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SINGULAR VALUES OF TOURNAMENT MATRICES*

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Abstract. Upper and lower bounds on both the largest and smallest singular values of a tournament matrix M of order n are obtained. For most values of n, the matrices M for which equality holds are characterized.

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1. Introduction. A *tournament matrix of order* n is an $n \times n$ $\{0,1\}$-matrix M_n such that $M_n + M_n^T = J_n - I_n$ where J_n is the $n \times n$ matrix of ones and I_n is the identity matrix of order n. Eigenvalues of tournament matrices have been studied extensively [5, 9, 13, 17, 12]. In this paper, the *singular values* are examined.

For an $n \times n$ complex matrix C, the singular values, $\sigma_1(C) \geq \sigma_2(C) \geq \cdots \geq \sigma_n(C)$ of C are the nonnegative square roots of the eigenvalues of C^*C or, equivalently, of CC^*. Thus, if eigenvalues are also taken in nonincreasing order, then $\sigma_i^2(C) = \lambda_i(C^*C) = \lambda_i(CC^*)$, $i = 1, \ldots, n$. In particular, $\sigma_1^2(C) = \rho(CC^*) = \rho(C^*C)$, where ρ denotes spectral radius. Also, using Rayleigh quotients, we have $\sigma_1^2(C) = \max_{x^*x=1} x^*C^*Cx$, while $\sigma_n^2(C) = \min_{x^*x=1} x^*C^*Cx$ where the maximimum and the minimum are taken over all vectors $x \in \mathbb{C}^n$ with $x^*x = \|x\|_2^2 = 1$. The largest singular value, $\sigma_1(C)$, is also called the *spectral norm* of C because $\sigma_1(C) = \max_{\|x\|_2=1} \|Cx\|_2 = |||C|||_2$, the operator norm induced by the usual Euclidean norm $\|\cdot\|_2$.

The singular values of C may also be defined as the n largest eigenvalues of the $2n \times 2n$ Hermitian matrix

$$\widetilde{C} = \begin{bmatrix} O & C \\ C^* & O \end{bmatrix},$$

the remaining n eigenvalues of $\widetilde{C}$ being the negatives of the singular values of C [10, p. 161]. Properties of the singular values of a tournament matrix have been examined from this viewpoint by Dedo, Zagaglia Salvi and Kirkland in [6].

Throughout the paper, $\mathit{1} = \mathit{1}_n$ will always denote a column vector of n ones, and $M = M_n$ a tournament matrix of order n with *score vector* $s = M\mathit{1}$ and *score variance* $\alpha_1^2 = \alpha_1^2(M) = \frac{1}{n}\sum_i (s_i - \frac{n-1}{2})^2 = \frac{s^Ts}{n} - (\frac{n-1}{2})^2$. If n is odd and each entry of s equals $\frac{n-1}{2}$, then M is said to be *regular.* It is said to be *almost regular* if n is even and $\frac{n}{2}$ of the entries of s equal $\frac{n}{2}$ and the other $\frac{n}{2}$ entries equal $\frac{n}{2} - 1$.

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If C is normal, that is if $C^*C = CC^*$, then the singular values of C are the moduli of its eigenvalues [11, p. 157]. It is easily seen that a tournament matrix M is nearly normal in the sense that the rank one perturbation, $M - \frac{1}{2}J$, is a normal matrix. It is perhaps surprising then to find in Section 2 that, for fixed n, $\sigma_1(M_n)$ is maximized precisely when $\rho(M_n)$ is minimized, that is, when M_n is the matrix of a transitive tournament. For n odd, we find in Section 2 that $\sigma_1(M_n)$ is minimized precisely when $\rho(M_n)$ is maximized, namely when M_n is regular. Further, for n even, we prove in Section 3 that if $\sigma_1(M_n)$ is minimized, then M_n must be almost regular. Again there appears to be a connection to $\rho(M_n)$; for n even and sufficiently large, a recent result supporting an outstanding conjecture of Brualdi and Li [3, Prob. 31(1)] asserts that M_n must be almost regular when $\rho(M_n)$ is maximum [15]. However, the tournament matrix conjectured by Brualdi and Li to maximize $\rho(M_n)$ need not minimize $\sigma_1(M_n)$.

2. Majorization and Singular Values. Let $x, y \in \mathbb{R}^n$. We say that x is *weakly majorized* by y and write $x \prec_w y$ if for each $k = 1, \dots, n$, the sum of the k largest entries of x is less than or equal to the sum of the k largest entries of y. We say that x is *majorized* by y and write $x \prec y$ if $x \prec_w y$ and $\sum x_i = \sum y_i$. These definitions may be rephrased using the matrix of a transitive tournament. Let

$$U = U_n = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

be the upper triangular tournament matrix of order n. Then U_n is the matrix of the transitive tournament with Hamilton path $1 \to 2 \to \cdots \to n$. If $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$, then $x \prec_w y$ if and only if $Ux \le Uy$ and $\sum x_i \le \sum y_i$, while $x \prec y$ if and only if and only if $Ux \le Uy$ and $\sum x_i = \sum y_i$.

A basic property of majorization asserts that $x \prec y$ if and only if the entries of x can be obtained from those of y by a finite number of transfers of the form $\{y_i, y_j\} \to \{y_i + d, y_j - d\}$ where $0 \le d \le y_j - y_i)$ [20, p. 6]. Since a tournament matrix is regular or almost regular if and only if every pair of entries of its score vector differ by at most one, it follows that a tournament matrix R of order n is regular or almost regular if and only if $R\mathit{1} \prec M\mathit{1}$ for all tournament matrices M of order n.

A similar defining property of majorization (resp. weak majorization) involves transfers of the form $\{y_i, y_j\} \to \{ay_i + by_j, by_i + ay_j\}$ where $0 \le a, b \le 1$ and $a + b = 1$. Such a transfer is called a T*-transform* [20, p. 21] and is *strict* if $0 < a < 1$. If $x, y \in \mathbb{R}^n$, then $x \prec y$ (resp. $x \prec_w y$) if and only if $x = T_1 \cdots T_k y$ (resp. $x \le T_1 \cdots T_k y$) for some finite sequence $T_1, \dots, T_k$ of T-transforms [20, p. 24,26].

