ m_\#(x, s) = \inf\{m_\#^\varepsilon(x, s) : \varepsilon > 0\}\,. \end{cases}$$

LEMMA 4.8. *For every* $x \in \Omega$, $s \in \mathbf{R}$

$$m_0(x, s) = \lim_{\varepsilon\to 0^+} m_0^\varepsilon(x, s)\,.$$

PROOF. Let $x \in \Omega$, $s \in \mathbf{R}$, $u \in W_0(s)$, $\varepsilon, \eta \in \mathbf{R}$ with $0 < \eta \leqslant \varepsilon$. Let v be the Y-periodic extension of u, that is the function which satisfies $v(x + y) = v(x)$ for every $x \in \mathbf{R}^n$, $y \in \mathbf{Z}^n$ and $v(x) = u(x)$ for every $x \in Y$. There exist $N \in \mathbf{N}$ and $\delta \in [0, 1[$ such that $\varepsilon = (N + \delta)\eta$. Define for every $y \in Y$

$$w(y) = \begin{cases} v\left(\dfrac{\varepsilon}{\eta} y\right) & \text{if } y \in N \dfrac{\eta}{\varepsilon} Y \\ s & \text{otherwise}. \end{cases}$$

Then $w \in W_0(s)$ and

$$\int_Y f(x, w(y), \eta^k D^k w(y))\, dy \leqslant \left(N \frac{\eta}{\varepsilon}\right)^n \int_Y f(x, u(y), \varepsilon^k D^k u(y))\, dy + n \frac{\delta\eta}{\varepsilon} f(x, s, 0)$$
$$\leqslant \int_Y f(x, u(y), \varepsilon^k D^k u(y))\, dy + n \frac{\eta}{\varepsilon} f(x, s, 0).$$

This implies that for every $\varepsilon, \eta \in \mathbf{R}$, with $0 < \eta \leqslant \varepsilon$

$$m_0^\eta(x, s) \leqslant m_0^\varepsilon(x, s) + n \frac{\eta}{\varepsilon} f(x, s, 0),$$

and from this inequality it follows that

$$\inf_{\varepsilon > 0} m_0^\varepsilon(x, s) = \lim_{\varepsilon \to 0^+} m_0^\varepsilon(x, s). \quad \blacksquare$$

LEMMA 4.9. *Suppose that the function f does not depend on the variable x and that*

$$\int_A \psi(u)\, dx = \Gamma(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{h \to \infty \\ v \to u}} F_{\varepsilon_h}(v, A)$$

for every $A \in \mathcal{A}$, $u \in L^p(A)$. Then m^ε, m_0^ε and m_0 do not depend on x and

$$\lim_{h \to \infty} m^{\varepsilon_h}(s) = m_0(s) = \psi(s)$$

for every $s \in \mathbf{R}$.

PROOF. Let $s \in \mathbf{R}$ and let (u_h) be a sequence converging to s weakly in $L^p(Y)$ such that $u_h - s \in W_0^{m,r}(Y)$; let $\varphi \in C_0^\infty(Y)$ with $\int \varphi\, dx = 1$; there exists a sequence (η_h) converging to 0 in $\mathbf{R}$ such that $\int_Y [u_h(y) + \eta_h \varphi(y)]\, dy = s$ for every $h \in \mathbf{N}$. Then by hypothesis (3.2) we have

$$m_0^{\varepsilon_h}(s) \leqslant F_{\varepsilon_h}(u_h + \eta_h \varphi, Y) \leqslant F_{\varepsilon_h}(u_h, Y) + \sigma(\eta_h M)\left[\int_Y a(x)\, dx + F_{\varepsilon_h}(u_h, Y)\right],$$

where $M = \sup \sum_{|\alpha| \leqslant m} |D^\alpha \varphi|$. Passing to the limit as $h \to +\infty$ we obtain

$$m_0(s) \leqslant \liminf_{h\to\infty} F_{\varepsilon_h}(u_h, Y) .$$

Since (u_h) is arbitrary, by Lemma 4.7 we get

$$(4.5) \qquad m_0(s) \leqslant \Gamma(\overline{\mathbf{N}}, w-L^p(A)^-) \lim_{\substack{h\to\infty \\ v\to s}} [F_{\varepsilon_h}(v, A) + T(v-s, A)] = \psi(s) .$$

Consider now a subsequence (ε_{h_k}) such that $\liminf_{h\to\infty} m^{\varepsilon_h}(s) = \lim_{h\to\infty} m^{\varepsilon_{h_k}}(s)$. For every $k \in \mathbf{N}$ there exists $w_k \in W(s)$ such that $F_{\varepsilon_{h_k}}(w_k, Y) \leqslant m^{\varepsilon_{h_k}}(s) + 1/k$. By hypothesis (3.1) the sequence (w_k) is bounded in $L^p(Y)$; thus for a suitable subsequence (w_{k_i}), we have that (w_{k_i}) converges weakly in $L^p(Y)$ to a function u such that $\int_Y u(y)\,dy = s$. Therefore, using Jensen's inequality, Remark 2.6, Lemma 4.8 and inequality (4.5), we get

$$\begin{aligned} m_0(s) \leqslant \psi(s) = \psi\left(\int_Y u(y)\,dy\right) \leqslant \int_Y \psi(u)\,dy = \Gamma(\overline{\mathbf{N}}, w-L^p(Y)^-) \lim_{\substack{h\to\infty \\ v\to u}} F_{\varepsilon_h}(v, Y) \\ \leqslant \liminf_{i\to\infty} F_{\varepsilon_{h_{k_i}}}(w_{k_i}, Y) \leqslant \lim_{k\to\infty} m^{\varepsilon_{h_k}}(s) = \liminf_{h\to\infty} m^{\varepsilon_h}(s) \\ \leqslant \limsup_{h\to\infty} m^{\varepsilon_h}(s) \leqslant \limsup_{h\to\infty} m_0^{\varepsilon_h}(s) = m_0(s) . \quad \blacksquare \end{aligned}$$

LEMMA 4.10. *Suppose that the function f does not depend on the variable x. Then there exists a convex function $\psi\colon \mathbf{R} \to [0, +\infty[$ such that*

$$(4.6) \qquad \begin{aligned} \int_A \psi(u)\,dx = \Gamma(\mathbf{R}, w-L^p(A)^-) \lim_{\substack{\varepsilon\to 0^+ \\ v\to u}} F_\varepsilon(v, A) \\ = \Gamma(\mathbf{R}, w-L^p(A)^-) \lim_{\substack{\varepsilon\to 0^+ \\ v\to u}} [F_\varepsilon(v, A) + T(v-w_0, A)] \end{aligned}$$

for every $A \in \mathcal{A}$, $u \in L^p(A)$, $w_0 \in W^{m,r}(A) \cap L^p(A)$.

