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SINGULAR SOLUTIONS TO A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM ORIGINATING FROM CORROSION MODELING

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Abstract. We consider a nonlinear elliptic boundary value problem on a planar domain. The exponential type nonlinearity in the boundary condition is one that frequently appears in the modeling of electrochemical systems. For the case of a disk, we construct a family of exact solutions that exhibit limiting logarithmic singularities at certain points on the boundary. Based on these solutions, we develop two criteria that we believe predict the possible locations of the boundary singularities on quite general domains.

1. Introduction. Let Ω be a bounded, simply connected, smooth domain in $\mathbb{R}^2$. The ultimate goal of the work we describe in this paper is to understand the behaviour of solutions to the nonlinear elliptic boundary value problem

$$\begin{aligned} \triangle u = 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda f(u) + g \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

with $f(u) = e^{u/2} - e^{-u/2}$, λ some real number, and g satisfying $\int_{\partial\Omega} g \, d\sigma = 0$. This problem, or slightly more complicated variations thereof, shows up quite frequently in connection with modeling of electrochemical systems, consisting of an electrolyte and an adjoining metal surface. The surface may be anodic or cathodic, corresponding to a corrosion or a deposition process, respectively. Models using this type of exponential boundary condition conditions are associated with the names of Butler and Volmer. For an in-depth discussion of the physical modeling, the significance of λ (and other physical parameters that enter into more complicated variations), we refer the reader to books such as [2] and [3]. We also refer to the introduction of [4], where there is a somewhat shorter discussion of some of these issues.

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Mathematically speaking, the case $\lambda \leq 0$ is very classical and straightforward: there is always a unique solution and this is characterized as the minimizer of the associated energy

$$E_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx\,dy - \lambda \int_{\partial\Omega} F(v)d\sigma - \int_{\partial\Omega} gv\,d\sigma,$$

where F is given by $F(v) = 2(e^{v/2} + e^{-v/2})$. For $\lambda = 0$, the solution is only unique up to a constant (it exists due to the fact that $\int_{\partial\Omega} g\,d\sigma = 0$).

The situation for $\lambda > 0$ is much more interesting. In [4] it was proven that there exists a finite constant $0 < \lambda^*$ such that the problem (1) also has a solution for $0 \leq \lambda < \lambda^*$. It was conjectured that the solution was not unique for $0 < \lambda$. We now conjecture that the problem (1) actually has infinitely many solutions for any value $0 < \lambda$. We base this conjecture on extensive numerical experimentation and on some theoretical results obtained for the case $g = 0$. The purpose of this paper is partially to present this evidence.

In Sec. 2 we describe one of our numerical experiments, performed in the case when the domain Ω is a disk. This example clearly exhibits infinitely many solutions for any $\lambda > 0$. In Sec. 3 we consider the special case $g = 0$, still on a circular domain. To our initial surprise we have been able to construct a family of closed form solutions to (1). Corresponding to any $0 < \lambda$, there are infinitely many solutions in this family. Each one of these solutions originates from a solution to the linearized eigenvalue problem

$$\begin{aligned} \triangle u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= \mu u \quad \text{on } \partial\Omega, \end{aligned} \tag{2}$$

for any of the infinitely (countably) many eigenvalues $\lambda < \mu_k$. We have computed the bifurcation diagram corresponding to our family of exact solutions, and it shows a striking similarity to those computed in the case $g \neq 0$, $\int_{\partial\Omega} g\,d\sigma = 0$. Members of our family of solutions (except for the trivial one) blow up as $\lambda \to 0^+$. This blow-up takes place as follows: the normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ converges to a sum of an even number of equidistantly distributed delta functions with alternating signs. The function u itself converges to the harmonic function with this distribution of delta functions as its Neumann data. Any even number of delta functions and any equidistant distribution is realizable. The solutions that we numerically calculate for $g \neq 0$ blow up in exactly the same way as $\lambda \to 0^+$; the blow-up appears largely independent of the inhomogeneity, although g seems to determine the rotation (thus rendering the singularity locations unique).

We have gained quite a good understanding of the mechanism by which equidistant locations of the limiting singularities emerge on the unit disk. In Sec. 4 we show that there are two (nearly equivalent) criteria forcing this selection: one is the criterium that the maximum of the residual $\frac{\partial u}{\partial \mathbf{n}} - \lambda f(u)$ be $O(\lambda)$ (as $\lambda \to 0$); the other is that an appropriately defined "renormalized energy" be stationary.

In the final section of this paper we apply these same two criteria to general domains (by means of a conformal mapping) and (for the case of two singularities) we provide a simple characterization of the possible locations of the limiting singularities.

One of the reasons we are quite interested in the asymptotic behaviour of the solutions to (1) for λ small but positive is that we think this behaviour may help explain different kinds of surface instabilities observed in real electrochemical systems. Due to the presence of the singularities, it is also very possible that overdetermined measurements (Cauchy data) from such solutions provide good information about the geometry of an inaccessible corroding surface.

2. Numerical approximation of the solution; qualitative behavior. We begin by numerically solving the boundary value problem of interest, in order to highlight some of the qualitative features of the solutions. We restrict our attention to the unit disk D in $\mathbb{R}^2$ and consider

$$\begin{aligned} \triangle u &= 0 \quad \text{in } D, \\ \frac{\partial u}{\partial \mathbf{n}} &= \lambda f(u) + g \quad \text{on } \partial D, \end{aligned} \tag{3}$$

where $f(u) = e^{u/2} - e^{-u/2}$, and $\int_{\partial D} g\, d\sigma = 0$. We parameterize ∂D as $(\cos(\theta), \sin(\theta))$ for $0 \leq \theta < 2\pi$. For simplicity let us consider solutions to (3) that are even with respect to the x-axis, i.e., we take g to be given by a cosine series, $g = \sum_{j=1}^{\infty} b_j \cos(j\theta)$, and we assume that the suitably smooth, harmonic function u may be expanded as

$$u(r, \theta) = \sum_{j=0}^{\infty} a_j r^j \cos(j\theta)$$

for some choice of coefficients a_j. By inserting this expansion into the boundary condition (3) (note that $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$) we obtain

$$\sum_{j=0}^{\infty} j a_j \cos(j\theta) = \lambda f\left(\sum_{j=0}^{\infty} a_j \cos(j\theta)\right) + g(\theta), \tag{4}$$

an equation which should be satisfied identically in θ. From this we may attempt to recover appropriate coefficients a_j.

A natural strategy is to choose a fixed n, truncate the infinite sum at $j = n$ and then project both sides of Eq. (4) onto the span of $\{\cos(k\theta)\}_{k=0}^{n}$ by integrating against $\cos(k\theta)$ for $k = 0$ to n. This yields

$$k c_k a_k = \lambda \int_0^{2\pi} f\left(\sum_{j=0}^{n} a_j \cos(j\theta)\right) \cos(k\theta) d\theta + c_k b_k \tag{5}$$

for $0 \leq k \leq n$, where $c_0 = 2\pi$, $c_k = \pi$ for $k \geq 1$, and $b_k = c_k^{-1} \int_0^{2\pi} g(\theta)\cos(k\theta) d\theta$ is the kth Fourier cosine coefficient of g ($b_0 = 0$, due to the fact that $\int_{\partial D} g\, d\sigma = 0$). We thus obtain $n+1$ nonlinear equations in $n+1$ unknowns $a_0, \ldots, a_n$, which can be solved using Newton's method.