According to a theorem of Landau [20, p. 186], an n-vector s of nonnegative integers is the score vector of some tournament matrix of order n if and only if $s \prec U\mathit{1}$. Let $\|\cdot\|_2$ denote the usual Euclidean norm. From the necessary condition in Landau's theorem and the T-transform characterization of majorization, a straightforward convexity argument implies that $\|M\mathit{1}\|_2 \le \|U\mathit{1}\|_2$ or, equivalently, that $\alpha_1^2(M) \le \alpha_1^2(U) = \frac{n^2-1}{12}$ for all tournament matrices M of order n. Moreover, equality holds if and only if M is the matrix of a transitive tournament. In Theorem 2.2, a similar result will be proved for the largest singular value $\sigma_1(M) = \max_{\|x\|_2=1} \|Mx\|_2$. The proof will require the following generalization of the necessary condition in Landau's theorem.

LEMMA 2.1. *Let U be the upper triangular tournament matrix of order n. If $x \prec_w y$ and $y_1 \leq y_2 \leq \cdots \leq y_n$, then $Mx \prec_w Uy$ for all tournament matrices M of order n.*

Proof. Since $Ux \prec_w Uy$ whenever the entries of x and y are nondecreasing and $x \prec_w y$, it is sufficient to prove that $Mx \prec_w Ux$ for all M whenever the entries of x are nondecreasing. Since M is arbitrary, this is equivalent to showing that $Mw \prec_w Ux$ for all M whenever the entries of w are a permutation of the entries of x. Also, since M is arbitrary, by permuting the rows and columns of M and the entries of w simultaneously, we may assume that the entries of Mw are nondecreasing. Then the sum of the k largest entries of Mw is $S_k = \sum_{i=1}^{k}(Mw)_i$.

If $k = n$, then $S_n = 1^T Mw = r_1 w_1 + \cdots r_n w_n$, where $r_1 + \cdots r_n = \binom{n}{2}$. Since the entries of x are a nondecreasing rearrangement of the entries of w, it follows that $S_n \leq x_2 + 2x_3 + \cdots + (n-1)x_n$, the sum of the n entries of Ux.

If $k < n$, let M_k be the $k \times k$ tournament submatrix of M obtained by selecting the first k rows and columns of M. Then $S_k \leq 1_k^T(M_k \hat{w} + J\tilde{w})$ where $\hat{w} = [w_1, \ldots, w_k]^T$, $\tilde{w} = [w_{k+1}, \ldots, w_n]^T$, and J is the $k \times (n-k)$ all-ones matrix. In the above upper bound on S_k, because M_k and J have k rows, each entry w_i of w appears at most $k-1$ times if $i \leq k$ and exactly k times if $i > k$. Consequently, because x is nondecreasing, the upper bound would not decrease if any entry x_i of $\tilde{w}$ with $i \leq k$ were swapped with any entry x_j of $\hat{w}$ with $j > k$. Thus, it may be assumed that $w = x$. By the same argument as that in the case where $k = n$, it follows that $1_k^T M_k \hat{x} \leq 1_k^T U_k \hat{x}$ where U_k is the upper triangular tournament matrix of order k. Thus, $S_k \leq 1_k^T(U_k\hat{x} + J\tilde{x})$, the sum of the k largest entries of Ux. ◻

Using Rayleigh quotients, we see that the smallest singular value of a tournament matrix M of order n is $\sigma_n(M) = \min_{\|x\|_2=1} \|Mx\|_2$. Thus, $\sigma_n(M) \geq 0$ with equality holding if and only if M is singular. A complete classification of the tournament matrices M with $\det M = 0$ seems difficult to obtain. We mention in passing, however, a striking necessary condition due to Shader [22]: if $\sigma_n(M) = 0$, then M has score variance $\alpha_1^2 \geq \frac{n-1}{4}$.

The following theorem characterizes the tournament matrices M whose spectral norm, $\sigma_1(M) = |||M|||_2$, is maximum.

THEOREM 2.2. *If U is the upper triangular tournament matrix of order $n \geq 2$ then, for all tournament matrices M of order n*

$$\sigma_1(M) \leq \sigma_1(U) = \frac{1}{2}\csc\frac{\pi}{4n-2}.$$

Equality holds if and only if M is the matrix of a transitive tournament.

Proof. Let x be a nonnegative eigenvector such that $M^T Mx = \sigma_1^2(M)x$ and let y be a nondecreasing rearrangement of the entries of x. By Lemma 2.1, $Mx \prec_w Uy$ and so $Mx \leq T_1 T_2 \cdots T_k Uy$ for some sequence $T_1, \ldots, T_k$ of T-transforms. Since the function $\phi(t) = |t|^2$ is strictly convex and is strictly increasing when $t \geq 0$, the Euclidean norm of a vector decreases whenever a strict T-transform is performed on it or whenever any of its entries is decreased in modulus. Therefore, $\|Mx\|_2 \leq \|Uy\|_2$ and equality will hold if and only if the entries of Mx are a permutation of the entries of My. Thus, $\sigma_1^2(M)x^Tx = x^TM^TMx \leq y^TU^TUy \leq \rho(U^TU)y^Ty = \sigma_1^2(U)x^Tx$. Therefore, $\sigma_1(M) \leq \sigma_1(U)$. Let

$$L_{n-1}^{-1} = \begin{bmatrix} n-1 & n-2 & n-3 & \cdots & 1 \\ n-2 & n-2 & n-3 & \cdots & 1 \\ n-3 & n-3 & n-3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \text{ and } D_{n-1} = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & \ddots & & \\ & & 1 & \ddots & 1 & \\ & & & \ddots & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix},$$

where D_{n-1} is the $(n-1)\times(n-1)$ tridiagonal matrix above with all remaining entries zero. Matrix multiplication shows that L_{n-1}^{-1} is the inverse of $L_{n-1} = 2I - D_{n-1}$. Since $UU^T = L_{n-1}^{-1} \oplus [0]$, it follows that $\sigma_1^2(U) = \rho(UU^T) = \rho(L_{n-1}^{-1}) = 1/(2-\rho(D_{n-1}))$. It is easily checked that the vector $u = (u_k)$, $u_k = \cos(\frac{2k-1}{2n-1}\frac{\pi}{2})$, $k = 1, ..., n-1$, is an eigenvector of D_{n-1} corresponding to the eigenvalue $2\cos\frac{\pi}{2n-1}$. Since u is the Perron vector of D_{n-1}, $\rho(D_{n-1}) = 2\cos\frac{\pi}{2n-1}$. Thus $\sigma_1^2(U) = 1/(2-\rho(D_{n-1})) = \frac{1}{2}(1-\cos\frac{\pi}{2n-1})^{-1} = \frac{1}{4}\csc^2\frac{\pi}{4n-2}$, as required.

