Singular values of products of random matrices and polynomial ensembles

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Abstract

Akemann, Ipsen, and Kieburg showed recently that the squared singular values of a product of M complex Ginibre matrices are distributed according to a determinantal point process. We introduce the notion of a polynomial ensemble and show how their result can be interpreted as a transformation of polynomial ensembles. We also show that the squared singular values of the product of $M-1$ complex Ginibre matrices with one truncated unitary matrix is a polynomial ensemble, and we derive a double integral representation for the correlation kernel associated with this ensemble. We use this to calculate the scaling limit at the hard edge, which turns out to be the same scaling limit as the one found by Kuijlaars and Zhang for the squared singular values of a product of M complex Ginibre matrices. Our final result is that these limiting kernels also appear as scaling limits for the biorthogonal ensembles of Borodin with parameter $\theta > 0$, in case θ or $1/\theta$ is an integer. This further supports the conjecture that these kernels have a universal character.

1. Introduction

1.1. Products of Ginibre matrices

There is a remarkable recent development in the understanding of the structure of eigenvalues and singular values of products of complex Ginibre matrices at the finite size level. Both the eigenvalues and the singular values turn out to have a determinantal structure. For the eigenvalues this was shown by Akemann and Burda [2]. Related results that involve also products with inverses of complex Ginibre matrices are in [23, 1], and products with truncated unitary matrices in [3, 20].

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The eigenvalue probability density function in these models takes the form

$$\frac{1}{Z_n} |\Delta(z)|^2 \prod_{j=1}^{n} w(z_j), \qquad (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

where $\Delta(z) = \prod_{j<k}(z_k - z_j)$ is the Vandermonde determinant, and w is a weight function that is expressed in terms of a Meijer G-function (see e.g. [7, 26] or the appendix for an introduction).

The determinantal structure also holds for the squared singular values of products $Y = G_M \cdots G_1$ of $M \geq 1$ complex Ginibre matrices. Suppose G_j has size $(n + \nu_j) \times (n + \nu_{j-1})$ with $\nu_0 = 0$, $\nu_1, \ldots, \nu_M \geq 0$. Then the joint probability density function takes the form

$$\frac{1}{Z_n} \Delta(y) \det\left[w_{k-1}(y_j)\right]_{j,k=1}^{n}, \qquad (y_1, \ldots, y_n) \in [0, \infty)^n, \tag{1.1}$$

where the y_j's are the squared singular values of Y, and

$$w_k(y) = G_{0,M}^{M,0}\left(\begin{matrix} - \\ \nu_M, \ldots, \nu_2, \nu_1 + k \end{matrix}\middle| y\right), \tag{1.2}$$

is again a Meijer G-function. This was shown by Akemann, Kieburg, and Wei [4] in the case of square matrices, and by Akemann, Ipsen, and Kieburg [5] for general rectangular matrices.

1.2. Polynomial ensembles

The density (1.1) defines a biorthogonal ensemble which is a special case of a determinantal process. Because of the Vandermonde determinant $\Delta(y)$ in (1.1) there is a connection with polynomials and we call (1.1) a **polynomial ensemble**. A general polynomial ensemble is of the form

$$\frac{1}{Z_n} \Delta(x) \det\left[f_{k-1}(x_j)\right]_{j,k=1}^{n}, \qquad (x_1, \ldots, x_n) \in \mathbb{R}^n, \tag{1.3}$$

for certain functions $f_0, \ldots, f_{n-1}$. In such an ensemble the correlation kernel is

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x) Q_k(y)$$

where P_k is a monic polynomial of degree k such that

$$\int_0^\infty P_k(x) f_j(x) dx = 0 \qquad \text{for } j = 0, 1, \ldots, k-1, \text{ and } k = 0, 1, \ldots, n, \tag{1.4}$$

and Q_k is in the linear span of $f_0, \ldots, f_k$ such that

$$\int_0^\infty x^j Q_k(x) dx = \delta_{j,k} \qquad \text{for } j = 0, 1, \ldots, k, \text{ and } k = 0, 1, \ldots, n-1.$$

2

If $f_k(x) = x^k f_0(x)$ for every k, then (1.3) is an orthogonal polynomial ensemble [24] and (1.4) reduces to the conditions for an orthogonal polynomial with respect to f_0. It is also worth noting that in a polynomial ensemble, P_n is the average characteristic polynomial

$$P_n(x) = \mathbb{E}\left[\prod_{j=1}^{n}(x - x_j)\right]$$

with the expectation taken over (1.3).

The first aim of the present work is to interpret the result of Akemann et al. as a transformation of polynomial ensembles. The result may be stated as follows: suppose X is a random matrix whose squared singular values form a polynomial ensemble and that G is a complex Ginibre matrix. Then the squared singular values of $Y = GX$ are also a polynomial ensemble. See Theorem 2.1 below for a precise formulation.

We can use the theorem repeatedly, and we obtain that the squared singular values of $Y = G_{M-1}\cdots G_1 X$ are also a polynomial ensemble, for any $M \geq 1$ and complex Ginibre matrices $G_1, \ldots, G_{M-1}$.

The theorem applies to any random matrix X whose squared singular values are a polynomial ensemble. Taking for X a complex Ginibre matrix itself, we obtain the result of Akemann et al., and by taking for X the inverse of a product of complex Ginibre matrices, we rederive a recent result of Forrester [15]. In both these examples, the functions in the polynomial orthogonal ensembles are expressed as Meijer G-functions.

We consider one new example where X is a truncation of a Haar distributed unitary matrix for which it is known that the squared singular values are a Jacobi ensemble on $[0, 1]$. We find explicit expressions that are once again in terms of Meijer G-functions, see Corollary 3.4.

1.3. Scaling limits at the hard edge

The correlation kernels K_n for the polynomial ensemble (1.1)-(1.2) have an interesting large n scaling limit at the hard edge

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(\frac{x}{n}, \frac{y}{n}\right) = K_{\nu_1,\ldots,\nu_M}(x, y)$$

with a limiting kernel that depends on M parameters, $\nu_1, \ldots, \nu_M$, see [25] and Theorem 4.1 below for a precise statement.

For $M = 1$ they reduce to the hard edge Bessel kernels, see e.g. [33] or [14, Section 7.2], and for $M = 2$ these kernels already appeared in work of Bertola et al. [8] on the Cauchy-Laguerre two matrix model. In [15] Forrester obtained the same family of limiting kernels for the squared singular values of a product of complex Ginibre matrices with the inverse of another product of complex Ginibre matrices. Differential equations for the gap probabilities are in [32].

For results on the global distribution of the points in (1.1)-(1.2), see e.g. [10, 16, 29, 31, 34].

The second aim of this paper is to provide two more examples of models with the kernels $K_{\nu_1,\ldots,\nu_M}$ as scaling limit. We show that the new example with the product

of Ginibre matrices with one truncated unitary matrix falls into this category. In the second example we consider the biorthogonal ensembles

$$\frac{1}{Z_n} \prod_{j<k}(x_k - x_j) \prod_{j<k}(x_k^\theta - x_j^\theta) \prod_{j=1}^n w(x_j) \tag{1.5}$$

with all $x_j > 0$ and $\theta > 0$. Borodin [9] found the hard edge scaling limits for the cases where $w(x) = x^\alpha e^{-x}$ or $w(x) = x^\alpha \chi_{[0,1]}(x)$. The scaling limit depends on the two parameters $\theta > 0$ and $\alpha > -1$. We show that, after suitable rescaling, the limiting kernels belong to the class of kernels $K_{\nu_1,\dots,\nu_M}$ provided that θ or $1/\theta$ is an integer, see Theorem 5.1.