Moreover m^ε, m_0^ε, m_0 do not depend on x and

$$\psi(s) = m_0(s) = \lim_{\varepsilon\to 0^+} m_0^\varepsilon(s) = \lim_{\varepsilon\to 0^+} m^\varepsilon(s)$$

for every $s \in \mathbf{R}$.

PROOF. Let (ε_h) be a sequence in $\mathbf{R}$ converging to 0 such that $\varepsilon_h > 0$ for every $h \in \mathbf{N}$. By Lemmas 4.4, 4.5 and 4.7 there exist a subsequence (ε_{h_k})

of (ε_h) and a Borel function $\psi(x, s)$, convex in s, such that

$$\int_A \psi(x, u)\,dx = \Gamma(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{k\to\infty \\ v\to u}} F_{\varepsilon_{h_k}}(v, A)$$
$$= \Gamma(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{k\to\infty \\ v\to u}} [F_{\varepsilon_{h_k}}(v, A) + T(v - w_0, A)]$$

for every $A \in \mathcal{A}$, $u \in L^p(A)$, $w_0 \in W^{m,r}(A) \cap L^p(A)$. Since f does not depend on x, it is easy to see that $\int_{y+A} \psi(x, u(x-y))\,dx = \int_A \psi(x, u(x))\,dx$ for every $A \in \mathcal{A}$, $u \in L^p(A)$ and for every $y \in \mathbb{R}^n$ such that $y + A \subseteq \Omega$. This implies that ψ does not depend on x, that is $\psi(x, s) = \psi(s)$.

By Lemma 4.9 we have

$$\psi(s) = m_0(s)$$

for every $s \in \mathbb{R}$. So the function ψ does not depend on the sequence (ε_h). By Proposition 2.8 this implies (4.6).

By Lemma 4.9 we have $m_0(s) = \lim_{k\to\infty} m^{\varepsilon_{h_k}}(s)$. Since the limit does not depend on the sequence (ε_h), we obtain

$$m_0(s) = \lim_{\varepsilon\to 0^+} m^\varepsilon(s)\,.$$

The equality $m_0(s) = \lim_{\varepsilon\to 0^+} m_0^\varepsilon(s)$ has already been proved in Lemma 4.8. ■

Proof of Theorem 3.1. Let (ε_h) be a sequence in $]0, +\infty[$ converging to 0. By Lemmas 4.4, 4.5 and 4.7 there exist a subsequence (ε_{h_k}) of (ε_h) and a Borel function $\psi\colon \Omega \times \mathbb{R} \to [0, +\infty[$, which satisfies condition (ii) of the theorem, such that

$$\int_A \psi(x, u)\,dx = \Gamma(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{k\to\infty \\ v\to u}} F_{\varepsilon_{h_k}}(v, A)$$
$$= \Gamma(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{k\to\infty \\ v\to u}} [F_{\varepsilon_{h_k}}(v, A) + T(v - w_0, A)]$$

for every $A \in \mathcal{A}$, $u \in L^p(A)$, $w_0 \in W^{m,r}(A) \cap L^p(A)$.

In order to prove (i), by Proposition 2.8 we have only to show that

$$\psi(x, s) = m_0(x, s) = \lim_{\varepsilon\to 0^+} m^\varepsilon(x, s) = m_f(x, s) \tag{4.7}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$, where m_0 and m^ε are defined by (4.4).

Let $N \geqslant 1$ be an integer; for every $j \in \mathbb{Z}^n$ we set $Y_N^j = (1/N)(Y + j)$ and $\Omega_N^j = \Omega \cap Y_N^j$ (here $Y =]0, 1[^n$). Define $f_N \colon \Omega \times \mathbb{R} \times \mathbb{R}^d \to [0, +\infty[$ by

$$f_N(x, s, z) = \fint_{\Omega_N^j} f(y, s, z)\,dy \qquad \text{for } x \in \Omega_N^j\,,$$

where $\fint_A$ denotes the average over the set A. Define

$$F_\varepsilon^N(u, A) = \int_A f_N(x, u, \varepsilon^k D^k u)\,dx$$

and let $(m_N)^\varepsilon(x, s)$, $(m_N)_0^\varepsilon(x, s)$, $(m_N)_0(x, s)$ be the functions related to f_N defined as in (4.4). Since f_N is piecewise constant with respect to the variable x, by Lemmas 4.5 and 4.10 there exists a Borel function $\psi_N(x, s)$, piecewise constant in x and convex in s, such that

$$\int_A \psi_N(x, u)\,dx = \Gamma(\overline{\mathbf{N}}, w - L^p(A)^-) \lim_{\substack{\varepsilon \to 0^+ \\ v \to u}} F_\varepsilon^N(v, A)$$

for every $A \in \mathcal{A}$, $u \in L^p(A)$; moreover

$$\psi_N(x, s) = (m_N)_0(x, s) = \lim_{\varepsilon \to 0^+} (m_N)_0^\varepsilon(x, s) = \lim_{\varepsilon \to 0^+} (m_N)^\varepsilon(x, s) \tag{4.8}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

Let $Q_N =]-1/N, 1/N[^n$. If $Y_N^j \subseteq \Omega$, using condition (3.2) we obtain for every $x \in Y_N^j$, $s \in \mathbf{N}$, $z \in \mathbb{R}^d$

$$\begin{aligned}
|f_N(x, s, z) - f(x, s, z)| &= \left| \fint_{Y_N^j - x} [f(x + y, s, z) - f(x, s, z)]\,dy \right| \\
&\leqslant 2^n \fint_{Q_N} |f(x + y, s, z) - f(x, s, z)|\,dy \leqslant 2^n \fint_{Q_N} \{\omega(x, y) + \sigma(|y|)[a(x) + f(x, s, z)]\}\,dy \\
&\leqslant 2^n \fint_{Q_N} \omega(x, y)\,dy + 2^n \sigma\left(\frac{\sqrt{n}}{N}\right)[a(x) + f(x, s, z)]\,.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\left[1 - 2^n \sigma\left(\frac{\sqrt{n}}{N}\right)\right] f(x, s, z) - 2^n \sigma\left(\frac{\sqrt{n}}{N}\right) a(x) - 2^n \fint_{Q_N} \omega(x, y)\,dy \\
&\qquad \leqslant f_N(x, s, z) \leqslant \left[1 + 2^n \sigma\left(\frac{\sqrt{n}}{N}\right)\right] f(x, s, z) + 2^n \sigma\left(\frac{\sqrt{n}}{N}\right) a(x) + 2^n \fint_{Q_N} \omega(x, y)\,dy
\end{aligned} \tag{4.9}$$

for every $s \in \mathbf{R}$, $z \in \mathbf{R}^d$ and for every $x \in \Omega$ such that $\operatorname{dist}(x, \mathbf{R}^n - \Omega) > \sqrt{n}/N$. Passing to the Γ-limit along the sequence (ε_{h_k}) we obtain