Suppose now that $\sigma_1(M) = \sigma_1(U)$. Let x and y be defined as in the first paragraph of the proof. From the inequalities there, it follows that the nondecreasing rearrangement y of the Perron vector x of M^TM must be a Perron vector of U^TU. Consequently, by permuting the rows and columns of M if necessary, we may assume that x is a Perron vector of both M^TM and U^TU with common Perron value $\rho = \sigma_1^2(M) = \sigma_1^2(U)$. Since the entries of U^Tw are strictly increasing whenever $w_i > 0$ for $i = 1, \ldots, n-1$, the entries of $x = \frac{1}{\rho}U^TUx$ must be strictly increasing. Therefore, by Lemma 2.1, $Mx \prec_w Ux$. But $\|Mx\|_2 = \|Ux\|_2$ so, as we observed in the first paragraph, there must be a permutation matrix P such that $PMx = Ux$. Because the entries of x are strictly increasing and $(Ux)_i = x_{i+1} + \cdots + x_n$, by successively comparing the rows of PM and U, it follows that $PM = U$. But M and U are tournament matrices. Thus, $M = U$. □

REMARKS 2.3. 1. The *algebraic connectivity* of a graph G with adjacency matrix A is defined as the second smallest eigenvalue, μ, of its Laplacian matrix $L = D - A$, where D is the diagonal matrix of vertex degrees. An interesting connection to the algebraic connectivity of a path with an odd number $N = 2n-1$ of consecutively adjacent vertices $1, 2, \ldots, N$ provides an alternate route to determining $\sigma_1^2(U) = \rho(L_{n-1}^{-1})$ in the proof of Theorem 2.2. Let $\rho = \rho(L_{n-1}^{-1})$. Then $1/\rho$ is the smallest eigenvalue of L_{n-1}. Thus, it is also the smallest eigenvalue of the direct sum $L_{n-1} \oplus L_{n-1}$, with multiplicity at least two. But the direct sum matrix is easily seen to be permutation similar to the matrix obtained by deleting the n^{th} row and column of L. Consequently, the eigenvalues of $L_{n-1} \oplus L_{n-1}$ interlace those of L [10, p. 186]. Since 0 is an eigenvalue of L, it follows that its second smallest eigenvalue, μ, must equal $1/\rho$.

Since the path on N vertices has algebraic connectivity $\mu = 2(1 - \cos\frac{\pi}{N})$ [7, p. 304], we see once more that $\sigma_1^2(U) = \rho(L_{n-1}^{-1}) = \frac{1}{2}(1-\cos\frac{\pi}{2n-1})^{-1}$.

2. There are other norms $||\cdot||$ on tournament matrices M_n of order n for which $||M_n|| \leq ||U_n||$. For $1 \leq p \leq \infty$, let $\|\cdot\|_p$ denote the usual p-norm on $\mathbb{C}^n$. Then $|||C|||_p = \max_{x\neq 0}\|Cx\|_p/\|x\|_p$, is the operator norm induced by $\|\cdot\|_p$ and so is submultiplicative [10, p. 293]. When $1 < p < \infty$, the function $\phi(t) = |t|^p$ is strictly convex and strictly increasing for $t \geq 0$. Thus, when $1 < p < \infty$, it follows as in the proof of Theorem 2.2 that $|||M_n|||_p \leq |||U_n|||_p$ with equality holding if and only if M_n is the matrix of a transitive tournament. Note that $|||C|||_1 = \max_j \sum_i |c_{ij}|$ is the maximum column sum norm of C while $|||C|||_\infty = \max_i \sum_j |c_{ij}|$ is the maximum row sum norm of C. Therefore, $|||M_n|||_1 \leq |||U_n|||_1 = n-1$ with equality holding if and only if M_n has some column sum equal to $n-1$, and $|||M_n|||_\infty \leq |||U_n|||_\infty = n-1$ with equality holding if and only if M_n has some row sum equal to $n-1$.

3. Some norms do not distinguish any of the tournament matrices of a given order n. The *numerical radius norm*, $r(C) = \max_{x^*x=1}|x^*Cx|$, and the *Frobenius norm*, $\|C\|_2 = (\sum_{i,j}|c_{ij}|^2)^{\frac{1}{2}}$ are examples: for every tournament matrix M of order n, $r(M) = \frac{n-1}{2}$, while $\|M\|_2 = (n(n-1)/2)^{\frac{1}{2}}$.

4. The inequality $||M_n|| \leq ||U_n||$ does not hold for every norm, even if we assume that the norm is unitarily invariant and submultiplicative. For example, take $|||C|||_{tr} = \sigma_1(C) + \cdots + \sigma_n(C)$, the *trace norm* of a complex $n \times n$ matrix C [10, p. 441], [11, p. 211]. For $n \leq 8$, a computer search shows that $|||U_n|||_{tr} \leq |||M_n|||_{tr}$ for all tournament matrices M_n. We have been unable to prove this inequality for all n, however.

Theorem 2.2 completed our analysis of the maximum value of $\sigma_1(M)$ for tournament matrices M of order n. The next proposition provides bounds on the minimum value of $\sigma_1(M)$ (and the maximum value of $\sigma_n(M)$). The bounds are easily verified but, unfortunately, are attained only for special orders n.

A tournament matrix M of order $n \geq 2$ is called *doubly regular* if every pair of vertices in the associated tournament jointly dominates the same number of vertices (necessarily, $\frac{n-3}{4}$). It follows that M is doubly regular if and only if $M^TM = \frac{n+1}{4}I + \frac{n-3}{4}J$. Such matrices are also called Hadamard tournament matrices since they are coexistent with skew Hadamard matrices of order $n+1$ [5]. They are the irreducible tournament matrices of order $n \geq 3$ that have precisely 3 distinct eigenvalues [5]. For a doubly regular tournament matrix of order n to exist, it is necessary that $n \equiv 3 \pmod 4$. The converse statement is a classical unsolved problem.