This last example in particular supports the conjecture that the kernels $K_{\nu_1,\dots,\nu_M}$ have a universal character and that they might appear as scaling limits in other interesting random models as well.

2. Transformation of Polynomial Ensembles

A complex Ginibre matrix G of size $m \times n$ has independent entries whose real and imaginary parts are independent and have a standard normal distribution. The probability distribution can be written as

$$\frac{1}{Z_n} e^{-\operatorname{Tr} G^* G} dG \tag{2.1}$$

where $dG = \prod_{j=1}^m \prod_{k=1}^n d\operatorname{Re} G_{j,k}\, d\operatorname{Im} G_{j,k}$ and Z_n is a normalization constant.

The main result of this section is the following:

Theorem 2.1. *Let $\nu \geq 0$ and $l \geq n \geq 1$ be integers and let G be a complex Ginibre random matrix of size $(n+\nu) \times l$. Let X be a random matrix of size $l \times n$, independent of G, such that the squared singular values $x_1, \dots, x_n$ of X have a joint probability density function on $[0,\infty)^n$ that is proportional to*

$$\Delta(x) \det[f_{k-1}(x_j)]_{j,k=1}^n, \tag{2.2}$$

for certain functions $f_0, \dots, f_{n-1}$. Then the squared singular values $y_1, \dots, y_n$ of $Y = GX$ are distributed with a joint probability density function on $[0,\infty)^n$ proportional to

$$\Delta(y) \det[g_{k-1}(y_j)]_{j,k=1}^n \tag{2.3}$$

with

$$g_k(y) = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right) \frac{dx}{x}, \qquad k = 0, \dots, n-1. \tag{2.4}$$

The theorem says that left multiplication by a complex Ginibre matrix maps polynomial ensembles to polynomial ensembles. Observe that g_k is the Mellin convolution [30, formula 1.14.39] of f_k with the function $x \mapsto x^\nu e^{-x}$.

Before we prove the theorem, we state an auxiliary result, which is essentially contained in [6, section 2.1] and also in [5]. For clarity, we give a detailed proof of this result.

4

Lemma 2.2. *Let ν, l, n and G be as in Theorem 2.1. Let X be a non-random matrix of size $l \times n$ with non-zero squared singular values $x_1, \ldots, x_n$. Then the squared singular values $y_1, \ldots, y_n$ of $Y = GX$ have a joint probability density function on $[0, \infty)^n$ that is proportional to*

$$\frac{\Delta(y)}{\Delta(x)} \det \left[\frac{y_j^\nu}{x_k^{\nu+1}} e^{-y_j/x_k} \right]_{j,k=1}^n \tag{2.5}$$

where the proportionality constant does not depend on X.

In case some of the x_k are the same, we have to interpret (2.5) in the appropriate limiting sense using l'Hôpital's rule.

Proof. First we show that we can reduce the proof to the case $l = n$. Assume $l > n$. Then any matrix X of size $l \times n$ can be written as

$$X = U \begin{pmatrix} X_0 \\ O \end{pmatrix}$$

where U is an $l \times l$ unitary, X_0 is an $n \times n$, and O is a zero matrix of size $(l - n) \times n$. Then by the unitary invariance of Ginibre random matrix ensembles, we have that $Y = GX$, has the same distribution of singular values as $Y_0 = G_0 X_0$, where G_0 is an $(n + \nu) \times n$ complex Ginibre matrix.

So in the rest of the proof we assume that $l = n$ and X is a square matrix of size $n \times n$ with squared singular values $x_1, \ldots, x_n$.

Then it is known that the change of variables $G \mapsto Y = GX$, with X being fixed, where G and Y are $(n + \nu) \times n$ matrices, has a Jacobian (see e.g. [27, Theorem 3.2])

$$\det(X^* X)^{-(n+\nu)} = \prod_{k=1}^n x_k^{-(n+\nu)}. \tag{2.6}$$

Thus under the mapping $G \mapsto Y$ the Ginibre probability distribution (2.1) transforms, up to a constant, into

$$e^{-\operatorname{Tr}(G^* G)} dG = \left(\prod_{k=1}^n x_k^{-(n+\nu)} \right) e^{-\operatorname{Tr}(Y^* Y (X^* X)^{-1})} dY. \tag{2.7}$$

Next we write $Y = U \Sigma V$ in its singular value decomposition. Thus Σ is a diagonal matrix with the singular values along the diagonal, V is a unitary matrix $n \times n$ and U is an $(n + \nu) \times n$ matrix with $U^* U = I$, that is, U belongs to the Stiefel manifold $V_{n,n+\nu}$. If we let $y_1, \ldots, y_n$ be the squared singular values of Y, then it is known that

$$dY \propto \left(\prod_{j=1}^n y_j^\nu \right) \Delta(y)^2 \, dU \, dV \, dy_1 \cdots dy_n \tag{2.8}$$

5

where dU is the invariant measure on the Stiefel manifold, and dV is the Haar measure on $U(n)$, see e.g. [13] and [35][1]. Combining (2.7) and (2.8) we obtain a probability measure proportional to

$$\left(\prod_{k=1}^{n} x_k^{-(n+\nu)}\right)\left(\prod_{j=1}^{n} y_j^{\nu}\right)\Delta(y)^2\, e^{-\operatorname{Tr}(V^*\Sigma^*\Sigma V(X^*X)^{-1})}dU\,dV\,dy_1\cdots dy_n. \qquad (2.9)$$

Since we are only interested in the squared singular values of Y, we integrate out the U and V part in (2.9). The integral over U only contributes to the constant. The integration over V is done by means of the Harish-Chandra/Itzykson-Zuber integral [19, 21]

$$\int_{U(n)} e^{-\operatorname{Tr}(V^*\Sigma^*\Sigma V(X^*X)^{-1})}dV = C_n\frac{\det\left[e^{-y_j/x_k}\right]_{j,k=1}^{n}}{\Delta(y)\Delta(x^{-1})}, \qquad (2.10)$$

where C_n is a (known) constant only depending on n. From (2.9) and (2.10) we obtain that the density of squared singular values of Y is proportional to

$$\left(\prod_{k=1}^{n} x_k^{-(n+\nu)}\right)\left(\prod_{j=1}^{n} y_j^{\nu}\right)\frac{\Delta(y)\,\det\left[e^{-y_j/x_k}\right]_{j,k=1}^{n}}{\Delta(x^{-1})}. \qquad (2.11)$$

Using $\Delta(x^{-1}) = (-1)^{n(n-1)/2}\prod_{k=1}^{n} x_k^{-n+1}\Delta(x)$ and bringing the products into the determinant, we immediately obtain (2.5) with a proportionality constant that is independent of $x_1,\ldots,x_n$. This proves the lemma. $\qquad\square$

We can now prove Theorem 2.1.

