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Singular Perturbations of Nonlinear Degenerate Parabolic PDEs: a General Convergence Result

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Abstract

The main result of the paper is a general convergence theorem for the viscosity solutions of singular perturbation problems for fully nonlinear degenerate parabolic PDEs (partial differential equations) with highly oscillating initial data. It substantially generalizes some results obtained previously in [2].

Under the only assumptions that the Hamiltonian is ergodic and stabilizing in a suitable sense, the solutions are proved to converge in a relaxed sense to the solution of a limit Cauchy problem with appropriate effective Hamiltonian and initial data. In its formulation, our convergence result is analogous to the stability property of Barles and Perthame. It should thus reveal a useful tool for studying general singular perturbation problems by viscosity solutions techniques. A detailed exposition of ergodicity and stabilization is given, with many examples. Applications to homogenization and averaging are also discussed.

1. Introduction

One of the major advantages of the theory of viscosity solutions of fully nonlinear degenerate elliptic equations is the stability property of the solutions. It allows us to pass to the limit of regular perturbations problems in an elementary way. This was applied successfully to prove the existence of viscosity solutions (by the vanishing viscosity method, which explains the name of the solutions), to the convergence of numerical schemes, to large deviations problems, ... For parabolic problems, the stability property can be stated as follows. For $\varepsilon > 0$, consider a viscosity solution of the equation

$$u_t^\varepsilon + H^\varepsilon(x, Du^\varepsilon, D^2u^\varepsilon) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \quad u^\varepsilon(0,x) = h^\varepsilon(x) \quad \text{on } \mathbb{R}^n.$$

The function u^ε is scalar; Du^ε and D^2u^ε denote respectively the gradient and the Hessian of u^ε with respect to the space variable x; H^ε is a fully nonlinear degenerate elliptic operator (i.e., is nonincreasing with respect to D^2u^ε). If $H^\varepsilon \to H$ and

$h^\varepsilon \to h$ as $\varepsilon \to 0$ uniformly on the compact sets and if u^ε converges uniformly on the compact sets to a function u, then u must be a viscosity solution of the limit equation

$$u_t + H(x, Du, D^2u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \quad u(0, x) = h(x) \quad \text{on } \mathbb{R}^n.$$

As it may be a delicate matter to show the local uniform convergence of u^ε, Barles and Perthame proved that we can assume simply that the family $\{u^\varepsilon\}$ is locally equibounded provided the comparison principle holds for the limit equation. We refer to the User's Guide [20] and the books [12] and [11] for a detailed exposition and for applications.

The purpose of this paper is to give a convergence result that is similar to the stability property but that applies to singular perturbations problems. For small $\varepsilon > 0$, we consider the solution u^ε of the following Cauchy problem for a degenerate parabolic equation

$$\begin{aligned} u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx} u^\varepsilon, \frac{D_{yy} u^\varepsilon}{\varepsilon}, \frac{D_{xy} u^\varepsilon}{\sqrt{\varepsilon}}\right) & \\ = 0 \qquad \text{in } (0, T) \times \mathbb{R}^n \times \mathbb{R}^m, & \\ u^\varepsilon(0, x, y) = h(x, y) \quad \text{on } \mathbb{R}^n \times \mathbb{R}^m. & \end{aligned} \tag{HJ$_\varepsilon$}$$

The state variable (x, y) splits into the slow variable $x \in \mathbb{R}^n$ and the fast variable $y \in \mathbb{R}^m$. The parameter ε only acts on the derivatives with respect to the fast variable. In order to simplify the presentation, all the data of the problem are assumed to be periodic in the fast variable, so that the solution u^ε itself is periodic in y, but the convergence result remains valid for other kind of boundary conditions (see the remarks after Theorems 2 and 3). We want to study the limit of the solution u^ε as $\varepsilon \to 0$.

The singular perturbation problem modelled by equation (HJ$_\varepsilon$) has several important motivations. The first is the reduction of dimension by scale separation in the optimal control of deterministic and stochastic systems. Basic references on this matter are the books by KOKOTOVIĆ, KHALIL & O'REILLY [31], which contains many problems arising in industry, BENSOUSSAN [15] and KUSHNER [32]. The PDE approach to this issue, based on the Hamilton-Jacobi-Bellman (briefly, HJB) equation, was started by JENSEN & LIONS [29] on quasilinear uniformly elliptic PDEs, and by LIONS on a first-order Hamilton-Jacobi (briefly, HJ) equation in a short section of his book [34]. More recent results for first-order equations with various boundary conditions are in [11, 9, 8, 2], and our paper [2] makes a systematic use of this approach for controlled degenerate diffusions in the periodic case. Recently LASRY & LIONS [33] found similar problems in some models arising in finance. The applications of the results of the present paper to HJB equations, to the more general Hamilton-Jacobi-Isaacs equations, and to the associated control and differential game problems, are presented in the companion paper [3] where more references are also given.

The second important motivation is the problem of periodic homogenization for first-order and parabolic operators not in divergence form. If we consider the

HJ equation

$$v_t^\varepsilon + G\left(x, \frac{x}{\varepsilon}, Dv^\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n,$$

and look for a solution of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{x}{\varepsilon})$, we find the singularly perturbed equation

$$u_t^\varepsilon + G\left(x, y, D_x u^\varepsilon + \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^n.$$

Similarly, the parabolic equation

$$v_t^\varepsilon + G\left(x, \frac{x}{\varepsilon}, D^2 v^\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n$$

is transformed into

$$u_t^\varepsilon + G\left(x, y, D_{xx}u^\varepsilon + \frac{D_{yy}u^\varepsilon}{\varepsilon^2} + \frac{D_{xy}u^\varepsilon}{\varepsilon} + \frac{(D_{xy}u^\varepsilon)^T}{\varepsilon}\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R}^n.$$

The last two equations for u^ε are both special cases of the PDE in (HJ_ε). If G depends on Dv^ε and D^2v^ε simultaneously, the equation for u^ε has a scaling different from (HJ_ε), but our method can be adapted to cover this case as well: this was showed in [2] for some HJB equations and is done systematically in [4]. We recall that the problem of homogenization arises in the study of the macroscopic properties of models with fast oscillations on a microscopic scale, and we refer to the books by BENSOUSSAN, J.-L. LIONS & PAPANICOLAOU [17] and JIKOV, KOZLOV & OLEINIK [30] for presentations of the general theory, mostly for linear and variational problems. The first result on quasilinear equations not in divergence form is in [16]. The approach to homogenization of fully nonlinear equations by viscosity solutions methods begins with the pioneering unpublished paper by LIONS, PAPANICOLAOU & VARADHAN [35] that introduced the effective Hamiltonian and gave the first convergence result for a HJ equation. The convergence proof was then simplified and extended to second-order equations by EVANS [21, 22] (following some suggestions of P.-L. Lions), who introduced for this purpose the perturbed test function method. Other contributions are [10, 28, 1, 38, 36, 2, 23, 24]; see also the references therein.

A third motivation is the averaging of PDEs with fast oscillations in the time variable. This is the problem of letting $\varepsilon \to 0$ in the degenerate parabolic equation

$$v_t^\varepsilon + F\left(x, \frac{t}{\varepsilon}, Dv^\varepsilon, D^2v^\varepsilon\right) = 0 \quad \text{in } (0,T)\times\mathbb{R}^n.$$

If we look for solutions of the form $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{t}{\varepsilon})$ we find the singular perturbation problem

$$u_t^\varepsilon + F\left(x, y, D_x u^\varepsilon, D_{xx}u^\varepsilon\right) + \frac{u_y^\varepsilon}{\varepsilon} = 0 \quad \text{in } (0,T)\times\mathbb{R}^n\times\mathbb{R},$$

which is of the form appearing in (HJ_ε). We refer again to [17] for the classical results on linear uniformly parabolic equations, to [22, 14] for the averaging of a first order HJB equation, and to [28] for the joint homogenization and averaging of HJ equations.

The question of the convergence of u^ε is delicate because the Hamiltonian in (HJ_ε) has no limit as $\varepsilon \to 0$. The desired result is

$$u^\varepsilon(t, x, y) \to u(t, x) \qquad \text{uniformly in } y, \quad \text{as } \varepsilon \to 0,$$

the limit u solving an appropriate effective Cauchy problem

$$u_t + \overline{H}(x, Du, D^2u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \quad u(0, x) = \overline{h}(x) \quad \text{on } \mathbb{R}^n. \qquad (\overline{\text{HJ}})$$

The effective Hamiltonian $\overline{H}$ and the effective initial data $\overline{h}$ are to be defined. It is an important feature of the problem that the limit should be independent of the fast variable y; it means that the we have got a macroscopic model in n space dimensions by eliminating the microscopic oscillations taking place in $\mathbb{R}^m$ on a faster scale. This can be informally justified by observing that sending $\varepsilon \to 0$ in (HJ_ε) should force the derivatives with respect to the fast variable y to vanish in order to prevent the terms $D_y u^\varepsilon/\varepsilon$ and $D_{yy}u^\varepsilon/\varepsilon$ from blowing up.

The principal virtue of our convergence result is the very mild assumptions on the Hamiltonian and the initial data. Precisely, we single out two crucial properties of the Hamiltonian with respect to the y variables, called *ergodicity* and *stabilization to a constant*. We give three equivalent definitions of ergodicity. The most suggestive of them considers, for frozen $\overline{x}, \overline{p}, \overline{X}$, the cell t-problem

$$w_t + H(\overline{x}, y, \overline{p}, D_y w, \overline{X}, D^2_{yy} w, 0) = 0 \text{ in } (0, +\infty) \times \mathbb{R}^m, \ w(0, y) = 0 \text{ on } \mathbb{R}^m. \qquad \text{(CP)}$$

The Hamiltonian is ergodic at $(\overline{x}, \overline{p}, \overline{X})$ if the solution $w(t, y; \overline{x}, \overline{p}, \overline{X})$ of this Cauchy problem, divided by t, converges to a constant as $t \to +\infty$, uniformly in y. If this is the case we define

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = - \lim_{t \to +\infty} \frac{w(t, y; \overline{x}, \overline{p}, \overline{X})}{t}.$$

The connection with the classical notion of ergodicity is transparent if H is linear with respect to $D_y w$ and $D^2_{yy} w$ [17, 26, 15]. In fact, in this case $w(t, y)$ can be represented as an integral over $[0, t]$ of a function of the trajectories of a dynamical system (deterministic or stochastic) with initial position y, and our definition requires that the time average of this function in the long run forgets the dependence on the initial position. Our definition of effective Hamiltonian is more general than the usual one based on the stationary cell problem [35, 21–23], that in our case is

$$H(\overline{x}, y, \overline{p}, D\chi, \overline{X}, D^2\chi, 0) = \overline{H} \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic.}$$

The two definitions coincide whenever the last cell problem has a continuous solution χ, but this does not necessarily happen in one of our main applications, the nonresonant case [6]. Our definition of stabilization to a constant and of $\overline{h}$ is based again on the asymptotic behaviour of the solution of a degenerate parabolic Cauchy problem in the fast variable with frozen slow variables. Now we replace the null initial condition in (CP) with $w(0, y) = h(\overline{x}, y)$ and H with its homogeneous part with respect to $D_y w$ and $D^2_{yy} w$; this is easy to define and interpret if H is linear in $D_y w$ and $D^2_{yy} w$. The general case is a bit technical and requires a suitable recession function; see Section 2.4.

The convergence result is the following. We assume the local equiboundedness of the family $\{u^\varepsilon\}$. If H is ergodic, then the relaxed semi-limits of u^ε are a sub- or a supersolution of the limit effective PDE. If the pair (H, h) is stabilizing to a constant, then the effective initial condition $\overline{h}$ is attained as $\varepsilon \to 0$ in a suitable sense. If, in addition, the comparison principle holds for ($\overline{\text{HJ}}$), then u^ε converges locally uniformly to the solution of ($\overline{\text{HJ}}$). Therefore we can conclude that, for small ε, the original problem in $\mathbb{R}^m \times \mathbb{R}^m$ decouples into three problems in lower dimensions: two problems in $\mathbb{R}^m$ corresponding to the fast scale that determine the effective data $\overline{H}, \overline{h}$, and the effective problem ($\overline{\text{HJ}}$) in $\mathbb{R}^n$ corresponding to the slow scale.

A model problem for testing the convergence theorem is

$$\begin{aligned} u^\varepsilon_t + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) - \frac{1}{\varepsilon} \operatorname{tr}(b(x, y) D_{yy} u^\varepsilon) \\ + \frac{c(x, y)}{\varepsilon} |D_y u^\varepsilon| - \frac{1}{\varepsilon} (g(x, y), D_y u^\varepsilon) = 0, \end{aligned}$$

with smooth and bounded coefficients, F degenerate elliptic, and b nonnegative semidefinite. In this model the Hamiltonian is ergodic and stabilizing, and therefore the semi-limits satisfy ($\overline{\text{HJ}}$), if, for any $\overline{x} \in \mathbb{R}^n$, one of the following conditions hold:

- for some $\nu > 0$, $b(\overline{x}, y) \geqq \nu I_m$ for all $y \in \mathbb{R}^m$;
- for some $\nu > 0$, $c(\overline{x}, y) \geqq |g(\overline{x}, y)| + \nu$ and $b(\overline{x}, y) = 0$ for all $y \in \mathbb{R}^m$;
- $b(\overline{x}, y)$, $c(\overline{x}, y)$, $g(\overline{x}, y)$ are constant in y, $c \geqq 0$, and $b(\overline{x})k \neq 0$ for all $k \in \mathbb{Z}^m \backslash \{0\}$.

Each of these three conditions is the prototype of a more general property of the Hamiltonian H in (HJ_ε), which we call, respectively, the *nondegenerate*, the *coercive*, and the *nonresonant* case.

Our convergence result improves upon the existing literature in three main directions. The most important contribution is that we single out the fact that ergodicity and stabilization alone are sufficient to guarantee some convergence. The second improvement is that we allow the initial data to depend on the fast variable and that we define the effective initial data. This issue was considered in [18] for the heat equation, and the relationship between stabilization to a constant and the definition of the effective data was shown by Zhikov for linear parabolic problems, see Section 10.4 of [30], but the extension to nonlinear equations seems completely new (see also [9, 2] for coercive first-order problems, [13] and the references therein for some related results on first-order equations). Finally, the use of the theory of viscosity solutions allows us to consider operators that are fully nonlinear with respect to all derivatives.

Section 2 is devoted to the statement of the convergence result. It contains also all the assumptions and the definitions of ergodicity, stabilization to a constant, effective Hamiltonian, and effective initial data. In Section 3, the proof of the convergence of u^ε in $(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ is given. In Section 4, we show the convergence of u^ε near $\{0\}\times\mathbb{R}^n\times\mathbb{R}^m$. Section 5 is devoted to illustrations of the definitions and of the use of the convergence result. It recalls and extends the sufficient conditions for the ergodicity of the Hamiltonian established by ARISAWA & LIONS [6] and studies the stabilization problem under similar assumptions, with a special emphasis on the model problem. Applications to homogenization and averaging are also given. Finally, the Appendix explains the relationship between the ergodic properties of a deterministic dynamical system and the associated linear first-order Hamiltonian. The section also contains a general nonlinear version of an Abelian-Tauberian theorem that provides several equivalent characterizations for the effective Hamiltonian.

2. The abstract convergence result

This section is devoted to the presentation of our main convergence result. The precise assumptions on the Hamiltonian H are given, as well as the definition of ergodicity and stabilization to a constant. The convergence result, stated in Section 2.5, will be proved in Sections 3 and 4.

2.1. The standing assumptions

We are given a Hamiltonian $H:\ \mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{S}^n\times\mathbb{S}^m\times\mathbb{M}^{n,m}\to\mathbb{R}$, where $\mathbb{S}^k$ denotes the space of $k\times k$ symmetric matrices and $\mathbb{M}^{n,m}$ the set of the $n\times m$ real matrices. We associate with H the function $\mathcal{H}:\ \mathbb{R}^{n+m}\times\mathbb{R}^{n+m}\times\mathbb{S}^{n+m}\to\mathbb{R}$ defined by

$$\mathcal{H}(\xi,P,\Theta):=H(x,y,p,q,X,Y,Z),$$

where $\xi:=(x,y)$, $P:=(p,q)$, and $\Theta:=\begin{pmatrix} X & Z\\ Z^T & Y\end{pmatrix}$ with $x,p\in\mathbb{R}^n$, $y,q\in\mathbb{R}^m$, $X\in\mathbb{S}^n$, $Y\in\mathbb{S}^m$, $Z\in\mathbb{M}^{n,m}$, and where Z^T denotes the transpose of Z. We make the following standing assumptions on H:

- H is continuous and degenerate elliptic (i.e., $\mathcal{H}(\xi,P,\Theta)\leqq\mathcal{H}(\xi,P,\Theta')$ for $\Theta\geqq\Theta'$);
- $\mathcal{H}$ satisfies the usual regularity condition for the comparison principle to hold in bounded domains [20]: for every $R>0$, there is a modulus ω_R such that, for every $\kappa>0$, $\xi,\xi'\in\mathbb{R}^{n+m}$ with $|\xi|,\ |\xi'|\leqq R$ and any $\Theta,\Theta'\in\mathbb{S}^{n+m}$ so that

$$-3\kappa\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\leqq\begin{pmatrix}\Theta & 0\\ 0 & -\Theta'\end{pmatrix}\leqq 3\kappa\begin{pmatrix} I & -I\\ I & I\end{pmatrix},$$

we have

$$\mathcal{H}(\xi',\kappa(\xi-\xi'),\Theta')\leqq\mathcal{H}(\xi,\kappa(\xi-\xi'),\Theta)+\omega_R(|\xi'-\xi|+\kappa|\xi'-\xi|^2). \tag{1}$$

– The Hamiltonian H satisfies the usual regularity with respect to the fast variables (y, q, Y) uniformly for bounded slow variables $(\tilde{x}, \tilde{p}, \tilde{X})$. Namely, for all $R > 0$ there is a concave modulus ω_R such that for every $\kappa > 0$, $|\tilde{x}|, |\tilde{p}|, |\tilde{X}| \leqq R$ and $Y, Y' \in \mathbb{S}^m$ so that

$$-3\kappa \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqq \begin{pmatrix} Y & 0 \\ 0 & -Y' \end{pmatrix} \leqq 3\kappa \begin{pmatrix} I & -I \\ I & I \end{pmatrix}, \tag{2}$$

we have

$$\begin{aligned} &H(\tilde{x}, y', \tilde{p}, \kappa(y - y'), \tilde{X}, Y', 0) \\ &\quad \leqq H(\tilde{x}, y, \tilde{p}, \kappa(y - y'), \tilde{X}, Y, 0) + \omega_R(|y' - y| + \kappa|y' - y|^2). \end{aligned} \tag{3}$$

– The Hamiltonian H is periodic in y, i.e., $H(x, y, p, q, X, Y, Z) = H(x, y + k, p, q, X, Y, Z)$ for all $k \in \mathbb{Z}^m$.
– The initial condition $h(x, y)$ is continuous in (x, y) and periodic in y.