Although we have not performed a rigorous convergence analysis of this numerical method, the solutions appear to be quite stable with respect to varying choices for n and the accuracy with which the integration is carried out, and we believe the solutions so constructed to be accurate, particularly in light of the results in the next section.

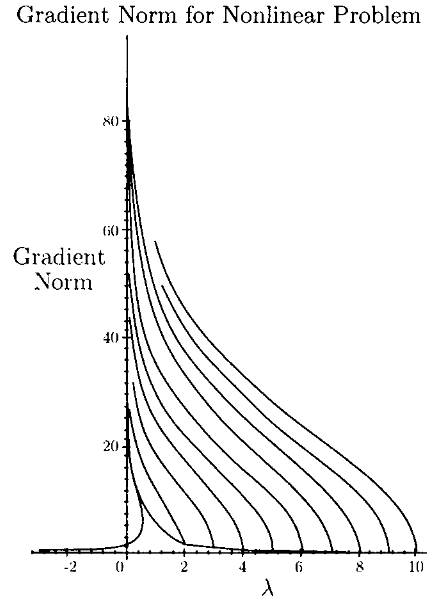

FIG. 1

We typically choose n to be between 20 and 50 and use the trapezoidal rule (rapidly convergent for this type of integrand) with 100 nodes for carrying out the integrations.

As stated in the introduction, for any choice of g and for any $\lambda < 0$, we will have a unique solution, but for $\lambda > 0$ numerical experiments suggest that the problem (3) possesses many solutions. In fact, for any positive λ, we believe there exists a countable number of solutions (in this case approximately one for every integer greater than λ). The solutions also appear to depend continuously on λ, as measured in the $H^1(D)$ norm. In short, there appear to be a countable number of branches of solutions. This behavior does not depend (qualitatively) on the function g.

An instructive way to summarize the situation is to construct a plot of λ versus the size of the solution, say as measured by $\|\nabla u\|_2$. Fig. 1 is such a plot for the case $g(\theta) = \cos(\theta)$. Fig. 1 illustrates that there is a branch of solutions for each nonnegative integer, with only one branch continuing into the left half plane. All branches have a part that is asymptotic to the vertical axis as $\lambda > 0$ approaches zero. For $n \geq 2$, we will refer to the branch originating near the λ-axis, at $\lambda = n$, as the "nth branch".

As $\lambda > 0$ approaches zero, the solutions along all growing branches develop singularities on the boundary (see Fig. 3 below). This behaviour is largely independent of the particular choice for the boundary data g.

For larger values of λ, the behaviour of the solutions is in all ways strongly dependent on g. For λ close to n, it is reasonable to gain insight into solutions on the nth branch by linearizing the boundary condition at $u = 0$, to obtain the boundary value problem $\triangle u = 0$ with boundary condition $\frac{\partial u}{\partial \mathbf{n}} = \lambda u + g$. For this linear problem, the solution u is

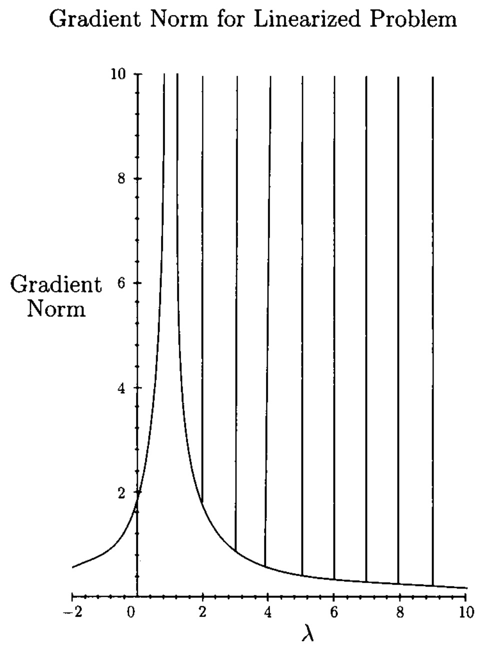

FIG. 2

of the form

$$u(r,\theta) = \sum_{j=1}^{\infty} \frac{b_j}{j-\lambda} r^j \cos(j\theta),$$

where b_j denotes the jth Fourier cosine coefficient of g (which we are still assuming is an even function). We thus obtain a unique solution for $\lambda < 0$ and for non-integral $\lambda \geq 0$. If $\lambda = k$ for some integer $k \geq 1$, then there is no solution, unless $b_k = 0$ in which case

$$u(r,\theta) = cr^k \cos(k\theta) + \sum_{j=1, j\neq k}^{\infty} \frac{b_j}{j-k} r^j \cos(j\theta)$$

is a solution for any real c.

As a specific example, consider the case in which $g(\theta) = \cos(\theta)$. One can compute $\|\nabla u\|_2 = \sqrt{\pi}/|1-\lambda|$ for $\lambda < 0$ and for non-integral $\lambda \geq 0$. If $\lambda = 1$, then we have no solution, while if $\lambda = k$ for an integer $k > 1$, then we have infinitely many solutions, and any energy is attainable. For this problem, a plot similar to Fig. 1 can be constructed. Fig. 2 shows $\|\nabla u\|_2$ versus λ for the linearized problem, in which the vertical lines represent the fact that by adding a suitable multiple of $r^k \cos(k\theta)$ one can obtain any energy. For λ sufficiently far from zero (when u is small), the behavior of the nonlinear problem is qualitatively quite similar to that of the linearized problem. Note that Figs. 1 and 2 are quite similar; in Fig. 1 the vertical lines arising from the eigenvalues for the linearized problem have merely been distorted.

As mentioned above, as λ approaches zero from the right, the solutions along all growing branches develop singularities on ∂D. Fig. 3 shows a solution with boundary

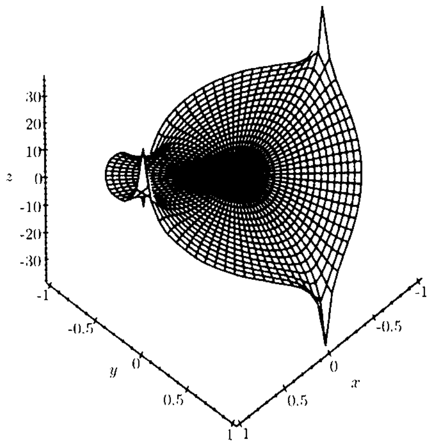

FIG. 3

data $g(\theta) = \cos(\theta)$ for $\lambda = 1.0 \times 10^{-5}$ on the $n = 2$ branch. Here, four singularities develop at $\theta = 0,\ \pi/2,\ \pi, 3\pi/2$. We find in general that as $\lambda > 0$ approaches zero, the solution on the nth branch develops $2n$ uniformly spaced singularities of alternating sign on ∂D. The singularities appear to be logarithmic, and this behavior does not seem to depend on g. The exact behavior of the solutions in the special case that $g \equiv 0$ is examined in the next section.