Proof. Suppose that the squared singular values of X have joint probability density function (2.2). Then the squared singular values are distinct almost surely, and we obtain from Lemma 2.2, after averaging out over X, that the squared singular values of $Y = GX$ have a joint probability density function that is proportional to

$$\Delta(y)\int_0^{\infty}\cdots\int_0^{\infty}\det\left[\frac{y_j^{\nu}}{x_k^{\nu+1}}e^{-y_j/x_k}\right]_{j,k=1}^{n}\det\left[f_{k-1}(x_j)\right]_{j,k=1}^{n}dx_1\cdots dx_n. \qquad (2.12)$$

The multiple integral in (2.12) can be evaluated with the Andreief identity, see e.g. [12, Chapter 3],

$$\int\cdots\int\det\left[\phi_j(x_k)\right]_{j,k=1}^{n}\det\left[\psi_k(x_j)\right]_{j,k=1}^{n}d\mu(x_1)\cdots d\mu(x_n)$$

$$= n!\det\left[\int\phi_j(x)\psi_k(x)d\mu(x)\right]_{j,k=1}^{n}$$

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[1]Note that in [13, page 10] the Jacobian is given in terms of the singular values $\sigma_j = \sqrt{y_j}$ with a factor $\prod_{j=1}^{n}\sigma_j^{2\nu+1}$. Since $2\sigma_j d\sigma_j = dy_j$ we obtain (2.8) with factor $\prod_{j=1}^{n} y_j^{\nu}$.

and the result is that (2.12) is proportional to (2.3) with functions

$$g_k(y) = \int_0^\infty \frac{y^\nu}{x^{\nu+1}} e^{-y/x} f_k(x) dx = \int_0^\infty x^\nu e^{-x} f_k\left(\frac{y}{x}\right) \frac{dx}{x}$$

and this completes the proof of Theorem 2.1. □

Remark 2.3. We emphasize that in Theorem 2.1 we do not assume that the probability distribution on X is invariant under (left or right) multiplication with unitary matrices.

Remark 2.4. It is of interest to find other random matrices A so that multiplication with A preserves the biorthogonal structure of squared singular values. In a forthcoming work we will show that this is the case for multiplication with truncated unitary matrices. The main issue is to find a suitable analogue of the Harish-Chandra/Itzykson Zuber formula (2.10).

Theorem 2.1 has a number of immediate consequences that we list now.

3. Corollaries

We can apply Theorem 2.1 to any random matrix X for which the squared singular values have a joint probability density function of the form (2.2).

In all the examples below, we will see the appearance of Meijer G-functions. This is actually quite naturally, because of its connections with the Mellin transform. In particular if f_k in Theorem 2.1 is a Meijer G-function, then also g_k in (2.4) is a Meijer G-function, see formula (A.2) in the appendix.

3.1. X is a Ginibre matrix

Suppose $X = G_1$ is itself a complex Ginibre random matrix of size $(n+\nu_1) \times n$, $\nu_1 \geq 0$. Then it is known that the squared singular values of X have a joint p.d.f. proportional to

$$\Delta(x)^2 \prod_{j=1}^n x_j^{\nu_1} e^{-x_j}$$

which can be written in the form (2.2) with

$$f_k(x) = x^{\nu_1+k} e^{-x} = G_{0,1}^{1,0}\left(\begin{matrix} - \\ \nu_1 + k \end{matrix} \middle| x\right). \tag{3.1}$$

Assume now that $Y = G_M \cdots G_1$ is the product of M independent complex Ginibre matrices where G_k has size $(n + \nu_k) \times (n + \nu_{k-1})$ and all $\nu_k \geq 0$ with $\nu_0 = 0$. Then we can apply Theorem 2.1 $M - 1$ times and using (A.3) and (3.1) we immediately find:

Corollary 3.1. *The joint probability density function of the squared singular values of $Y = G_M \cdots G_1$ is proportional to*

$$\Delta(y) \det [w_{k-1}(y_j)]_{j,k=1}^n \tag{3.2}$$

7

where

$$w_k(y) = G_{0,M}^{M,0}\left(\begin{matrix} - \\ \nu_M, \dots, \nu_1 + k \end{matrix} \middle| y\right). \tag{3.3}$$

This is the result of Akemann, Ipsen and Kieburg [5] mentioned in the introduction.

3.2. X is the inverse of a product of Ginibre matrices

A second application of Theorem 2.1 is inspired by the recent work of Forrester [15] who considered the product

$$Y = G_M \cdots G_1 (\tilde{G}_K \cdots \tilde{G}_1)^{-1} \tag{3.4}$$

of M Ginibre random matrices with the inverse of a product of K Ginibre random matrices. Here it is assumed that $\tilde{G}_j$ has size $(n + \tilde{\nu}_j) \times (n + \tilde{\nu}_{j-1})$ with all $\tilde{\nu}_j \geq 0$, and $\tilde{\nu}_0 = \tilde{\nu}_K = 0$, so that $\tilde{G}_K \cdots \tilde{G}_1$ is a square matrix.

From Corollary 3.1 we know that the squared singular values of $\tilde{G}_K \cdots \tilde{G}_1$ have a joint probability density function proportional to

$$\Delta(x) \det\left[\phi_{k-1}(x_j)\right]_{j,k=1}^n, \qquad \phi_k(x) = G_{0,K}^{K,0}\left(\begin{matrix} - \\ \tilde{\nu}_K, \dots, \tilde{\nu}_1 + k \end{matrix} \middle| x\right). \tag{3.5}$$

The following simple lemma shows that the squared singular values of $(\tilde{G}_K \cdots \tilde{G}_1)^{-1}$ then also have the structure of a polynomial ensemble.

Lemma 3.2. *Let X be a random matrix of size $n \times n$ such that the squared singular values $x_1, \dots, x_n$ of X have a joint probability density function proportional to*

$$\Delta(x) \det\left[\phi_k(x_j)\right]_{j,k=1}^n, \tag{3.6}$$

for certain functions ϕ_k. Then the squared singular values of $Y = X^{-1}$ have a joint probability density function proportional to

$$\Delta(y) \det\left[\psi_k(y_j)\right]_{j,k=1}^n \tag{3.7}$$

with

$$\psi_k(y) = y^{-n-1}\phi_k\left(y^{-1}\right), \qquad k = 1, \dots, n. \tag{3.8}$$

Proof. The squared singular values of X^{-1} are given by $y_j = x_j^{-1}$, $j = 1, \dots, n$. making the change of variables $x_j \mapsto y_j = x_j^{-1}$ in (3.6) gives us the joint probability density function of the squared singular values of X^{-1}. The Jacobian of this change of variables is $(-1)^n \prod_{j=1}^n y_j^{-2}$. Noting also that

$$\Delta(y^{-1}) = (-1)^{n(n-1)/2} \prod_{j=1}^n y_j^{-n+1} \Delta(y),$$

we find that the joint probability density function of $y_1, \ldots, y_n$ is therefore proportional to

$$\prod_{j=1}^{n} y_j^{-2} \prod_{j=1}^{n} y_j^{-n+1} \Delta(y) \det\left[\phi_{k-1}(y_j^{-1})\right]$$

which is indeed (3.7) with ψ_k given by (3.8). $\qquad\square$

The class of Meijer G-functions is closed under inversion of the argument and under multiplication by a power of the independent variable, see (A.4) and (A.5). It follows that if ϕ_k in Lemma 3.2 is a Meijer G-function, then so is ψ_k. To be precise, if

$$\phi_k(x) = G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle|\, x\right)$$

then

$$\psi_k(y) = G_{q,p}^{n,m}\left(\begin{matrix} -b_1 - n, \ldots, -b_q - n \\ -a_1 - n, \ldots, -a_p - n \end{matrix}\middle|\, x\right).$$