Let us make a few comments on the regularity assumption (3) on H. Note that the inequality follows from the assumption (1) on $\mathcal{H}$ in the special case $\tilde{p} = 0$, $\tilde{X} = 0$. For partially separated operators of the form $H = G_1(x, y, p, X, Z) + G_2(x, y, q, Y)$ with G_i continuous, (3) follows from the usual regularity condition on G_2 for the comparison in $\mathbb{R}^m$, provided this property is uniform with respect to x bounded. Moreover, if a parametrized family of operators satisfy the condition (3) with the same modulus, and if an operator obtained by taking the sup or the inf over the parameters is finite, then it satisfies (3) as well. For linear operators depending on parameters $\alpha \in A$ and $\beta \in B$

$$\begin{aligned} &L_{\alpha,\beta}(x, y, p, q, X, Y, Z) \\ &\quad := -\operatorname{tr}(a_{\alpha,\beta}(x, y)X) - \operatorname{tr}(b_{\alpha,\beta}(x, y)Y) - \operatorname{tr}(c_{\alpha,\beta}(x, y)Z) \\ &\qquad - \operatorname{tr}(Zc_{\alpha,\beta}(x, y)) - (p, f_{\alpha,\beta}(x, y)) - (q, g_{\alpha,\beta}(x, y)) - l_{\alpha,\beta}(x, y), \end{aligned} \tag{4}$$

where tr denotes the trace and $(\cdot, \cdot)$ the scalar product, both regularity conditions (1) and (3) are satisfied if the matrices are of the form

$$a_{\alpha,\beta} = \sigma_{\alpha,\beta}\sigma_{\alpha,\beta}^T/2, \quad b_{\alpha,\beta} = \tau_{\alpha,\beta}\tau_{\alpha,\beta}^T/2, \quad c_{\alpha,\beta} = \tau_{\alpha,\beta}\sigma_{\alpha,\beta}^T/2$$

and if $f_{\alpha,\beta}$, $g_{\alpha,\beta}$, $\sigma_{\alpha,\beta}$, $\tau_{\alpha,\beta}$, $l_{\alpha,\beta}$ are functions in $\mathbb{R}^n \times \mathbb{R}^m$ with values, respectively, in $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{M}^{n,r}$, $\mathbb{M}^{m,r}$ and $\mathbb{R}$, with $l_{\alpha,\beta}$ continuous and $f_{\alpha,\beta}$, $g_{\alpha,\beta}$, $\sigma_{\alpha,\beta}$, $\tau_{\alpha,\beta}$ Lipschitz continuous in (x, y). The associated Hamilton-Jacobi-Bellman-Isaacs (briefly, HJBI) operator

$$H(x, y, p, q, X, Y, Z) := \sup_{\alpha \in A} \inf_{\beta \in B} L_{\alpha,\beta}(x, y, p, q, X, Y, Z) \tag{5}$$

also verifies the assumptions (1) and (3) if all the local bounds and all the moduli of continuity of the data $f_{\alpha,\beta}$, $g_{\alpha,\beta}$, $\sigma_{\alpha,\beta}$, $\tau_{\alpha,\beta}$, $l_{\alpha,\beta}$ are uniform in α and β. This class of examples is the reason for calling H the Hamiltonian.

2.2. *The ε-problem*

Our problem is to pass to the limit as $\varepsilon > 0$ tends to 0 in the degenerate parabolic equation

$$u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx} u^\varepsilon, \frac{D_{yy} u^\varepsilon}{\varepsilon}, \frac{D_{xy} u^\varepsilon}{\sqrt{\varepsilon}}\right) = 0 \text{ in } (0,T) \times \mathbb{R}^n \times \mathbb{R}^m,$$
$$u^\varepsilon(0, x, y) = h(x, y) \text{ on } \mathbb{R}^n \times \mathbb{R}^m. \tag{HJ$_\varepsilon$}$$

The notation $D_{xx}u$ and $D_{yy}u$ is used for the Hessian matrices of a function $u = u(t, x, y)$ with respect to the x and y variables, respectively, while $D_{xy}u$ denotes the $n \times m$ matrix of mixed derivatives; throughout the paper the solutions to PDEs will be always meant in the viscosity sense.

In Sections 2, 3, and 4, we assume that this Cauchy problem has a solution $u^\varepsilon \in C([0, T] \times \mathbb{R}^n \times \mathbb{R}^m)$ that is periodic in y and that the family $\{u^\varepsilon\}$ is locally equibounded. We give in Section 2.6 some additional mild conditions on H implying these properties if $h \in BUC(\mathbb{R}^{n+m})$, the space of bounded and uniformly continuous functions $\mathbb{R}^{n+m} \to \mathbb{R}$. For instance, the assumed existence and local equiboundedness of u^ε hold for HJBI operators with data satisfying the conditions at the end of Section 2.1 if $h \in BUC(\mathbb{R}^{n+m})$ and all the other functions are bounded and uniformly continuous (or Lipschitz continuous) uniformly in the parameters. In this case the solution is also globally bounded and unique [27, 25]. For our purpose the uniqueness of the solution u^ε is not essential. The regularity assumptions we are making on H and the periodicity in y imply the comparison principle, and therefore uniqueness, only in domains of the form $[0, T] \times \Omega \times \mathbb{R}^m$ with $\Omega \subset \mathbb{R}^n$ bounded, with prescribed boundary and initial data [20].

2.3. *Ergodicity and effective Hamiltonian*

We give two definitions of ergodicity of H. The first will be used in the proof of convergence of the singular perturbation problem. The second makes the connection with classical ergodic theory as explained in the Appendix. This accounts for the name. The equivalence between the definitions is proved in the Abelian-Tauberian Theorem 4 of the Appendix.

Fix $(\overline{x}, \overline{p}, \overline{X})$. The first definition is based on the *cell δ-problem*, for $\delta > 0$,

$$\delta w_\delta + H(\overline{x}, y, \overline{p}, D w_\delta, \overline{X}, D^2 w_\delta, 0) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic.} \tag{CP$_\delta$}$$

It has a unique viscosity solution (see Lemma 1 in Section 3) that we denote with $w_\delta(y; \overline{x}, \overline{p}, \overline{X})$ so as to display the dependence of the solution on the slow variables. We say that the operator, or the Hamiltonian, is (uniformly) *ergodic* in the fast variable at $(\overline{x}, \overline{p}, \overline{X})$ if

$$\delta w_\delta(y; \overline{x}, \overline{p}, \overline{X}) \to \text{const} \quad \text{as } \delta \to 0, \text{ uniformly in } y.$$

We say that it is ergodic at $\overline{x}$ if it is ergodic at $(\overline{x}, \overline{p}, \overline{X})$ for every $(\overline{p}, \overline{X})$, and that it is ergodic in a set if it is ergodic at every points of this set. When the operator is ergodic at $(\overline{x}, \overline{p}, \overline{X})$, we put

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = -\text{const}.$$

The function $\overline{H}$ is called the *effective operator*, or *effective Hamiltonian*.

The second definition of ergodicity is based on the *cell t-problem*, that is,

$$\begin{aligned} &w_t + H(\overline{x}, y, \overline{p}, D_y w, \overline{X}, D^2_{yy} w, 0) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m, \\ &w(0, y) = 0 \quad \text{on } \mathbb{R}^m, \qquad w \text{ periodic.} \end{aligned} \tag{CP}$$

If $w(t, y; \overline{x}, \overline{p}, \overline{X})$ denotes the solution of this Cauchy problem, the Hamiltonian is ergodic at $(\overline{x}, \overline{p}, \overline{X})$ if and only if

$$\frac{w(t, y; \overline{x}, \overline{p}, \overline{X})}{t} \to \text{const} \quad \text{as } t \to +\infty, \text{ uniformly in } y,$$

and when this occurs the two constants coincide. Therefore

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = -\lim_{t \to +\infty} \frac{w(t, y; \overline{x}, \overline{p}, \overline{X})}{t}.$$

Remark 1. In Theorem 4 of the Appendix, we also characterize the property of ergodicity of H in terms of the *true cell problem*

$$\lambda + H(\overline{x}, y, \overline{p}, D\chi, \overline{X}, D^2\chi, 0) = 0 \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic} \tag{6}$$

for some constant λ, provided this is interpreted in a relaxed sense. The constant λ must coincide with $-\overline{H}$. For the moment, we only observe that the Hamiltonian is ergodic whenever there is a solution (λ, χ) to (6). Indeed, applying the comparison principle to (CP), we see that $\|w(t, \cdot) - \lambda t - \chi\|_{L^\infty(\mathbb{R}^m)} \leqq \|\chi\|_{L^\infty(\mathbb{R}^m)}$. Sending $t \to +\infty$, we get

$$\frac{w(t, y)}{t} \to \lambda \quad \text{as } t \to +\infty, \text{ uniformly in } y.$$

Therefore, the Hamiltonian is ergodic and $\lambda = -\overline{H}$. The true cell problem also comes up in studying the singular perturbation by a formal asymptotic expansion, as we show in Section 3.1.

2.4. Stabilization to a constant and effective initial data

For the definition of stabilization to a constant we need a mild additional assumption on the Hamiltonian. We say that $H(x, y, p, q, X, Y, Z)$ has a recession function in the fast derivatives (q, Y) in a neighbourhood of $\overline{x}$ if there is a function $H'(x, y, q, Y)$ that is positively 1-homogeneous in (q, Y), i.e.,

$$H'(x, y, \lambda q, \lambda Y) = \lambda H'(x, y, q, Y), \qquad \lambda > 0,$$

with the following property: for every $\overline{p} \in \mathbb{R}^n$, $\overline{X} \in \mathbb{S}^n$, there is a constant C such that

$$|H(x, y, p, q, X, Y, 0) - H'(x, y, q, Y)| \leqq C \quad \text{for all } (y, q, Y) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{S}^m, \tag{7}$$

for every (x, p, X) in a neighbourhood of $(\overline{x}, \overline{p}, \overline{X})$. The Hamiltonian H' is called the *recession function* of H or the *homogeneous part* of H in (q, Y). Since it satisfies

$$H'(x, y, q, Y) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} H(x, y, 0, \lambda q, 0, \lambda Y, 0) \tag{8}$$

uniformly, it is continuous and degenerate elliptic.

For example, in the HJBI equations described in Section 2.1 (H defined by (5)) it is easy to see that the recession function is

$$H'(x, y, q, Y) = \sup_{\alpha \in A} \inf_{\beta \in B} \{-\operatorname{tr}(b_{\alpha,\beta}(x, y)Y) - (q, g_{\alpha,\beta}(x, y))\}.$$

Fix $\overline{x}$ and assume that H has a recession function in (q, Y) in a neighbourhood of $\overline{x}$. Since $h(\overline{x}, \cdot)$ is $\mathbb{Z}^m$-periodic, the *cell Cauchy problem* for the homogeneous Hamiltonian H'

$$\begin{aligned} &w'_t + H'(\overline{x}, y, D_y w', D^2_{yy} w') = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m, \\ &w'(0, y) = h(\overline{x}, y) \quad \text{on } \mathbb{R}^m, \quad w' \text{periodic}, \end{aligned} \tag{CP$'$}$$

has a unique viscosity solution $w'(t, y; \overline{x})$ (see Lemma 2 in Section 4). By the comparison principle, it is bounded by $\|h(\overline{x}, \cdot)\|_{L^\infty(\mathbb{R}^m)}$.

We say that the pair (H, h) is *stabilizing* (to a constant) at $\overline{x}$ if

$$w'(t, y; \overline{x}) \to \text{const} \quad \text{as } t \to +\infty, \text{ uniformly in } y.$$

We call the pair stabilizing in a given set if it is stabilizing at every point of the set. We say that the Hamiltonian is stabilizing if the pair (H, h) is stabilizing for every continuous initial data h. Finally, if the pair is stabilizing at $\overline{x}$, we put

$$\overline{h}(\overline{x}) := \text{const}.$$

The function $\overline{h}$ is called the *effective initial data*.

Remark 2. If $h(\overline{x}, y) = h(\overline{x})$ is a constant with respect to y, then for any Hamiltonian H the pair (H, h) is stabilizing at $\overline{x}$ and $\overline{h}(\overline{x}) = h(\overline{x})$. In fact, $w'(t, y) \equiv h(\overline{x})$ is the solution of (CP$'$) because H' is homogeneous in (q, Y).

2.5. The convergence result

Let $\{u^\varepsilon\}$ be a locally equibounded family of solutions of $(\mathrm{HJ}_\varepsilon)$. The upper semi-limit $\overline{u} = \limsup_{\varepsilon\to 0} u^\varepsilon$ is defined as follows

$$\overline{u}(t,x) := \limsup_{\varepsilon\to 0,\ (t',x')\to(t,x)} \sup_y u^\varepsilon(t',x',y) \quad \text{if } t>0,$$
$$\overline{u}(0,x) := \limsup_{(t',x')\to(0,x),\ t'>0} \overline{u}(t',x') \quad \text{if } t=0.$$

We define analogously the lower semi-limit $\underline{u}$ by replacing lim sup with lim inf and sup with inf. The two-steps definition of the semi-limit for $t=0$ permits us to sweep away an expected initial layer.

The main result of the paper is the following convergence result; its proof is split between Sections 3 and 4.

Theorem 1. *Assume that the Hamiltonian is ergodic and the pair (H,h) is stabilizing. Assume also that the family $\{u^\varepsilon\}$ is locally equibounded. Then the semi-limits $\overline{u} = \limsup_{\varepsilon\to 0} u^\varepsilon$ and $\underline{u} = \liminf_{\varepsilon\to 0} u^\varepsilon$ are, respectively, a subsolution and a supersolution of the effective Cauchy problem*

$$u_t + \overline{H}(x, Du, D^2u) = 0 \ \text{ in } (0,T)\times\mathbb{R}^n, \quad u(0,x) = \overline{h}(x) \quad \text{on } \mathbb{R}^n. \qquad (\overline{\mathrm{HJ}})$$

The convergence result is stated with the help of the semi-limits. As explained in the introduction, this form is the most tractable one; it also has the advantage of focusing on the key assumptions of ergodicity and stabilization. Under a mild additional hypothesis, the theorem can actually be expressed in terms of the more familiar local uniform convergence of u^ε.

In a first corollary, we assume that u^ε converges uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n$ to some function u. We extend u for $t=0$ by setting $u(0,\cdot) = \overline{h}$. Then, we see that $\underline{u} = \overline{u} = u$ in $[0,T)\times\mathbb{R}^n$, by using the convergence of u^ε for $t>0$ and the theorem for $t=0$. This implies that u is continuous in $[0,T)\times\mathbb{R}^n$ and that it is a viscosity solution of $(\overline{\mathrm{HJ}})$. We have therefore proved the following corollary.

Corollary 1. *Suppose that, in addition to the assumptions of Theorem 1, u^ε converges uniformly on the compact subsets of $(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ to some function $u(t,x)$ and extend u by $\overline{h}$ at $t=0$. Then u is a viscosity solution of $(\overline{\mathrm{HJ}})$.*

The second corollary is most useful because it proves the local uniform convergence of u^ε. It supposes that the comparison principle holds for the limit equation $(\overline{\mathrm{HJ}})$ in the sense that every upper-semicontinuous viscosity subsolution must be smaller than every lower-semicontinuous viscosity supersolution. The theorem says that $\overline{u}$ is an u.s.c. subsolution and that $\underline{u}$ is a l.s.c. supersolution. Hence, the comparison principle gives that $\overline{u} \leqq \underline{u}$ in $[0,T)\times\mathbb{R}^n$. The reverse inequality is obvious by the definition of the semi-limits. Therefore, we actually have $\overline{u} = \underline{u}$ in $[0,T)\times\mathbb{R}^n$. This implies that u^ε converges locally uniformly to the function $\overline{u} = \underline{u}$ and Corollary 1 ensures that the limit is a viscosity solution of $(\overline{\mathrm{HJ}})$. It is the unique solution because of the comparison principle. We therefore have shown the second corollary to Theorem 1.

Corollary 2. *Suppose that, in addition to the assumptions of Theorem 1, $\overline{H}$ and $\overline{h}$ satisfy the usual regularity assumptions so as to get comparison for* ($\overline{\text{HJ}}$). *Then u^ε converges uniformly on the compact subsets of* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *to the unique viscosity solution of* ($\overline{\text{HJ}}$).