PROPOSITION 2.4. *Let M be a tournament matrix of order n and let $\sigma_1(M)$ and $\sigma_n(M)$ denote the largest and smallest singular values of M, respectively. Then*

(i) $\sigma_1(M) \geq \frac{n-1}{2}$ *with equality holding if and only if M is regular; and,*

(ii) $\sigma_n(M) \leq \frac{\sqrt{n+1}}{2}$ *with equality holding if and only if M is doubly regular.*

Proof. Since M^TM is symmetric, $\sigma_n^2(M) \leq \frac{x^TM^TMx}{x^Tx} \leq \sigma_1^2(M)$ for all $x \neq 0$. If $x = \mathit{1}$ and s denotes the score vector $M\mathit{1}$ then, by the Cauchy-Schwartz inequality,

$$\sigma_1^2 \geq \frac{\sum s_i^2}{n} \geq \left(\frac{\sum s_i}{n}\right)^2 = \left(\frac{n-1}{2}\right)^2. \tag{2.1}$$

If equality holds in (2.1), then the entries of s must all be equal, that is, M must be regular. Conversely, if M is regular, then $M^TM\mathit{1} = (J-I-M)M\mathit{1} = (\frac{n-1}{2})^2\mathit{1}$ and so, by the Perron-Frobenius theorem, $\sigma_1^2 = (\frac{n-1}{2})^2$.

Since the trace of M^TM is the sum of its eigenvalues,

$$(2.2) \qquad (n-1)\sigma_n^2 + \sigma_1^2 \leq \operatorname{tr}(M^TM) = \operatorname{tr}(MM^T) = \sum s_i = \frac{n(n-1)}{2}.$$

Thus, by (2.1),

$$(2.3) \qquad \sigma_n^2 \leq \frac{n}{2} - \frac{\sigma_1^2}{n-1} \leq \frac{n}{2} - \frac{n-1}{4} = \frac{n+1}{4}.$$

Equality holds in (2.3) if and only if it holds in both (2.1) and (2.2), that is, if and only if $\lambda_1(M^TM) = (\frac{n-1}{2})^2$ with multiplicity 1 and $\lambda_n(M^TM) = \frac{n+1}{4}$ with multiplicity $n-1$. If the latter conditions hold then λ_1 is an eigenvalue with associated eigenspace projection $\frac{1}{n}\mathit{1}\mathit{1}^T = \frac{1}{n}J$, while the remaining eigenvalue, λ_n has the orthogonal eigenspace projection $I - \frac{1}{n}J$. Consequently, we have the spectral resolution $M^TM = \frac{\lambda_1}{n}J + \lambda_n(I - \frac{1}{n}J) = \frac{n+1}{4}I + \frac{n-3}{4}J$; that is, M is doubly regular. Conversely, if M is doubly regular, the spectral resolution above implies that the eigenvalues of M^TM are $(\frac{n-1}{2})^2$ with multiplicity 1 and $\frac{n+1}{4}$ with multiplicity $n-1$. $\square$

The *spread*, $\operatorname{sp}(C)$, of an $n \times n$ complex matrix C is defined as $\max|\lambda - \mu|$ where the maximum is taken over all eigenvalues λ, μ of C.

PROPOSITION 2.5. *Let M be a tournament matrix of order n. Then*

(i) $\operatorname{sp}(M^TM) \geq \frac{n(n-3)}{4}$ *with equality holding if and only if M is doubly regular; and,*

(ii) $\operatorname{sp}(M^TM) \leq \frac{1}{4}\csc^2\frac{\pi}{4n-2}$ *with equality holding if and only if M is the matrix of a transitive tournament.*

Proof. Since $\operatorname{sp}(M^TM) = \sigma_1^2(M) - \sigma_n^2(M)$, the proof of (i) follows directly from Proposition 2.4. By Theorem 2.2, $\operatorname{sp}(M^TM) \leq \frac{1}{4}\csc^2\frac{\pi}{4n-2}$ and, because the matrix of a transitive tournament is singular, equality holds if and only if M is the matrix of a transitive tournament. $\square$

The following theorem is proved in [6, p. 26] under the additional assumption that M is irreducible.

THEOREM 2.6. *Let M be a tournament matrix of order $n \geq 4$. Then M has precisely two distinct singular values if and only if M is doubly regular.*

Proof. The sufficiency can be seen from the spectral resolution in the proof of Proposition 2.4(ii).

Suppose now that M has precisely two singular values, that is, suppose that M^TM has precisely two eigenvalues: $\rho = \rho(M^TM)$ and $\lambda_n = \lambda_n(M^TM)$, where $\rho > \lambda_n \geq 0$. We first show that $\lambda_n > 0$ and that the matrix $\widetilde{M} = \begin{bmatrix} O & M \\ M^T & O \end{bmatrix}$ is irreducible. The necessity will then follow from a result in [6].

Recall that if an eigenvalue λ of M has geometric multiplicity 2 or more, then $\operatorname{Re}\lambda = -\frac{1}{2}$ [21]. Since $\operatorname{Nul} M = \operatorname{Nul} M^TM$, it follows that if $\lambda_n = 0$, then λ_n has multiplicity 1 and so the sum of the eigenvalues of M^TM is $(n-1)\rho = \operatorname{tr} M^TM = \frac{n(n-1)}{2}$.

Thus $\rho = \frac{n}{2}$, so $\sigma_1(M) = \sqrt{n/2}$. Since $n \geq 4$, this contradicts Proposition 2.4(i). Thus, $\lambda_n > 0$.

Suppose $\widetilde{M}$ is reducible. Then by [6, Thm. 2.6], either M has a zero row or column or else, by permuting rows and columns if necessary, it may be assumed that there is a tournament matrix M_{n-2} of order $n-2$ such that

$$M = \begin{bmatrix} M_{n-2} & \mathit{1} & \mathit{0} \\ \mathit{0}^T & 0 & 1 \\ \mathit{1}^T & 0 & 0 \end{bmatrix}, \quad \text{and so} \quad M^T M = \begin{bmatrix} M_{n-2}^T M_{n-2} + J & r \\ r^T & n-2 \end{bmatrix} \oplus [1] \ ,$$

where $r = M_{n-2}^T \mathit{1}$ is the score vector of M_{n-2}^T. Since $\lambda_n > 0$, M has no zero row or column and so only the latter case may occur. Because the first summand has at most 2 zero entries and order $n-1 \geq 3$, it must be irreducible and have a Perron value $\rho \geq 2$ with multiplicity 1. Since the remaining eigenvalues all equal 1, $M^T M - I$ has rank 1 and all subdeterminants of order at least 2 must equal zero. The principal 2×2 subdeterminants of $M^T M$ determined by indices $i = 1, \ldots, n-2$ and $n-1$ are $(n-3)r_i - r_i^2 = 0$. Thus, $r_i = 0$ or $n-3$ for each $i = 1, \ldots, n-2$; that is, for each i, either row i or column i of M_{n-2} is zero. Since M_{n-2} has at most one zero row and at most one zero column, M_{n-2} must be a 2×2 tournament matrix. But then it is easily checked that $M^T M$ is a 4×4 matrix with 4 distinct eigenvalues. Thus $\widetilde{M}$ is irreducible.