If we apply this to (3.5) we see that the squared singular values of $X = (\tilde{G}_K \cdots \tilde{G}_1)^{-1}$ have a joint p.d.f. of the form (2.2) with

$$f_k(x) = G_{K,0}^{0,K}\left(\begin{matrix} -\tilde{\nu}_K - n, \ldots, -\tilde{\nu}_2 - n, -\tilde{\nu}_1 - n - k \\ - \end{matrix}\middle|\, x\right).$$

Then a repeated application of Theorem 2.1 and formula (A.3) gives the following result of [15, Proposition 3].

Corollary 3.3. *Let* $\tilde{G}_1, \cdots \tilde{G}_K$ *and* $G_1, \ldots, G_M$ *be independent complex Ginibre matrices where* $\tilde{G}_j$ *has size* $(n + \tilde{\nu}_j) \times (n + \tilde{\nu}_{j-1})$ *and* G_j *has size* $(n + \nu_j) \times (n + \nu_{j-1})$ *with* $\tilde{\nu}_1, \ldots, \tilde{\nu}_{K-1} \geq 0$, $\nu_1, \ldots, \nu_M \geq 0$ *and* $\nu_0 = \tilde{\nu}_0 = \tilde{\nu}_K = 0$. *Then the squared singular values* $y_1, \ldots, y_n$ *of* Y *given in (3.4) with* $K \geq 1$, *have a joint probability density function proportional to*

$$\Delta(y) \det\left[w_{k-1}(y_j)\right]_{j,k=1}^{n} \tag{3.9}$$

where

$$w_k(y) = G_{K,M}^{M,K}\left(\begin{matrix} -\tilde{\nu}_K - n, \ldots, -\tilde{\nu}_2 - n, -\tilde{\nu}_1 - n - k \\ \nu_M, \ldots, \nu_1 \end{matrix}\middle|\, y\right). \tag{3.10}$$

3.3. X is a truncation of a random unitary matrix

As a third application we consider a new example, where we start from a matrix X which is a truncated unitary matrix. Let U be an $l \times l$ Haar distributed unitary matrix and let X be the $(n + \nu_1) \times n$ upper left block of U, where $\nu_1 \geq 0$ and $l \geq 2n + \nu_1$. Then the squared singular values of X are in $(0, 1)$ with a joint p.d.f that is proportional to (see e.g. [22, Proposition 2.1])

$$\prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^{n} x_j^{\nu_1} (1 - x_j)^{l - 2n - \nu_1}, \qquad \text{all } x_j \in (0, 1). \tag{3.11}$$

This is a Jacobi ensemble with parameters ν_1 and $l - 2n - \nu_1$. Note that in the case $l < 2n + \nu_1$ the truncation X always has 1 as a singular value, and (3.11) is not valid. We can rewrite (3.11) as (2.2) with functions

$$f_k(x) = \begin{cases} x^{\nu_1+k}(1-x)^{l-2n-\nu_1} & \text{for } 0 < x < 1, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.12}$$

Then

$$f_k(x) = \Gamma(l - 2n - \nu_1 + 1)G_{1,1}^{1,0}\left(\begin{matrix} l - 2n + k + 1 \\ \nu_1 + k \end{matrix}\middle| x\right). \tag{3.13}$$

Let $M \geq 1$ and form the product $Y = G_{M-1} \cdots G_1 X$ where G_j is a complex Ginibre matrix of size $(n + \nu_j) \times (n + \nu_{j-1})$ for $j = 1, \ldots, M - 1$. Then Theorem 2.1 together with (A.3) and (3.13) gives the following.

Corollary 3.4. *Let X be the $(n + \nu_1) \times n$ truncation of a unitary matrix of size $l \times l$ with $l \geq 2n + \nu_1$. Let $M \geq 1$ and let $Y = G_{M-1} \cdots G_1 X$ where each G_j is a complex Ginibre matrix of size $(n + \nu_j) \times (n + \nu_{j-1})$ with $\nu_1, \ldots, \nu_M \geq 0$. Then the squared singular values $y_1, \ldots, y_n$ of Y have a joint probability density function proportional to*

$$\Delta(y) \det[w_{k-1}(y_j)]_{j,k=1}^n \tag{3.14}$$

where

$$w_k(y) = G_{1,M}^{M,0}\left(\begin{matrix} l - 2n + 1 + k \\ \nu_M, \ldots, \nu_2, \nu_1 + k \end{matrix}\middle| y\right), \qquad y > 0. \tag{3.15}$$

4. Integral Representations and Hard Edge Scaling Limit

4.1. Kernels $K_{\nu_1,\ldots,\nu_M}$

For any set of functions $w_0, \ldots, w_{n-1}$, the probability density function (1.1) is a polynomial ensemble and we already noted in the introduction that the correlation kernel takes the form

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x)Q_k(y) \tag{4.1}$$

where, for each $k = 0, \ldots, n - 1$, P_k is a polynomial of degree k and Q_k is in the span of $w_0, \ldots, w_k$ such that

$$\int_0^\infty P_j(x)Q_k(x)dx = \delta_{j,k}. \tag{4.2}$$

For the case of weight functions (1.2) that are associated with the product of M Ginibre matrices, it was shown in [5] and [25] that the functions P_j and Q_k have contour integral representations, which was used in [25] to derive a double integral representation of the correlation kernel (4.1). Based on this double integral representation the following scaling limit was obtained in [25, Theorem 5.3].
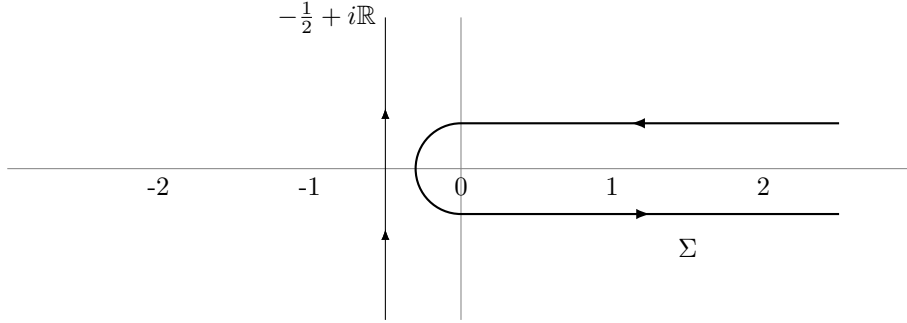
Figure 1: The contours $-\frac{1}{2} + i\mathbb{R}$ and Σ in the double integral (4.3)

Theorem 4.1. *Let $M \geq 1$ and $\nu_1, \ldots, \nu_M \geq 0$ be fixed integers. Then the kernels K_n have the scaling limit*

$$\lim_{n \to \infty} \frac{1}{n} K_n \left(\frac{x}{n}, \frac{y}{n} \right) = K_{\nu_1, \ldots, \nu_M}(x, y).$$

The limiting kernel has a double integral represention

$$K_{\nu_1, \ldots, \nu_M}(x, y) = \frac{1}{(2\pi i)^2} \int_{-1/2 - i\infty}^{-1/2 + i\infty} ds \oint_\Sigma dt \prod_{i=0}^{M} \frac{\Gamma(s + \nu_i + 1)}{\Gamma(t + \nu_i + 1)} \frac{\sin(\pi s)}{\sin(\pi t)} \frac{x^t y^{-s-1}}{s - t} \quad (4.3)$$

where Σ is a contour in $\{t \in \mathbb{C} \mid \operatorname{Re} t > -1/2\}$ encircling the positive real axis, as illustrated in Figure 1.