Remark 3. In the case when the initial data h is independent of y, we only have to assume the ergodicity of the Hamiltonian. The effective initial data will of course be h, as explained at the end of Section 2.4. In this case the convergence of Corollary 2 is indeed uniform on the compact subsets of $[0, T) \times \mathbb{R}^n \times \mathbb{R}^m$.

2.6. An important illustration

The convergence result Theorem 1 is most useful in the form of Corollary 2. In this subsection, we give sufficient conditions for the family $\{u^\varepsilon\}$ to be equibounded and for the effective data to have enough regularity so that the comparison principle for ($\overline{\text{HJ}}$) holds true. Then Corollary 2 shows that if the effective Hamiltonian is ergodic and stabilizing, the family $\{u^\varepsilon\}$ converges uniformly on the compact sets to the solution of the effective Cauchy problem ($\overline{\text{HJ}}$). Several explicit examples ensuring the ergodicity and the stabilization of the Hamiltonian and the initial data will be given in Section 5.

Besides the standing hypotheses on $H(x, y, p, q, X, Y, Z) = \mathcal{H}(\xi, P, \Theta)$ and h of Section 2.1,we suppose the existence of a recession function H' in the (q, Y) variables for H as defined in Section 2.4. In order to ensure that the family $\{u^\varepsilon\}$ is equibounded, we add the following mild hypotheses:

- $h \in BUC(\mathbb{R}^{n+m})$, i.e., it is bounded and uniformly continuous;
- $\mathcal{H}$ is uniformly continuous in (P, Θ), uniformly in $\xi = (x, y)$, i.e.,

$$|\mathcal{H}(\xi, P, \Theta) - \mathcal{H}(\xi, P', \Theta')| \leqq \omega(|P - P'| + |\Theta - \Theta'|) \tag{9}$$

  for some modulus ω;
- there exists a constant M such that $|\mathcal{H}(\xi, 0, 0)| \leqq M$ for all ξ.

The last two conditions are satisfied for HJBI operators (5) provided the various data are bounded (namely f, g, σ, τ and l in (4)).

Proposition 1. *Under the previous assumption, for any $\varepsilon > 0$, there exists a unique bounded solution $u^\varepsilon \in C([0, T] \times \mathbb{R}^n \times \mathbb{R}^m)$ of the ε-problem* (HJ_ε). *Moreover u^ε is periodic in y and the family $\{u^\varepsilon\}$ is equibounded with the estimate*

$$-Mt - \sup|h| \leqq u^\varepsilon(t, x, y) \leqq Mt + \sup|h| \qquad \forall\, t, x, y, \quad \forall\, \varepsilon > 0.$$

Proof. We fix $\varepsilon > 0$ and drop it temporarily in the notation. The Cauchy problem to be solved is

$$u_t + \mathcal{H}(\xi, Du, D^2u) = 0 \text{ in } (0, T) \times \mathbb{R}^n \times \mathbb{R}^m, \; u(0, \xi) = h(\xi) \text{ on } \mathbb{R}^n \times \mathbb{R}^m. \tag{10}$$

Let us first prove the comparison principle, by adapting an argument in ISHII [27]. Consider an u.s.c. bounded subsolution u and a l.s.c. bounded supersolution v such that $u(0,\xi) \leqq h(\xi) \leqq v(0,\xi)$. Put $g(\xi) := \log(1+|\xi|^2)$ and note that its first two derivatives are bounded by a constant K. For every $\delta > 0$, consider the function

$$u^\delta(t,\xi) := u(t,\xi) - t\omega(2K\delta) - \delta g(\xi),$$

where ω is the modulus of uniform continuity of $\mathcal{H}$ with respect to (P,Θ). It is a subsolution of the equation as the initial condition trivially holds and

$$\begin{aligned} u_t^\delta + \mathcal{H}(\xi, Du^\delta, D^2u^\delta) &= u_t - \omega(2K\delta) + \mathcal{H}(\xi, Du - \delta Dg, D^2u - \delta D^2 g) \\ &\leqq u_t - \omega(2K\delta) + \mathcal{H}(\xi, Du, D^2u) + \omega(2K\delta) \\ &\leqq 0. \end{aligned}$$

As $\lim_{|\xi|\to+\infty} u^\delta(\xi) = -\infty$, we know that $u^\delta \leqq v$ for $|\xi|$ large. By the assumptions on $\mathcal{H}$ of Section 2.1 we can apply a standard comparison principle in a bounded cylinder large enough [20] and deduce that $u^\delta \leqq v$ in $[0,T]\times\mathbb{R}^n$. Sending $\delta \to 0$, we conclude that $u \leqq v$.

We now turn to the proof of the existence of a solution. If h has first and second derivatives bounded by C_1, the assumptions made above on $\mathcal{H}$ imply $C := \sup_y |\mathcal{H}(\xi, Dh, D^2h)| \leqq M + \omega(2C_1)$ is finite. Then $h(\xi) - Ct$ and $h(\xi) + Ct$ are a sub- and a supersolution of the Cauchy problem (10) attaining the initial data, and then Perron's method give the existence of a continuous solution of (10) (see [27, 20]). For $h \in BUC(\mathbb{R}^{n+m})$ we approximate uniformly on $\mathbb{R}^{n+m}$ with functions h_k with bounded derivatives, take the corresponding sequence of solution u_k of (10), and use the comparison principle to see that it is a Cauchy sequence in the sup-norm. Therefore it converges uniformly on $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m$ to the desired solution of (10).

The periodicity of u^ε follows from the uniqueness because $u^\varepsilon(t,x,y+k)$ is a solution of (HJ$_\varepsilon$) for any $k \in \mathbb{Z}^m$. Finally, $Mt + \sup|h|$ is a supersolution and $-Mt - \sup|h|$ is a subsolution of (HJ$_\varepsilon$) for any ε, so the comparison principle gives the uniform estimate on u^ε. This ends the proof of the proposition.

The comparison principle for the effective Cauchy problem ($\overline{\text{HJ}}$) for bounded sub- and supersolutions can be established in the same way, under analogous hypotheses for $\overline{H}$. Namely, the assumptions on the effective Hamiltonian are

$$|\overline{H}(x,p,X) - \overline{H}(x,p',X')| \leqq \omega(|p-p'| + |X-X'|),$$

for all x, p, p', X, X', and the usual regularity condition

$$\overline{H}(x', \kappa(x-x'), X') \leqq \overline{H}(x, \kappa(x-x'), X) + \omega(|x'-x| + \kappa|x'-x|^2) \quad (11)$$

whenever X and X' satisfy

$$-3\kappa\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leqq \begin{pmatrix} X & 0 \\ 0 & -X' \end{pmatrix} \leqq 3\kappa\begin{pmatrix} I & -I \\ I & I \end{pmatrix}. \quad (12)$$

The first condition is easy to verify.The delicate issue is to determine when the effective Hamiltonian satisfies (11).An important observation of [3] is that this property is not automatic. In that paper an example is given of a first-order Hamiltonian that is ergodic but for which the effective Hamiltonian is not regular. As a consequence, the limit equation ($\overline{\mathrm{HJ}}$) has no continuous solution so the family u^ε cannot converge uniformly on the compact subsets.

An ad hoc analysis is therefore required to prove that the effective Hamiltonian satisfies (11). A thorough discussion of the sufficient conditions is given in the papers [2] and [3] for operators arising, respectively, in optimal control and differential games. For an illustrative purpose, we mention here a straightforward assumption that guarantees (11):

– H satisfies the usual regularity property with respect to the slow variables (x, p, X) with x bounded, uniformly in the fast variables (y, q, Y). Namely, for all $R > 0$ there is a concave modulus ω_R such that for every $\kappa > 0$, $|x|, |x'| \leqq R$ and $X, X' \in \mathbb{S}^n$ satisfying (12) and every (y, q, Y), we have

$$H(x', y, \kappa(x - x'), q, X', Y, 0) \leqq H(x, y, \kappa(x - x'), q, X, Y, 0) + \omega_R(|x' - x| + \kappa|x' - x|^2). \quad (13)$$

Though restrictive, this condition covers many cases of interest. For instance, in the case of HJBI operators (5), the condition is satisfied under the assumptions of Sections 2.1 and 2.2 provided the functions driving the dynamics of the fast variable y (namely, g and τ) are independent of the slow variable x. Two other examples are given in Section 5, see Corollaries 4 and 10.

Proposition 2. *In addition to the hypotheses of Proposition 1, assume that the Hamiltonian is ergodic, the pair* (H, h) *is stabilizing, and* (13) *holds. Then* u^ε *converges uniformly on the compact subsets* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *to the unique viscosity solution of* ($\overline{\mathrm{HJ}}$).

Proof. In order to apply Corollary 2, we have to check that the comparison principle holds for the limit equation ($\overline{\mathrm{HJ}}$). To apply the comparison principle of the proof of Proposition 1, we must only check that the effective Hamiltonian satisfies the same properties as H.

We first observe that $\overline{H}$ is always continuous: this general property will be proved later in Proposition 3. If $w(t, y; x, p, X)$ denotes the solution of (CP) for the slow variable (x, p, X), then for a different (p', X'), we get

$$|w_t + H(x, y, p', D_y w, X', D^2_{yy} w, 0)| \leqq \omega(|p - p'| + |X - X'|),$$

where ω is the modulus of continuity in (9). By the comparison principle, we deduce that

$$|w(t, y; x, p, X) - w(t, y; x, p', X')| \leqq t\omega(|p - p'| + |X - X'|).$$

Sending $t \to +\infty$, we conclude that $\overline{H}$ satisfies (9) with the same modulus of continuity.

Finally, let (x, p, X) and (x', p, X') be slow variables with $|x|, |x'| \leqq R$, $p = \kappa(x - x')$ and X, X' satisfying (12). Because of assumption (1), we know that

$$H(x', y, \kappa(x - x'), q, X', Y, 0) \leqq H(x, y, \kappa(x - x'), q, X, Y, 0) + \omega_R(|x' - x| + \kappa|x' - x|^2)$$

uniformly in (y, q, Y). Arguing as in the preceding paragraph, we immediately find that $\overline{H}$ satisfies (11) with the same modulus. This completes the proof of the Proposition.

3. Ergodicity implies convergence in the interior

In this section we prove that the effective Hamiltonian coming from the ergodicity assumption gives the correct PDE solved by the limit of the solution u^ε of $(\mathrm{HJ}_\varepsilon)$ when $\varepsilon \to 0$.

3.1. Heuristics by formal expansions

In this subsection, we assume that the Hamiltonian is ergodic and we explain, in an informal manner, why the solution u^ε of $(\mathrm{HJ}_\varepsilon)$ should converge to the solution of $(\overline{\mathrm{HJ}})$. To concentrate on the equation we take the initial data h to be 0. We freely assume that the Hamiltonian is Lipschitz continuous and that all the functions are smooth with bounded derivatives.

Let u be the solution of the effective equation $(\overline{\mathrm{HJ}})$ with $\overline{h} \equiv 0$. Supposing that the ergodicity of the Hamiltonian holds in the slightly stronger sense of Remark 1 in Section 2.3, then, for every x, there is a solution $\chi(x, y)$ of the true cell problem

$$H(\overline{x}, y, \overline{p}, D_y\chi, \overline{X}, D^2_{yy}\chi, 0) = \overline{H}(\overline{x}, \overline{p}, \overline{X}) \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic},$$

with $\overline{x} = x$, $\overline{p} = D_x u(t, x)$, $\overline{X} = D^2_{xx} u(t, x)$. The function

$$v^\varepsilon(t, x, y) = u(t, x) + \varepsilon\chi(t, x, y)$$

then solves the equation

$$\begin{aligned}
v^\varepsilon_t &+ H(x, y, D_x v^\varepsilon, \varepsilon^{-1} D_y v^\varepsilon, D_{xx} v^\varepsilon, \varepsilon^{-1} D_{yy} v^\varepsilon, \varepsilon^{-1/2} D_{xy} v^\varepsilon) \\
&= u_t(t, x) + H\big(x, y, Du, D_y\chi, D^2 u, D^2_{yy}\chi, 0\big) + O(\varepsilon) \\
&= u_t(t, x) + \overline{H}(x, Du, D^2 u) + O(\varepsilon) \\
&= O(\varepsilon),
\end{aligned}$$

with initial data $v^\varepsilon(0, x, y) = O(\varepsilon)$. By the comparison principle, we deduce that the solution of $(\mathrm{HJ}_\varepsilon)$ satisfies

$$u^\varepsilon(t, x, y) = v^\varepsilon(t, x, y) + O(\varepsilon) = u(t, x) + O(\varepsilon).$$

Therefore, the function u^ε should converge as $\varepsilon \to 0$ to the solution u of $(\overline{\mathrm{HJ}})$.

3.2. Local convergence

The next result is the main one of this section. It is a local convergence theorem under an assumption of ergodicity at a single point. In particular, it gives the first half of Theorem 1 if the Hamiltonian is ergodic everywhere.

Theorem 2. *Fix a point $\overline{x}$ and assume that the Hamiltonian is ergodic at $\overline{x}$. Fix a neighbourhood U of $(\overline{t}, \overline{x})$ with $\overline{t} > 0$. Let u^ε be a subsolution (respectively, supersolution) of the equation*

$$u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx} u^\varepsilon, \frac{D_{yy} u^\varepsilon}{\varepsilon}, \frac{D_{xy} u^\varepsilon}{\sqrt{\varepsilon}}\right) = 0 \quad \text{in } U \times \mathbb{R}^m \tag{14}$$

and assume that the family $\{u_\varepsilon\}$ is equibounded in $U \times \mathbb{R}^m$. Then, the semi-limit $\overline{u} = \limsup_{\varepsilon \to 0} u^\varepsilon$ (or $\underline{u} = \liminf_{\varepsilon \to 0} u^\varepsilon$) is a subsolution (respectively, supersolution) of the effective equation

$$u_t + \overline{H}(x, Du, D^2 u) = 0$$

at the point $(\overline{t}, \overline{x})$.

Proof. The proof makes rigorous the heuristics of the preceding subsection. We only show the result for subsolutions, the case of supersolutions being analogous. Let φ be a test function and $(\overline{t}, \overline{x})$ be a point of strict maximum of $\overline{u} - \varphi$ such that $\overline{u}(\overline{t}, \overline{x}) = \varphi(\overline{t}, \overline{x})$. We assume for contradiction there exists $\eta > 0$ such that

$$\varphi_t(\overline{t}, \overline{x}) + \overline{H}(\overline{x}, D\varphi(\overline{t}, \overline{x}), D^2\varphi(\overline{t}, \overline{x})) \geqq 3\eta.$$

We shorten the notation by setting $\overline{H} := \overline{H}(\overline{x}, D\varphi(\overline{t}, \overline{x}), D^2\varphi(\overline{t}, \overline{x}))$. For $r > 0$ we define

$$\begin{aligned} &H_r(y, q, Y) \\ &\quad := \min\{H(x, y, D\varphi(t, x), q, D^2\varphi(t, x), Y, 0) \mid |t - \overline{t}| \leqq r, \\ &\qquad\qquad |x - \overline{x}| \leqq r\}. \end{aligned} \tag{15}$$

We claim that, for $r > 0$ small enough, there exists a periodic viscosity solution $\chi(y)$ of

$$H_r(y, D_y\chi, D_{yy}^2\chi) \geqq \overline{H} - 2\eta \quad \text{in } \mathbb{R}^m.$$

To prove the claim we first observe that, by definition of $\overline{H}$, we can find $\delta > 0$ such that the solution w_δ of (CP$_\delta$) with $\overline{p} = D\varphi(\overline{t}, \overline{x})$ and $\overline{X} = D^2\varphi(\overline{t}, \overline{x})$ verifies $\|\delta w_\delta + \overline{H}\|_{L^\infty} \leqq \eta$. Next we consider the problem

$$\delta w_{\delta,r} + H_r(y, D_y w_{\delta,r}, D_{yy}^2 w_{\delta,r}) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_{\delta,r} \text{ periodic.} \tag{16}$$

Using assumption (3), we can prove that this problem has exactly one solution (see Lemma 1 below). Since

$$H_r(y, q, Y) \to H(\overline{x}, y, D\varphi(\overline{t}, \overline{x}), q, D^2\varphi(\overline{t}, \overline{x}), Y, 0)$$

uniformly on compact sets as $r \to 0+$, and since the cell problem (CP_δ) has a unique solution (see Lemma 1), we deduce from the stability results for viscosity solutions that $w_{\delta,r} \to w_\delta$ uniformly on compact sets. The convergence is uniform by periodicity. In particular, we can choose $r > 0$ small enough so that $\|\delta w_{\delta,r} + \overline{H}\|_{L^\infty} \leqq 2\eta$. The function $\chi = w_{\delta,r}$ has the required properties.

Now we consider the perturbed test function

$$\psi^\varepsilon(t,x,y) = \varphi(t,x) + \varepsilon\chi(y).$$

We fix $r > 0$ as in the preceding paragraph so that $|\varphi_t(t,x) - \varphi_t(\bar{t},\bar{x})| \leqq \eta$ as $|t-\bar{t}| < r$, $|x-\bar{x}| \leqq r$. In the cylinder $Q_r =]\bar{t}-r, \bar{t}+r[\times B_r(\bar{x}) \times \mathbb{R}^m$ the function ψ^ε is a supersolution of

$$\begin{aligned}
&\psi^\varepsilon_t + H(x,y,D_x\psi^\varepsilon, \varepsilon^{-1}D_y\psi^\varepsilon, D_{xx}\psi^\varepsilon, \varepsilon^{-1}D_{yy}\psi^\varepsilon, \varepsilon^{-1/2}D_{xy}\psi^\varepsilon)\\
&\quad = \varphi_t(t,x) + H\big(x,y,D\varphi(t,x), D_y\chi(y), D^2\varphi(t,x), D^2_{yy}\chi(y), 0\big)\\
&\quad \geqq \varphi_t(t,x) + H_r\big(y, D_y\chi(y), D^2_{yy}\chi(y)\big)\\
&\quad \geqq \varphi_t(t,x) + \overline{H} - 2\eta\\
&\quad \geqq \varphi_t(\bar{t},\bar{x}) + \overline{H} - 3\eta\\
&\quad \geqq 0.
\end{aligned}$$

The verification that this formal computation is true in the viscosity sense is deferred to the end of the proof.