3. A family of exact solutions on the unit disk. In this section, we consider the boundary value problem (3) with the special boundary condition $\frac{\partial v}{\partial \mathbf{n}} = \lambda f(v)$, i.e., with $g \equiv 0$. In the numerical experiments detailed in the last section, we saw that solutions develop uniformly spaced logarithmic singularities of equal strength and alternating sign around the boundary of the circle. Motivated by this observation, let us seek solutions v_λ, $\lambda > 0$, of the form

$$v_\lambda(x, y) = c \sum_{k=0}^{2n-1} (-1)^k K((x, y) - \mu(\lambda) p_k), \tag{6}$$

where $K(x, y) = \log(x^2 + y^2)$ is 4π times the standard Green's function for the Laplacian, $p_k = (x_k, y_k) = (\cos(k\pi/n), \sin(k\pi/n))$ for $k = 0$ to $k = 2n - 1$, and $\mu(\lambda)$ is a function of λ with the property that $\mu(\lambda) > 1$ for $\lambda > 0$ and $\mu(\lambda) \to 1$ as $\lambda \to 0$. The constant c is to be determined. For any c and $\mu(\lambda) > 1$, the function v_λ is harmonic inside the unit circle and develops logarithmic singularities at the boundary points (x_k, y_k) as λ approaches zero. Note that v_λ also depends on the integer $n \geq 1$, although we do not explicitly indicate this. We will show that, with an appropriate choice for c and $\mu(\lambda)$, the formula (6) yields an exact, nontrivial family of solutions to the boundary value problem.

In what follows we make use of the identities

$$\begin{aligned}\prod_{k=0,\,\text{even}}^{2n-2}(1+\mu^2-2\mu\cos(\theta-\theta_k)) &= \mu^{2n}-2\mu^n\cos(n\theta)+1,\\ \prod_{k=1,\,\text{odd}}^{2n-1}(1+\mu^2-2\mu\cos(\theta-\theta_k)) &= \mu^{2n}+2\mu^n\cos(n\theta)+1.\end{aligned} \tag{7}$$

These can be proved by noting that in each equation the left and right sides are polynomials in μ of degree $2n$, with the same roots and the same leading coefficients. We will also make use of

$$\begin{aligned}\sum_{k=0,\,\text{even}}^{2n-2}\frac{1}{1+\mu^2-2\mu\cos(\theta-\theta_k)} &= \frac{n(\mu^{2n}-1)}{(\mu^2-1)(\mu^{2n}-2\mu^n\cos(n\theta)+1)},\\ \sum_{k=1,\,\text{odd}}^{2n-1}\frac{1}{1+\mu^2-2\mu\cos(\theta-\theta_k)} &= \frac{n(\mu^{2n}-1)}{(\mu^2-1)(\mu^{2n}+2\mu^n\cos(n\theta)+1)}.\end{aligned} \tag{8}$$

These can be proved by taking the logarithm of both sides of the respective product identities (7), then differentiating with respect to μ and rearranging.

We begin by computing $\frac{\partial v_\lambda}{\partial \mathbf{n}}$ and $\lambda f(v_\lambda)$ explicitly. First,

$$\begin{aligned}\frac{\partial v_\lambda}{\partial x} &= 2c\sum_{k=0}^{2n-1}(-1)^k\frac{x-\mu x_k}{(x-\mu x_k)^2+(y-\mu y_k)^2}\\ &= 2c\sum_{k=0}^{2n-1}(-1)^k\frac{x-\mu x_k}{1+\mu^2-2\mu\cos(\theta-\theta_k)},\end{aligned}$$

where $\theta_k = k\pi/n$ and (x,y) lies at angle θ. Here we use the facts that $x^2+y^2 = x_k^2+y_k^2 = 1$ and $xx_k + yy_k = \cos(\theta-\theta_k)$. A similar computation can be made for $\frac{\partial v_\lambda}{\partial y}$ and, since $\mathbf{n} = (x, y)$ on the unit circle, we obtain

$$\begin{aligned}\frac{\partial v_\lambda}{\partial \mathbf{n}} = x\frac{\partial v_\lambda}{\partial x} + y\frac{\partial v_\lambda}{\partial y} &= 2c\sum_{k=0}^{2n-1}(-1)^k\frac{1-\mu\cos(\theta-\theta_k)}{1+\mu^2-2\mu\cos(\theta-\theta_k)}\\ &= 2c\sum_{k=0}^{2n-1}(-1)^k\left(\frac{1}{2}+\frac{1-\mu^2}{2(1+\mu^2-2\mu\cos(\theta-\theta_k))}\right)\\ &= c(1-\mu^2)\sum_{k=0}^{2n-1}(-1)^k\frac{1}{1+\mu^2-2\mu\cos(\theta-\theta_k)}.\end{aligned} \tag{9}$$

We can make use of the identities (8) in Eq. (9) to find that

$$\frac{\partial v_\lambda}{\partial \mathbf{n}} = \frac{-4cn\mu^n(\mu^{2n}-1)\cos(n\theta)}{(\mu^{2n}-2\mu^n\cos(n\theta)+1)(\mu^{2n}+2\mu^n\cos(n\theta)+1)}. \tag{10}$$

Now consider the quantity $\lambda f(v_\lambda)$. Using $f(v) = e^{v/2} - e^{-v/2}$ as well as $(x - \mu x_k)^2 + (y - \mu y_k)^2 = 1 + \mu^2 - 2\mu\cos(\theta - \theta_k)$, we find that

$$f(v_\lambda) = \left(\frac{\prod_{k=0,\,\mathrm{even}}^{2n-2}(1 + \mu^2 - 2\mu\cos(\theta - \theta_k))}{\prod_{k=1,\,\mathrm{odd}}^{2n-1}(1 + \mu^2 - 2\mu\cos(\theta - \theta_k))} \right)^{c/2} - \left(\frac{\prod_{k=1,\,\mathrm{odd}}^{2n-1}(1 + \mu^2 - 2\mu\cos(\theta - \theta_k))}{\prod_{k=0,\,\mathrm{even}}^{2n-2}(1 + \mu^2 - 2\mu\cos(\theta - \theta_k))} \right)^{c/2}. \tag{11}$$

We now make use of the identities (7). Inserting these into Eq. (11) yields

$$\lambda f(v_\lambda) = \lambda \frac{(\mu^{2n} - 2\mu^n\cos(n\theta) + 1)^c - (\mu^{2n} + 2\mu^n\cos(n\theta) + 1)^c}{(\mu^{2n} - 2\mu^n\cos(n\theta) + 1)^{c/2}(\mu^{2n} + 2\mu^n\cos(n\theta) + 1)^{c/2}}. \tag{12}$$

Comparing Eqs. (10) and (12) in the case that $c = 2$ shows that, remarkably,

$$\frac{\frac{\partial v_\lambda}{\partial \mathbf{n}}}{\lambda f(v_\lambda)} = \frac{n(\mu^{2n} - 1)}{\lambda(\mu^{2n} + 1)},$$

where the right side above has no θ dependence. As a consequence, we obtain an *exact* solution to the boundary value problem of interest if we choose μ so that

$$\frac{n(\mu^{2n} - 1)}{\lambda(\mu^{2n} + 1)} = 1.$$

This yields

$$\mu(\lambda) = \left(\frac{n + \lambda}{n - \lambda} \right)^{\frac{1}{2n}}. \tag{13}$$

Note that $\mu(\lambda) > 1$ and $\mu(\lambda) \to 1$ as $\lambda \to 0$.

PROPOSITION 1. The function defined by Eq. (6) with $\mu(\lambda)$ chosen according to Eq. (13) and $c = 2$ defines a harmonic function in the disk with $\frac{\partial v_\lambda}{\partial \mathbf{n}} = \lambda f(v_\lambda)$. For any given $n \geq 1$, this solution exists for $0 < \lambda < n$, and it develops $2n$ logarithmic singularities as $\lambda > 0$ approaches zero. Since the Laplacian is rotationally invariant, any rotation of such a solution through a fixed angle yields another solution to (3) (with $g = 0$).