The kernels (4.3) have the alternative representation in terms of Meijer G-functions

$$K_{\nu_1, \ldots, \nu_M}(x, y)$$
$$= \int_0^1 G_{0,M+1}^{1,0} \left(\begin{matrix} - \\ -\nu_0, -\nu_1, \ldots, -\nu_M \end{matrix} \middle| ux \right) G_{0,M+1}^{M,0} \left(\begin{matrix} - \\ \nu_1, \ldots, \nu_M, \nu_0 \end{matrix} \middle| uy \right) du, \quad (4.4)$$

with $\nu_0 = 0$, see also [25].

In this section, we consider the polynomial ensemble (3.14) from Corollary 3.4 that is associated with the product of complex Ginibre random matrices with one truncated unitary matrix. Following [25] and [15] we are able to obtain integral representations for P_k and Q_k in this case as well, and from this a double integral representation for the correlation kernel. While keeping $\nu_1, \ldots, \nu_M$ fixed and letting l grow at least as $2n$, we obtain the limiting kernel $K_{\nu_1, \ldots, \nu_M}$ at the hard edge also in this case.

4.2. Integral representations for Q_k and P_k

So in the rest of this section, we assume that we work with the functions w_k in (3.15). The corresponding polynomials P_k are such that P_k is monic of degree k with

$$\int_0^\infty P_k(x) w_j(x) dx = 0 \qquad \text{for } j = 0, \ldots, k - 1 \quad (4.5)$$

11

and the functions Q_k satisfy

$$\int_0^\infty x^j Q_k(x)dx = \delta_{j,k} \qquad \text{for } j = 0, \ldots, k, \tag{4.6}$$

with Q_k in the linear span of $w_0, \ldots, w_k$. The polynomials P_k and functions Q_k have the following integral representation.

Proposition 4.2. *We have*

$$Q_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q_k(s) \frac{\prod_{j=1}^M \Gamma(s+\nu_j)}{\Gamma(s+l-2n+1)} x^{-s} ds \tag{4.7}$$

where $c > 0$ and

$$q_k(s) = \frac{\Gamma(l-2n+2k+2)}{\prod_{j=0}^M \Gamma(k+1+\nu_j)} \frac{(s-k)_k}{(s+l-2n+1)_k}. \tag{4.8}$$

Recall that $\nu_0 = 0$ and that the Pochhammer symbol is given by

$$(a)_k = a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

Proof. The functions w_k from (3.15) have the integral representation

$$w_k(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(s+\nu_1)_k}{(s+l-2n+1)_k} \frac{\prod_{j=1}^M \Gamma(s+\nu_j)}{\Gamma(s+l-2n+1)} x^{-s} ds, \qquad x > 0. \tag{4.9}$$

Then it is easy to see that the linear span of $w_0, \ldots, w_k$ consists of all functions as in the right-hand side of (4.7) with q_k being a rational function in s such that $(s+l-2n+1)_k q_k(s)$ is a polynomial of degree $\leq k$. Since (4.8) is of that type, we see that Q_k belongs to the linear span of $w_0, \ldots, w_k$.

By the properties of the Mellin transform, we have from (4.7) that

$$\int_0^\infty x^{s-1} Q_k(x)dx = q_k(s) \frac{\prod_{j=1}^M \Gamma(s+\nu_j)}{\Gamma(s+l-2n+1)}, \qquad s > 0. \tag{4.10}$$

Since q_k has zeros in $1, \ldots, k$ we find from (4.10) that

$$\int_0^\infty x^j Q_k(x)dx = 0, \qquad \text{for } j = 0, \ldots, k-1.$$

The prefactor in (4.8) has been chosen so that

$$\int_0^\infty x^k Q_k(x)dx = 1$$

as can be readily verified from (4.8) and (4.10). Thus (4.6) holds and the proposition is proved. $\qquad\square$

Notice that we can rewrite Q_k as a Meijer G-function

$$Q_k(x) = \frac{\Gamma(l - 2n + 2k + 2)}{\prod_{i=0}^{M} \Gamma(k + 1 + \nu_i)} G_{2,M+1}^{M+1,0} \left(\begin{matrix} -k, l - 2n + k + 1 \\ \nu_0, \nu_1, \ldots, \nu_M \end{matrix} \middle| x \right).$$

There is a similar integral representation for P_k.

Proposition 4.3. *We have*

$$P_k(x) = \frac{\prod_{j=0}^{M} \Gamma(k + 1 + \nu_j)}{\Gamma(l - 2n + 2k + 1)} \frac{1}{2\pi i} \oint_{\Sigma_k} \Gamma(t - k) \frac{\Gamma(t + l - 2n + k + 1)}{\prod_{j=0}^{M} \Gamma(t + 1 + \nu_j)} x^t dt \qquad (4.11)$$

where Σ_k is a closed contour encircling the interval $[0, k]$ once in the positive direction and such that $\mathrm{Re}\, t > -1$ for $t \in \Sigma_k$.

Proof. The integrand in the right-hand side of (4.11) is a meromorphic function with simple poles $0, 1, \ldots, k$ inside the contour. Since $\mathrm{Re}\, t > -1$ for $t \in \Sigma_k$, we have that the other poles are outside. Hence, by the residue theorem we have that the right-hand side of (4.11) defines a polynomial of degree at most k, and in fact

$$P_k(x) = \frac{\prod_{j=0}^{M} \Gamma(k + 1 + \nu_j)}{\Gamma(l - 2n + 2k + 1)} \sum_{t=0}^{k} \mathrm{Res}_t \left(\Gamma(t - k) \frac{\Gamma(t + l - 2n + k + 1)}{\prod_{j=0}^{M} \Gamma(t + 1 + \nu_j)} \right) x^t$$

$$= \sum_{t=0}^{k} \frac{(-1)^{k-t}}{(k-t)!} \frac{\Gamma(l - 2n + k + t + 1)}{\Gamma(l - 2n + 2k + 1)} \prod_{j=0}^{M} \frac{\Gamma(k + 1 + \nu_j)}{\Gamma(t + 1 + \nu_j)} x^t. \qquad (4.12)$$

The leading coefficient is 1 and so P_k is indeed a monic polynomial of degree k.