Since $\{\psi_\varepsilon\}$ converges uniformly to φ on $\overline{Q_r}$, we have

$$\limsup_{\varepsilon\to 0,\, t'\to t,\, x'\to x} \sup_y (u_\varepsilon - \psi_\varepsilon)(t',x',y) = \overline{u}(t,x) - \varphi(t,x).$$

But $(\bar{t},\bar{x})$ is a strict maximum point of $\overline{u} - \varphi$, so the above relaxed upper limit is < 0 on ∂Q_r. By compactness (recall that u_ε and ψ_ε are periodic in y), we can find $\eta' > 0$ such that $u_\varepsilon - \psi_\varepsilon \leqq -\eta'$ on ∂Q_r for ε small, i.e., $\psi_\varepsilon \geqq u_\varepsilon + \eta'$ on ∂Q_r. Since ψ_ε is a supersolution of (14) in Q_r, we deduce from the comparison principle that $\psi_\varepsilon \geqq u_\varepsilon + \eta'$ in Q_r for ε small. Taking the upper semi-limit, we get $\varphi \geqq \overline{u} + \eta'$ in $(\bar{t}-r, \bar{t}+r) \times B(\bar{x}, r)$. This is impossible, for $\varphi(\bar{t},\bar{x}) = \overline{u}(\bar{t},\bar{x})$, and we have reached the desired conclusion.

To complete the proof, we justify that ψ^ε is indeed a viscosity supersolution of the equation. A few fundamental notions of the theory of viscosity solutions need to be recalled. We refer to the User's Guide [20] for complements. Given a point (t,x,y) and a lower semicontinuous function u, the parabolic subdifferential $J^-u(t,x,y)$ is the set consisting of the generalized gradient $(\pi, P) \in \mathbb{R}^{1+n+m}$ and Hessian $\Theta \in \mathbb{S}^{n+m}$ with $P = (p,q)$ and $\Theta := \begin{pmatrix} X & Z \\ Z^T & Y \end{pmatrix}$ satisfying the Taylor inequality

$$\begin{aligned}
u(t+h_t, x+h_x, y+h_y) &\geqq u(t,x,y) + \pi h_t + (p,h_x) + (q,h_y)\\
&\quad + \tfrac{1}{2}(Xh_x,h_x) + (Zh_y,h_x) + \tfrac{1}{2}(Yh_y,h_y) - o\big(|h_t| + |h_x|^2 + |h_y|^2\big).
\end{aligned}$$

The closure $\overline{J^-}u(t,x,y)$ of the subdifferential consists of the limit points

$$(\pi, P, \Theta) = \lim_{n\to+\infty}(\pi_n, P_n, \Theta_n)$$

with $(\pi_n, P_n, \Theta_n) \in J^-u(t_n, x_n, y_n)$ and

$$\big(t_n, x_n, y_n, u(t_n, x_n, y_n)\big) \to \big(t, x, y, u(t, x, y)\big).$$

To prove that ψ^ε is a viscosity supersolution of the equation in Q_r, we have to verify that the differential inequality holds when the derivatives are replaced by the subdifferential. In other words, for every $(t,x,y) \in Q_r$ and every $(\pi, P, \Theta) \in J^-\psi^\varepsilon(t,x,y)$, we have to show that

$$\pi + H(x, y, p, \varepsilon^{-1}q, X, \varepsilon^{-1}Y, \varepsilon^{-1/2}Z) \geqq 0. \tag{17}$$

The key tool is the fundamental characterization of the subdifferential of the sum of two functions with independent variables (Theorem 3.2 and its parabolic version Theorem 8.3 of [20]). It shows that, for every $\delta > 0$, there are $\widetilde{X} \in \mathbb{S}^n$ and $\widetilde{Y} \in \mathbb{S}^m$ so that $(\pi, p, \widetilde{X}) \in \overline{J^-}\varphi(t,x)$, $(q, \widetilde{Y}) \in \varepsilon\overline{J^-}\chi(y)$ and $\begin{pmatrix} \widetilde{X} & 0 \\ 0 & \widetilde{Y}\end{pmatrix} \geqq \Theta - \delta\Theta^2$. Because φ is smooth, we know that $\pi = \varphi_t(t,x)$, $p = D\varphi(t,x)$ and $\widetilde{X} \leqq D^2\varphi(t,x)$. Putting $\Theta^2 = \begin{pmatrix} X' & Z' \\ (Z')^T & Y'\end{pmatrix}$, the formal string of differential inequalities above has to be replaced by the following ones, that we justify below:

$$\begin{aligned}
\pi &+ H(x, y, p, \varepsilon^{-1}q, X - \delta X', \varepsilon^{-1}Y - \varepsilon^{-1}\delta Y', \varepsilon^{-1/2}Z - \varepsilon^{-1/2}\delta Z') \\
&\geqq \pi + H(x, y, p, \varepsilon^{-1}q, \widetilde{X}, \varepsilon^{-1}\widetilde{Y}, 0) \\
&\geqq \varphi_t(t,x) + H\big(x, y, D\varphi(t,x), \varepsilon^{-1}q, D^2\varphi(t,x), \varepsilon^{-1}\widetilde{Y}, 0\big) \\
&\geqq \varphi_t(t,x) + H_r\big(y, \varepsilon^{-1}q, \varepsilon^{-1}\widetilde{Y}\big) \\
&\geqq \varphi_t(t,x) + \overline{H} - 2\eta \\
&\geqq \varphi_t(\bar{t}, \bar{x}) + \overline{H} - 3\eta \\
&\geqq 0.
\end{aligned}$$

The first and second inequalities used the ellipticity of the Hamiltonian and the matrix inequalities (after rescaling); the fourth one applied the fact that χ is chosen as a supersolution of the appropriate equation and the fact that $\varepsilon^{-1}(q, \widetilde{Y}) \in \overline{J^-}\chi(y)$. Sending $\delta \to 0$, we obtain the inequality (17). This completes the proof.

Remark 4. As we mentioned in the introduction, the periodicity of the problem in the fast variable y is not essential for the convergence result to hold. The result is still valid with other boundary conditions in the fast variable provided the fast variable lies in a compact set and the auxiliary problems introduced in the proof enjoy certain existence and uniqueness properties. Specifically, what matters are the existence of a unique viscosity solution to (CP_δ) in order to define the effective Hamiltonian; the comparison principle for (CP_δ); the existence of viscosity solutions to (16) for r small (after having performed an elementary small perturbation on the boundary operator in the fast variable, if needed); the comparison principle for (HJ_ε) in Q_r for r small with Dirichlet boundary conditions on ∂Q_r.

The proof used the next Lemma.

Lemma 1. *For a given test function φ and $r > 0$ consider the Hamiltonian H_r defined by* (15). *Then, for each $\delta > 0$, there exists a unique viscosity solution to the problem* (16).

The same conclusion holds if H_r is replaced by $H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0)$, i.e., the cell problem (CP_δ) *has a unique solution.*

Proof. We need a modulus ω' such that, for every $\kappa > 0$ and $y, y', p \in \mathbb{R}^m$,

$$H_r(y', \kappa(y - y'), Y') \leqq H_r(y, \kappa(y - y'), Y) + \omega'(|y' - y| + \kappa|y' - y|^2) \quad (18)$$

for all $Y, Y' \in \mathbb{S}^m$ satisfying (2). We take $\widetilde{t}, \widetilde{x}$ such that

$$H_r(y, \kappa(y - y'), Y) = H(\widetilde{x}, y, D\varphi(\widetilde{t}, \widetilde{x}), \kappa(y - y'), D^2\varphi(\widetilde{t}, \widetilde{x}), Y, 0)$$

and set $\widetilde{p} := D\varphi(\widetilde{t}, \widetilde{x})$, $\widetilde{X} := D^2\varphi(\widetilde{t}, \widetilde{x})$. Note that $\widetilde{x}$, $\widetilde{p}$, $\widetilde{X}$ remain in a bounded set. Then

$$\begin{aligned} &H_r(y', \kappa(y - y'), Y') - H_r(y, \kappa(y - y'), Y) \\ &\quad \leqq H(\widetilde{x}, y', \widetilde{p}, \kappa(y - y'), \widetilde{X}, Y', 0) - H(\widetilde{x}, y, \widetilde{p}, \kappa(y - y'), \widetilde{X}, Y, 0), \end{aligned}$$

and the regularity assumption on H in Section 2.1 gives the desired inequality with $\omega' = \omega_R$ for a suitable R. Therefore the comparison principle in the User's guide [20] ensures that the problem (16) has at most one solution.

In order to prove the existence of a solution to (16) we observe that H_r periodic with respect to y implies $|H_r(y, 0, 0)| \leqq C$, so $-C/\delta$ and C/δ are, respectively, a subsolution and a supersolution of the PDE in (16). We follow Perron's method and define the periodic function $w_{\delta,r}(y)$ as the supremum of $w(y)$ as w varies among the periodic functions such that $-C/\delta \leqq w \leqq C/\delta$ and the u.s.c. envelope of w is a subsolution of the PDE in (16). By a standard argument of Ishii (see, e.g., the User's guide [20]), the u.s.c. and l.s.c. envelopes of $w_{\delta,r}$ are, respectively, a sub- and a supersolution of the PDE in (16). Because of the periodicity we can apply the comparison principle and show that $w_{\delta,r}$ is continuous and it is the solution of (16). Therefore the Lemma is proved.

The proof of Theorem 2 can be modified to give the continuity of the effective Hamiltonian.

Proposition 3. (i) *If H is ergodic in a neighbourhood of $(\overline{x}, \overline{p}, \overline{X})$, then $\overline{H}$ is continuous at $(\overline{x}, \overline{p}, \overline{X})$.*
(ii) *If H is ergodic at $\overline{x}$, then $\overline{H}$ is degenerate elliptic, i.e., $\overline{H}(\overline{x}, p, X) \leqq \overline{H}(\overline{x}, p, X')$ for all $X \geqq X'$ and all p.*

Proof. Define

$$\begin{aligned} &H^{(r)}(y, q, Y) \\ &\quad := \min\{H(x, y, p, q, X, Y, 0) \mid |x - \overline{x}| \leqq r,\ |p - \overline{p}| \leqq r, \\ &\qquad\qquad ||X - \overline{X}|| \leqq r\}. \end{aligned}$$

Lemma 1 holds with H_r replaced by $H^{(r)}$ and we repeat the argument in the proof of Theorem 2 to get for any $\eta > 0$ the existence of $r > 0$ such that there is periodic solution $\chi \in C(\mathbb{R}^m)$ of

$$H^{(r)}(y, D\chi, D^2\chi) \geqq \overline{H} - 2\eta \quad \text{in } \mathbb{R}^m,$$

where $\overline{H} := \overline{H}(\overline{x}, \overline{p}, \overline{X})$. Then

$$H(x, y, p, D\chi, X, D^2\chi, 0) \geqq \overline{H} - 2\eta \quad \text{in } \mathbb{R}^m$$

for all x, p, X such that $|x - \overline{x}| \leqq r,\ |p - \overline{p}| \leqq r,\ ||X - \overline{X}|| \leqq r$. By Theorem 4 in the Appendix, this implies

$$\overline{H}(x, p, X) \geqq \overline{H}(\overline{x}, \overline{p}, \overline{X}) - 2\eta$$

for all such x, p, X. The inequality

$$\overline{H}(x, p, X) \leqq \overline{H}(\overline{x}, \overline{p}, \overline{X}) + 2\eta$$

can be obtained in a similar way and gives the continuity of $\overline{H}$ at $(\overline{x}, \overline{p}, \overline{X})$.

For the degenerate ellipticity of $\overline{H}$ we denote by $w(t, y; \overline{x}, p, X)$ the solution of (CP) corresponding to the parameters $(\overline{x}, p, X)$. If $X \geqq X'$, the degenerate ellipticity of H and the comparison principle imply $w(t, y; \overline{x}, p, X) \leqq w(t, y; \overline{x}, p, X')$. By the ergodicity at $\overline{x}$ we can divide by t and let $t \to +\infty$ to obtain $\overline{H}(\overline{x}, p, X) \leqq \overline{H}(\overline{x}, p, X')$ for all p.

4. Stabilization implies convergence at $t = 0$

The question we address in this section is the definition of the initial condition for the limit of the solution u^ε of $(\mathrm{HJ}_\varepsilon)$ when $\varepsilon \to 0$. We require that the limit be independent of the fast variable y. The crucial assumption for this is that the recession Hamiltonian $H'(x, y, q, Z)$ stabilizes to a constant. To explain why this assumption provides the correct initial data we first work out an example with explicit probabilistic formulas for the solution. Then in Section 4.2 we prove the convergence theorem.

4.1. Heuristics on a model problem

We consider the linear equation in the fast variable,

$$u_t^\varepsilon + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) - \frac{1}{\varepsilon}\operatorname{tr}(b(x, y) D_{yy} u^\varepsilon) - \frac{1}{\varepsilon}(g(x, y), D_y u^\varepsilon) = 0,$$

with $b = \tau\tau^T/2$, and we also freeze x, so that we remain with the linear problem

$$\begin{aligned} u_t^\varepsilon - \frac{1}{\varepsilon}\operatorname{tr}(b(y) D_{yy} u^\varepsilon) - \frac{1}{\varepsilon}(g(y), D_y u^\varepsilon) - l(y) = 0 \qquad & \text{in } (0, +\infty) \times \mathbb{R}^m, \\ u^\varepsilon(0, y) = h(y) \quad & \text{on } \mathbb{R}^m, \end{aligned}$$

for some bounded running cost l. By Ito's formula, the solution is the value function

$$u^{\varepsilon}(t, y) = Eh(y_t) + E\int_0^t l(y_s)\,ds,$$

with the dynamics

$$dy_s = \varepsilon^{-1} g(y_s)\,ds + \varepsilon^{-1/2}\tau(y_s)\,dW_s, \quad y_0 = y$$

for a Brownian motion W. We rescale the time by setting $T = t/\varepsilon$ and $y'_T = y_{\varepsilon T}$. Then we have

$$u^{\varepsilon}(t, y) = Eh(y'_{t/\varepsilon}) + \varepsilon E\int_0^{t/\varepsilon} l(y'_\sigma)\,d\sigma,$$

with

$$dy'_\sigma = g(y'_\sigma)\,d\sigma + \tau(y'_\sigma)\,dW^{\varepsilon}_\sigma, \quad y_0 = y,$$

where $W^{\varepsilon}_\sigma = \varepsilon^{-1/2} W_{\varepsilon\sigma}$ is a Brownian motion. When $t = 0$ the initial data $h(y)$ is not a constant. However, if for $t > 0$ we let $\varepsilon \to 0$, $u^{\varepsilon}(t, y)$ behaves as the limit as $T \to +\infty$ of $Eh(y'_T) + O(t)$, with y'_σ solving

$$dy'_\sigma = g(y'_\sigma)\,d\sigma + \tau(y'_\sigma)\,dW'_\sigma, \quad y_0 = y, \tag{19}$$

for some Brownian motion W'_σ that we may take independent of ε. So

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0} u^{\varepsilon}(t, y) = \lim_{T\to+\infty} w'(T, y), \qquad w'(T, y) := Eh(y'_T),$$

where y'_T is given by (19). Note that the value function w' solves

$$\begin{aligned} &w'_t - \mathrm{tr}(b(y)D_{yy}w') - (g(y), D_y w') = 0 \quad \text{in } (0, +\infty)\times\mathbb{R}^m,\\ &w'(0, y) = h(y) \quad \text{on } \mathbb{R}^m, \quad w' \text{periodic;}\end{aligned}$$

this is (CP′) in this special case, with the homogeneous Hamiltonian

$$H'(y, q, Y) = -\mathrm{tr}(b(y)Y) - (g(y), q).$$

Therefore, the definition of pair (H, h) stabilizing to a constant means exactly that $\lim_{T\to+\infty} w'(T, y)$ is a constant, so it is the right condition for $\lim_{t\to 0}\lim_{\varepsilon\to 0} u^{\varepsilon}$ (t, y) to be independent of y as well. Of course, since $u^{\varepsilon}(0, y) = h(y)$, the convergence of u^{ε} to a limit independent of y cannot be uniform for $t \geqq 0$ but only for $t \geqq r$, with $r > 0$ arbitrary. An initial layer is therefore expected.

4.2. Local convergence

The next result is the main one of this section. It is a local convergence theorem under the assumption that the pair (H, h) is stabilizing at a single point. In particular, it gives the second half of Theorem 1 if the Hamiltonian is stabilizing to a constant everywhere. We assume throughout this section that the recession function H' of H exists so that (7) holds.