$$(4.10) \qquad \left[1 - 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\right]\int_A \psi(x, u)\,dx - 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\int_A a(x)\,dx - 2^n\int_A dx \fint_{Q_N} \omega(x, y)\,dy$$
$$\leqslant \int_A \psi_N(x, u)\,dx \leqslant \left[1 + 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\right]\int_A \psi(x, u)\,dx$$
$$+ 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\int_A a(x)\,dx + 2^n\int_A dx \fint_{Q_N} \omega(x, y)\,dy$$

for every $A \in \mathcal{A}$ with $d(A, \mathbf{R}^n - \Omega) > \sqrt{n}/N$ and for every $u \in L^p(A)$. By (3.2) we have

$$(4.11) \qquad \lim_{N\to\infty} \int_A dx \fint_{Q_N} \omega(x, y)\,dy = \lim_{N\to\infty} \fint_{Q_N} dy \int_A \omega(x, y)\,dx = 0$$

for every $A \in \mathcal{A}$ with $A \subset\subset \Omega$. Thus, passing to the limit in (4.10) as $N \to +\infty$ we get

$$(4.12) \qquad \int_A \psi(x, u)\,dx = \lim_{N\to\infty} \int_A \psi_N(x, u)\,dx$$

for every $A \in \mathcal{A}$ with $A \subset\subset \Omega$ and for every $u \in L^p(A)$.

Using the definitions of m^ε and $(m_N)^\varepsilon$, from (4.9) we obtain that

$$\left[1 - 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\right] m^\varepsilon(x, s) - 2^n\sigma\left(\frac{\sqrt{n}}{N}\right) a(x) - 2^n \fint_{Q_N} \omega(x, y)\,dy$$
$$\leqslant (m_N)^\varepsilon(x, s) \leqslant \left[1 + 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\right] m^\varepsilon(x, s) + 2^n\sigma\left(\frac{\sqrt{n}}{N}\right) a(x) + 2^n \fint_{Q_N} \omega(x, y)\,dy$$

for every $x \in \Omega$ with $\operatorname{dist}(x, \mathbf{R}^n - \Omega) > \sqrt{n}/N$ and for every $s \in \mathbf{R}$. Letting $\varepsilon \to 0^+$ and using (4.8) we get

$$(4.13) \qquad \left[1 - 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\right] \limsup_{\varepsilon\to 0^+} m^\varepsilon(x, s) - 2^n\sigma\left(\frac{\sqrt{n}}{N}\right) a(x) - 2^n \fint_{Q_N} \omega(x, y)\,dy$$
$$\leqslant \lim_{\varepsilon\to 0^+} (m_N)^\varepsilon(x, s) = \psi_N(x, s) \leqslant \left[1 + 2^n\sigma\left(\frac{\sqrt{n}}{N}\right)\right] \liminf_{\varepsilon\to 0^+} m^\varepsilon(x, s)$$
$$+ 2^n\sigma\left(\frac{\sqrt{n}}{N}\right) a(x) + 2^n \fint_{Q_N} \omega(x, y)\,dy\,.$$

Equality (4.11) implies that there exists an increasing sequence of integers (N_k) such that $\lim_{k\to\infty} \int_{Q_{N_k}} \omega(x, y)dy = 0$ for a.a. $x \in \Omega$. Letting $N \to +\infty$ in (4.13) along the sequence (N_k), we get that there exists

$$\lim_{\varepsilon\to 0^+} m^\varepsilon(x, s) = m(x, s)$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$, and that

$$m(x, s) = \lim_{k\to\infty} \psi_{N_k}(x, s)$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$. In the same way we prove that

$$m_0(x, s) = \lim_{k\to\infty} \psi_{N_k}(x, s) .$$

Using (4.12) we obtain

$$\int_A m(x, s)\,dx = \int_A m_0(x, s)\,dx = \lim_{k\to\infty} \int_A \psi_{N_k}(x, s)\,dx = \int_A \psi(x, s)\,dx$$

for every $A \in \mathcal{A}$ with $A \subset\subset \Omega$ and for every $s \in \mathbb{R}$.

Since m, m_0, ψ are continuous in s (indeed they are convex), this implies that

$$m(x, s) = m_0(x, s) = \psi(x, s) \tag{4.14}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

In order to prove (4.7) it is enough to show that

$$\psi(x, s) = m_\#(x, s) \tag{4.15}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

Since $W_0(s) \subseteq W_\#(s) \subseteq W(s)$ we have

$$m^\varepsilon(x, s) \leqslant m^\varepsilon_\#(x, s) \leqslant m^\varepsilon_0(x, s);$$

thus from (4.14) it follows that

$$\psi(x, s) = \lim_{\varepsilon\to 0^+} m^\varepsilon_\#(x, s) . \tag{4.16}$$

By a change of variables, it is easy to verify that $m^{2\varepsilon}_\#(x, s) \geqslant m^\varepsilon_\#(x, s)$ for every $\varepsilon > 0$. Therefore (4.16) yields

$$\psi(x, s) = \lim_{\varepsilon \to 0^+} m^\varepsilon_\#(x, s) = \inf_{\varepsilon > 0^+} m^\varepsilon_\#(x, s).$$

This proves (4.15).

It remains to prove property (iii). The inequality $\psi(x, s) \leqslant f^+(x, s, 0)$ follows from Lemma 4.1 and from the convexity of $\psi(x, \cdot)$.

Let $x \in \Omega$, $s \in \mathbb{R}$, $u \in W_0(s)$, $\varepsilon > 0$; by Jensen's inequality we have

$$f^-(x, s, 0) = f^-\Big(x, \int_Y u(y)\,dy, \varepsilon^k \int_Y D^k u(y)\,dy\Big) \leqslant \int_Y f^-\big(x, u(y), \varepsilon^k D^k u(y)\big)\,dy \leqslant \int_Y f\big(x, u(y), \varepsilon^k D^k u(y)\big)\,dy.$$

Thus by the representation formula for ψ we have

$$f^-(x, s, 0) \leqslant \psi(x, s). \quad \blacksquare$$

5. – Some examples.

In this section we give some examples and applications of Theorem 3.1. In particular we show that the inequalities

$$f^-(x, s, 0) \leqslant \psi(x, s) \leqslant f^+(x, s, 0) \tag{5.1}$$

cannot be improved; in fact, there are some examples where $\psi(x, s) = f^-(x, s, 0)$ (see Proposition 5.9 and Remark 5.10), and some other examples where $\psi(x, s) = f^+(x, s, 0)$ (see Proposition 5.2). In the case $f^-(x,s,0) = f^+(x, s, 0)$ the integrand $\psi(x, s)$ is determined by the inequalities (5.1); this allows us to generalize some results of A. Bensoussan [2] and V. Komornik [13] (see Proposition 5.5 and Proposition 5.6).