Because $M^T M$ is nonsingular with exactly two distinct eigenvalues, ρ, λ_n, it follows that $\widetilde{M}$ is nonsingular with exactly four distinct eigenvalues, $\pm\sqrt{\rho}, \pm\sqrt{\lambda_n}$. Also, $\widetilde{M}$ is irreducible. Thus, by [6, Prop. 5.5], M is doubly regular. $\square$

Theorem 2.2 gave the exact maximum of the spectral norms $\sigma_1(M)$ of the tournament matrices M for each order n. Finding the minimum spectral norm for each n is more elusive. We have only been able to do this in the cases where n or $n/2$ is an odd integer. Results on walk spaces that appear in [17] will be used. Because of some sign errors in that paper, we first redevelop the results needed.

3. Walk Spaces and Minimum Spectral Norms. An eigenvector of an $n \times n$ complex matrix A is called *normal* if it is also an eigenvector of A^*. It follows that a nonzero vector $x \in \mathbb{C}^n$ is a normal eigenvector of A if and only if there is a scalar $\lambda \in \mathbb{C}$ such that $Ax = \lambda x$ and $A^* x = \bar{\lambda} x$.

Let M be a tournament matrix of order n. Then $M + M^T = J - I$. The ones vector $\mathit{1}$ is an eigenvector of M if and only if M (and hence M^T) is regular. Thus $\mathit{1}$ is an eigenvector of M if and only if it is a normal eigenvector. If $Mx = \lambda x$, then $(1 + 2\,\mathrm{Re}\,\lambda) x^* x = x^*(M + M^T + I)x = x^* J x = |\mathit{1}^T x|^2$. Also, if $Mx = \lambda x$, then $M^T x = \lambda x$ if and only if $(1 + \lambda + \bar{\lambda})x = (\mathit{1}^T x)\mathit{1}$. Thus if x is a λ-eigenvector of M and x is not a multiple of $\mathit{1}$, then x is normal if and only if $\mathrm{Re}\,\lambda = -\frac{1}{2}$, equivalently, if and only if $\mathit{1}^T x = 0$. Let $N(M)$ be the subspace spanned by the normal eigenvectors x of M with $\mathit{1}^T x = 0$. Then N_M is invariant under multiplication by M and M^T and, consequently, so is its orthogonal complement, $N_M^\perp$. Since the normal eigenvectors in N_M are orthogonal to $\mathit{1}$, it follows that $N_M^\perp \supseteq W_M$ where

$W_M = \mathrm{Span}\{\, M^j \mathit{1} \mid j = 0, 1, 2, \ldots \}$. The subspace W_M, called the *walkspace* of M, has already been examined in [17] and is shown there to be equal to $W_M = N_M^{\perp}$. Thus, $\mathbb{C}^n = W_M \oplus N_M$.

Let $\dim W_M = k$ and let $e^1, e^2, \ldots, e^k$ be the orthonormal basis of W_M obtained by applying the Gram-Schmidt process to the vectors $\mathit{1}, M\mathit{1}, \ldots, M^{k-1}\mathit{1}$. As in [9], it will be convenient to work with the *payoff matrix* $A = \frac{1}{2}(M - M^T)$. Since $A = M - \frac{1}{2}J + \frac{1}{2}I$ and $J\mathit{1}$ is a multiple of $e^1 = \frac{1}{\sqrt{n}}\mathit{1}$, it follows that $e^1, e^2, \ldots, e^k$ may also be obtained by applying the Gram-Schmidt process to the vectors $\mathit{1}, A\mathit{1}, \ldots, A^{k-1}\mathit{1}$. Then, for each $j = 1, \ldots, k+1$, $\mathrm{Span}\{\, \mathit{1}, A\mathit{1}, \ldots, A^{j-2} \,\} = \mathrm{Span}\{\, e^1, e^2, \ldots, e^{j-1} \,\}$. Applying A and including $e^1 = \frac{1}{\sqrt{n}}\mathit{1}$ gives

$$\mathrm{Span}\{\, \mathit{1}, A\mathit{1}, \ldots, A^{j-1}\mathit{1} \,\} = \mathrm{Span}\{\, e^1, Ae^1, \ldots, Ae^{j-1} \,\} \quad \text{for} \quad j = 1, \ldots, k.$$

Thus we may also obtain $e^1, e^2, \ldots, e^k$ by successively using $e^1, Ae^1, \ldots Ae^{k-1}$. Because A is skew-symmetric and e^{j-1} is orthogonal to

$$\mathrm{Span}\{\, e^1, \ldots, e^{j-2} \,\} \supseteq \mathrm{Span}\{\, e^1, Ae^1, \ldots, Ae^{j-3} \,\},$$

we have $(e^i)^T A e^{j-1} = -(e^{j-1})^T A e^i = 0$ for $i = 1, \ldots, j-3$ and for $i = j-1$. Thus, letting $e^0 = \mathit{0}$, we have $e^1 = \frac{1}{\sqrt{n}}\mathit{1}$, and

$$e^j = \frac{1}{\alpha_{j-1}} \left(Ae^{j-1} - ((e^{j-2})^T A e^{j-1}) e^{j-2} \right), \quad j = 2, \ldots, k$$

where $\alpha_{j-1} > 0$ is the norm of the numerator. Then

$$Ae^{j-1} = \alpha_{j-1} e^j + \left((e^{j-2})^T A e^{j-1} \right) e^{j-2},$$

and taking scalar products with e^j gives $\alpha_{j-1} = (e^j)^T A e^{j-1}$ for $j = 2, \ldots, k$. Thus, we have the recursion

$$\text{(3.1)} \qquad e^1 = \frac{1}{\sqrt{n}}\mathit{1} \quad \text{and} \quad e^j = \frac{1}{\alpha_{j-1}}(Ae^{j-1} + \alpha_{j-2} e^{j-2}), \quad j = 2, \ldots, k$$

where $\alpha_{j-1} = \|Ae^{j-1} + \alpha_{j-2} e^{j-2}\|_2$ is the norm of the numerator in (3.1) for $j = 2, \ldots, k$ and $\alpha_k = \|Ae^k + \alpha_{k-1} e^{k-1}\|_2 = 0$. In particular, in terms of the score vector $s = M\mathit{1}$, we have $e^2 = \frac{1}{\alpha_1} Ae^1$ where $Ae^1 = \frac{1}{\sqrt{n}}(s - \frac{n-1}{2}\mathit{1})$ and $\alpha_1^2 = \|Ae^1\|_2^2 = \frac{s^T s}{n} - (\frac{n-1}{2})^2$ is the variance of s.