Theorem 3. *Fix a point $\overline{x}$ and assume that the pair (H, h) is stabilizing at $\overline{x}$. Fix a neighbourhood U of $(0, \overline{x})$ and put $U_0 = U \cap \{t = 0\}$ and $U_+ = U \cap \{t > 0\}$. Let u^ε be a subsolution (respectively, supersolution) of the equation*

$$u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}, D_{xx} u^\varepsilon, \frac{D_{yy} u^\varepsilon}{\varepsilon}, \frac{D_{xy} u^\varepsilon}{\sqrt{\varepsilon}}\right) = 0 \qquad \text{in } U_+ \times \mathbb{R}^m,$$
$$u^\varepsilon(0, x, y) = h(x, y) \quad \text{on } U_0 \times \mathbb{R}^m$$

and assume that the family $\{u^\varepsilon\}$ is equibounded in $(U_+ \cup U_0) \times \mathbb{R}^m$. Then the semi-limit $\overline{u} = \limsup_{\varepsilon \to 0} u^\varepsilon$ (or $\underline{u} = \limsup_{\varepsilon \to 0} u^\varepsilon$) satisfies

$$\overline{u}(0, \overline{x}) \leqq \overline{h}(\overline{x})$$

(respectively, $\underline{u}(0, \overline{x}) \geqq \overline{h}(\overline{x})$).

Proof. With every $r > 0$ such that $[-r, r] \times \overline{B}_r(\overline{x}) \subset U$, we associate the homogeneous Hamiltonian in the fast derivatives (q, Y)

$$H_r'(y, q, Y) := \inf\{H'(x, y, q, Y) \mid |x - \overline{x}| \leqq r\} \tag{20}$$

as well as the continuous and periodic initial data

$$h_r(y) := \sup\{h(x, y) \mid |x - \overline{x}| \leqq r\}.$$

By Lemma 2 below, there is a unique solution $w_r'(t, y)$ of the equation

$$\begin{aligned} &\partial_t w_r' + H_r'(y, D_y w_r', D_{yy}^2 w_r') = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m, \\ &w_r'(0, y) = h_r(y) \quad \text{on } \mathbb{R}^m, \quad w_r' \text{ periodic.} \end{aligned} \tag{21}$$

Moreover, $H_r' \to H'$ and $h_r \to h$ as $r \to 0$ uniformly on the compact sets; for H' this follows from the fact that $H_r(y, 0, \lambda q, 0, \lambda Y, 0)/\lambda$ converges to $H_r'(y, q, Y)$ uniformly in (r, q, Y) as $\lambda \to +\infty$.

We claim that

$$\lim_{r \to 0,\, t \to \infty} \sup_y |w_r'(t, y) - \overline{h}(\overline{x})| = 0.$$

Indeed, let w' be the solution of the equation

$$\begin{aligned} &\partial_t w' + H'(\overline{x}, y, D_y w', D_{yy}^2 w') = 0 \quad \text{in } \mathbb{R}_+^* \times \mathbb{R}^m, \\ &w'(0, y) = h(\overline{x}, y) \quad \text{on } \mathbb{R}^m, \quad w' \text{ periodic.} \end{aligned}$$

Fix $\eta > 0$. By the definition of $\overline{h}(\overline{x})$, we can find some time $T > 0$ such that $|w'(T, y) - \overline{h}(\overline{x})| \leqq \eta/2$ for every y. By the stability properties of viscosity solutions, we know that $w'_r \to w'$ uniformly on the compact sets as $r \to 0$. Therefore there is r_0 such that $|w'_r(T, y) - w'(T, y)| \leqq \eta/2$ for every y and every $r < r_0$. In particular, $|w'_r(T, y) - \overline{h}(\overline{x})| \leqq \eta$ for every y and every $r < r_0$. Noting that $H'_r(\cdot, 0, 0) \equiv 0$, we deduce from the comparison principle that $|w'_r(t, y) - \overline{h}(\overline{x})| \leqq \eta$ for every y, every $r < r_0$ and every $t \geqq T$. This proves our claim.

From now on $\eta > 0$ is fixed and $r > 0$ is such that

$$\sup_{t \geqq T} \sup_{y} |w'_r(t, y) - \overline{h}(\overline{x})| \leqq \eta \qquad \text{for some } T > 0. \tag{22}$$

We consider the cylinder $Q =]0, r[\times B_r(\overline{x})$ with parabolic boundary

$$\partial' Q = (\{0\} \times B_r(\overline{x})) \cup (]0, r[\times \partial B_r(\overline{x})).$$

Fix a constant M so that $M \geqq u^\varepsilon$ on $[0, r] \times \overline{B}_r(\overline{x})$ for every ε. Let ψ_0 be a nonnegative smooth function in $\mathbb{R}^m$ such that $\psi_0(\overline{x}) = 0$ and $\psi_0 \geqq M - \inf h$ on $\partial B_r(\overline{x})$. By the definition of H', there is a constant $C > 0$ such that

$$|H(x, y, D\psi_0(x), q, D^2\psi_0(x), Y, 0) - H'(x, y, q, Y)| \leqq C$$

for every (y, q, Y), $x \in B_r(\overline{x})$. We claim that for every $\varepsilon > 0$ the function

$$\psi^\varepsilon(t, x, y) = w'_r\left(\frac{t}{\varepsilon}, y\right) + \psi_0(x) + Ct$$

is a supersolution of

$$\partial_t \psi^\varepsilon + H\left(x, y, D_x\psi^\varepsilon, \frac{D_y\psi^\varepsilon}{\varepsilon}, D^2_{xx}\psi^\varepsilon, \frac{D^2_{yy}\psi^\varepsilon}{\varepsilon}, \frac{D^2_{xy}\psi^\varepsilon}{\sqrt{\varepsilon}}\right) = 0 \text{ in } Q \times \mathbb{R}^m,$$
$$\psi^\varepsilon = h \text{ on } \{0\} \times B_r(\overline{x}) \times \mathbb{R}^m, \quad \psi^\varepsilon = M \text{ on }]0, r[\times \partial B_r(\overline{x}) \times \mathbb{R}^m. \tag{23}$$

The proof that ψ^ε satisfies the boundary condition

$$\psi^\varepsilon \geqq M \quad \text{on }]0, r[\times \partial B_r(\overline{x}) \times \mathbb{R}^m$$

follows at once from the inequalities $w'_r \geqq \inf h$ and $\psi_0 \geqq M - \inf h$ on $\partial B_r(\overline{x})$. The initial condition is clear, as

$$\psi^\varepsilon(0, x, y) = w'_r(0, y) + \psi_0(x) \geqq h_r(y) \geqq h(x, y).$$

The proof that ψ^ε is a supersolution of the equation results from the inequalities

$$H(x, y, D\psi_0(x), q, D^2\psi_0(x), Y, 0) \geqq H'(x, y, q, Y) - C \geqq H'_r(y, q, Y) - C$$

whenever $x \in B_r(\overline{x})$. In fact, by the homogeneity of H'_r in (q, Y),

$$\begin{aligned}\partial_t \psi^\varepsilon + H\left(x, y, D_x\psi^\varepsilon, \frac{D_y\psi^\varepsilon}{\varepsilon}, D^2_{xx}\psi^\varepsilon, \frac{D^2_{yy}\psi^\varepsilon}{\varepsilon}, \frac{D^2_{xy}\psi^\varepsilon}{\sqrt{\varepsilon}}\right)\\ = \frac{1}{\varepsilon}\partial_t w'_r + C + H\left(x, y, D\psi_0, \frac{D_y w'_r}{\varepsilon}, D^2\psi_0, \frac{D^2_{yy}w'_r}{\varepsilon}, 0\right)\\ \geqq \frac{1}{\varepsilon}\left(\partial_t w'_r + H'_r(y, D_y w'_r, D^2_{yy}w'_r)\right) = 0.\end{aligned}$$

This proves that ψ^ε is a supersolution of (23) if w'_r is smooth. The general case is easily handled by means of test functions, thus completing the proof of the claim.

Now we recall that u^ε is a subsolution of (23). By the comparison principle, we obtain the inequality

$$u^\varepsilon(t, x, y) \leqq \psi^\varepsilon(t, x, y) = w'_r\left(\frac{t}{\varepsilon}, y\right) + \psi_0(x) + Ct$$

for all $\varepsilon > 0$, $y \in \mathbb{R}^m$, $(t, x) \in Q$. Taking the upper limit as $(\varepsilon, t', x') \to (0, t, x)$ for $t > 0$ and $(t, x) \in Q$, we deduce from (22) that $\overline{u}(t, x) \leqq \overline{h}(\overline{x}) + \eta + \psi_0(x) + Ct$. Taking now the upper limit as $(t, x) \to (0, \overline{x})$, we obtain $\overline{u}(0, \overline{x}) \leqq \overline{h}(\overline{x}) + \eta$. The arbitrariness of η yields $\overline{u}(0, \overline{x}) \leqq \overline{h}(\overline{x})$. This completes the proof of the theorem.

Remark 5. A remark similar to the one following Theorem 2 is possible when we want to replace the periodicity assumption in the fast variable by a suitable boundary condition on the boundary of a compact set. The main assumptions needed in this case are: the cell Cauchy problem (CP′) (with suitable boundary conditions for the fast variable) has a unique viscosity solution and satisfies the comparison principle; the auxiliary problem (21) has a viscosity solution for r small; the comparison principle holds for (23).

Lemma 2. *The cell problem for the homogeneous Hamiltonian* (CP′) *and the Cauchy problem* (21) *have a unique viscosity solution.*

Proof. For the comparison principle and uniqueness in (CP′) we need for each fixed x a modulus ω' such that, for every $\kappa > 0$ and $y, y', p \in \mathbb{R}^m$,

$$H'(x, y', \kappa(y - y'), Y') - H'(x, y, \kappa(y - y'), Y) \leqq \omega'(|y' - y| + \kappa|y' - y|^2) \tag{24}$$

for all $Y, Y' \in \mathbb{S}^m$ satisfying (2). We are going to use the formula (8) for H' and the regularity property (3) with $R = |x|$. Since the modulus ω_R is concave, there are C_R, S_R such that $\omega_R(s) \leqq C_R s$ for all $s \geqq S_R$. Then we get

$$\begin{aligned}H(x, y', 0, \lambda\kappa(y - y'), 0, \lambda Y', 0)/\lambda - H(x, y, 0, \lambda\kappa(y - y'), 0, \lambda Y, 0)/\lambda\\ \leqq \omega_R(|y' - y| + \lambda\kappa|y' - y|^2)/\lambda \leqq C_R(|y' - y|/\lambda + \kappa|y' - y|^2),\end{aligned}$$

where the last inequality holds for $|y' - y| > 0$ and λ large enough. By letting $\lambda \to +\infty$ we obtain (24) with the right-hand side $C_R\kappa|y' - y|^2$. This inequality remains valid for $|y' - y| = 0$ because (2) implies $Y' \geqq Y$ and H' is degenerate elliptic.

To prove the existence of a solution of (CP′) we first assume the initial data $h = h(\overline{x}, \cdot)$ is smooth. For $C := \sup_y |H'(\overline{x}, y, D_y h, D^2_{yy} h)|$ the functions $h(y) - Ct$ and $h(y) + Ct$ are, respectively, a sub- and a supersolution of the Cauchy problem, periodic in y. Then we can use Perron's method as in Lemma 1 to obtain the desired solution of (CP′), continuous in $[0, +\infty) \times \mathbb{R}^m$. Now we approximate h uniformly by a sequence of smooth periodic h_k, and denote by w'_k the corresponding solutions of (CP′). By the comparison principle

$$\sup_{[0,+\infty)\times\mathbb{R}^m} |w'_k - w'_n| \leqq \max_{\mathbb{R}^m} |h_k - h_n|.$$

Since the right-hand side goes to 0 as $k, n \to \infty$, (w'_k) is a Cauchy sequence and therefore converges uniformly on $[0, +\infty) \times \mathbb{R}^m$. By the stability of viscosity solutions the limit is the solution of (CP′).

Now we turn to the Cauchy problem (21). A regularity estimate like (24) holds for the Hamiltonian H'_r, with right-hand side $C_{|\overline{x}|+r}\kappa|y' - y|^2$ (this follows easily from the corresponding estimate for H', as in the proof of Lemma 1). Therefore the comparison principle holds for (21), and the proof of the existence of the solution is exactly the same as for (CP′). The Lemma is proved.

Proposition 4. *If the pair (H, h) is stabilizing in a neighbourhood of $\overline{x}$, then, the effective initial condition $\overline{h}$ is continuous at $\overline{x}$.*

Proof. We keep the notation of the preceding proof. Fix $\eta > 0$ arbitrary and $r > 0$ so that (22) holds. For every $x \in B_r(\overline{x})$, w'_r is a supersolution of

$$\partial_t w'_r + H'(x, y, D_y w'_r, D^2_{yy} w'_r) \geqq \partial_t w'_r + H'_r(y, D_y w'_r, D^2_{yy} w'_r) = 0$$

with $w'_r(y) = h_r(x, y) \geqq h(x, y)$. By the comparison principle, we get $w'_r(t, y) \geqq w'(t, y; x)$ for every (t, y). Sending $t \to +\infty$, we obtain $\overline{h}(\overline{x}) \geqq \overline{h}(x) - \eta$. Since η is arbitrary, we deduce that

$$\overline{h}(\overline{x}) \geqq \limsup_{x\to\overline{x}} \overline{h}(x).$$

The reverse inequality is proved in the same way.

5. Sufficient conditions for convergence and examples

This section is devoted to the illustration of the convergence results Theorem 1 and Corollary 2. We present three types of sufficient conditions on the Hamiltonian ensuring its ergodicity and stabilization to a constant. The first is a non-degeneracy (or uniform ellipticity) assumption of H as an operator on the fast variables y, the second is a coercivity assumption with respect to $q = D_y u$, and the third is a non-resonance condition related to the classical theorem of Jacobi on ergodic dynamical systems on the torus. The first two conditions are rather classical for the ergodicity of the Hamiltonian (see, e.g., [35, 22, 11]). A thorough discussion of the ergodicity of HJB operators is given in [6]. We refer to the companion paper [3] for a systematic presentation of ergodicity and stabilization for HJBI operators.

These conditions are tested on the model problem

$$u_t^\varepsilon + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) - \frac{1}{\varepsilon} \operatorname{tr}(b(x, y) D_{yy} u^\varepsilon) + \frac{c(x, y)}{\varepsilon} |D_y u^\varepsilon| - \frac{1}{\varepsilon}(g(x, y), D_y u^\varepsilon) = 0. \tag{25}$$

The coefficients b, c and g are bounded, periodic in y, and Lipschitz continuous; the matrix $b \in \mathbb{S}^m$ is nonnegative semidefinite; and the function c is nonnegative. Moreover, the function $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ and the initial data h satisfy the following assumptions:

- F is continuous, degenerate elliptic, and periodic in y;
- F is uniformly continuous in (p, X), uniformly in (x, y), i.e., for some modulus ω,
$$|F(x, y, p, X) - F(x, y, p', X')| \leqq \omega(|p - p'| + |X - X'|);$$
- F satisfies the usual regularity conditions in x for the comparison principle in bounded domains [20] uniformly in y: for every $R > 0$, there is a modulus ω_R such that, for every $\kappa > 0$, $x, x' \in \mathbb{R}^n$ with $|x|$, $|x'| \leqq R$, every $y \in \mathbb{R}^m$ and every $X, X' \in \mathbb{S}^n$ satisfying (12), we have
$$F(x', y, \kappa(x - x'), X') \leqq F(x, y, \kappa(x - x'), X) + \omega_R(|x' - x| + \kappa|x' - x|^2); \tag{26}$$
- there exists a constant M such that $|F(x, y, 0, 0)| \leqq M$ for all (x, y);
- $h \in BUC(\mathbb{R}^{n+m})$, i.e., it is bounded and uniformly continuous, and periodic in y.

These assumptions ensure that the Hamiltonian

$$H(x, y, p, q, X, Y, Z) = F(x, y, p, X) - \operatorname{tr}\big(b(x, y)Y\big) - c(x, y)|q| - \big(g(x, y), q\big)$$

fulfils the requirements of Section 2.1 and those of Section 2.6 before Proposition 1. Therefore, there is a unique bounded viscosity solution of (HJ$_\varepsilon$) and the family u^ε is equibounded. Moreover, H satisfies the regularity condition (13) if b, c, and g are independent of x. In this case, and when $\overline{H}$ and $\overline{h}$ exist, the effective Cauchy problem ($\overline{\text{HJ}}$) has a unique solution and satisfies the comparison principle.

In some cases, we give an explicit formula for the effective Hamiltonian $\overline{H}$ and the effective initial condition $\overline{h}$. In the nondegenerate and coercive cases, we also characterize $-\overline{H}$ as the unique constant λ such that there exists a continuous solution of the true cell problem

$$\lambda + H(\overline{x}, y, \overline{p}, D_y\chi, \overline{X}, D^2_{yy}\chi, 0) = 0 \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic.} \tag{27}$$

Moreover, in each subsection we give at least one example where $\overline{H}$ is regular enough to ensure the uniqueness of the solution of the effective Cauchy problem ($\overline{\text{HJ}}$) and the local uniform convergence of u^ε to it, following Corollary 2.

In all the results of this section on the general problem (HJ$_\varepsilon$) we are tacitly assuming the standing assumptions of Section 2.1.

5.1. The nondegenerate case

In this subsection we consider the case of Hamiltonians that are uniformly elliptic with respect to the "large derivatives" $\varepsilon^{-1}D_{yy}$ (i.e., with respect to the fast variables). This means there are positive constants $\nu,\ \nu'$ depending only on $\overline{x},\ \overline{p},\ \overline{X}$ such that

$$\nu \operatorname{tr} W \leqq H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0) - H(\overline{x}, y, \overline{p}, q, \overline{X}, Y + W, 0) \leqq \nu' \operatorname{tr} W \quad (28)$$

for all $W \in \mathbb{S}^m$, $W \geqq 0$ and all y, q, Y. This condition is easily readable on the model problem (25) where it becomes

$$b(\overline{x}, y) \geqq \nu I_m \quad \forall\, y \in \mathbb{R}^m, \quad (29)$$

where I_m denotes the m-dimensional identity matrix.