For every $p \geqslant 2$ we denote by $\mathcal{G}_p$ the class of functions $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$|g(s)| \leqslant c\big(1 + |s|^{p/2}\big) \tag{5.2}$$

$$|g(t) - g(s)| \leqslant \varrho\big(|t - s|\big)\big(1 + |s|^{p/2}\big) \tag{5.3}$$

for every $s, t \in \mathbb{R}$, where c is a positive constant and $\varrho\colon [0, +\infty[\to [0, +\infty[$

is an increasing continuous function with $\varrho(0) = 0$. Examples of functions of the class $\mathcal{G}_p$ are the polynomials of degree less than or equal to $p/2$.

Let $N > 0$, $b \in L^p(\Omega)$, $g \in \mathcal{G}_p$; after some simple calculations (see section 6) one can verify that the functionals

$$F_\varepsilon(u, A) = \int_A [N|\varepsilon^2 \Delta u + g(u)|^2 + |u - b(x)|^p] dx$$

satisfy all hypotheses of Theorem 3.1, with $m = r = 2$,

$$f(x, s, z) = N\left|\sum_{i=1}^n z_{ii} + g(s)\right|^2 + |s - b(x)|^p \qquad \left(\text{here } z = (z_{ij})_{1 \leq i+j \leq 2}\right),$$

$$\gamma(s, z) = c_1\left[\left|\sum_{i=1}^n z_{ii}\right|^2 + s^2\right],$$

$$\lambda(A', A) = c_2 \max\{1, \operatorname{dist}(A', \mathbf{R}^n - A)^{-4}\},$$

where c_1, c_2 are suitable positive constants.

Let $\psi(x, s)$ be the function, convex in s, such that

$$\int_A \psi(x, u) dx = \Gamma(\mathbf{R}, w - L^p(A)^-) \lim_{\substack{\varepsilon \to 0^+ \\ v \to u}} F_\varepsilon(v, A)$$

for every $A \in \mathcal{A}$, $u \in L^p(A)$.

PROPOSITION 5.1. *If g is an affine function, then*

$$\psi(x, s) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^p$$

for a.a. $x \in \Omega$ and for all $s \in \mathbf{R}$.

PROOF. Since in this case $f(x, s, z) = f^-(x, s, z) = f^+(x, s, z)$, the proposition follows from (5.1). ■

In the following proposition we give a new proof of a result due to A. Haraux and F. Murat [11].

PROPOSITION 5.2. *Let g be a decreasing function of the class $\mathcal{G}_p$, let $b \in L^p(\Omega)$, and let $N > 0$. Then*

$$\psi(x, s) = f^+(x, s, 0)$$

for a.a. $x \in \Omega$ and for all $s \in \mathbf{R}$.

PROOF. Let $x \in \Omega$, $s \in \mathbb{R}$, $\varepsilon > 0$, $u \in W_0(s)$ (see (4.4)). Then

$$(5.4)\quad \int_Y [N|\varepsilon^2 \Delta u(y) + g(u(y))|^2 + |u(y) - b(x)|^p]\, dy$$
$$= \int_Y [N\varepsilon^4 |\Delta u(y)|^2 + N|g(u(y))|^2 + 2N\varepsilon^2 \Delta u(y) g(u(y)) + |u(y) - b(x)|^p]\, dy\,.$$

Let us prove that

$$(5.5)\qquad \int_Y g(u)\Delta u\, dy \geqslant 0\,.$$

There exists a sequence (g_h) of decreasing functions of class C^1, with bounded derivatives, such that $g(s) = \lim_h g_h(s)$ for every $s \in \mathbb{R}$, and $|g_h(s)| \leqslant c(1+|s|^{p/2})$ for every $h \in \mathbb{N}$, $s \in \mathbb{R}$.

By the dominated convergence theorem

$$\int_Y g(u)\Delta u\, dy = \lim_h \int_Y g_h(u)\Delta u\, dy\,.$$

Since $u - s \in W_0^{2,2}(Y)$ we have

$$\int_Y g_h(u)\Delta u\, dy = -\int_Y g_h'(u)|Du|^2 dy \geqslant 0\,,$$

so (5.5) is proved. From (5.4), (5.5) and Jensen's inequality it follows that

$$\int_Y [N|\varepsilon^2 \Delta u(y) + g(u(y))|^2 + |u(y) - b(x)|^p]\, dy$$
$$\geqslant \int_Y [N|g(u(y))|^2 + |u(y) - b(x)|^p]\, dy \geqslant \int_Y f^+(x, u(y), 0)\, dy \geqslant f^+(x, s, 0)\,.$$

Since $\varepsilon > 0$ and $u \in W_0(s)$ are arbitrary, the representation formula for ψ implies $\psi(x, s) \geqslant f^+(x, s, 0)$. The opposite inequality follows from (5.1). ■

We construct now an example which shows that the equality $\psi(x, s) = f^+(x, s, 0)$ does not hold for an arbitrary function $g \in \mathcal{G}_p$.

PROPOSITION 5.3. *Let $n = 1$, $m = p = r = 2$, $\Omega =]0, 1[$ and let g be defined by*

$$g(s) = \begin{cases} s & \text{if } s < 0 \\ s/4 & \text{if } s \geqslant 0\,. \end{cases}$$

If $N > 6\pi^2 - 16$ *and* $b \in L^2(\Omega)$, *then*

$$\psi(x, s) < f^+(x, s, 0) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^2$$

for a.a. $x \in \Omega$ *and for all* $s > 0$. *If in addition* $b(x) > 0$ *for a.a.* $x \in \Omega$, *then*

$$\lim_{\varepsilon \to 0^+} \inf\{F_\varepsilon(u, \Omega) : u \in W^{2,2}(\Omega)\} < \min\left\{\int_\Omega f^+(x, u, 0)\, dx : u \in L^2(\Omega)\right\}.$$

Proof. Define on $[-\pi, 2\pi]$

$$u(x) = \begin{cases} \dfrac{k}{2} \sin x & \text{if } x \in [-\pi, 0] \\ k \sin \dfrac{x}{2} & \text{if } x \in [0, 2\pi] \end{cases}$$