We can compute the energy $\int_D |\nabla v_\lambda|^2\,dx$ on the disk D versus λ for various n as in Fig. 1, and so determine the rate at which $\|\nabla v_\lambda\|_2$ grows as λ approaches zero. Since v_λ is harmonic, we have

$$\begin{aligned}\int_D |\nabla v_\lambda|^2\,dx &= \int_{\partial D} v_\lambda \frac{\partial v_\lambda}{\partial \mathbf{n}}\,d\sigma_x \\ &= \int_0^{2\pi} \frac{-16n\mu^n(\mu^{2n} - 1)\cos(n\theta)\log\left(\frac{\mu^{2n} - 2\mu^n\cos(n\theta) + 1}{\mu^{2n} + 2\mu^n\cos(n\theta) + 1}\right)}{(\mu^{2n} - 2\mu^n\cos(n\theta) + 1)(\mu^{2n} + 2\mu^n\cos(n\theta) + 1)}\,d\theta,\end{aligned} \tag{14}$$

in which we have made use of Eq. (10) and the product identities above to simplify v_λ on ∂D. Plotting the value of this integral (computed numerically) versus λ for a variety of n produces Fig. 4, which is quite similar to Fig. 1. Note that in the case $g \equiv 0$ we

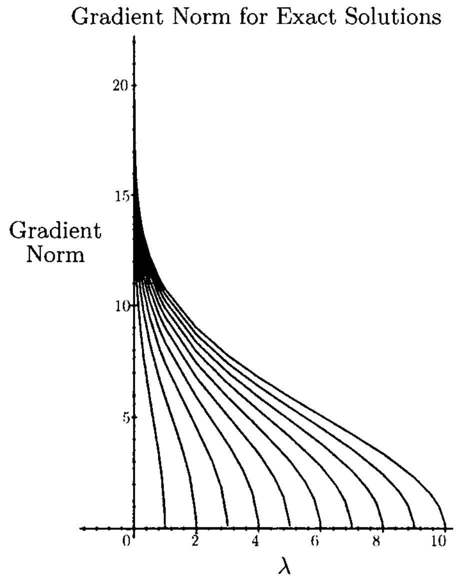

FIG. 4

have a solution $v_\lambda \equiv 0$ for all λ, so that the λ-axis is the branch that extends to the left half plane.

We can determine the precise asymptotic behavior of the integral in Eq. (14) as λ approaches zero. Let $N_\varepsilon = (\partial D) \cap (\bigcup_{k=0}^{2n-1} B_\varepsilon(p_k))$ (where $B_\varepsilon(p)$ is an open disk of radius ε around p) denote a neighborhood of the p_k in ∂D and set $\partial D_\varepsilon = \partial D \backslash N_\varepsilon$; so ∂D_ε is a subset of ∂D that excludes a small interval around each p_k. It is easy to see that as λ approaches zero, the function v_λ remains uniformly bounded on ∂D_ε, and Eq. (10) makes it clear that $\frac{\partial v_\lambda}{\partial \mathbf{n}}$ approaches zero uniformly on ∂D_ε (since the denominator on the right in (10) is bounded away from zero on ∂D_ε, and $\mu \to 1$ as $\lambda \to 0^+$). As a consequence, $\int_{\partial D_\varepsilon} v_\lambda \frac{\partial v_\lambda}{\partial \mathbf{n}}\, d\sigma_x$ approaches 0 as λ approaches 0 (for any fixed ε).

From this observation and the symmetry of the solution with respect to the $2n$ poles, we see that the asymptotic behavior of the integral on the right in Eq. (14) will be the same as that of

$$2n \int_{-\varepsilon}^{\varepsilon} \frac{-16n\mu^n(\mu^{2n}-1)\cos(n\theta)\log\left(\frac{\mu^{2n}-2\mu^n\cos(n\theta)+1}{\mu^{2n}+2\mu^n\cos(n\theta)+1}\right)}{(\mu^{2n}-2\mu^n\cos(n\theta)+1)(\mu^{2n}+2\mu^n\cos(n\theta)+1)}\, d\theta,$$

for fixed small ε, in which the integral above is the contribution of the pole near $(1,0)$. Making the approximations $\mu^n = 1 + O(\lambda)$, $\mu^{2n} + 2\mu^n\cos(n\theta) + 1 = 4 + O(\lambda + \varepsilon)$, and $\cos(n\theta) = 1 + O(\varepsilon)$, we find that

$$\int_{\partial D} v_\lambda \frac{\partial v_\lambda}{\partial \mathbf{n}}\, d\sigma_x \sim -8n^2(\mu^{2n}-1)\int_{-\varepsilon}^{\varepsilon} \frac{\log\left(\frac{\mu^{2n}-2\mu^n\cos(n\theta)+1}{4}\right)}{(\mu^{2n}-2\mu^n\cos(n\theta)+1)}\, d\theta.$$

Now use $\mu^{2n} - 2\mu^n \cos(n\theta) + 1 = (\mu^n - 1)^2 + 2\mu^n(1 - \cos(n\theta)) = (\mu^n - 1)^2 + \mu^n n^2\theta^2 + O(n^4\theta^4)$, and $\mu(\lambda)$ as defined in Eq. (13) to find that as $\lambda \to 0$ we have

$$\int_{\partial D} v_\lambda \frac{\partial v_\lambda}{\partial \mathbf{n}}\, d\sigma_x = -32\pi n \log(\lambda) + O(1), \tag{15}$$

so that $\|\nabla v_\lambda\|_2^2$ grows logarithmically, with energy proportional to the number of singularities.

4. Equivalent criteria for the location of the singularities as $\lambda \to 0$. The case of the unit disk. As briefly mentioned in the introduction, the study of the behaviour of the solutions as $\lambda \to 0$ is of particular interest to us. Based on a comparison of the exact solutions from the previous section and several numerical solutions, we conjecture that, as far as "blow-up" is concerned, the asymptotic behaviour of any solution (with $g \neq 0$, $\int_{\partial\Omega} g\, d\sigma = 0$) mimics that of one of the special solutions (with $g = 0$). In particular, we conjecture that if Ω is a disk, then the boundary currents $\lambda f(u_\lambda)$ converge to a set of $2n$ equidistant Dirac δ-functions, of alternating signs. The corresponding voltages u_λ converge to the solution u_0, with this set of δ-functions $+g$ as Neumann data. We conjecture that an analogous result holds for an arbitrary domain, with the proviso that the δ-functions no longer be equidistant, nor even in number. The natural selection of the singularity locations is far from obvious, but we are convinced that we have a very good understanding of the mechanism by which it happens.

In order to explain this mechanism on an arbitrary domain, we first need to study, in more detail, the situation with $g = 0$ on a disk. We restrict attention to solutions with two "singularities" at angles θ_0 and θ_1, with $\theta_0 \neq \theta_1$. A slight extension of the ansatz (6) (with $c = 2$) now takes the form

$$v_\lambda(x, y) = 2K(x - \alpha(\lambda)x_0, y - \alpha(\lambda)y_0) - 2K(x - \beta(\lambda)x_1, y - \beta(\lambda)y_1), \tag{16}$$

with $(x_0, y_0) = (\cos(\theta_0), \sin(\theta_0))$, $(x_1, y_1) = (\cos(\theta_1), \sin(\theta_1))$, and $\alpha(\lambda) = 1 + O(\lambda)$, $\beta(\lambda) = 1 + O(\lambda)$, with $O(\lambda) > 0$ for $\lambda > 0$. Note that we allow the singularities to approach the boundary at different rates, given by the functions $\alpha(\lambda)$ and $\beta(\lambda)$. Performing calculations similar to those that led to (9) and (11), we now get

$$\frac{\partial v_\lambda}{\partial \mathbf{n}} = \frac{2(1-\alpha^2)}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))} - \frac{2(1-\beta^2)}{(1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))}$$

and

$$\lambda f(v_\lambda) = \lambda\frac{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))}{(1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))} - \lambda\frac{(1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))}.$$