We begin with the ergodicity result which is a generalization of Lemma 3.1 in Evans [22] (see also Arisawa & Lions [6]).

Proposition 5. *Assume that* (28) *holds for* $(\overline{x}, \overline{p}, \overline{X})$ *fixed. Then* H *is ergodic at* $(\overline{x}, \overline{p}, \overline{X})$.

We refer to [3] for the complete proof of the proposition. Here we simply give a sketch of it. It can actually be shown that there is a solution to the true cell problem (27). The idea of the proof is the following. Let w_δ be the solution of (CP$_\delta$). Then, by the comparison principle, the family $\{\delta w_\delta\}$ is equibounded. Moreover, using the regularity theory of uniformly elliptic equations, we can show that the family $\{w_\delta - w_\delta(0)\}$ is equicontinuous. Therefore, along a subsequence, δw_δ converges uniformly to a constant λ and $w_\delta - w_\delta(0)$ converges to a continuous function χ. By the stability results of viscosity solutions, we can pass to the limit in (CP$_\delta$). This ensures that (λ, χ) solves the true cell problem (27). As observed at the end of Section 2.3, the solvability of the true cell problem yields the ergodicity of the Hamiltonian.

We can also prove, in the current nondegenerate case, that H stabilizes to a constant any continuous initial data h periodic in y. The result, which is related to Theorem II.2 of Arisawa & Lions [6] for HJB equations, is proved in [3]. As the proof of ergodicity, it relies deeply on the comparison principle and on the regularity theory for solutions of uniformly elliptic and parabolic equations. Observe that the non-degeneracy condition (28) implies the uniform ellipticity of the recession function H', that is, there are positive constants ν, ν' such that

$$\nu \operatorname{tr} W \leqq H'(\overline{x}, y, q, Y) - H'(\overline{x}, y, q, Y + W) \leqq \nu' \operatorname{tr} W \quad (30)$$

for all $W \in \mathbb{S}^m$, $W \geqq 0$ and all y, q, Y.

Proposition 6. *Assume that* (30) *holds for* $\overline{x}$ *fixed. Then, for every continuous* h*, the pair* (H, h) *is stabilizing at* $\overline{x}$.

As a consequence of the preceding two propositions, we can restate Theorem 1 for uniformly elliptic Hamiltonians.

Corollary 3. *Assume that for all $\overline{x}$, $\overline{p}$, $\overline{X}$ there exist $\nu', \nu > 0$ such that* (28) *holds, and suppose that the family $\{u^\varepsilon\}$ of solutions of* (HJ_ε) *is locally equibounded. Then there exist a continuous degenerate elliptic $\overline{H}$ and a continuous $\overline{h}$ such that the semi-limits $\overline{u} = \limsup_{\varepsilon\to 0} u^\varepsilon$ and $\underline{u} = \liminf_{\varepsilon\to 0} u^\varepsilon$ are, respectively, a subsolution and a supersolution of the effective Cauchy problem* ($\overline{HJ}$).

We now turn to the model equation (25). As noted above, the uniform ellipticity amounts to assumption (29), so Corollary 3 applies in this case. Under a further condition the convergence is actually uniform, in view of Proposition 2.

Corollary 4. *Assume* (29) *and that b, c, and g are independent of x. Then the solution u^ε of* (31) *with initial data $u^\varepsilon(0, x, y) = h(x, y)$ converges uniformly on compact subsets of* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as $\varepsilon \to 0$ to the unique viscosity solution of* ($\overline{HJ}$).

In the special case that $c \equiv 0$, we can derive some explicit formulas for $\overline{H}$ and $\overline{h}$. Note that the model problem (25) becomes linear in all the "large derivatives" $\varepsilon^{-1} D_{yy}$ and $\varepsilon^{-1} D_y$:

$$u_t^\varepsilon + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) - \frac{1}{\varepsilon}\operatorname{tr}(b(x, y) D_{yy} u^\varepsilon) - \frac{1}{\varepsilon}(g(x, y), D_y u^\varepsilon) = 0. \tag{31}$$

Following BENSOUSSAN, J.-L. LIONS & PAPANICOLAOU [17], JENSEN & LIONS [29] and EVANS [21] we consider the invariant measure μ_x associated with the diffusion process defined by the matrix b and the vector field g, that is, the solution of the adjoint equation

$$-\sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j}(b_{ij}(x, y)\mu_x) + \sum_i \frac{\partial}{\partial y_i}(g_i(x, y)\mu_x) = 0 \quad \text{in } \mathbb{R}^m, \qquad \mu_x \text{ periodic} \tag{32}$$

with mean $\int_{(0,1)^m} \mu_x(y)\, dy = 1$.

Proposition 7. *Assume b and g are smooth in y and, for all $\overline{x} \in \mathbb{R}^n$, there is $\nu > 0$ such that the ellipticity condition* (29) *holds. Then the invariant measure μ_x exists and is unique. The Hamiltonian in* (31) *is ergodic with*

$$\overline{H}(x, p, X) = \int_{(0,1)^m} F(x, y, p, X)\mu_x(y)\, dy, \tag{33}$$

and the pair (H, h) is stabilizing to the constant

$$\overline{h}(x) = \int_{]0,1[^m} h(x, y)\mu_x(y)\, dy.$$

Proof. Note that the cell problem (27) associated with (31) is linear and uniformly elliptic, as well as the recession Hamiltonian $H'(x, y, q, Y) = -\operatorname{tr}(b(x, y)Y) - (q, g(x, y))$.

From classical results based on the Fredholm alternative (see, for instance, [17] or [16]), for b and g smooth in y, there is a unique solution μ_x of (32) with average 1. Then (33) is a necessary and sufficient condition for the true cell problem (27) to have a solution. This is a known result that follows formally from multiplying (27) by μ_x and integrating by parts.

To prove the formula for $\overline{h}$ we multiply by μ_x the PDE in (CP′)

$$w_t - \operatorname{tr}(b(x,y)D^2_{yy}w) - (g(x,y), D_y w) = 0, \qquad w(0,y) = h(x,y), \qquad (34)$$

and integrate over $(0,1)^m$. We see that the function

$$\varphi(t) := \int_{(0,1)^m} w(t,y)\mu_x(y)\,dy$$

has $\dot{\varphi} \equiv 0$. Therefore $\lim_{t\to+\infty}\varphi(t) = \varphi(0)$, which gives the desired formula.

When the invariant measure is independent of x we can use these formulas and Corollary 2 in Section 2 to give another case of uniform convergence of u^ε.

Corollary 5. *In addition to the hypotheses of Proposition 7 assume*

$$\frac{\partial}{\partial y_i}\left(g_i - \sum_j \frac{\partial b_{ij}}{\partial y_j}\right) = 0$$

(*e.g., b and g independent of y*). *Then the solution u^ε of* (31) *with initial data $u^\varepsilon(0,x,y) = h(x,y)$ converges uniformly on compact subsets of $(0,T)\times\mathbb{R}^n\times\mathbb{R}^m$ as $\varepsilon \to 0$ to the unique viscosity solution of*

$$\begin{aligned} &u_t + \int_{(0,1)^m} F(x,y,D_x u, D_{xx}u)\,dy = 0 \quad \textit{in } (0,T)\times\mathbb{R}^n,\\ &u(x,0) = \int_{(0,1)^m} h(x,y)\,dy \quad \textit{on } \mathbb{R}^n. \end{aligned} \qquad (35)$$

Proof. Under the current assumption it is easy to see that the solution of (32) is $\mu_x \equiv 1$. Therefore in this case the effective Hamiltonian and initial data are the averages with respect to the Lebesgue measure

$$\overline{H}(x,p,X) = \int_{(0,1)^m} F(x,y,p,X)\,dy, \qquad \overline{h}(x) = \int_{(0,1)^m} h(x,y)\,dy.$$

Then the assumptions on f and h ensure that the comparison principle holds among bounded sub- and supersolutions of the effective Cauchy problem (35). Indeed, the regularity property (11) of $\overline{H}$ immediately follows from (26) for F. Corollary 2 in Section 2.5 then gives the local uniform convergence of u^ε.

For the linear model problem (31) we can also give some simple probabilistic formulas for $\overline{H}$ and $\overline{h}$. They involve the diffusion process

$$dy_s = g(x,y_s)ds + \tau(x,y_s)dW_s, \quad y_0 = y, \qquad (36)$$

where x is frozen, $\tau\tau^T = 2b$, and W_s is a Brownian motion.

Proposition 8. *Assume there is* $\nu > 0$ *such that the ellipticity condition* (29) *holds for all* $\overline{x} \in \mathbb{R}^n$. *Then the Hamiltonian in* (31) *is ergodic with*

$$\overline{H}(x, p, X) = \lim_{T\to+\infty} \frac{1}{T} E \int_0^T F(x, y_s, p, X)\, ds \quad \text{for all } y_0 = y \in \mathbb{R}^m, \tag{37}$$

where E *denotes the expectation, and the pair* (H, h) *is stabilizing to the constant*

$$\overline{h}(x) = \lim_{s\to+\infty} Eh(x, y_s) \quad \text{for all } y_0 = y \in \mathbb{R}^m.$$

Proof. The cell t-problem (CP) in the second definition of ergodicity is

$$w_t + F(x, y, p, X) - \operatorname{tr}(b(x, y)D_{yy}w) - (g(x, y), D_y w) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m,$$

with $w(0, y) = 0$ on $\mathbb{R}^m$. By standard results on linear parabolic equations the solution is $w(t, y) = -E \int_0^t F(x, y_s, p, X)\, ds$. The existence of the limit in the right-hand side of (37) independent of y comes from Proposition 5 or from the classical ergodic theory of nondegenerate diffusions (see, e.g., [26]).

Similarly, the cell Cauchy problem (CP′) in the definition of stabilization is (34) for the current model problem. It involves a linear parabolic PDE whose solution is $w(t, y) = Eh(x, y_t)$. The existence of the long-time limit independent of y comes, for instance, from Proposition 6.

Remark 6. The consistency of the formulas for $\overline{H}$ in the Propositions 7 and 8, i.e., the equality of the limit in (37) and the average with respect to the invariant measure, is a classical result in the theory of nondegenerate diffusions, see, for instance, [26]. Note also that the linearity of the elliptic PDE in the cell δ-problem allows us to compute the solution $w_\delta(y)$ along the paths y_s and leads to the formula

$$\overline{H}(x, p, X) = \lim_{\delta\to 0+} \delta E \int_0^{+\infty} F(x, y_s, p, X)e^{-\delta s}\, ds \quad \text{for all } y_0 = y \in \mathbb{R}^m.$$

5.2. The coercive case

In this subsection, we make a coercivity assumption on the Hamiltonian with respect to the gradient q in the fast variables y, following [35, 6]. More precisely, we assume that there are constants $\nu > 0$ and C depending only on $\overline{x}$, $\overline{p}$, $\overline{X}$ such that

$$\begin{aligned} H_2(\overline{x}, y, \overline{p}, q, \overline{X}, Y) &:= H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0) - H(\overline{x}, y, \overline{p}, 0, \overline{X}, 0, 0) \\ &\geqq \nu|q| - C \end{aligned} \tag{38}$$

for all y, q, Y. Adapting the proof for uniformly elliptic operators, it can be shown that the true cell problem (27) has a solution (see [35, 22, 6, 3]). This ensures the ergodicity of the Hamiltonian.

Proposition 9. *Assume that* (38) *holds for* $(\overline{x}, \overline{p}, \overline{X})$ *fixed. Then* H *is ergodic at* $(\overline{x}, \overline{p}, \overline{X})$. *Moreover, if* $C = 0$ *in* (38), *then the following explicit formula holds:*

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = \max_y H(\overline{x}, y, \overline{p}, 0, \overline{X}, 0, 0). \tag{39}$$

Proof. We only show that the effective Hamiltonian is given by (39) whenever $C = 0$ and refer to the above mentioned references for a complete proof of the ergodicity of the Hamiltonian. We set $H_1(y) := H(\overline{x}, y, \overline{p}, 0, \overline{X}, 0, 0)$. If w is the solution of the cell t-problem (CP), then the comparison principle gives at once $w(t, y) \geqq -t \max_y H_1$. Sending $t \to +\infty$, we deduce that $\overline{H} \leqq \max_y H_1$.

To prove the reverse inequality, we assume for contradiction that $\overline{H} < H_1(y)$ in a neighbourhood of a maximum point of H_1. The true cell problem (27) now reads

$$H_2(\overline{x}, y, \overline{p}, D\chi, \overline{X}, D^2_{yy}\chi, 0) = \overline{H} - H_1(y).$$

Thus, (38) with $C = 0$ gives $\nu|D_y\chi| < 0$ in an open set. Since this is impossible the proof is complete.

Next we prove that in the current coercive case H stabilizes to a constant any continuous initial data h. We note that the coercivity (38) of H yields the coercivity of the recession function H', that is, the existence of a constant $\nu > 0$ such that

$$H'(\overline{x}, y, q, Y) \geqq \nu|q|, \quad \forall\, y, q, Y. \tag{40}$$

Proposition 10. *Assume that* (40) *holds for* $\overline{x}$ *fixed. Then, for every continuous* h*, the pair* (H, h) *is stabilizing at* $\overline{x}$*. Moreover, we have the explicit formula*

$$\overline{h}(\overline{x}) = \min_{y\in\mathbb{R}^m} h(\overline{x}, y).$$

Proof. We provide a complete proof of the proposition to justify the explicit formula for $\overline{h}$. Put $h_1 = \min_{y\in\mathbb{R}^m} h(\overline{x}, y)$. Consider the solution of

$$z_t + \nu|D_y z| = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^m, \quad z(0, y) = h(\overline{x}, y) \quad \text{on } \mathbb{R}^m.$$

It is a supersolution of the cell problem (CP′) by (40). Denoting by w' the solution of (CP′), we get by the comparison principle that $h_1 \leqq w' \leqq z$ on $[0, +\infty) \times \mathbb{R}^m$. Moreover, z can be represented as the value function of a deterministic control problem

$$z(t, y) = \inf\{h(\overline{x}, y_t) \mid y_0 = y,\ |\dot{y}_s| \leqq \nu\}.$$

It is easy to see from this formula that $z(t, \cdot) \equiv h_1$ for t large. So, $w(t, \cdot) \equiv h_1$ for t large, and thus $\overline{h}(\overline{x}) = h_1$.

As a consequence of the preceding two propositions, we can restate Theorem 1 for coercive Hamiltonians.

Corollary 6. *Assume that, for all* $\overline{x}$, $\overline{p}$, $\overline{X}$*, there exists* $\nu > 0$ *such that* (38) *holds, and suppose that the family* $\{u^\varepsilon\}$ *of solutions of* (HJ$_\varepsilon$) *is locally equibounded. Then there exist a continuous degenerate elliptic* $\overline{H}$ *and a continuous* $\overline{h}$ *such that the semi-limits* $\overline{u} = \limsup_{\varepsilon\to 0} u^\varepsilon$ *and* $\underline{u} = \liminf_{\varepsilon\to 0} u^\varepsilon$ *are, respectively, a subsolution and a supersolution of the effective Cauchy problem* ($\overline{\text{HJ}}$).

As an illustration of the use of the preceding corollary, we state a local uniform convergence of the solution to the model equation (25). We assume that

$$c(\overline{x}, y) \geqq |g(\overline{x}, y)| + \nu \quad \text{and} \quad b(\overline{x}, y) = 0 \quad \text{for all } y \in \mathbb{R}^m. \tag{41}$$

Observe that this implies that the related Hamiltonian satisfies the coercivity assumption (38) with $C = 0$.

Corollary 7. *Under* (41)*, the solution* u^ε *of the model equation* (25) *with initial data* $u^\varepsilon(0, x, y) = h(x, y)$ *converges uniformly on compact subsets of* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *as* $\varepsilon \to 0$ *to the unique viscosity solution of the effective Cauchy problem*

$$u_t + \max_{y \in [0,1]^m} F(x, y, Du, D^2u) = 0 \quad \textit{in } (0, T) \times \mathbb{R}^n,$$
$$u(0, x) = \min_{y \in [0,1]^m} h(x, y) \quad \textit{on } \mathbb{R}^n. \tag{42}$$

Proof. We have to check that the effective Cauchy problem (42) satisfies the comparison principle. The effective Hamiltonian is $\overline{H}(x, p, X) = \max_y F(x, y, p, X)$. The uniform continuity of F in (p, X) yields the same property for $\overline{H}$. Moreover, the inequality (26) for F implies that the effective Hamiltonian satisfies (11) for all $\kappa > 0$, $x, x' \in \mathbb{R}^n$, $X, X' \in \mathbb{S}^n$ satisfying (12). Hence, the limit problem (42) satisfies the comparison principle.

5.3. The nonresonant cases

We limit ourselves in this subsection to the model problem (25). All terms are possibly nonzero, so that the equation is nonlinear also in the y-derivatives, but the coefficients are independent of y

$$u_t^\varepsilon + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) - \frac{1}{\varepsilon} \operatorname{tr}(b(x) D_{yy} u^\varepsilon)$$
$$+ \frac{c(x)}{\varepsilon} |D_y u^\varepsilon| - \frac{1}{\varepsilon} (g(x), D_y u^\varepsilon) = 0. \tag{43}$$

We recall that $c \geqq 0$. Moreover, when $c(x) = 0$, we shall make one of the following non-resonance assumption:

$$b(x)k \neq 0 \qquad \text{for all } k \in \mathbb{Z}^m \backslash \{0\}. \tag{44}$$

or

$$\big(g(x), k\big) \neq 0 \qquad \text{for all } k \in \mathbb{Z}^m \backslash \{0\}. \tag{45}$$

This last condition, which is the classical Jacobi necessary and sufficient condition for a constant vector field to be ergodic in the torus, will be further discussed in the appendix.