($k > 0$ is a parameter) and extend u to $\mathbb{R}$ by periodicity (the period is 3π). Set $u_\varepsilon(x) = u(x/\varepsilon)$; as $\varepsilon \to 0^+$ we have that (u_ε) converges to k/π and $(|u_\varepsilon|^2)$ converges to $\frac{3}{8}k^2$ weakly in $L^2(0,1)$. Since $\varepsilon^2 u''_\varepsilon + g(u_\varepsilon) = 0$, for every $A \in \mathcal{A}$, $b \in L^2(A)$ we have

$$\int_A \psi\left(x, \frac{k}{\pi}\right) dx \leq \liminf_{\varepsilon \to 0^+} \int_A [N|\varepsilon^2 u'' + g(u_\varepsilon)|^2 + |u_\varepsilon - b(x)|^2]\, dx$$
$$= \lim_{\varepsilon \to 0^+} \int_A [|u_\varepsilon|^2 - 2u_\varepsilon b(x) + b(x)|^2]\, dx = \int_A \left[\frac{3}{8} k^2 - \frac{2k}{\pi} b(x) + |b(x)|^2\right] dx.$$

Therefore, for a.a. $x \in]0, 1[$ and for all $s > 0$, we have

$$\psi(x, s) \leq \frac{3}{8} \pi^2 s^2 - 2sb(x) + |b(x)|^2.$$

On the other hand

$$f^+(x, s, 0) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^2$$
$$= \begin{cases} (N+1)\, s^2 - 2sb(x) + |b(x)|^2 & \text{if } s < 0 \\ \left(\dfrac{N}{16} + 1\right) s^2 - 2sb(x) + |b(x)|^2 & \text{if } s \geq 0. \end{cases}$$

Therefore, if $N > 6\pi^2 - 16$, then $\psi(x, s) < f^+(x, s, 0)$ for a.a. $x \in \Omega$ and for all $s > 0$. If in addition $b(x) > 0$, we obtain from Corollary 3.2

$$\lim_{\varepsilon\to 0^+} \inf\{F_\varepsilon(u, \Omega): u \in W^{2,2}(\Omega)\} = \lim_{\varepsilon\to 0^+} \inf\{F_\varepsilon(u, \Omega): u \in W_0^{2,2}(\Omega)\}$$
$$= \min\left\{\int_\Omega \psi(x, u)\,dx: u \in L^2(\Omega)\right\} \leqslant \left(1 - \frac{8}{3\pi^2}\right)\int_\Omega |b(x)|^2 dx$$
$$< \left(1 - \frac{16}{N+16}\right)\int_\Omega |b(x)|^2 dx = \min\left\{\int_\Omega f^+(x, u, 0)\,dx: u \in L^2(\Omega)\right\}. \quad \blacksquare$$

We give now another example where g is a polynomial and the equality $\psi(x, s) = f^+(x, s, 0)$ is not satisfied.

PROPOSITION 5.4. *Let $n = 1$, $m = r = 2$, $p = 6$, $\Omega =]0, 1[$, and let g be defined by*

$$g(s) = s^3 + s - \frac{5}{8}.$$

Then there exist $s_0 \in]0, \frac{1}{2}[$ and $K \in]0, +\infty[$ with the following property: if $b \in L^\infty(\Omega)$ and $N \geqslant K[1 + \|b\|^4_{L^\infty(\Omega)}]$, then

$$\psi(x, s_0) < f^+(x, s_0, 0) = f(x, s_0, 0) = N|g(s_0)|^2 + |s_0 - b(x)|^6$$

for a.a. $x \in \Omega$.

PROOF. Let u be the solution of the Cauchy problem

$$\begin{cases} u'' + u^3 + u - \dfrac{5}{8} = 0 \\ u(0) = u'(0) = 0\,. \end{cases}$$

The function u is periodic with period $2T$ where

$$T = \int_0^\sigma \left(\frac{5}{4}s - s^2 - \frac{1}{2}s^4\right)^{-\frac{1}{2}} ds$$

and σ is the unique positive solution of $\frac{5}{4}s - s^2 - \frac{1}{2}s^4 = 0$. Let s_0 be defined by

$$s_0 = \frac{1}{2T}\int_0^{2T} u(x)\,dx = \frac{1}{T}\int_0^T u(x)\,dx\,.$$

Since

$$u' = \left(\frac{5}{4}u - u^2 - \frac{1}{2}u^4\right)^{\frac{1}{2}} \quad \text{in } [0, T]$$

we have

$$\int_0^T u(x)\,dx = \int_0^\sigma s\left(\frac{5}{4}s - s^2 - \frac{1}{2}s^4\right)^{-\frac{1}{2}} ds\,.$$

We prove that $s_0 < \frac{1}{2}$; this is equivalent to show that

$$\int_0^\sigma \left(s - \frac{1}{2}\right)\left(\frac{5}{4}s - s^2 - \frac{1}{2}s^4\right)^{-\frac{1}{2}} ds < 0\,. \tag{5.6}$$

Let $v(s) = (\frac{5}{4}s - s^2 - \frac{1}{2}s^4)^{\frac{1}{2}}$; the function v is increasing in $[0, \frac{1}{2}]$ and decreasing in $[\frac{1}{2}, \sigma]$. Let $v_0 = \sqrt{11/32}$, let $w_1\colon [0, v_0] \to [0, \frac{1}{2}]$ be the inverse of the function $v|_{[0,\frac{1}{2}]}$ and let $w_2\colon [0, v_0] \to [\frac{1}{2}, \sigma]$ be the inverse of the function $v|_{[\frac{1}{2},\sigma]}$; then (5.6) is equivalent to

$$\begin{aligned} \int_0^{v_0} 2\left(w_1(t) - \frac{1}{2}\right)\left[\frac{5}{4} - 2w_1(t) - 2\big(w_1(t)\big)^3\right]^{-1} dt \\ < \int_0^{v_0} 2\left(w_2(t) - \frac{1}{2}\right)\left[\frac{5}{4} - 2w_2(t) - 2\big(w_2(t)\big)^3\right]^{-1} dt\,. \end{aligned} \tag{5.7}$$

Since the function $(s - \frac{1}{2})(\frac{5}{4} - 2s - 2s^3)^{-1}$ is increasing in $[0, +\infty[$ and $0 < w_1(t) < w_2(t)$, we obtain (5.7). This proves that $s_0 < \frac{1}{2}$, hence

$$(s_0^3 + s_0 - \tfrac{5}{8})^2 > 0\,.$$

Let $u_T(x) = u(2Tx)$; note that u_T is 1-periodic and $s_0 = \int_0^1 u_T(x)\,dx$; by the representation formula for ψ we get for every $b \in L^6(\Omega)$

$$\begin{aligned} \psi(x, s_0) &\leqslant \int_0^1 \left[N\left|\frac{1}{(2T)^2}u_T''(y) + \big(u_T(y)\big)^3 + u_T(y) - \frac{5}{8}\right|^2 + |u_T(y) - b(x)|^6\right] dy \\ &= \int_0^1 |u_T(y) - b(x)|^6\,dy\,. \end{aligned} \tag{5.8}$$