As a consequence,

$$\begin{aligned}\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda) &= \lambda\frac{2(1-\alpha^2)/\lambda + (1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))} \\ &\quad - \lambda\frac{2(1-\beta^2)/\lambda + (1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))}{(1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))}.\end{aligned} \tag{17}$$

Consider now the first term of this residual. A necessary and sufficient condition that this first term be uniformly bounded (in θ and λ) is that the numerator of the fraction vanishes at $(\lambda, \theta) = (0, \theta_0)$, i.e.,

$$-4\alpha'(0) + 2(1 - \cos(\theta_0 - \theta_1)) = 0. \tag{18}$$

To see this, note that the denominator of this fraction is bounded from below by $c[(1 - \alpha)^2 + (\theta - \theta_0)^2]$. Furthermore, note that if (18) is satisfied, then $\alpha'(0) \neq 0$, and thus

$$\begin{aligned}|\text{First term of residual}| &\leq \lambda \frac{C(\lambda + |\theta - \theta_0|)}{c[(1-\alpha)^2 + (\theta - \theta_0)^2]} \\ &\leq C \frac{\lambda^2 + \lambda|\theta - \theta_0|}{(\alpha'(0)\lambda)^2 + (\theta - \theta_0)^2} \leq C.\end{aligned}$$

Similarly, the second term of the residual (17) is uniformly bounded if and only if

$$-4\beta'(0) + 2(1 - \cos(\theta_0 - \theta_1)) = 0.$$

In summary, the entire residual $\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda)$ is uniformly bounded in θ and λ exactly when

$$\alpha'(0) = \beta'(0) = \tfrac{1}{2}(1 - \cos(\theta_0 - \theta_1)).$$

We now proceed further and ask the question: "exactly when is the residual (17) bounded by $C\lambda$ uniformly in θ?" The first term of (17) is clearly bounded by $C\lambda$ uniformly in θ, exactly when the numerator (in addition to vanishing) has first derivatives with respect to λ and θ that vanish at $(\lambda, \theta) = (0, \theta_0)$. The corresponding constraints are

$$-2[(\alpha'(0))^2 + \alpha''(0)] + 2\beta'(0)(1 - \cos(\theta_0 - \theta_1)) = 0$$

and

$$2\sin(\theta_0 - \theta_1) = 0.$$

Similarly, the second term of (17) is bounded by $C\lambda$ uniformly in θ, exactly when

$$-2[(\beta'(0))^2 + \beta''(0)] + 2\alpha'(0)(1 - \cos(\theta_0 - \theta_1)) = 0$$

and

$$2\sin(\theta_0 - \theta_1) = 0.$$

Combining these two sets of constraints, we see that the residual $\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda)$ is uniformly bounded by $C\lambda$, uniformly in θ, exactly when

$$\theta_1 = \theta_0 + \pi, \quad \alpha'(0) = \beta'(0) = 1, \quad \text{and} \quad \alpha''(0) = \beta''(0) = 1.$$

OBSERVATION 1. From the above analysis, we conclude that it is the requirement that the residual $\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda)$ be uniformly bounded by $C\lambda$ (and not the fact that it be identically zero) that forces the selection of opposite (equidistant) points for the singularities. We note that the requirements $\alpha(0) = \alpha'(0) = \alpha''(0) = 1$ and $\beta(0) = \beta'(0) = \beta''(0) = 1$ are consistent with the choice $\alpha(\lambda) = \beta(\lambda) = \left(\frac{1+\lambda}{1-\lambda}\right)^{1/2}$ associated with the exact solution.

We want to describe another (energy related) criterium, which also selects the correct singularity locations. Let E_λ denote the energy

$$E_\lambda(v) = \frac{1}{2}\int_D |\nabla v|^2 \, dx\, dy - \lambda \int_{\partial D} F(v) d\sigma,$$

where $F(\cdot)$ denotes the function $F(v) = 2(e^{v/2} + e^{-v/2})$ (an integral of f). The boundary value problem we study is indeed formally the "Euler Lagrange Equations" associated with E. For $\lambda < 0$, the function that is constantly zero (the only solution to the "Euler Lagrange equations") is indeed the unique minimizer of $E_\lambda(\cdot)$.

LEMMA 1. Let v_λ, $\lambda > 0$, be given by the ansatz (16) with $\alpha'(0)$, $\beta'(0) > 0$. Then

$$\begin{aligned} E_\lambda(v_\lambda) = &-16\pi \log \lambda - 8\pi[\log \alpha'(0) + \log \beta'(0)] + 8\pi \log[2 - 2\cos(\theta_0 - \theta_1)] \\ &- 4\pi[1 - \cos(\theta_0 - \theta_1)] \left(\frac{1}{\alpha'(0)} + \frac{1}{\beta'(0)} \right) \\ &- 16 \int_0^{+\infty} \frac{1}{s^2+1} \log(s^2+1) ds + o(1), \end{aligned}$$

where $o(1)$ denotes a term that approaches zero as $\lambda > 0$ approaches zero.

Proof. After integration by parts,

$$E_\lambda(v_\lambda) = \frac{1}{2}\int_{\partial D} \frac{\partial v_\lambda}{\partial \mathbf{n}} v_\lambda \, d\sigma - \lambda \int_{\partial D} F(v_\lambda) d\sigma. \tag{19}$$

The desired formula for $E_\lambda(v_\lambda)$ thus follows immediately by insertion of

$$\begin{aligned} \frac{1}{2}\int_{\partial D} \frac{\partial v_\lambda}{\partial \mathbf{n}} v_\lambda \, d\sigma = &-16\pi \log \lambda - 8\pi \log \alpha'(0) - 8\pi \log \beta'(0) \\ &+ 8\pi \log[2 - 2\cos(\theta_0 - \theta_1)] \\ &- 16 \int_0^{+\infty} \frac{1}{s^2+1} \log(s^2+1) ds + o(1) \end{aligned} \tag{20}$$

and

$$\lambda \int_{\partial D} F(v_\lambda) d\sigma = 4\pi[1 - \cos(\theta_0 - \theta_1)] \left(\frac{1}{\alpha'(0)} + \frac{1}{\beta'(0)} \right) + o(1) \tag{21}$$

into (19). We now proceed to verify the identities (20) and (21). To that end

$$\begin{aligned} &\frac{1}{2}\int_{\partial D} \frac{\partial v_\lambda}{\partial \mathbf{n}} v_\lambda \, d\sigma \\ &= 2\int_0^{2\pi} \left[\frac{1-\alpha^2}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))} - \frac{1-\beta^2}{(1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))} \right] \\ &\times \Big[\log[(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))] \\ &\qquad - \log[(1-\beta)^2 + 2\beta(1-\cos(\theta-\theta_1))] \Big] d\theta. \end{aligned} \tag{22}$$

Let us first calculate

$$\int_0^{2\pi} \frac{1-\alpha^2}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))] d\theta = I + II,$$

where

$$I = \int_{S_1} \frac{1-\alpha^2}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))]d\theta,$$

with $S_1 = (0, 2\pi) \cap \{1 - \cos(\theta - \theta_0) \geq \lambda^{1/4}\}$, and

$$II = \int_{S_2} \frac{1-\alpha^2}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))]d\theta,$$

with $S_2 = (0, 2\pi) \cap \{1 - \cos(\theta - \theta_0) < \lambda^{1/4}\}$. For $\theta \in S_1$ we immediately get

$$\left|\frac{1-\alpha^2}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))}\right| \leq C\lambda^{3/4}.$$