For optimal control problems, the relevance of the non-resonance condition for ergodic problems on the torus was pointed out by ARISAWA & LIONS [6]. The following proposition is a special case of one of their results.

Proposition 11. *For $\overline{x}$ fixed, assume that either $c(\overline{x}) > 0$ or $b(\overline{x})$ satisfies* (44) *or $g(\overline{x})$ verifies* (45)*. Then the Hamiltonian in* (43) *is ergodic at $\overline{x}$.*

Proof. The cell t-problem in this case can be written as

$$w_t + F(x, y, p, X) - \operatorname{tr}(b(x) D_{yy} w) + \max_{|\alpha| \leqq 1} (c(x)\alpha - g(x), D_y w) = 0.$$

It is a Hamilton-Jacobi-Bellman equation and the convergence of $w(t, y)/t$ to a constant corresponds to a stochastic ergodic control problem. Under the assumption of the proposition, Theorem IV.1 in [6] ensures the desired convergence.

Proposition 12. *For $\overline{x}$ fixed assume that either $c(\overline{x}) > 0$ or $b(\overline{x})$ satisfies* (44)*. Then, for every continuous h, the pair (H, h) in* (43) *is stabilizing at $\overline{x}$.*

Proof. The homogeneous PDE in the cell Cauchy problem is also a Hamilton-Jacobi-Bellman equation and the stabilization to a constant is a special case of a result in our paper [3].

Comparing the issues of ergodicity and stabilization in the nonresonant case, we see that this is one of the few situations where the former property may hold without the latter. Indeed, whenever $c(\overline{x}) = 0$ and $b(\overline{x}) = 0$, the Jacobi condition (45) ensures ergodicity but stabilization does not occur (unless h is independent of y). To see this, we note that the associated homogeneous cell Cauchy problem (CP′) is the linear transport equation

$$w_t - \big(g(x), D_y w\big) = 0 \;\text{ in } \mathbb{R}_+^* \times \mathbb{R}^m, \quad w(0, y; x) = h(x, y) \;\text{ on } \mathbb{R}^m.$$

The solution is $w(t, y; x) = h(x, y_t)$ with $y_t = y + g(x)t$. For every fixed t, the mapping $y \mapsto y_t$ is a bijection. This implies that $\sup_y w(t, y; x)$ and $\inf_y w(t, y; x)$ are constant in t. Therefore $w(t, \cdot; x)$ cannot converge uniformly as $t \to +\infty$ unless $h(x, \cdot)$ is constant.

As usual, the preceding two propositions can be coupled with Theorem 1 to obtain a weak convergence result for solutions of $(\mathrm{HJ}_\varepsilon)$. In the general case, we do not get explicit formulas for the effective data. However, when $c \equiv 0$, the effective Hamiltonian and the effective initial condition can be easily computed. For the resulting equation

$$\begin{aligned} u_t^\varepsilon + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) - \frac{1}{\varepsilon} \operatorname{tr}(b(x) D_{yy} u^\varepsilon) \\ - \frac{1}{\varepsilon}(g(x), D_y u^\varepsilon) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n \times \mathbb{R}^m, \end{aligned} \tag{46}$$

we therefore obtain the following stronger convergence result.

Corollary 8. *Assume that* (44) *holds for all $\overline{x} \in \mathbb{R}^n$. Then the solution u^ε of* (46) *with initial data $u^\varepsilon(0, x, y) = h(x, y)$ converges uniformly on compact subsets of $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ as $\varepsilon \to 0$ to the unique viscosity solution of*

$$\begin{aligned} u_t + \int_{(0,1)^m} F(x, y, D_x u, D_{xx} u)\, dy = 0 \quad \textit{in } (0, T) \times \mathbb{R}^n, \\ u(x, 0) = \int_{(0,1)^m} h(x, y)\, dy \quad \textit{on } \in \mathbb{R}^n. \end{aligned} \tag{47}$$

We remark that this theorem has exactly the same conclusions as Corollary 5 for a nondegenerate diffusion, although here the matrix b can be very degenerate. For instance, $b = \xi\xi^T$ with $\xi \in \mathbb{R}^n$ has rank one and (44) holds if $(\xi, k) \neq 0$ for all $k \in \mathbb{Z}^m \setminus \{0\}$.

Proof. The cell t-problem associated with (46) is the linear degenerate parabolic equation

$$w_t + F(x, y, p, X) - \operatorname{tr}(b(x) D_{yy} w) - (g(x), D_y w) = 0, \qquad w(0, y) = 0.$$

By Proposition 11, the non-resonance condition (44) implies that $w(t, y)/t$ converges to the constant $-\overline{H}(x, p, X)$ as $t \to +\infty$. To prove a formula for $\overline{H}$ we mollify w and assume, without loss of generality, that the PDE is satisfied almost everywhere. Then we use the PDE and the periodicity of w to compute

$$\begin{aligned}
&\frac{d}{dt} \int_{(0,1)^m} w(t, y)\, dy \\
&\quad = -\int_{(0,1)^m} F(x, y, p, X)\, dy + \int_{(0,1)^m} (\operatorname{tr}(b(x) D_{yy} w) + (g(x), D_y w))\, dy \\
&\quad = -\int_{(0,1)^m} F(x, y, p, X)\, dy.
\end{aligned}$$

We integrate and get $\int_{(0,1)^m} w(t, y)\, dy = -t \int_{(0,1)^m} F(x, y, p, X)\, dy$. Therefore

$$\overline{H}(x, p, X) = \int_{(0,1)^m} F(x, y, p, X)\, dy.$$

Next, we have to look at the associated homogeneous cell Cauchy problem (CP′), which is the linear degenerate parabolic equation

$$w_t - \operatorname{tr}(b(x) D_{yy} w) - (g(x), D_y w) = 0, \quad w(0, y) = h(x, y).$$

The convergence of $w(t, y)$ to $\overline{h}(x)$ as $t \to +\infty$ under the non-resonance condition (44) follows from Proposition 12. To prove a formula for $\overline{h}$ we proceed as above and compute $\frac{d}{dt} \int_{(0,1)^m} w(t, y)\, dy = 0$. Then

$$\int_{(0,1)^m} w(t, y)\, dy = \int_{(0,1)^m} h(x, y)\, dy$$

and we obtain

$$\overline{h}(x) = \int_{(0,1)^m} h(x, y)\, dy.$$

The explicit formulas for $\overline{H}$ and $\overline{h}$ and the assumptions on F and h ensure the comparison principle for the effective cauchy problem (47), as in the proof of Corollary 5. Then Corollary 2 in Section 2.5 gives the local uniform convergence of u^ε.

5.4. Concluding remarks and further applications

1. Convergence under mixed conditions. We recall that the notions of ergodicity and stabilization to a constant are pointwise in $\overline{x} \in \mathbb{R}^n$ and therefore the different sufficient conditions presented in this section can be combined to obtain more general results. Here is an explicit example on the model problem (25).

Corollary 9. *Let u^ε be the solution of* (25) *in* $(0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ *with initial condition* $u^\varepsilon(0, x, y) = h(x, y)$. *Assume that for any* $\overline{x} \in \mathbb{R}^n$ *either* (29) *holds for some* $\nu > 0$, *or* (41) *is valid, or* $b(\overline{x}, y)$, $c(\overline{x}, y)$, $g(\overline{x}, y)$ *are constant in y and* (44) *holds. Then there exist a continuous degenerate elliptic* $\overline{H}$ *and a continuous* $\overline{h}$ *such that the semi-limits* $\overline{u} = \limsup_{\varepsilon\to 0} u^\varepsilon$ *and* $\underline{u} = \liminf_{\varepsilon\to 0} u^\varepsilon$ *are, respectively, a subsolution and a supersolution of the effective Cauchy problem*

$$u_t + \overline{H}(x, Du, D^2u) = 0 \quad \textit{in } (0, T) \times \mathbb{R}^n, \quad u(0, x) = \overline{h}(x) \quad \textit{on } \mathbb{R}^n.$$

2. On uniform convergence. We gave four results,at least one for each subsection, where the convergence of u^ε is locally uniform and not only obtained via the semi-limits, namely the Corollaries 4, 5, 7, and 8. All of them are obtained by the approach of Corollary 2: since the effective Hamiltonian $\overline{H}$ is always continuous and degenerate elliptic, it is enough to show it has sufficient regularity with respect to x to be able to use a comparison principle for the effective cauchy problem and get simultaneously the uniqueness of the limit and the local uniform convergence. We believe this is more powerful than an approach based on equicontinuity estimates on u^ε. More results on the regularity of $\overline{H}$ are given in [2] for Hamilton-Jacobi-Bellman equations, and in [3] for Bellman-Isaacs equations. They always require some additional condition on the dependence of H on x and on the derivatives with respect to x, such as (13) in Proposition 2. Without extra assumptions, the effective Hamiltonian may not satisfy the comparison principle and the uniform convergence may fail. We show this on an example in [3] where u^ε converges pointwise to a discontinuous function.

3. Periodic homogenization. From the theory developed so far, we can easily deduce new results on the periodic homogenization of parabolic equations not in divergence form

$$v_t^\varepsilon + G\left(\frac{x}{\varepsilon}, D^2 v^\varepsilon\right) = l\left(x, \frac{x}{\varepsilon}\right) \quad \text{in } (0, T) \times \mathbb{R}^n,$$
$$v^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right) \quad \text{on } \mathbb{R}^n,$$

where G is periodic in the first entry, continuous, degenerate elliptic, and satisfies the assumptions of the comparison principle. In fact, we can look for solutions of the form $v^\varepsilon(t, x) = u^\varepsilon(t, x, \frac{x}{\varepsilon})$ and observe that the Cauchy problem for u^ε is

$$u_t^\varepsilon + G\left(y, D_{xx}u^\varepsilon + \frac{D_{yy}u^\varepsilon}{\varepsilon^2} + \frac{D_{xy}u^\varepsilon}{\varepsilon} + \frac{(D_{xy}u^\varepsilon)^T}{\varepsilon}\right)$$
$$= l(x, y) \quad \text{in } (0, T) \times \mathbb{R}^n \times \mathbb{R}^n,$$
$$u^\varepsilon(0, x) = h(x, y) \quad \text{on } \mathbb{R}^n \times \mathbb{R}^n,$$

for $y = x/\varepsilon$. After replacing ε with $\sqrt{\varepsilon}$ this problem becomes a special case of our singular perturbation problem (HJ_ε).

The first example is an application of Propositions 2, 5, and 6 in the case of uniformly elliptic G

$$\nu \operatorname{tr} W \leqq G(y, X) - G(y, X + W) \leqq \nu' \operatorname{tr} W \tag{48}$$

for all $W \in \mathbb{S}^m$, $W \geqq 0$, and all y, X.

Corollary 10. *Assume* (48). *Then there exist a continuous degenerate elliptic* $\overline{H}$ *and a continuous* $\overline{h}$ *such that* v^ε *converges uniformly on the compact subsets of* $(0, T) \times \mathbb{R}^n$ *as* $\varepsilon \to 0$ *to the unique viscosity solution of the effective Cauchy problem*

$$v_t + \overline{H}(x, D^2 v) = 0 \quad \text{in } (0, T) \times \mathbb{R}^n, \qquad v(0, x) = \overline{h}(x) \quad \text{on } \mathbb{R}^n.$$

The second example is an application of Corollary 8 to the problem

$$v_t^\varepsilon - \operatorname{tr}(bD^2 v^\varepsilon) = l\left(x, \frac{x}{\varepsilon}\right) \ \text{in } (0, T) \times \mathbb{R}^n, \quad v^\varepsilon(0, x) = h\left(x, \frac{x}{\varepsilon}\right) \ \text{on } \mathbb{R}^n,$$

where b is a constant nonnegative matrix, and l is bounded, uniformly continuous, and periodic in the second entry. Although the PDE is now linear with constant coefficients (but degenerate!) the next result seems to be new.

Corollary 11. *Assume* $bk \neq 0$ *for all* $k \in \mathbb{Z}^m \setminus \{0\}$. *Then* v^ε *converges uniformly on compact subsets of* $(0, T) \times \mathbb{R}^n$ *as* $\varepsilon \to 0$ *to the unique viscosity solution of*

$$\begin{aligned} v_t - tr(bD^2 v) &= \int_{(0,1)^m} l(x, y)\, dy \quad \text{in } (0, T) \times \mathbb{R}^n, \\ v(x, 0) &= \int_{(0,1)^m} h(x, y)\, dy \quad \text{on } \mathbb{R}^n. \end{aligned}$$

Many more applications of the methods of this paper to periodic homogenization are given in our article [4].

4. Periodic averaging. Consider the degenerate parabolic equation with fast oscillations in time

$$v_t^\varepsilon + F\left(x, \frac{t}{\varepsilon}, Dv^\varepsilon, D^2 v^\varepsilon\right) = 0 \ \text{in } (0, T) \times \mathbb{R}^n, \quad v^\varepsilon(0, x) = h(x) \ \text{on } \mathbb{R}^n,$$

where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ is 1-periodic in the second entry and satisfies the assumptions at the beginning of Section 5, and $h \in BUC(\mathbb{R}^n)$. This is equivalent to the singular perturbation problem

$$\begin{aligned} u_t^\varepsilon + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) + \frac{u_y^\varepsilon}{\varepsilon} &= 0 \qquad \text{in } (0, T) \times \mathbb{R}^n \times \mathbb{R}, \\ u^\varepsilon(0, x, y) &= h(x) \quad \text{on } \mathbb{R}^n \times \mathbb{R}. \end{aligned}$$

In fact, by the uniqueness of the solutions, $v^\varepsilon(t,x) = u^\varepsilon(t,x,\frac{t}{\varepsilon})$. The Hamiltonian of the last problem is easily checked and found to be ergodic with effective Hamiltonian

$$\overline{H}(x,p,X) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T F(x,t_0-s,p,X)\,ds = \int_0^1 F(x,s,p,X)\,ds.$$

By combining Corollary 2 and the discussion in Section 2.6, we obtain a general convergence theorem for the time-averaging problem. Earlier results for linear uniformly parabolic equations are in [16] and for first-order equations in [22, 14, 28]; see also the references therein.

Corollary 12. *Under the previous assumptions v^ε converges uniformly on compact subsets of* $[0,T]\times\mathbb{R}^n$ *as* $\varepsilon\to 0$ *to the unique viscosity solution of*

$$v_t + \int_0^1 F(x,s,Dv,D^2v)\,ds = 0 \;\text{ in } (0,T)\times\mathbb{R}^n, \quad v(0,x) = h(x) \;\text{ on } \mathbb{R}^n.$$

6. Appendix

The purpose of this appendix is twofold. The first is to present briefly some elements of the ergodic properties of a deterministic dynamical system on the torus in order to clarify the relationship with our definitions of ergodic Hamiltonian. A general introduction to ergodic theory can be found in the books by ARNOLD & AVEZ [7] and CORNFELD, FOMIN & SINAI [19], and some connections with HJB equations in the Ph. D. thesis of M. Arisawa and in her paper [5]. For the connections among parabolic PDEs and the ergodic properties of diffusion processes we refer to [17, 26, 15, 32]. The second goal of the section is the proof of the equivalence of the three definitions of ergodicity of a Hamiltonian given in Section 3, which is a sort of generalized nonlinear version of the classical Abelian-Tauberian theorem, see, e.g., [37].

6.1. Connections with the classical ergodic theory

Let $\mathbb{T}^m$ denote the torus obtained by identifying the opposite faces of $(0,1)^m$. Consider the flow $\Phi = \{\Phi_\tau \mid \tau\in\mathbb{R}\}$ on $\mathbb{T}^m$ associated with the ordinary differential equation

$$\dot{y} = g(y),$$

where g is a Lipschitzean vector field on $\mathbb{R}^m$ such that $g(y) = g(y+k)$ for all $k\in\mathbb{Z}^m$. The point $\Phi_\tau y = y_\tau$ is therefore the value at time τ of the solution of the ordinary differential equation starting from y at time 0. A Radon measure μ on the torus is said to be *invariant* for the flow Φ if, for every measurable bounded set A and every $\tau\in\mathbb{R}$, we have $\mu(\Phi_\tau A) = \mu(A)$. This is equivalent to asking that

$$\int \psi\circ\Phi_\tau\,d\mu = \int \psi\,d\mu \quad \text{for every } \psi\in C(\mathbb{T}^m) \text{ and every } \tau\in\mathbb{R}.$$

Given an invariant measure μ, the dynamical system associated with the flow is said to be *ergodic* if, for every $\psi \in L^1(\mu)$, we have

$$\frac{1}{T}\int_0^T \psi(\Phi_t y)\,dt \to \int \psi\,d\mu \quad \text{as } T \to +\infty, \quad \text{for } \mu\text{-almost every } y.$$

This is equivalent to saying that all the invariant measurable sets must have zero or full μ measure. The dynamical system is said to be *uniquely ergodic* if there exists a unique invariant probability measure. The next proposition states that this is equivalent to the property

$$\frac{1}{T}\int_0^T \psi(\Phi_t y)\,dt \to \text{const} \qquad \text{as } T \to +\infty, \text{ uniformly in } y$$

for every $\psi \in C(\mathbb{T}^m)$.