Let $\widehat{M}$ be the matrix with respect to the basis $e^1, \ldots, e^k$, of the restriction to W_M of the linear transformation corresponding to M. Then $\widehat{M}$ has entries $\widehat{M}_{i,j} = (e^i)^T M e^j$, $1 \le i, j \le k$. Thus $\widehat{M}$ is the following $k \times k$ tridiagonal matrix:

$$\widehat{M} = \begin{bmatrix} \frac{n-1}{2} & -\alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & -\frac{1}{2} & -\alpha_2 & 0 & \cdots & 0 \\ 0 & \alpha_2 & -\frac{1}{2} & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & -\alpha_{k-2} & 0 \\ 0 & \vdots & \ddots & \alpha_{k-2} & -\frac{1}{2} & -\alpha_{k-1} \\ 0 & 0 & \cdots & 0 & \alpha_{k-1} & -\frac{1}{2} \end{bmatrix}.$$

Of course, if an orthonormal basis of eigenvectors of M is used in N_M then, with respect to that basis, the matrix of the restriction to N_M of the linear transformation corresponding to M will be a diagonal matrix with the $n-k$ eigenvalues of M with real part $-\frac{1}{2}$ on the diagonal.

The following proposition provides a lower bound on the spectral norm, $\sigma_1(M)$, of a tournament matrix of order n. When n is odd, it agrees with Proposition 2.4 and the regular tournament matrices are those that give equality. When n is even, it will yield the lower bound in Corollary 3.2 below. In that lower bound, equality holds only in the special case that $n = 2m$ where m is odd.

PROPOSITION 3.1. *Let M be a tournament matrix of order $n \geq 2$ and let*

$$B = \begin{bmatrix} (\frac{n-1}{2})^2 + \alpha_1^2 & \frac{n\alpha_1}{2} \\ \frac{n\alpha_1}{2} & \alpha_1^2 + \frac{1}{4} \end{bmatrix},$$

where α_1^2 is the score variance of M. Then

$$\sigma_1^2(M) \geq \rho(B) = \alpha_1^2 + \frac{1}{8}\left(n^2 - 2n + 2 + n\sqrt{(n-2)^2 + 16\alpha_1^2}\right).$$

Equality holds if and only if M has at least $n-2$ eigenvalues with real part $-\frac{1}{2}$.

Proof. The proposition is easily verified for $n = 2$. We assume then that $n \geq 3$.

By the preceeding discussion, M is unitarily equivalent to the direct sum of a $k \times k$ matrix $\widehat{M}$ and an $(n-k) \times (n-k)$ diagonal matrix N whose diagonal entries are the $n-k$ eigenvalues of M with real part $-\frac{1}{2}$. Thus, M^TM is unitarily equivalent to the direct sum $\widehat{M}^T\widehat{M} \oplus N^TN$. The first summand is shown in Figure 1.

$$\begin{bmatrix} (\frac{n-1}{2})^2 + \alpha_1^2 & -\frac{n\alpha_1}{2} & -\alpha_1\alpha_2 & 0 & \cdots & 0 \\ -\frac{n\alpha_1}{2} & \alpha_1^2+\alpha_2^2+\frac{1}{4} & 0 & -\alpha_2\alpha_3 & \ddots & \vdots \\ -\alpha_1\alpha_2 & 0 & \alpha_2^2+\alpha_3^2+\frac{1}{4} & 0 & \ddots & 0 \\ 0 & -\alpha_2\alpha_3 & 0 & \ddots & \ddots & -\alpha_{k-2}\alpha_{k-1} \\ \vdots & \ddots & \ddots & \ddots & \alpha_{k-1}^2+\alpha_{k-2}^2+\frac{1}{4} & 0 \\ 0 & \cdots & 0 & -\alpha_{k-2}\alpha_{k-1} & 0 & \alpha_{k-1}^2+\frac{1}{4} \end{bmatrix}$$

FIGURE 1. The matrix $\widehat{M}^T\widehat{M}$.

The eigenvalues of N^TN are all of the form $|\lambda|^2 = \frac{1}{4}+\beta^2$ where β is the imaginary part of an eigenvalue λ of M. By Bendixson's theorem [19, p. 140], $\beta^2 \le \frac{1}{8}n(n-1)$. Thus, $|\lambda|^2 \le \frac{1}{8}(n^2-n+2) \le \left(\frac{n-1}{2}\right)^2 \le \rho(M^TM)$, by Proposition 2.4(i). Consequently, $\rho(M^TM) = \rho(\widehat{M}^T\widehat{M})$. If $k = 1$, equivalently, if M is regular, then $\alpha_1 = 0$ and the statement of the lemma is easily checked. Suppose then that $k \ge 2$. Then $\alpha_i > 0$ for $i = 1, \ldots, k-1$ and $\alpha_k = 0$. Examining $D\widehat{M}^T\widehat{M}D$ where D is the $k \times k$ diagonal matrix with diagonal entries $[1,-1,-1,1,1,-1,-1,1,1,\cdots]$, we see that $\widehat{M}^T\widehat{M}$ can be signed so that all of its entries are nonnegative. It follows from the monotone property of the Perron value [1, p. 27] that $\sigma_1^2 = \rho(M^TM)$ is at least as large as the Perron value of the principal submatrix

$$\begin{bmatrix} (\frac{n-1}{2})^2 + \alpha_1^2 & \frac{n\alpha_1}{2} \\ \frac{n\alpha_1}{2} & \alpha_1^2 + \alpha_2^2 + \frac{1}{4} \end{bmatrix},$$

which, in turn, is at least as large as the Perron value of B, with strict inequality if $\alpha_2 > 0$. A routine calculation shows that the Perron value $\rho(B)$ is the lower bound in the statement of the lemma. Since $\alpha_2 = 0$ if and only if $k = 2$, the result follows for $k \ge 2$. □

COROLLARY 3.2. *If M is a tournament matrix of even order $n = 2m$, then*

$$\sigma_1^2(M) \ge \frac{1}{8}\left((n-2)^2 + n\sqrt{(n-2)^2+4}\right).$$

Equality is attained if and only if m is odd, and M is permutation similar to a matrix of the form

$$\begin{bmatrix} R & X \\ J - X^T & S \end{bmatrix},$$

where R and S are regular tournament matrices of order m and X is an $m \times m$ $\{0,1\}$-matrix with constant row and column sums $(m-1)/2$.