Therefore,

$$|I| \leq C\lambda^{3/4} \int_0^{2\pi} |\log[(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))]|\, d\theta \leq C\lambda^{3/4}. \tag{23}$$

We may, without loss of generality, suppose that the polar coordinate system has been chosen so that θ_0 and θ_1 are both different from 0 (and 2π). For sufficiently small λ, the set $S_2 = (0, 2\pi) \cap \{1 - \cos(\theta - \theta_0) < \lambda^{1/4}\}$ then splits into two equal parts S_2^+ and S_2^-: one in which $\theta_0 < \theta$ and one in which $\theta < \theta_0$. The contribution to the integral is the same from the two sets. We introduce the new variable of integration

$$s = \frac{\sqrt{2\alpha(1-\cos(\theta-\theta_0))}}{\alpha - 1},$$

for which

$$\frac{ds}{d\theta} = \frac{1 + O(\lambda)}{\alpha - 1}$$

on $S_2^+ = (0, 2\pi) \cap \{1 - \cos(\theta - \theta_0) < \lambda^{1/4}\} \cap \{\theta_0 < \theta\}$. By this change of variable we obtain

$$\begin{aligned}
II &= 2\int_{S_2^+} \frac{1-\alpha^2}{(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2 + 2\alpha(1-\cos(\theta-\theta_0))]d\theta \\
&= 2(1-\alpha^2)\frac{\alpha-1}{1+O(\lambda)} \int_{M_\lambda} \frac{1}{(1-\alpha)^2(1+s^2)} \log[(1-\alpha)^2(1+s^2)]ds \\
&= \frac{2(1-\alpha^2)}{\alpha-1}(1+O(\lambda)) \left[\log[(1-\alpha)^2] \int_{M_\lambda} \frac{1}{1+s^2}\, ds \right. \\
&\qquad \left. + \int_{M_\lambda} \frac{1}{1+s^2} \log[1+s^2] ds\right]
\end{aligned}$$

with $M_\lambda = \{0 < s < \sqrt{2\alpha}\frac{\lambda^{1/2}}{\alpha-1}\}$. Since $0 < \alpha - 1 \leq C\lambda$, we have that $M_\lambda \to (0, \infty)$ as $\lambda \to 0$, and from the above identity we now immediately get

$$\begin{aligned}
II &= -4\left[\log[(\alpha'(0)\lambda)^2] \int_0^\infty \frac{1}{1+s^2}\, ds + \int_0^\infty \frac{1}{1+s^2} \log[1+s^2] ds\right] \\
&= -4\pi \log\lambda - 4\pi \log\alpha'(0) - 4\int_0^\infty \frac{1}{1+s^2} \log[1+s^2] ds + o(1).
\end{aligned}$$

In summary,

$$\int_0^{2\pi} \frac{1-\alpha^2}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]d\theta$$
$$= -4\pi\log\lambda - 4\pi\log\alpha'(0) - 4\int_0^\infty \frac{1}{1+s^2}\log[1+s^2]ds + o(1). \quad (24)$$

We similarly get

$$\int_0^{2\pi} \frac{1-\beta^2}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))} \log[(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))]d\theta$$
$$= -4\pi\log\lambda - 4\pi\log\beta'(0) - 4\int_0^\infty \frac{1}{1+s^2}\log[1+s^2]ds + o(1). \quad (25)$$

The expression

$$\int_0^{2\pi} \frac{1-\alpha^2}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))} \log[(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))]d\theta \quad (26)$$

may alternatively be written

$$\int_{\partial D} \left[\frac{2((x,y)-\alpha(x_0,y_0))\cdot\mathbf{n}_{(x,y)}}{|(x,y)-\alpha(x_0,y_0)|^2} - 1\right] \log|(x,y)-\beta(x_1,y_1)|^2\, d\sigma,$$

in which form we recognize part of the integral as a double layer potential with density $\log|(x,y)-\beta(x_1,y_1)|^2$ evaluated at the point $\alpha(x_0,y_0)$ (which lies outside D). Since $(x_0,y_0)\neq(x_1,y_1)$, it now follows immediately from the "jump relations" for double layer potentials that the term (26) converges to

$$\int_{\partial D} \left[\frac{2((x,y)-(x_0,y_0))\cdot\mathbf{n}_{(x,y)}}{|(x,y)-(x_0,y_0)|^2} - 1\right] \log|(x,y)-(x_1,y_1)|^2\, d\sigma$$
$$- 2\pi\log|(x_0,y_0)-(x_1,y_1)|^2 \quad (27)$$

as $\lambda\to 0^+$. Since ∂D is the unit circle, it is easy to see that

$$\frac{2((x,y)-(x_0,y_0))\cdot\mathbf{n}_{(x,y)}}{|(x,y)-(x_0,y_0)|^2} = 1, \quad (x,y)\in\partial D.$$

Inserting this into (27), we conclude that the term (26) converges to

$$-2\pi\log|(x_0,y_0)-(x_1,y_1)|^2 \quad (28)$$

as $\lambda\to 0^+$. By the exact same argument, we get that

$$\int_0^{2\pi} \frac{1-\beta^2}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))} \log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]d\theta$$

converges to

$$-2\pi\log|(x_0,y_0)-(x_1,y_1)|^2 \quad (29)$$

as $\lambda \to 0^+$. Substituting (24), (25), (28), and (29) into (22) we arrive at

$$\begin{aligned}\frac{1}{2}\int_{\partial D}\frac{\partial v_\lambda}{\partial \mathbf{n}}v_\lambda\,d\sigma = &-16\pi\log\lambda - 8\pi\log\alpha'(0) - 8\pi\log\beta'(0)\\ &-16\int_0^{+\infty}\frac{1}{s^2+1}\log(s^2+1)ds\\ &+8\pi\log|(x_0,y_0)-(x_1,y_1)|^2 + o(1),\end{aligned}$$

which is exactly the same as (20).

It only remains to verify (21). We calculate

$$\begin{aligned}\lambda\int_{\partial D}F(v_\lambda)d\sigma &= 2\lambda\int_{\partial D}(e^{v_\lambda/2}+e^{-v_\lambda/2})d\sigma\\ &= 2\lambda\int_0^{2\pi}\frac{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}\,d\theta\\ &\quad+2\lambda\int_0^{2\pi}\frac{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}\,d\theta.\end{aligned}\tag{30}$$

Upon replacing $1-\beta^2 = -2\beta'(0)\lambda + O(\lambda^2)$ by 2λ, and $\log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]$ by $(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))$, we may use the exact same procedure that was used to compute the formula (29) for

$$\int_0^{2\pi}\frac{1-\beta^2}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}\log[(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))]d\theta$$

to derive

$$2\lambda\int_0^{2\pi}\frac{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}\,d\theta = 4\pi(1-\cos(\theta_1-\theta_0))\frac{1}{\beta'(0)}+o(1).$$

Similarly we also get

$$2\lambda\int_0^{2\pi}\frac{(1-\beta)^2+2\beta(1-\cos(\theta-\theta_1))}{(1-\alpha)^2+2\alpha(1-\cos(\theta-\theta_0))}\,d\theta = 4\pi(1-\cos(\theta_0-\theta_1))\frac{1}{\alpha'(0)}+o(1).$$

Substitution of the above two formulas into (30) immediately leads to (21). □

As was done in the context of the Ginzburg-Landau equations (see [1]), we define the "renormalized" energy

$$\begin{aligned}E^*(\theta_0,\theta_1,\alpha'(0),\beta'(0)) := &-8\pi[\log\alpha'(0)+\log\beta'(0)] + 8\pi\log[2-2\cos(\theta_0-\theta_1)]\\ &-4\pi[1-\cos(\theta_0-\theta_1)]\left(\frac{1}{\alpha'(0)}+\frac{1}{\beta'(0)}\right)\\ &-16\int_0^{+\infty}\frac{1}{s^2+1}\log(s^2+1)ds.\end{aligned}$$

In view of Lemma 1, E^* is simply the zeroth-order term of the energy E_λ (which itself tends to infinity as λ approaches 0, due to the fact that the "limiting" function $v_0(x,y) = 2K(x-x_0,y-y_0)-2K(x-x_1,y-y_1)$ is not in $H^1(\Omega)$).