Using the facts that $s_0 = \int_0^1 u_T(y)\,dy$ and that $0 \leqslant u_T(y) \leqslant \sigma < 1$, we obtain

$$\int_0^1 |u_T(y) - b(x)|^6 dy = |s_0 - b(x)|^6 + \sum_{i=0}^{6} \binom{6}{i} (-b(x))^i \Big[\int_0^1 u_T(y)^{6-i} dy - s_0^{6-i} \Big]$$
$$\leqslant |s_0 - b(x)|^6 + \sum_{i=0}^{4} \binom{6}{i} |b(x)|^i < |s_0 - b(x)|^6 + 56[1 + \|b\|^4_{L^\infty(\Omega)}] .$$

Let $K = 56(s_0^3 + s_0 - \frac{5}{8})^{-2}$; if $N \geqslant K[1 + \|b\|_{L^\infty(\Omega)}]$ we obtain from (5.8)

$$\psi(x, s_0) < |s_0 - b(x)|^6 + N\left(s_0^3 + s_0 - \frac{5}{8}\right)^2 = f^+(x, s_0, 0) = f(x, s_0, 0) ,$$

and the proposition is proved. ∎

REMARK 5.5. For every $N > 0$ let $b_N = s_0 + [(N/3)(s_0^3 + s_0 - \frac{5}{8})(3s_0^2 + 1)]^{\frac{1}{5}}$. There exists $N_0 > 0$ such that for every $N \geqslant N_0$ we have $N \geqslant K[1 + b_N^4]$. If in the previous proposition we take $N \geqslant N_0$ and $b(x) = b_N$ for every $x \in \Omega$, then we obtain from Corollary 3.2

$$\lim_{\varepsilon \to 0^+} \inf \{F_\varepsilon(u, \Omega) \colon u \in W^{2,2}(\Omega)\} = \lim_{\varepsilon \to 0^+} \inf \{F_\varepsilon(u, \Omega) \colon u \in W_0^{2,2}(\Omega)\}$$
$$= \min\left\{ \int_\Omega \psi(x, u)\,dx \colon u \in L^6(\Omega)\right\} \leqslant \int_\Omega \psi(x, s_0)\,dx < \int_\Omega f(x, s_0, 0)\,dx$$
$$= \min\left\{ \int_\Omega f(x, u, 0)\,dx \colon u \in L^6(\Omega)\right\} .$$

The following proposition generalizes some results proved by V. Komornik in [13].

PROPOSITION 5.6. *Let g be a non-negative convex function of the class $\mathcal{G}_p$, let $b \in L^p(\Omega)$, and let $N > 0$. Then for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$*

$$\psi(x, s) = f^-(x, s, 0) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^p .$$

PROOF. Since $f^-(x, s, 0) \leqslant \psi(x, s) \leqslant f(x, s, 0)$, it is enough to prove that for a.a. $x \in \Omega$ and for all $s_0 \in \mathbb{R}$ we have

$$f^-(x, s_0, 0) = f(x, s_0, 0) . \tag{5.9}$$

In order to prove (5.9) we show that

$$f(x, s, z) \geqslant f(x, s_0, 0) + \frac{\partial f}{\partial s}(x, s_0^+, 0)(s - s_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial z_{ii}}(x, s_0, 0) z_{ii} \tag{5.10}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$, $s_0 \in \mathbb{R}$, $z \in \mathbb{R}^d$. Inequality (5.10) is equivalent to

$$\begin{aligned} N\left(\sum_{i=1}^{n} z_{ii}\right)^2 &+ 2N[g(s) - g(s_0)] \sum_{i=1}^{n} z_{ii} \\ &+ \{|s - b(x)|^p + N|g(s)|^2 - |s_0 - b(x)|^p - N|g(s_0)|^2 \\ &- [p|s_0 - b(x)|^{p-1} \operatorname{sign}(s_0 - b(x)) + 2Ng(s_0)g'(s_0^+)](s - s_0)\} \geqslant 0 . \end{aligned} \tag{5.11}$$

Since the left hand side of (5.11) is a polynomial of the second order in $\sum_{i=1}^{n} z_{ii}$, inequality (5.11) is equivalent to

$$\begin{aligned} |s - b(x)|^p - p|s_0 - b(x)|^{p-1} \operatorname{sign}(s_0 - b(x))(s - s_0) - |s_0 - b(x)|^p \\ + 2Ng(s_0)[g(s) - g'(s_0^+)(s - s_0) - g(s_0)] \geqslant 0 . \end{aligned} \tag{5.12}$$

Putting $\varphi(s) = |s - b(x)|^p + 2Ng(s_0)g(s)$, inequality (5.12) can be written in the form $\varphi(s) - \varphi'(s_0^+)(s - s_0) - \varphi(s_0) \geqslant 0$ which is always satisfied because the function φ is convex. ■

The following proposition generalizes some results proved by A. Bensoussan in [2].

PROPOSITION 5.7. *Suppose that g is a function which is convex and non-negative for $s \geqslant 0$, concave and non-positive for $s \leqslant 0$, and which satisfies $|g(s)| \leqslant c|s|^{p/2}$ for every $s \in \mathbb{R}$. Then there exists $N_0 > 0$ (depending only on the constants p and c) such that for every $N \in]0, N_0]$ and for every $b \in L^p(\Omega)$ we have*

$$\psi(x, s) = f^-(x, s, 0) = f(x, s, 0) = N|g(s)|^2 + |s - b(x)|^p$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

PROOF. As in Proposition 5.6 we have only to prove that

$$\begin{aligned} |s - b|^p - p|s_0 - b|^{p-1} \operatorname{sign}(s_0 - b)(s - s_0) - |s_0 - b|^p \\ + 2Ng(s_0)[g(s) - g'(s_0^+)(s - s_0) - g(s_0)] \geqslant 0 \end{aligned} \tag{5.13}$$

for all $s, s_0, b \in \mathbb{R}$. Let $\varphi(s) = |s - b|^p + 2Ng(s_0)g(s)$; if $s_0 \geqslant 0$ the function φ

is convex on $[0, +\infty[$; if $s_0 \leqslant 0$ the function φ is convex on $]-\infty, 0]$. Therefore, if $ss_0 \geqslant 0$

$$\varphi(s) - \varphi'(s_0^+)(s - s_0) - \varphi(s_0) \geqslant 0, \tag{5.14}$$

hence (5.13) is proved in the case $ss_0 \geqslant 0$. Suppose now $s_0 > 0$ and $s < 0$; let

$$\begin{aligned}\alpha(s, b) = |s - b|^p - p|s_0 - b|^{p-1} \operatorname{sign}(s_0 - b)(s - s_0) - |s_0 - b|^p \\ + 2Ng(s_0)[g(s) - g'(s_0^+)(s - s_0) - g(s_0)];\end{aligned}$$