Proof. Since n is even, the score vector $s = M\mathit{1}$ must satisfy $|s_i - \frac{n-1}{2}| \ge \frac{1}{2}$ for $i = 1, \ldots, n$. Thus its variance is $\alpha_1^2 = \frac{1}{n}\sum_i(s_i - \frac{n-1}{2})^2 \ge \frac{1}{4}$. Moreover, $\alpha_1^2 = \frac{1}{4}$ if and only if $s_i = \frac{n}{2}$ or $\frac{n-2}{2}$ for each i, that is, if and only if M is almost regular. The lower bound on $\sigma_1^2(M)$ now follows from Proposition 3.1 with $\alpha_1^2 = \frac{1}{4}$.

Suppose now that equality is attained in the lower bound on $\sigma_1^2(M)$. Since M is almost regular, by permuting the rows and columns of M, we may assume that each of the first m rows of M sum to $m-1$ and each of the last m rows sum to m. Let M be partitioned into four $m \times m$ blocks. Then M must have the form in the statement of the theorem where, at the moment, we only know that R and S are tournament matrices. Because M is not regular, Proposition 3.1 implies that M must have $n-2$ eigenvalues with real part $-\frac{1}{2}$, equivalently, $\dim W_M = 2$. Applying M and M^2 to $\mathit{1}_n$, we obtain

$$M\begin{bmatrix} \mathit{1}_m \\ \mathit{1}_m \end{bmatrix} = (m-1)\begin{bmatrix} \mathit{1}_m \\ \mathit{1}_m \end{bmatrix} + \begin{bmatrix} \mathit{0}_m \\ \mathit{1}_m \end{bmatrix}, \quad M^2\begin{bmatrix} \mathit{1}_m \\ \mathit{1}_m \end{bmatrix} = (m-1)M\begin{bmatrix} \mathit{1}_m \\ \mathit{1}_m \end{bmatrix} + \begin{bmatrix} X\mathit{1}_m \\ S\mathit{1}_m \end{bmatrix}$$

Since $\dim W_M = 2$, the vector $M^2\mathit{1}_n$ must be a linear combination of $\mathit{1}_n$ and $M\mathit{1}_n$. Thus the entries of $X\mathit{1}_m$ must all be equal, and the entries of $S\mathit{1}_m$ must all be equal. Therefore, X and S must each have constant row sums and, consequently, so must R and $J - X^T$. Therefore, the tournament matrices R and S must be regular with row sums $(m-1)/2$ and X must have row and column sums $(m-1)/2$. Conversely, it is straightforward to check that if M is a partitioned matrix that satisfies these constraints, then $\dim W_M = 2$, $\alpha_1^2 = \frac{1}{4}$ and equality holds in the bound in Proposition 3.1. $\square$

If $n = 2m$ where m is odd, then the minimum spectral norm for tournament matrices of order n is given by the lower bound in Proposition 3.2. Although we do not know the minimum spectral norm for all cases where $n = 2m$, we will prove in Theorem 3.5 that any tournament matrix of even order that attains the minimum spectral norm must be almost regular. The following corollary to Proposition 3.1 will be needed in the proof.

COROLLARY 3.3. *If M is a tournament matrix of even order n and M is not almost regular, then*

$$\sigma_1^2(M) \geq \rho(B) = \frac{1}{8n}\left(n^3 - 4n^2 + 4n + 16 + n\sqrt{n^4 - 4n^3 + 8n^2 + 32n}\right),$$

where B is the matrix in Proposition 3.1 with $\alpha_1^2 = \frac{1}{4} + \frac{2}{n}$.

Proof. Since n is even and M is not almost regular, the score vector $s = M\mathit{1}$ must satisfy $|s_i - \frac{n-1}{2}| \geq \frac{1}{2}$ for $i = 1, \ldots, n$ and $|s_i - \frac{n-1}{2}| \geq \frac{3}{2}$ for at least one i. Thus $\alpha_1^2 = \frac{1}{n}\sum_i (s_i - \frac{n-1}{2})^2 \geq \frac{1}{4} + \frac{2}{n}$. The corollary now follows from Proposition 3.1. $\square$

We will require the following lemma in the proof of Theorem 3.5.

LEMMA 3.4. *Let R be a regular tournament matrix of odd order m and let*

$$M = \begin{bmatrix} R & \mathit{0} & R^T & \mathit{1} \\ \mathit{1}^T & 0 & \mathit{0}^T & 0 \\ R^T{+}I & \mathit{1} & R & \mathit{0} \\ \mathit{0}^T & 1 & \mathit{1}^T & 0 \end{bmatrix}$$

where $\mathit{1}$ and $\mathit{0}$ are column m-vectors. Then M is an almost regular tournament matrix of order $n = 2(m+1)$, $\dim W_M = 4$, and $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \sqrt{m}$, $\alpha_3 = \frac{1}{2}$.

Proof. Clearly, M is an almost regular tournament matrix. Let $A = \frac{1}{2}(M - M^T)$. Applying the Gram-Schmidt process to the vectors $\mathit{1}, A\mathit{1}, A^2\mathit{1}, A^3\mathit{1}$, recursion (3.1) yields numerator norms $\alpha_1 = \frac{1}{2}, \alpha_2 = \sqrt{mn}, \alpha_3 = \frac{1}{2}$ and orthonormal vectors

$$e^1 = \frac{1}{\sqrt{n}}\begin{bmatrix} \mathit{1} \\ 1 \\ \mathit{1} \\ 1 \end{bmatrix}, \; e^2 = \frac{1}{\sqrt{n}}\begin{bmatrix} -\mathit{1} \\ -1 \\ \mathit{1} \\ 1 \end{bmatrix}, \; e^3 = \frac{1}{\sqrt{mn}}\begin{bmatrix} \mathit{1} \\ -m \\ -\mathit{1} \\ m \end{bmatrix}, \; e^4 = \frac{1}{\sqrt{mn}}\begin{bmatrix} \mathit{1} \\ -m \\ \mathit{1} \\ -m \end{bmatrix}.$$

Since $Ae^4 = -\frac{1}{2}e^3$, it follows that these four vectors are a basis for W_M. $\square$

THEOREM 3.5. *If M is a tournament matrix of even order n with minimum spectral norm, then M is almost regular.*

Proof. Let $n = 2(m+1)$. If m is even, the theorem follows from Corollary 3.2 and the subsequent remark. Also, the theorem is easily checked for $m = 1$. Referring to Corollary 3.3, we see that it is sufficient to show that the squared spectral norm $\sigma_1^2(M)$ of the nearly regular tournament matrix M in Lemma 3.4 is less than the lower bound $\rho(B)$ in Corollary 3.3 for odd $m \geq 3$.