Now, calculate the stationarity conditions for E^*:

$$\alpha'(0) = \beta'(0) = \tfrac{1}{2}[1-\cos(\theta_0-\theta_1)]\tag{31}$$

and

$$\frac{8\pi}{1-\cos(\theta_0-\theta_1)}\sin(\theta_0-\theta_1) = 4\pi\sin(\theta_0-\theta_1)\left(\frac{1}{\alpha'(0)}+\frac{1}{\beta'(0)}\right).$$

These conditions are easily seen to be equivalent to

$$\theta_1 = \theta_0 + \pi \quad \text{and} \quad \alpha'(0) = \beta'(0) = 1. \tag{32}$$

OBSERVATION 2. We thus conclude that the condition of stationarity of the "renormalized energy" E^* also selects the correct (equidistant) singularity locations (even though it only selects the correct values of $\alpha'(0)$ and $\beta'(0)$, and not those of the second derivatives).

If one inserts the formulae (31) into the expression for E^*, then one arrives at a modified "renormalized" energy depending only on θ_0 and θ_1; the values defined by (32) are indeed minimizers for this energy.

In the next section, we shall apply the two criteria mentioned in Observations 1 and 2 to the case in which the domain is an arbitrary simply connected, smooth domain (and not "only" the unit disk).

5. Location of singularities on arbitrary domains. Consider now the boundary value problem

$$\begin{aligned} \triangle u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= \lambda f(u) + g \quad \text{on } \partial\Omega \end{aligned} \tag{33}$$

on a smooth, bounded, simply connected domain $\Omega \subset \mathbb{R}^2$. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by identifying the point (x, y) with the complex number $z = x + iy$. According to Riemann's Mapping Theorem, there exists an analytic function $\Phi(\cdot)$, such that the mapping $z \to \Phi(z)$ maps Ω one-to-one onto the unit disk, D. From elliptic regularity theory, we know that Φ has a smooth extension to $\overline{\Omega}$, and we furthermore know that the extension of $|\frac{d\Phi(z)}{dz}| = |\det[D\Phi(x,y)]|^{1/2}$ does not vanish on $\overline{\Omega}$. The function $w = u \circ \Phi^{-1}$ now satisfies

$$\begin{aligned} \triangle w &= 0 \quad \text{in } D, \\ h\frac{\partial w}{\partial \mathbf{n}} &= \lambda f(w) + g \quad \text{on } \partial D, \end{aligned} \tag{34}$$

where h denotes the (boundary) function $h(\cdot) = |\det[D\Phi(\Phi^{-1}(\cdot))]|^{1/2}$. We may think of h as a function of the angular variable $\theta : h(\theta) = h(e^{i\theta})$. Recall that $\int_{\partial D} gh^{-1}\,d\sigma = \int_{\partial\Omega} g\,d\sigma = 0$.

We believe that singularities will develop in w at specific points on ∂D as $\lambda > 0$ approaches 0. We also believe that these singularities and their locations will mimic those that develop in one of the nontrivial solutions to the homogeneous boundary value problem

$$\begin{aligned} \triangle v &= 0 \quad \text{in } D, \\ h\frac{\partial v}{\partial \mathbf{n}} &= \lambda f(v) \quad \text{on } \partial D. \end{aligned} \tag{35}$$

In the previous section, we identified two criteria that correctly selected the possible locations of the limiting singularities for solutions to (35) in the case when Ω was the unit disk, i.e., when $h(\theta) = 1$. We shall now calculate what these criteria predict concerning the singularity locations in the case when h is not identically 1 (and two singularities develop). Fortunately, the two predictions coincide—and we do conjecture that these locations are those that will appear in solutions to (35) (and (34)) in the limit as $\lambda \to 0^+$ (when two singularities develop). By the conformal mapping Φ^{-1}, these locations get carried to the (conjectured) locations for the limiting singularities of solutions to the problem (33). We note that even though we here only consider the case of two singularities, similar calculations could be carried out for any number of singularities. As mentioned earlier, we do expect that it is possible to develop an odd number of singularities for a general domain.

With v_λ given by the ansatz (16), we calculate

$$h\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda) = \lambda \frac{2h(\theta)(1-\alpha^2(\lambda))/\lambda + (1-\beta(\lambda))^2 + 2\beta(\lambda)(1-\cos(\theta-\theta_1))}{(1-\alpha(\lambda))^2 + 2\alpha(\lambda)(1-\cos(\theta-\theta_0))}$$
$$- \lambda \frac{2h(\theta)(1-\beta^2(\lambda))/\lambda + (1-\alpha(\lambda))^2 + 2\alpha(\lambda)(1-\cos(\theta-\theta_0))}{(1-\beta(\lambda))^2 + 2\beta(\lambda)(1-\cos(\theta-\theta_1))}.$$

Just as in the previous section, we see that a necessary and sufficient condition that this residual be bounded by $C\lambda$, uniformly in θ, is that the two numerators together with their first derivatives (in λ and θ) vanish at the points $(\lambda, \theta) = (0, \theta_0)$ and $(\lambda, \theta) = (0, \theta_1)$. This leads to the algebraic conditions

$$-4h(\theta_0)\alpha'(0) + 2(1-\cos(\theta_0-\theta_1)) = 0,$$
$$-4h(\theta_1)\beta'(0) + 2(1-\cos(\theta_0-\theta_1)) = 0,$$

$$-2h(\theta_0)[(\alpha'(0))^2 + \alpha''(0)] + 2\beta'(0)(1-\cos(\theta_0-\theta_1)) = 0,$$
$$-2h(\theta_1)[(\beta'(0))^2 + \beta''(0)] + 2\alpha'(0)(1-\cos(\theta_0-\theta_1)) = 0,$$

and

$$-4h'(\theta_0)\alpha'(0) + 2\sin(\theta_0-\theta_1) = 0,$$
$$-4h'(\theta_1)\beta'(0) - 2\sin(\theta_0-\theta_1) = 0.$$

Simple manipulations give that these three sets of conditions are equivalent to

$$\alpha'(0) = \frac{1-\cos(\theta_0-\theta_1)}{2h(\theta_0)}, \qquad \beta'(0) = \frac{1-\cos(\theta_0-\theta_1)}{2h(\theta_1)}, \tag{36}$$

$$\alpha''(0) = \alpha'(0)^2\left(2\frac{h(\theta_0)}{h(\theta_1)} - 1\right), \qquad \beta''(0) = \beta'(0)^2\left(2\frac{h(\theta_1)}{h(\theta_0)} - 1\right), \tag{37}$$

$$\frac{h'(\theta_0)}{h(\theta_0)} = \frac{\sin(\theta_0-\theta_1)}{1-\cos(\theta_0-\theta_1)}, \qquad \frac{h'(\theta_1)}{h(\theta_1)} = -\frac{\sin(\theta_0-\theta_1)}{1-\cos(\theta_0-\theta_1)}, \tag{38}$$

which therefore are exactly the conditions that guarantee that the residual $h\frac{\partial v_\lambda}{\partial \mathbf{n}} - \lambda f(v_\lambda)$ is uniformly bounded by $C\lambda$ on the boundary ∂D.