To check the orthogonality condition (4.5), we recall that by (4.9) and the properties of the Mellin transform

$$\int_0^\infty x^{s-1} w_j(x) dx = \frac{(s + \nu_1)_j \prod_{i=1}^{M} \Gamma(s + \nu_i)}{(s + l - 2n + 1)_j \Gamma(s + l - 2n + 1)}.$$

And so, if we use (4.11) and interchange the integrals

$$\int_0^\infty P_k(x) w_j(x) dx$$

$$= \frac{c_k}{2\pi i} \oint_{\Sigma_k} \Gamma(t - k) \frac{\Gamma(t + l - 2n + k + 1)}{\prod_{i=0}^{M} \Gamma(t + 1 + \nu_i)} \frac{(t + 1 + \nu_1)_j}{(t + l - 2n + 2)_j} \frac{\prod_{i=1}^{M} \Gamma(t + 1 + \nu_i)}{\Gamma(t + l - 2n + 2)} dt \qquad (4.13)$$

with $c_k = \frac{\prod_{i=0}^{M} \Gamma(k + 1 + \nu_i)}{\Gamma(l - 2n + 2k + 1)}$. In case $j = 0, \ldots, k - 1$, the integrand in (4.13) simplifies to

$$\frac{(t + 1 + \nu_1)_j (t + l - 2n + j + 2)_{k-j-1}}{(t - k)_{k+1}}$$

13

which is a rational function in t with poles at $t = 0, 1, \ldots, k$ only, and these are inside the contour Σ_k. In addition it is $O(t^{-2})$ as $t \to \infty$. Thus by moving the contour Σ_k to infinity, we see that (4.13) vanishes for $j = 0, 1, \ldots, k - 1$, and we obtain (4.5). $\square$

Formula (4.12) shows that P_k is a hypergeometric polynomial, namely

$$P_k(x) = (-1)^k \prod_{i=1}^{M} \frac{\Gamma(k + 1 + \nu_i)}{\Gamma(\nu_i + 1)} \frac{\Gamma(l - 2n + k + 1)}{\Gamma(l - 2n + 2k + 1)} {}_2F_M\left(\begin{matrix} -k, l - 2n + k + 1 \\ 1 + \nu_1, \ldots, 1 + \nu_M \end{matrix} \middle| x \right).$$

Finally, we notice that P_k can also be identified as a Meijer G-function

$$P_k(x) = -\frac{\prod_{i=0}^{M} \Gamma(k + 1 + \nu_i)}{\Gamma(l - 2n + 2k + 1)} G_{2,M+1}^{0,2}\left(\begin{matrix} k + 1, -(l - 2n + k) \\ -\nu_0, \ldots, -\nu_M \end{matrix} \middle| x \right).$$

4.3. Hard edge limit

We proceed to obtain a double integral representation for the kernel K_n.

Proposition 4.4. *We have*

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} ds \oint_{\Sigma} dt \prod_{j=0}^{M} \frac{\Gamma(s + 1 + \nu_j)}{\Gamma(t + 1 + \nu_j)}$$
$$\frac{\Gamma(t + 1 - n)\Gamma(t + l - n + 1)}{\Gamma(s + 1 - n)\Gamma(s + l - n + 1)} \frac{x^t y^{-s-1}}{s - t} \quad (4.14)$$

where Σ is a closed contour encircling $0, 1, \ldots, n$ once in the positive direction such that $\operatorname{Re} t > -1/2$ for $t \in \Sigma$.

Proof. The correlation kernel (4.1) can be written as

$$K_n(x, y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} ds \oint_{\Sigma} dt \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j)}{\Gamma(t + 1 + \nu_j)}$$
$$\sum_{k=0}^{n-1} (l - 2n + 2k + 1) \frac{\Gamma(t - k)}{\Gamma(s - k)} \frac{\Gamma(t + l - 2n + k + 1)}{\Gamma(s + l - 2n + k + 1)} x^t y^{-s}$$

where we used the representations (4.11) and (4.7) for $P_k(x)$ and $Q_k(y)$. By using the functional equation $\Gamma(z + 1) = z\Gamma(z)$, one can see that

$$(s - t - 1)(l - 2n + 2k + 1) \frac{\Gamma(t - k)}{\Gamma(s - k)} \frac{\Gamma(t + l - 2n + k + 1)}{\Gamma(s + l - 2n + k + 1)}$$
$$= \frac{\Gamma(t - k)}{\Gamma(s - k - 1)} \frac{\Gamma(t + l - 2n + k + 2)}{\Gamma(s + l - 2n + k + 1)} - \frac{\Gamma(t - k + 1)}{\Gamma(s - k)} \frac{\Gamma(t + l - 2n + k + 1)}{\Gamma(s + l - 2n + k)}$$

14

which means that we have a telescoping sum

$$(s - t - 1) \sum_{k=0}^{n-1} (l - 2n + 2k + 1) \frac{\Gamma(t - k)}{\Gamma(s - k)} \frac{\Gamma(t + l - 2n + k + 1)}{\Gamma(s + l - 2n + k + 1)}$$

$$= \frac{\Gamma(t - n + 1)}{\Gamma(s - n)} \frac{\Gamma(t + l - n + 1)}{\Gamma(s + l - n)} - \frac{\Gamma(t + 1)}{\Gamma(s)} \frac{\Gamma(t + l - 2n + 1)}{\Gamma(s + l - 2n)}. \quad (4.15)$$

By taking $c = 1/2$ and letting Σ encircle $0, 1, \ldots, n$ such that $\text{Re}(t) > -1/2$ for $t \in \Sigma$, we ensure that $s - t - 1 \neq 0$ whenever $s \in c + i\mathbb{R}$ and $t \in \Sigma$. And so we obtain that

$$K_n(x, y) =$$

$$\frac{1}{(2\pi i)^2} \int_{1/2 - i\infty}^{1/2 + i\infty} ds \oint_\Sigma dt \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j)}{\Gamma(t + 1 + \nu_j)} \frac{\Gamma(t - n + 1)}{\Gamma(s - n)} \frac{\Gamma(t + l - n + 1)}{\Gamma(s + l - n)} \frac{x^t y^{-s}}{s - t - 1}$$

$$- \frac{1}{(2\pi i)^2} \int_{1/2 - i\infty}^{1/2 + i\infty} ds \oint_\Sigma dt \prod_{j=0}^{M} \frac{\Gamma(s + \nu_j)}{\Gamma(t + 1 + \nu_j)} \frac{\Gamma(t + 1)}{\Gamma(s)} \frac{\Gamma(t + l - 2n + 1)}{\Gamma(s + l - 2n)} \frac{x^t y^{-s}}{s - t - 1}.$$

The integrand of the second double integral has no singularities inside Σ and hence the t-integral vanishes by Cauchy's theorem. By finally making the change of variable $s \mapsto s + 1$ in the first double integral, we obtain (4.14). $\qquad\square$

Remark 4.5. The proofs in subsections 4.2 and 4.3 are modelled after those in [25], but there are slight differences in all proofs.

We want to emphasize one difference which has to do with the telescoping sum (4.15). The left-hand side of (4.15) has the factors $l - 2n + k + 1$ which come from the product of the prefactors in (4.8) and (4.11). The corresponding prefactors in [25, formulas (3.2) and (3.8)] are each other inverses, and as a consequence there is no such factor. However it is remarkable that the factors $l - 2n + k + 1$ are actually necessary for the telescoping sum (4.15) to hold.