we want to prove that $(\partial\alpha/\partial s)(s^+, b) \leqslant 0$. We have

$$\begin{aligned}\frac{\partial\alpha}{\partial s}(s^+, b) = p|s - b|^{p-1} \operatorname{sign}(s - b) - p|s_0 - b|^{p-1} \operatorname{sign}(s_0 - b) \\ + 2Ng(s_0)[g'(s^+) - g'(s_0^+)];\end{aligned}$$

therefore

$$\begin{aligned}\max_{b\in\mathbf{R}} \frac{\partial\alpha}{\partial s}(s^+, b) = \frac{\partial\alpha}{\partial s}\left(s^+, \frac{s + s_0}{2}\right) &= -p2^{2-p}|s - s_0|^{p-1} + 2Ng(s_0)[g'(s^+) - g'(s_0^+)] \\ &\leqslant -p^{2-p}(s_0 + |s|)^{p-1} + 2NK(c, p)\, s_0^{p/2}|s|^{-1+p/2} \\ &\leqslant \big(-p2^{2-p} + 2NK(c, p)\big)(s_0 + |s|)^{p-1}\end{aligned}$$

where $K(c,p) = c(p/2)\big(p/(p-2)\big)^{-1+p/2}$ if $p > 2$, $K(c,p) = c$ if $p = 2$.

If $0 < N \leqslant \big(p2^{1-p}/K(c,p)\big)$ we have $(\partial\alpha/\partial s)(s^+, b) \leqslant 0$ for every $s < 0$, $b \in \mathbf{R}$. This implies that $\alpha(s, b) \geqslant \alpha(0, b) = \varphi(0) + \varphi'(s_0^+)s_0 - \varphi(s_0)$; therefore by (5.14) we get $\alpha(s, b) \geqslant 0$, hence (5.1) is proved for $s_0 > 0$, $s < 0$. The case $s_0 < 0$, $s > 0$ can be proved in the same way. ∎

The previous proposition applies for instance to the case $g(s) = s|s|^{-1+p/2}$ and to the case considered in Proposition 5.3.

If $b \in L^p(\Omega)$ and if g satisfies the conditions of Proposition 5.7, it is possible to prove that the set

$$\{N \in]0, +\infty[: \psi(x, s) = N|g(s)|^2 + |s - b(x)|^p \text{ for a.a. } x \in \Omega \text{ and for all } s \in \mathbf{R}\}$$

is an interval. In fact the following result holds.

PROPOSITION 5.8. *Let $f(x, s, z)$ be a function satisfying* (3.1), (3.2), (3.3) *and let $b \in L^p(\Omega)$. For every $\lambda > 0$ let*

$$f_\lambda(x, s, z) = f(x, s, z) + \lambda|s - b(x)|^p,$$

and let $\psi_\lambda(x, s)$ *be the integrand of the* Γ*-limit associated to* f_λ *by Theorem* 3.1. *If there exists* $\lambda_0 > 0$ *such that* $\psi_{\lambda_0}(x, s) = f_{\lambda_0}(x, s, 0)$ *for a.a.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$, *then for all* $\lambda \geq \lambda_0$ *we have* $\psi_\lambda(x, s) = f_\lambda(x, s, 0)$ *for a.a.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$.

PROOF. Let $\lambda > \lambda_0$; by Proposition 2.3 and by (5.1)

$$\begin{aligned} f_\lambda(x, s, 0) &= f_{\lambda_0}(x, s, 0) + (\lambda - \lambda_0)|s - b(x)|^p \\ &= \psi_{\lambda_0}(x, s) + (\lambda - \lambda_0)|s - b(x)|^p \leq \psi_\lambda(x, s) \leq f_\lambda(x, s, 0) \end{aligned}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$. ■

We show now a situation where $\psi(x, s) = f^{-}(x, s, 0)$.

PROPOSITION 5.9. *Let* $n = 1$ *(hence* $d = m$*) and let* $f(x, s, z)$ *be a function satisfying* (3.1), (3.2), (3.3). *Suppose that*

$$f(x, s, z) = f_1(x, s) + f_2(x, z_m) \qquad \text{for all } x \in \Omega,\ s \in \mathbb{R},\ z \in \mathbb{R}^m .$$

Then, if $\psi(x, s)$ *is the integrand of the* Γ*-limit associated to* f *by Theorem* 3.1, *we have for a.a.* $x \in \Omega$ *and for all* $s \in \mathbb{R}$

$$\psi(x, s) = f^{-}(x, s, 0) = \bar{f}_1(x, s) + \bar{f}_2(x, 0) ,$$

where $\bar{f}_1(x, s)$ *denotes the greatest function convex in* s *which is less than or equal to* $f_1(x, s)$ *and* $\bar{f}_2(x, z_m)$ *denotes the greatest function convex in* z_m *which is less than or equal to* $f_2(x, z_m)$.

PROOF. By (5.1) it is enough to prove that

$$\psi(x, s) \leq f_1(x, s) + \bar{f}_2(x, 0) \tag{5.15}$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$. Fix $x \in \Omega$, $s \in \mathbb{R}$, $\eta > 0$; there exist $z > 0$, $w < 0$, and $0 < \lambda < 1$ such that $\lambda z + (1 - \lambda) w = 0$ and

$$\lambda f_2(x, z) + (1 - \lambda) f_2(x, w) < \eta + \bar{f}_2(x, 0) .$$

For every $h \in \mathbb{N}$ set

$$I_h = \bigcup_{k=-\infty}^{+\infty} \left] \frac{k}{h}, \frac{k + \lambda}{h} \right[\quad \text{and} \quad J_h = \bigcup_{k=-\infty}^{+\infty} \left] \frac{k + \lambda}{h}, \frac{k + 1}{h} \right[;$$

it is easy to prove that there exists a unique 1-periodic function u_h such that

$$\int_0^1 u_h(y)\,dy = s \qquad \text{and} \qquad u_h^{(m)} = \begin{cases} z & \text{on } I_h \\ w & \text{on } J_h \,. \end{cases}$$

By the representation formula for ψ we have

$$\psi(x, s) \leqslant \int_0^1 [f_1(x, u_h(y)) + f_2(x, u_h^{(m)}(y))]\,dy$$
$$= \int_0^1 f_1(x, u_h(y))\,dy + \lambda f_2(x, z) + (1-\lambda) f_2(x, w)\,.$$

Since (u_h) converges to s uniformly and $f_1(x, s)$ is continuous in s we have

$$\psi(x, s) \leqslant f_1(x, s) + \bar{f}_2(x, 0) + \eta\,.$$

Since η was arbitrary we obtain (5.15) and so the proposition is proved. ■

REMARK 5.10. The previous proposition applies for example to the case

$$F_\varepsilon(u, A) = \int_A [|(\varepsilon^2 u'')^2 - a(x)|^2 + |u - b(x)|^4]\,dx$$

with $a \in L^2(\Omega)$ and $b \in L^4(\Omega)$. In this case we obtain

$$\psi(x, s) = f^-(x, s, 0) = (a(x) \wedge 0)^2 + |s - b(x)|^4$$

while $f^+(x, s, 0) = |a(x)|^2 + |s - b(x)|^4$.