This allows us to show the connection between ergodic theory and our definitions of ergodicity in Section 2.3 in the following special case. Consider a Hamiltonian linear in the fast first derivatives q and independent of the second derivatives Y, Z, i.e., take the ε-problem

$$u_t^\varepsilon + F(x, y, D_x u^\varepsilon, D_{xx} u^\varepsilon) - \frac{1}{\varepsilon}(g(y), D_y u^\varepsilon) = 0.$$

The associated cell t-problem is the linear transport equation

$$w_t + \psi(y) - (g(y), D_y w) = 0, \qquad w(0, y) = 0,$$

where $\psi(y) = F(\overline{x}, y, \overline{p}, \overline{X})$. Its solution is $w(t, y) = -\int_0^t \psi(\Phi_t y)\,ds$. By the second definition in Section 2.3 the current Hamiltonian is ergodic if and only if

$$\frac{1}{T} w(T, y) \to \text{const} \qquad \text{as } T \to +\infty, \text{ uniformly in } y,$$

and therefore this occurs for all continuous F if and only if the dynamical system is uniquely ergodic.

As for the first definition of ergodicity of H in Section 2.3, we recall that

$$\lim_{t\to+\infty} \frac{1}{T}\int_0^T \psi(\Phi_t y)\,dt = \lim_{\delta\to 0+} \delta \int_0^{+\infty} \psi(\Phi_t y) e^{-\delta t}\,dt$$

if either one of the two limits exists, by the classical Abelian-Tauberian theorem [37]. The cell δ-problem associated with the current ε-problem is

$$\delta w_\delta + \psi(y) - (g(y), D_y u_\delta) = 0 \quad \text{in } \mathbb{R}^m, \qquad w_\delta \text{ periodic},$$

whose solution is $w_\delta(y) = -\int_0^{+\infty} \psi(\Phi_t y) e^{-\delta t}\,dt$. The first definition in Section 2.3 says that the current Hamiltonian is ergodic if and only if $\delta w_\delta(y) \to \text{const}$ as $\delta \to 0+$, uniformly in y, so we get the equivalence of the two definitions in this special case. Moreover we have the formula for the effective Hamiltonian

$$\overline{H}(\overline{x}, \overline{p}, \overline{X}) = \int_{(0,1)^m} F(\overline{x}, y, \overline{p}, \overline{X})\,d\mu(y).$$

The next proposition is a classical result for discrete dynamical systems, and we adapt to the continuous-time case the proof in [19] for the discrete-time case.

Proposition 13. *There exists a unique invariant probability Radon measure for the flow Φ if and only if, for every $\psi \in C(\mathbb{T}^m)$,*

$$\frac{1}{T}\int_0^T \psi(\Phi_t y)\,dt \to \mathit{const} \qquad \text{as } T \to +\infty, \text{ uniformly in } y. \tag{49}$$

Moreover, the constant is $\int \psi\,d\mu$.

Proof. Note first that, by using the density of the continuous functions in $L^1(\mu)$ and the inequality $\|\frac{1}{T}\int_0^T \psi \circ \Phi_t\,dt\|_{L^1(\mu)} \leqq \|\psi\|_{L^1(\mu)}$, it is an easy exercise to check that a uniquely ergodic dynamical system is ergodic for the invariant probability measure.

We start by assuming that there is a unique invariant probability Radon measure μ and prove (49). As we are working with Radon measures, we shall write $\int \psi\,d\mu$ as $\mu(\psi)$ when we see μ as an element of the dual space of $C(\mathbb{T}^m)$. We define the subset of $C(\mathbb{T}^m)$

$$G = \{\chi - \chi \circ \Phi_\tau \mid \tau \in \mathbb{R},\ \chi \in C(\mathbb{T}^m)\} \cup \{\text{constants}\}.$$

For every $\chi \in C(\mathbb{T}^m)$ and every τ, we have

$$\begin{aligned}
&\frac{1}{T}\int_0^T \chi(\Phi_t y)\,dt - \frac{1}{T}\int_0^T \chi \circ \Phi_\tau(\Phi_t y)\,dt \\
&\quad = \frac{1}{T}\int_0^T \chi(\Phi_t y)\,dt - \frac{1}{T}\int_\tau^{\tau+T} \chi(\Phi_t y)\,dt \\
&\quad = \frac{1}{T}\left(\int_0^\tau \chi(\Phi_t y)\,dt - \int_T^{\tau+T} \chi(\Phi_t y)\,dt\right).
\end{aligned}$$

Then, for $\psi = \chi - \chi \circ \Phi_\tau$ we see that

$$\frac{1}{T}\int_0^T \psi(\Phi_t y)\,dt \to 0 \qquad \text{as } T \to +\infty, \text{ uniformly in } y.$$

Since (49) is trivially true when ψ is constant, we have proved it for every $\psi \in G$. Since the linear mapping $\nu_T(\psi) = \frac{1}{T}\int_0^T \psi \circ \Phi_t\,dt$ satisfies the uniform bound $\|\nu_T(\psi)\|_{C(\mathbb{T}^m)} \leqq \|\psi\|_{C(\mathbb{T}^m)}$, we deduce that (49) must be true on a closed vector subspace of $C(\mathbb{T}^m)$. It therefore holds on $H := \overline{\operatorname{span}(G)}$.

We shall prove by duality that $H = C(\mathbb{T}^m)$. This will give the "only if" part of the proposition. Let ν be a Radon measure on $\mathbb{T}^m$ so that $\nu(\psi) = 0$ for every $\psi \in G$. By the Hahn-Banach theorem, the desired equality will follow if we show that ν must be 0. First we note that $0 = \int d\nu = \nu(\mathbb{T}^m)$ because $1 \in G$. Next, by definition of G, $\nu(\chi \circ \Phi_\tau) = \nu(\chi)$ for every $\chi \in C(\mathbb{T}^m)$ and every τ. This means that the measure ν is invariant by the flow Φ. The positive variation ν^+ of ν is a positive invariant measure. Indeed, for every $\psi \geqq 0$, we have $\nu^+(\psi) = \sup\{\nu(\chi) \mid 0 \leqq \chi \leqq \psi\}$; hence

$$\begin{aligned}
&\nu^+(\psi\circ\Phi_\tau)\\
&\quad=\sup\{\nu(\chi)\mid 0\leqq\chi\leqq\psi\circ\Phi_\tau\}\\
&\quad=\sup\{\nu(\chi)\mid 0\leqq\chi\circ\Phi_{-\tau}\leqq\psi\}\\
&\quad=\sup\{\nu(\chi'\circ\Phi_\tau)\mid 0\leqq\chi'\leqq\psi\}\\
&\quad=\sup\{\nu(\chi')\mid 0\leqq\chi'\leqq\psi\}\\
&\quad=\nu^+(\psi).
\end{aligned}$$

The fact that $\nu^+(\psi\circ\Phi_\tau)=\nu^+(\psi)$ for every ψ follows by linearity. Since μ is the unique invariant probability measure, we get $\nu^+=\nu^+(\mathbb{T}^m)\mu$. Similarly, it can be shown that the negative variation ν^- of ν is an invariant positive measure so that $\nu^-=\nu^-(\mathbb{T}^m)\mu$. We conclude that $\nu=\nu^+-\nu^-=\nu(\mathbb{T}^m)\mu$. But $\nu(\mathbb{T}^m)=0$, so $\nu=0$.

We now prove the converse and therefore we assume that (49) holds for every $\psi\in C(\mathbb{T}^m)$. The linear functional

$$\mu(\psi)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T\psi(\Phi_t 0)\,dt$$

defines a Radon probability measure on $\mathbb{T}^m$. It is invariant by the flow Φ. Indeed, for every $\psi\in C(\mathbb{T}^m)$ and every τ ,

$$\frac{1}{T}\int_0^T\psi(\Phi_t 0)\,dt-\frac{1}{T}\int_0^T\psi\circ\Phi_\tau(\Phi_t 0)\,dt=\frac{1}{T}\left(\int_0^\tau\psi(\Phi_t 0)\,dt-\int_T^{\tau+T}\psi(\Phi_t 0)\,dt\right).$$

Sending $T\to+\infty$, we see that $\mu(\psi\circ\Phi_\tau)=\mu(\psi)$. We have therefore proved the existence of an invariant probability Radon measure.

Now, let ν be an invariant probability Radon measure. Integrating (49) with respect to ν, we find that the constant must be equal to

$$\lim_{T\to+\infty}\int\frac{1}{T}\int_0^T\psi(\Phi_t y)\,dt\,d\nu(y)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T\int\psi\,d\nu\,dt=\int\psi\,d\nu.$$

Therefore, we must have

$$\int\psi\,d\nu=\lim_{T\to+\infty}\frac{1}{T}\int_0^T\psi(\Phi_t 0)\,dt=\mu(\psi).$$

This means that the invariant Radon probability measure is determined uniquely.

Example 1. A classical example of a vector field that is uniformly ergodic is

$$g(y)=\xi,\qquad(\xi,k)\neq 0\quad\forall\,k\in\mathbb{Z}^m\setminus\{0\}.$$

Indeed, it is well known (see, e.g., [7]) that the non-resonance condition on ξ is necessary and sufficient for the constant vector field to be ergodic, a result going back to Jacobi. Moreover, for ψ Riemann integrable the convergence of $\frac{1}{T}\int_0^T\psi(y_s)\,ds$ to a constant occurs for all initial positions y_0 (and not just a.e.): this is the theorem of equipartition module 1 of Bohl, Serpinskii, and Weyl; see [7]. For continuous

ψ it is easy to give an equicontinuity estimate with respect to y_0 that implies the uniform convergence. Under the Jacobi condition, the unique invariant probability measure is of course the Lebesgue measure.

6.2. A general Abelian-Tauberian theorem

This subsection provides three equivalent characterizations of the ergodicity of the Hamiltonian H at the point $(\overline{x}, \overline{p}, \overline{X})$ and therefore gives different interpretations of the effective Hamiltonian. The proof of Proposition 3 used the characterization of the effective Hamiltonian provided by (iii). The equivalence (i) $\Leftrightarrow$ (ii) can be viewed as a generalized Abelian-Tauberian theorem [37]. It was proved in [5] (see also [11]) for first-order HJB equations and extended in [6] to second-order HJB equations; these papers exploited the optimal control interpretations of the solutions and used the dynamic programming principle. Our proof is valid for an arbitrary Hamiltonian and only uses the comparison principle and the theory of viscosity solutions.

For $(\overline{x}, \overline{p}, \overline{X})$ fixed, we set

$$G(y, q, Y) = H(\overline{x}, y, \overline{p}, q, \overline{X}, Y, 0).$$

Theorem 4. *The following statements are equivalent.*

(i) *If w_δ is the solution of the stationary problem*

$$\delta w_\delta + G(y, Dw_\delta, D^2 w_\delta) = 0 \quad \text{in } \mathbb{R}^m, \tag{50}$$

then $\delta w_\delta \to$const uniformly in y as $\delta \to 0$.

(ii) *If w is the solution of the Cauchy problem*

$$w_t + G(y, Dw, D^2 w) = 0 \ \text{ in } (0, +\infty) \times \mathbb{R}^m, \quad w(0, \cdot) = 0 \ \text{ on } \mathbb{R}^m, \tag{51}$$

then $w(t, \cdot)/t \to$ const uniformly in y as $t \to +\infty$.

(iii) *Consider the following cell problem for a constant λ*

$$\lambda + G(y, Dv, D^2 v) = 0 \quad \text{in } \mathbb{R}^m, \qquad v \text{ periodic.} \tag{52}$$

Then

$$\sup\{\lambda \mid \text{there is a subsolution of } (52)\} \\ = \inf\{\lambda \mid \text{there is a supersolution of } (52)\}. \tag{53}$$

If one of the above assertion is true, then the constants in (i) *and* (ii) *are equal and they coincide with the number defined by* (53). *Moreover, the Hamiltonian H is ergodic at $(\overline{x}, \overline{p}, \overline{X})$ and the constant is $-\overline{H}(\overline{x}, \overline{p}, \overline{X})$.*

Proof. In (iii), we denote by λ_1 the value of the supremum and by λ_2 the value of the infimum. We first observe that the inequality $\lambda_1 \leqq \lambda_2$ is always true. Indeed assume for contradiction that there exist $\mu_1 > \mu_2$ and periodic functions $\chi_1, \chi_2 \in C(\mathbb{R}^m)$ such that

$$\mu_1 + G(y, D\chi_1, D^2\chi_1) \leqq 0 \quad \text{and} \quad \mu_2 + G(y, D\chi_2, D^2\chi_2) \geqq 0 \quad \text{in } \mathbb{R}^m.$$

We fix a constant $C < \min(\chi_1 - \chi_2)$ and $\delta > 0$ such that $-\mu_1 + \delta\chi_1 \leqq -\mu_2 + \delta(\chi_2 + C)$. Since

$$\delta(\chi_2 + C) + G(y, D(\chi_2 + C), D^2(\chi_2 + C)) \geqq -\mu_2 + \delta(\chi_2 + C) \quad \text{in } \mathbb{R}^m$$

and

$$\delta\chi_1 + G(y, D\chi_1, D^2\chi_1) \leqq -\mu_1 + \delta\chi_1 \quad \text{in } \mathbb{R}^m,$$

the comparison principle gives $\chi_1 \leqq \chi_2 + C$, a contradiction with the choice of C.

We now assume that (iii) holds. Let λ be such that there is a subsolution v of (52). By subtracting a constant, we can assume that v is nonpositive. This implies that $v + \lambda/\delta$ is a subsolution of (50). By the comparison principle, we get $\delta v + \lambda \leqq \delta w_\delta$. Sending $\delta \to 0$, we deduce that $\liminf_{\delta\to 0} \inf_y \delta w_\delta \geqq \lambda$. Taking the supremum over λ, we get

$$\liminf_{\delta\to 0} \inf_y \delta w_\delta \geqq \lambda_1.$$

we can show similarly that $\limsup_{\delta\to 0} \sup_y \delta w_\delta \leqq \lambda_2$. By virtue of (iii), we conclude that $\delta w_\delta \to \lambda_1 = \lambda_2$ uniformly on y as $\delta \to 0$. This is (i).

In the same way, given a nonpositive subsolution v of (52), we see that $v + \lambda t$ is a subsolution of (51). By the comparison principle, we get $v + \lambda t \leqq w$. Dividing by t, sending t to $+\infty$ and taking the supremum over λ, we deduce that $\liminf_{t\to+\infty} \inf_y (w(t,\cdot)/t) \geqq \lambda_1$. In a similar way, we get $\limsup_{t\to+\infty} \sup_y (w(t,\cdot)/t) \leqq \lambda_2$. If (iii) holds, we deduce that $w(t,\cdot)/t \to \lambda_1 = \lambda_2$ uniformly on y as $t \to +\infty$. This yields (ii).

We now assume (i) and prove (iii). We recall that δw_δ converges uniformly to a constant, call it μ, and that w_δ solves

$$\delta w_\delta + G(y, Dw_\delta, D^2 w_\delta) = 0 \quad \text{in } \mathbb{R}^m.$$

Then for all $\lambda < \mu$ there is a periodic subsolution of $\lambda + G(y, Dv, D^2v) \leqq 0$, namely $v = w_\delta$ for δ small enough. This proves the inequality $\lambda_1 \geqq \mu$. In a similar way, we get $\lambda_2 \leqq \mu$. We conclude that $\lambda_1 = \lambda_2 = \mu$.

We finally assume (ii) and prove (iii). Define $\mu = \lim_{t\to+\infty}(w(t,\cdot)/t)$ and pick $\lambda < \mu$ arbitrary. We can construct a smooth function $\zeta : [0, +\infty] \to \mathbb{R}$ so that

$$\zeta(0) = 0, \qquad \zeta' \geqq \lambda \quad \text{in } [0, +\infty),$$
$$\inf_y w(t, y) > \zeta(t) \text{ for some } t > 0, \qquad \sup_y w(t, y) < \zeta(t) \text{ for } t \text{ large.}$$

Define the function

$$v(y) = \sup\{w(t, y) - \zeta(t) \mid t \in [0, +\infty)\}.$$

Then, by the construction of ζ, for every y, the supremum is achieved at some point t_y lying in a compact subset of $(0, +\infty)$ that is independent of y. The function v is therefore well defined and continuous. Moreover, if y is a maximum point of $v - \varphi$ for a smooth test function φ, we get

$$0 \geqq \zeta'(t_y) + G(y, D\varphi(y), D^2\varphi(y)) \geqq \lambda + G(y, D\varphi(y), D^2\varphi(y)).$$

Therefore, v is a viscosity subsolution of (52). Since $\lambda < \mu$ was arbitrary, we conclude that $\lambda_1 \geqq \mu$. We can show in the same way that $\lambda_2 \leqq \mu$. We deduce that $\lambda_1 = \lambda_2 = \mu$.

Remark 7. It is not hard to see that the existence of a solution w of the PDE in (51) with $w(0, \cdot)$ bounded and such that $w(t, \cdot)/t \to$ const uniformly in y as $t \to +\infty$ is also equivalent to the statements (i), (ii), and (iii) of the previous Proposition, and the constant is always the same.

Remark 8. In many important cases the inf and the sup in the formula (53) are attained and $\overline{H}(\overline{x}, \overline{p}, \overline{X})$ is the unique constant such that the *true cell problem*

$$H(\overline{x}, y, \overline{p}, D\chi, \overline{X}, D^2\chi, 0) = \overline{H} \quad \text{in } \mathbb{R}^m, \qquad \chi \text{ periodic,}$$

has a continuous viscosity solution $\chi = \chi(y)$ (depending also on the parameters $\overline{x}, \overline{p}, \overline{X}$), see, e.g., [35, 6, 11] and the Propositions 5 and 9 in Section 5. In general, however, the true cell problem may have no solution, as shown in Arisawa & Lions [6] on an example.

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