The energy, after the conformal change of variables, becomes

$$E_\lambda(v) = \frac{1}{2}\int_D |\nabla v|^2\,dx - \lambda\int_{\partial D} F(v)\frac{1}{h}\,d\sigma_x$$

and we calculate the corresponding "renormalized energy" (the zeroth-order term of $E(v_\lambda)$) to be

$$\begin{aligned} E^*(\theta_0,\theta_1,\alpha'(0),\beta'(0)) &= -8\pi[\log\alpha'(0)+\log\beta'(0)] + 8\pi\log[2-2\cos(\theta_0-\theta_1)] \\ &\quad - 4\pi[1-\cos(\theta_0-\theta_1)]\left[\frac{1}{\alpha'(0)h(\theta_0)}+\frac{1}{\beta'(0)h(\theta_1)}\right] \\ &\quad - 16\int_0^\infty \frac{1}{s^2+1}\log(s^2+1)ds. \end{aligned} \tag{39}$$

It is now very simple to calculate that the stationarity conditions of the "renormalized energy" $E^*(\theta_0,\theta_1,\alpha'(0),\beta'(0))$ with respect to $\alpha'(0)$, $\beta'(0)$, θ_0 and θ_1 (in that order) amount to exactly the conditions (36) and (38). The equations (38) determine the locations of the limiting singularities; given these locations, the equations (36) then determine $\alpha'(0)$ and $\beta'(0)$. Just as we experienced in the last section, $\alpha''(0)$ and $\beta''(0)$ are not determined by stationarity of the "renormalized energy".

OBSERVATION 3. As a consequence of the calculations carried out above and in the previous section, we conjecture that in the case when two limiting singularities develop in a solution to (33), then these singularities will be located at points $(x_0,y_0) = \Phi^{-1}(\cos(\theta_0),\sin(\theta_0))$ and $(x_1,y_1) = \Phi^{-1}(\cos(\theta_1),\sin(\theta_1))$, where θ_0 and θ_1 satisfy

$$\frac{h'(\theta_0)}{h(\theta_0)} = \frac{\sin(\theta_0-\theta_1)}{1-\cos(\theta_0-\theta_1)} = -\frac{h'(\theta_1)}{h(\theta_1)}. \tag{40}$$

We illustrate the assertion of this conjecture with a simple numerical example.

EXAMPLE. Let Ω_a be the image of the unit disk under the mapping $z \to \exp(az)$ for $0 < a < \pi$. As the mapping $\Phi_a : \Omega_a \to D$ we may thus take $\Phi_a(z) = \frac{1}{a}\log(z)$. Simple calculations give that

$$\left|\frac{d\Phi}{dz}\right| = \frac{1}{a|z|},$$

$$h(\theta) = h(\cos\theta + i\sin\theta) = \frac{1}{a|\exp(a\cos\theta + ia\sin\theta)|} = \frac{1}{a}\exp(-a\cos\theta),$$

and

$$h'(\theta) = \sin\theta\exp(-a\cos\theta).$$

It follows immediately from (40) that the conjectured limiting singularity locations correspond to angles θ_0 and θ_1 that satisfy

$$a\sin\theta_0 = \frac{\sin(\theta_0-\theta_1)}{1-\cos(\theta_0-\theta_1)} = -a\sin\theta_1. \tag{41}$$

We note that if (θ_0,θ_1) satisfies these equations, so does (θ_1,θ_0) (in terms of solutions to (35), this just reflects the fact that if v is a solution, so is $-v$). Let us therefore, to

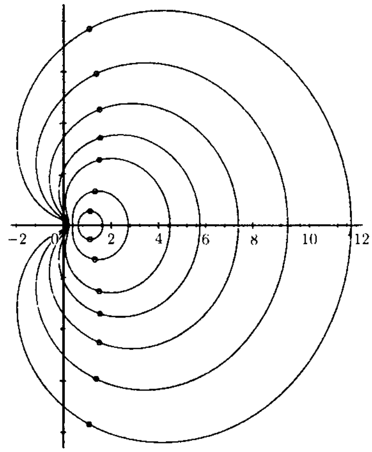

FIG. 5

eliminate this trivial symmetry, for the moment adopt the convention that $0 \le \theta_0 < \theta_1 < 2\pi$. The equations (41) have the solutions

$$(\theta_0, \theta_1) = (0, \pi) \tag{42}$$

and

$$(\theta_0, \theta_1) \quad \text{where } a \sin\theta_0 = \frac{\sin(2\theta_0)}{1 - \cos(2\theta_0)} \quad \text{and} \quad \theta_1 = 2\pi - \theta_0. \tag{43}$$

There is exactly one solution coming from (43). By inserting the formulas (36) into (39) (subtracting the constant $24\pi \log 2 - 16\pi - 16 \int_0^\infty \frac{1}{s^2+1} \log(s^2+1)ds$ and dividing by 8π), we get the following modified "renormalized energy":

$$\widetilde{E}^*(\theta_0, \theta_1) = \log h(\theta_0) + \log h(\theta_1) - \log[1 - \cos(\theta_0 - \theta_1)],$$

which, with $h(\theta) = \frac{1}{a}\exp(-a\cos\theta)$, reads

$$\widetilde{E}^*(\theta_0, \theta_1) = -2\log a - a\cos\theta_0 - a\cos\theta_1 - \log[1 - \cos(\theta_0 - \theta_1)].$$

A simple computation shows that $(\theta_0, \theta_1) = (0, \pi)$ is a saddle point for $\widetilde{E}^*$, and that the solution coming from (43) is the minimizer of $\widetilde{E}^*$ (the maximum of $\widetilde{E}^*$ is ∞, and is achieved for $\theta_1 = \theta_0 \bmod 2\pi$). Figure 5 shows the domains Ω_a and the "minimizing" limiting singularity locations $\exp(a\cos\theta_k + ia\sin\theta_k)$, $k = 0, 1$, for seven values of a ($a = 2.5$, 2.25, 2, 1.75, 1.5, 1, 0.5, with larger values corresponding to larger domains).

Figure. 6 shows the surface plot of the "renormalized energy" $\widetilde{E}^*$ as a function of $(\theta_0, \theta_1) \in [0, 2\pi]^2$, in the case when $a = 2$. The point $(\theta_0, \theta_1) = (0.67489,\ 5.60830)$ (and $(\theta_0, \theta_1) = (5.60830\ 0.67489)$) is the minimizer. The point $(0, \pi)$ (as well as $(2\pi, \pi)$, $(\pi, 0)$, and $(\pi, 2\pi)$) is a saddle point.

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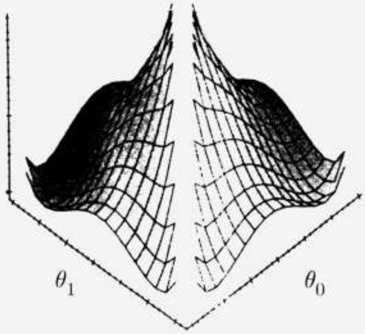

FIG. 6

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