We notice that we can rewrite the kernel in terms of Meijer G-functions :

Corollary 4.6. *We have*

$$K_n(x, y) = \int_0^1 G_{2,M+1}^{0,2} \left(\begin{matrix} n, -(l - n) \\ -\nu_0, \ldots, -\nu_M \end{matrix} \middle| ux \right) G_{2,M+1}^{M+1,0} \left(\begin{matrix} -n, l - n \\ \nu_0, \ldots, \nu_M \end{matrix} \middle| uy \right) du. \quad (4.16)$$

Proof. Since

$$\frac{x^t y^{-s-1}}{s - t} = -\int_0^1 (ux)^t (uy)^{-s-1} du,$$

the kernel (4.14) can be rewritten as

$$K_n(x, y) = -\int_0^1 \left(\frac{1}{2\pi i} \oint_\Sigma \frac{\Gamma(t + 1 - n)\Gamma(t + l - n + 1)}{\prod_{j=0}^{M} \Gamma(t + 1 + \nu_j)} (ux)^t dt \right)$$

$$\times \left(\frac{1}{2\pi i} \int_{-1/2 - i\infty}^{-1/2 + i\infty} \frac{\prod_{j=0}^{M} \Gamma(s + 1 + \nu_j)}{\Gamma(s + 1 - n)\Gamma(s + l - n + 1)} (uy)^{-s-1} ds \right) du.$$

15

By the definition of a Meijer G-function and making the change of variables $t \mapsto -t$ and $s \mapsto s - 1$, we obtain the identity (4.16). $\qquad \square$

Using the integral representation for K_n, we can derive the scaling limit at the hard edge.

Theorem 4.7. *With $\nu_1, \ldots, \nu_M$ being fixed and with l growing at least as $2n$, we have*

$$\lim_{n \to \infty} \frac{1}{(l-n)n} K_n \left(\frac{x}{(l-n)n}, \frac{y}{(l-n)n} \right) = K_{\nu_1, \ldots, \nu_M}(x, y). \qquad (4.17)$$

uniformly for x, y in compact subsets of the real positive axis, where

$$K_{\nu_1, \ldots, \nu_M}(x, y)$$
$$= \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} ds \oint_\Sigma dt \prod_{j=0}^{M} \frac{\Gamma(s+1+\nu_j)}{\Gamma(t+1+\nu_j)} \frac{\sin(\pi s)}{\sin(\pi t)} \frac{x^t y^{-s-1}}{s-t}. \qquad (4.18)$$

The contour Σ starts at $+\infty$ in the upper half plane and returns to $+\infty$ in the lower half plane encircling the positive real axis such that $\operatorname{Re} t > -1/2$ for $t \in \Sigma$ (see also Figure 1 on page 11).

Proof. By using identity (4.14), we know

$$\frac{1}{(l-n)n} K_n \left(\frac{x}{(l-n)n}, \frac{y}{(l-n)n} \right) = \frac{1}{(2\pi i)^2} \int_{-1/2-i\infty}^{-1/2+i\infty} ds \oint_\Sigma dt \prod_{j=0}^{M} \frac{\Gamma(s+1+\nu_j)}{\Gamma(t+1+\nu_j)}$$
$$\frac{\Gamma(t+1-n)\Gamma(t+l-n+1)}{\Gamma(s+1-n)\Gamma(s+l-n+1)} \frac{x^t y^{-s-1}}{s-t} (l-n)^{s-t} n^{s-t}.$$

Euler's reflection formula tells us that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

and so we see that

$$\frac{\Gamma(t-n+1)}{\Gamma(s-n+1)} = \frac{\Gamma(n-s)}{\Gamma(n-t)} \frac{\sin(\pi s)}{\sin(\pi t)}.$$

Furthermore, as $n \to \infty$ we also know [30, formula 5.11.13]

$$\frac{\Gamma(n-s)}{\Gamma(n-t)} = n^{t-s} \left(1 + O(n^{-1}) \right)$$

and similarly

$$\frac{\Gamma(t+l-n+1)}{\Gamma(s+l-n+1)} = (l-n)^{t-s} \left(1 + O((l-n)^{-1}) \right).$$

Hence, if we deform the contour Σ to a two sided, unbounded contour as in Figure 1 and apply the identities above, we immediately obtain identity (4.17), provided that we can take the limit inside the integral. This can be justified by using the dominated convergence theorem (see [25, Theorem 5.3] for details). $\qquad \square$

5. Borodin Biorthogonal Ensembles

In this final section we consider the biorthogonal ensembles (1.5) that were studied by Borodin in [9], see also [11, 28]. These are determinantal point process on $[0, \infty)$, whose correlation kernels are expected to have interesting scaling limits at the hard edge $x = 0$. This was proved in [9] for the cases where w is either a special Jacobi weight

$$w(x) = \begin{cases} x^\alpha, & 0 < x \leq 1, \\ 0, & x > 1, \end{cases} \qquad \alpha > -1, \tag{5.1}$$

or a Laguerre weight

$$w(x) = x^\alpha e^{-x} \qquad x > 0, \qquad \alpha > -1. \tag{5.2}$$

In both cases it was shown that a scaling limit at the origin leads to the following correlation kernel that depends on α and θ,

$$K^{(\alpha,\theta)}(x,y) = \theta x^\alpha \int_0^1 J_{\frac{\alpha+1}{\theta}, \frac{1}{\theta}}(xu) J_{\alpha+1,\theta}((yu)^\theta) u^\alpha du. \tag{5.3}$$

where $J_{a,b}$ is Wright's generalization of the Bessel function given by

$$J_{a,b}(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j! \Gamma(a + jb)}. \tag{5.4}$$

The kernels (5.3) are related to the Meijer G-kernel $K_{\nu_1,\ldots,\nu_M}$ in case θ or $1/\theta$ is an integer. This is our final result.

Theorem 5.1. *Let $M \geq 1$ be an integer.*

(a) *Then we have*

$$M^M K^{(\alpha, \frac{1}{M})}(M^M x, M^M y) = \left(\frac{x}{y} \right)^\alpha K_{\nu_1,\ldots,\nu_M}(x,y) \tag{5.5}$$

with parameters

$$\nu_j = \alpha + \frac{j-1}{M}, \qquad j = 1, \ldots, M. \tag{5.6}$$

(b) *We also have*

$$x^{\frac{1}{M}-1} K^{(\alpha,M)}(M x^{\frac{1}{M}}, M y^{\frac{1}{M}}) = K_{\tilde{\nu}_1,\ldots,\tilde{\nu}_M}(y,x) \tag{5.7}$$

with parameters

$$\tilde{\nu}_j = \frac{\alpha}{M} - 1 + \frac{j}{M}, \qquad j = 1, \ldots, M. \tag{5.8}$$

17

The parameters (5.6) and (5.8) come in an arithmetic progression with step size $1/M$, and therefore they cannot all be integers if $M \geq 2$. This is in contrast to the limiting kernels obtained from the products of random matrices where the ν_j are necessarily integers.