Let M be the tournament matrix of order $n = 2(m+1)$ in Lemma 3.4. By the results in the proof of Proposition 3.1, it follows that $\sigma_1^2(M) = \rho(C)$, where $\rho(C)$ is the Perron value of the matrix

$$C = D\widehat{M}^T\widehat{M}D = \frac{1}{2}\begin{bmatrix} mn+1 & \frac{n}{2} & \sqrt{m} & 0 \\ \frac{n}{2} & n-1 & 0 & \sqrt{m} \\ \sqrt{m} & 0 & n-1 & 0 \\ 0 & \sqrt{m} & 0 & 1 \end{bmatrix}$$

and D is the diagonal matrix with diagonal entries $[1, -1, -1, 1]$. Let $x = \rho(B)$ be the lower bound in Corollary 3.3. We wish to show that $\sigma_1^2(M) < x$ for $n \geq 8$. This will the case if the matrix $xI - C$ is positive definite. Selecting the rows and columns of $xI - C$ in the order $2, 3, 4, 1$, we obtain the matrix

$$xI - \frac{1}{2}\begin{bmatrix} n-1 & 0 & \sqrt{m} & \frac{n}{2} \\ 0 & n-1 & 0 & \sqrt{m} \\ \sqrt{m} & 0 & 1 & 0 \\ \frac{n}{2} & \sqrt{m} & 0 & mn+1 \end{bmatrix}.$$

Since this matrix is permutation similar to $xI - C$, it is sufficient to prove that it is positive definite, equivalently, that its leading principal subdeterminants of orders 1, 2, 3, 4 are all positive. The first 3 of these are easily verified by noting that $x > \frac{n}{2}$. It remains only to show that the full 4×4 determinant is positive. It follows from [10, p. 22] that $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB)$ when A commutes with C. Thus the 4×4 determinant is equal to the determinant of the matrix

$$\left(x - \frac{n-1}{2}\right)\begin{bmatrix} x - \frac{1}{2} & 0 \\ 0 & x - \frac{1}{2}(mn+1) \end{bmatrix} - \frac{1}{8}\begin{bmatrix} n-2 & n\sqrt{m} \\ n\sqrt{m} & \frac{n^2}{2} + n - 2 \end{bmatrix}.$$

Evaluating this determinant, we see that it remains to show that

$$\left(x^2 - \frac{n^2}{4}x + \frac{1}{16}(2n^3 - 7n^2 + 6n)\right)\left(x^2 - \frac{n}{2}x + \frac{n}{8}\right) - \frac{n^2(n-2)}{128} \; > \; 0. \tag{3.2}$$

Because x is a root of the characteristic polynomial of the matrix B in Corollary 3.3,

$$x^2 = \frac{1}{4n}(n^3 - 2n^2 + 4n + 16)x - \frac{1}{16n^2}(n^4 - 4n^3 - 12n^2 + 32n + 64). \tag{3.3}$$

From the expression for $x = \rho(B)$ in Corollary 3.3, it is straightforward to verify that $\frac{1}{4}(n^2 - 2n + 3) \leq x \leq \frac{1}{4n}(n^3 - 2n^2 + 4n + 16)$. Substituting the expression (3.3) for x^2 into each of the two factors in the first term on the left of the desired inequality (3.2), and using these bounds on x in each factor, we find that the left hand side of (3.2) is greater than or equal to

$$\begin{aligned}
&\frac{(10n^3-36n^2-32n+192)}{16n^2}\frac{(n^6-6n^5+14n^4+2n^3-8n^2+12n-64)}{16n^2} - \frac{n^2(n-2)}{128} \\
\geq\ & \frac{1}{256n^4}(10n^3 - 36n^2 - 32n + 192)(n^6 - 6n^5 + 14n^4) - \frac{n^2(n-2)}{128} \\
=\ & \frac{1}{256}(10n^5 - 96n^4 + 322n^3 - 116n^2 - 1600n + 2088).
\end{aligned}$$

The last expression is positive when $n = 8$ and is easily seen to be positive when $n \geq 10$. Thus, if a tournament matrix of even order has minimum spectral norm, then it must be almost regular. □

A computer search shows that the minimum spectral norm of the tournament matrices of order 8 is approximately 3.588 and that the minimum is attained only by tournament matrices that are permutation similar to

$$\left[\begin{array}{cccc|cccc}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{array}\right].$$

By comparison, when $n = 8$, Corollary 3.2 along with the construction in Lemma 3.4 imply only that the minimum spectral norm is between 3.29 and 3.61, approximately.

We do not know which tournament matrices of order n minimize the spectral norm in the cases $n = 12$, $n = 16$. The cases $n = 4m$, m odd, seem tractable, but very difficult. Determining the minimum spectral norm for all n may hinge on the cases $n = 2^k$. Unfortunately, we are unable to suggest a pattern that yields a reasonable conjecture for the cases $n = 2^k$.

Let S_n be the matrix obtained from the upper triangular matrix U_n by replacing u_{1n} by 0 and u_{n1} by 1. Thus, S_n is the matrix of the tournament obtained by reversing the $1 \rightarrow n$ arc of the transitive tournament with Hamilton path $1 \rightarrow 2 \rightarrow \cdots \rightarrow n$. A theorem of Moser [4, p. 220], implies that an n-vector s of nonnegative integers is the score vector of some irreducible tournament matrix of order n if and only if

$s \prec S_n\mathit{1}$. This leads us to conjecture that an irreducible tournament matrix M_n of order n which maximizes $\sigma_1(M_n)$ must be permutation similar to S_n. This conjecture has been verified for $n \leq 8$. By comparison, a recent result resolving a 1983 conjecture of Brualdi and Li [3, Prob. 31(2)] asserts that an irreducible tournament matrix M_n of order n which minimizes $\rho(M_n)$ must be the matrix of a tournament obtained from the transitive tournament by reversing each of the arcs on its Hamilton path. The latter matrix is not permutation similar to S_n for $n \geq 4$, but has the same set of scores.

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