6. – Appendix.

In this section we prove that the function

$$f(x, s, z) = N\left|\sum_{i=1}^n z_{ii} + g(s) + a(x)\right|^2 + |s - b(x)|^p$$

$(z = (z_{ij})_{1 \leqslant i+j \leqslant 2})$ satisfies condition (3.2) whenever $N > 0$, $p \geqslant 2$, $g \in \mathcal{G}_p$, $a \in L^2(\Omega)$, $b \in L^p(\Omega)$, where $\mathcal{G}_p$ is the class of functions defined in section 5. Condition (3.1) is trivial for f and condition (3.3) follows from well known estimates for the Laplace operator.

First of all we extend the functions a and b to all of $\mathbb{R}^n$, by setting $a(x) = b(x) = 0$ for $x \in \mathbb{R}^n - \Omega$; so the function f is extended to $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$.

We shall use the following elementary inequalities, which hold for every $\alpha > 0$, $\beta > 0$, $p > 1$:

$$\alpha\beta \leqslant \frac{1}{p}\alpha^p + \frac{1}{q}\beta^q$$

$$|\alpha^p - \beta^p| \leqslant p(1 \vee 2^{p-2})(\alpha^{p-1}|\alpha - \beta| + |\alpha - \beta|^p)$$

$$(\alpha + \beta)^p \leqslant 2^{p-1}\alpha^p + 2^{p-1}\beta^p .$$

The last inequality implies that

$$|s|^p \leqslant 2^{p-1} f(x, s, z) + 2^{p-1}|b(x)|^p .$$

In what follows $q = p/(p-1)$ and c_1, c_2, c_3 are positive constants independent of x, y, s, t, z, w. Let $\eta\colon \mathbb{R}^n \to [0, +\infty[$ be an arbitrary function with $\eta(0) = 0$ and $\eta(y) > 0$ for $y \neq 0$, and let $\eta^*\colon \mathbb{R}^n \to [0, +\infty[$ be defined by $\eta^*(0) = 0$ and $\eta^*(y) = \eta(y)^{-1}$ for $y \neq 0$. For every $x, y \in \mathbb{R}^n$, $s, t \in \mathbb{R}$, $z, w \in \mathbb{R}^d$ we have

$$
\begin{aligned}
(6.1)\quad & |f(x+y, s+t, z+w) - f(x, s, z)| \\
& \leqslant c_1 \Big\{ \Big| \sum_{i=1}^n z_{ii} + g(s) + a(x) \Big| \Big[\sum_{i=1}^n |w_{ii}| + \varrho(|t|)(1 + |s|)^{p/2} + |a(x+y) - a(x)| \Big] \\
& + \Big[\sum_{i=1}^n |w_{ii}| + \varrho(|t|)(1 + |s|)^{p/2} + |a(x+y) - a(x)| \Big]^2 \\
& + |s - b(x)|^{p-1} [|t| + |b(x+y) - b(x)|] + [|t| + |b(x+y) - b(x)|^p] \Big\} \\
& \leqslant c_2 \Big\{ f(x, s, z)^{\frac12} \sum_{i=1}^n |w_{ii}| + f(x, s, z)^{\frac12} \varrho(|t|)(f(x, s, z) + |b(x)|^p + 1)^{\frac12} \\
& + f(x, s, z)^{\frac12} |a(x+y) - a(x)| + \Big(\sum_{i=1}^n |w_{ii}| \Big)^2 \\
& + \varrho(|t|)^2 (f(x, s, z) + |b(x)|^p + 1) + |a(x+y) - a(x)|^2 \\
& + f(x, s, z)^{1/q}|t| + f(x, s, z)^{1/q}|b(x+y) - b(x)| + |t|^p + |b(x+y) - b(x)|^p \Big\} \\
& \leqslant c_3 \Big\{ (f(x, s, z) + |b(x)|^p + 1) \Big[\sum_{i=1}^n |w_{ii}| + \varrho(|t|) + \Big(\sum_{i=1}^n |w_{ii}| \Big)^2 \\
& + \varrho(|t|)^2 + |t| + |t|^p \Big] + \eta(y) f(x, s, z) \\
& + \eta^*(y)|a(x+y) - a(x)|^2 + |a(x+y) - a(x)|^2 \\
& + \eta(y)^{q/p} f(x, s, z) + \eta^*(y)|b(x+y) - b(x)|^p + |b(x+y) - b(x)|^p \Big\} \\
& \leqslant (f(x, s, z) + |b(x)|^p + 1)\lambda(y, t, w) \\
& + c_3(1 + \eta^*(y))[|a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p]
\end{aligned}
$$

where

$$\lambda(y, t, w)$$
$$= c_3\left[\sum_{i=1}^{n} |w_{ii}| + \varrho(|t|) + \left(\sum_{i=1}^{n} |w_{ii}|\right)^2 + \varrho(|t|)^2 + |t| + |t|^p + \eta(y) + \eta(y)^{q/p}\right].$$

Since $a \in L^2(\mathbf{R}^n)$ and $b \in L^p(\mathbf{R}^n)$, we have

$$\lim_{y\to 0} \int_{\mathbf{R}^n} [|a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p]\, dx = 0 .$$

Therefore there exists a continuous function $\eta: \mathbf{R}^n \to [0, +\infty[$ such that $\eta(0) = 0$, $\eta(y) > 0$ for $y \neq 0$, and

$$\lim_{y\to 0} (1 + \eta^*(y)) \int_{\mathbf{R}^n} [|a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p]\, dx = 0 .$$

For every $x, y \in \mathbf{R}^n$ we set

$$\omega(x, y) = c_3(1 + \eta^*(y))\, [|a(x+y) - a(x)|^2 + |b(x+y) - b(x)|^p] .$$

Since λ is continuous and $\lambda(0, 0, 0) = 0$, there exists an increasing continuous function $\sigma: [0, +\infty[\to [0, +\infty[$, with $\sigma(0) = 0$, such that

$$\lambda(y, t, w) \leqslant \sigma(|y| + |t| + |w|)$$

for every $y \in \mathbf{R}^n$, $t \in \mathbf{R}$, $w \in \mathbf{R}^d$.

Therefore from (6.1) it follows that

$$|f(x+y, s+t, z+w) - f(x, s, z)|$$
$$\leqslant \sigma(|y| + |t| + |w|)(f(x, s, z) + |b(x)|^p + 1) + \omega(x, y)$$

for every $x, y \in \mathbf{R}^n$, $s, t \in \mathbf{R}$, $z, w \in \mathbf{R}^d$. This shows that condition (3.2) is satisfied.

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