Proof. If b is a rational number, then (5.4) can be expressed as a Meijer G-function, see [17, formula (22)] and [18, formula (13)]. For the case when $b = M$ with $M \geq 2$ an integer, we have

$$
J_{a,M}(x) = (2\pi)^{\frac{M-1}{2}} M^{-a+1/2} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{\prod_{j=0}^{M-1} \Gamma\left(\frac{a}{M} - s + \frac{j}{M}\right)} \left(\frac{x}{M^M}\right)^{-s} ds
$$

$$
= (2\pi)^{\frac{M-1}{2}} M^{-a+1/2} G_{0,M+1}^{1,0}\left(0, -\frac{a}{M} + \frac{1}{M}, -\frac{a}{M} + \frac{2}{M}, \ldots, -\frac{a}{M} + 1 \,\middle|\, \frac{x}{M^M}\right)
$$

$$(5.9)$$

and for $b = 1/M$,

$$
J_{a,\frac{1}{M}}(x) = (2\pi)^{-\frac{M-1}{2}} M^{1/2} \frac{1}{2\pi i} \oint_\Sigma \frac{\prod_{k=0}^{M-1} \Gamma\left(t + \frac{k}{M}\right)}{\Gamma(a-t)} \left(\frac{x^M}{M^M}\right)^{-t} dt
$$

$$
= (2\pi)^{-\frac{M-1}{2}} M^{1/2} G_{0,M+1}^{M,0}\left(0, \frac{1}{M}, \ldots, \frac{M-1}{M}, 1-a \,\middle|\, \frac{x^M}{M^M}\right) \qquad (5.10)
$$

where Σ is a contour encircling the negative real axis.

Inserting (5.9) and (5.10) into (5.3) we obtain for $\theta = 1/M$ with a positive integer M,

$$
K^{(\alpha,\frac{1}{M})}(x,y)
$$

$$
= M^{-(\alpha+1)M} x^\alpha \int_0^1 G_{0,M+1}^{1,0}\left(0, -\alpha, -\alpha - \frac{1}{M}, \ldots, -\alpha - \frac{M-1}{M} \,\middle|\, \frac{ux}{M^M}\right)
$$

$$
\times G_{0,M+1}^{M,0}\left(0, \frac{1}{M}, \ldots, \frac{M-1}{M}, -\alpha \,\middle|\, \frac{uy}{M^M}\right) u^\alpha du, \quad (5.11)
$$

and after a rescaling of variables $x \mapsto M^M x$, $y \mapsto M^M y$,

$$
M^M K^{(\alpha,\frac{1}{M})}(M^M x, M^M y)
$$

$$
= x^\alpha \int_0^1 G_{0,M+1}^{1,0}\left(0, -\alpha, -\alpha - \frac{1}{M}, \ldots, -\alpha - \frac{M-1}{M} \,\middle|\, ux\right)
$$

$$
\times G_{0,M+1}^{M,0}\left(0, \frac{1}{M}, \ldots, \frac{M-1}{M}, -\alpha \,\middle|\, uy\right) u^\alpha du. \quad (5.12)
$$

Then we realize that for a Meijer G-function $G(z)$, we have that $z^\alpha G(z)$ is again a

Meijer G-function with parameters shifted by α, see formula (A.4). Thus by (5.12)

$$
M^M K^{(\alpha, \frac{1}{M})}(M^M x, M^M y)
$$
$$
= \left(\frac{x}{y}\right)^\alpha \int_0^1 G_{0,M+1}^{1,0}\left(\begin{matrix} - \\ 0, -\alpha, -\alpha - \frac{1}{M}, \ldots, -\alpha - \frac{M-1}{M} \end{matrix}\middle| ux\right)
$$
$$
\times G_{0,M+1}^{M,0}\left(\begin{matrix} - \\ \alpha, \alpha + \frac{1}{M}, \ldots, \alpha + \frac{M-1}{M}, 0 \end{matrix}\middle| uy\right) du
$$

which proves part (a) of the theorem because of (4.4).

Part (b) follows in a similar way. Alternatively it can be obtained from part (a) because of the formula

$$
\frac{1}{\theta} x^{\frac{1}{\theta}-1} K^{(\alpha,\theta)}(x^{\frac{1}{\theta}}, y^{\frac{1}{\theta}}) = \left(\frac{x}{y}\right)^{\alpha'} K^{(\alpha', \frac{1}{\theta})}(y, x), \qquad \alpha' = \frac{\alpha+1}{\theta} - 1
$$

which can be easily deduced from (5.3). $\qquad\square$

Acknowledgements

A. The Meijer G-function

For ease of reference, we collect in this appendix the definition and properties of the Meijer G-function that are used in this paper. By definition, the Meijer G-function is given by the following contour integral:

$$
G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| z\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(s + b_j) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(s + a_j)} z^{-s} ds,
$$

(A.1)

where the branch cut of z^{-s} is taken along the negative x-axis. Furthermore, it is also assumed that

- m, n, p, q are integers such that $0 \leq p \leq n$ and $0 \leq q \leq m$;

- the real (or complex) parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ satisfy the conditions

$$
a_k - b_j \neq 1, 2, 3, \ldots \text{ for } k = 1, \ldots, n \text{ and } j = 1, \ldots, m.
$$

I.e., none of the poles of $\Gamma(b_j + u)$ coincides with any of the poles of $\Gamma(1 - a_k - u)$.

The contour L is such that all the poles of $\Gamma(u + b_j)$ are on the left of the path while the poles of $\Gamma(1 - a_j + u)$ are on the right of the path. In typical situations the contour is a vertical line $c + i\mathbb{R}$ with $c > 0$.

The Mellin transform of an integrable function w on $[0, \infty)$ is

$$(\mathcal{M}w)(s) = \int_0^\infty x^{s-1} w(x) dx, \qquad a < \operatorname{Re} s < b.$$

The inverse Mellin transform is

$$w(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\mathcal{M}w)(s) x^{-s} ds,$$

where $a < c < b$. Thus for a Meijer G-function which is defined and integrable on the positive half-line we have

$$\int_0^\infty x^{s-1} G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| x\right) dx = \frac{\prod_{j=1}^m \Gamma(s + b_j) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^q \Gamma(1 - b_j - s) \prod_{j=n+1}^p \Gamma(s + a_j)}. \quad \text{(A.2)}$$

The Mellin convolution of two Meijer G-functions is again a Meijer G-function. A special case of this is

$$\int_0^\infty x^{\nu-1} e^{-x} G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| \frac{y}{x}\right) dx = G_{p,q+1}^{m+1,n}\left(\begin{matrix} a_1, \ldots, a_p \\ \nu, b_1, \ldots, b_q \end{matrix}\middle| y\right), \quad \text{(A.3)}$$

provided that the integral in the left-hand side converges.

Further identities are

$$x^\alpha G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| x\right) = G_{p,q}^{m,n}\left(\begin{matrix} a_1 + \alpha, \ldots, a_p + \alpha \\ b_1 + \alpha, \ldots, b_q + \alpha \end{matrix}\middle| x\right) \quad \text{(A.4)}$$

and

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| x^{-1}\right) = G_{q,p}^{n,m}\left(\begin{matrix} 1 - b_1, \ldots, 1 - b_q \\ 1 - a_1, \ldots, 1 - a_p \end{matrix}\middle| x\right). \quad \text{(A.5)}$$

For more details, we refer the reader to [7, 26].

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