MATH. SCAND. 66 (1990), 73–90

SINGULAR VECTORS CORRESPONDING TO IMAGINARY ROOTS IN VERMA MODULES OVER AFFINE LIE ALGEBRAS

F. MALIKOV

Introduction.

Let $\mathfrak{g}_A = \mathfrak{n}_A^- \oplus \mathfrak{f}_A \oplus \mathfrak{n}_A^+$ be a Kac-Moody algebra corresponding to a symmetrizable Cartan matrix A, $M(\lambda)$ the Verma module corresponding to a functional $\lambda \in \mathfrak{f}_A^*$. Kac and Kazhdan [1] described the set of λ for which $M(\lambda)$ is reducible. This set is the union of countably many hyperplanes $H_{n,\alpha} \in \mathfrak{f}_A^*$ labeled by the pairs (n, α), where n is a positive integer, α a positive root of $\mathfrak{g}_A$. If $\lambda \in H_{n,\alpha}$ then $M(\lambda)$ contains at least one singular (annihilated by $\mathfrak{n}_A^+$) vector of weight $\lambda - n\alpha$. If α is a real root then the singular vector is unique up to a scalar for almost all $\lambda \in H_{n,\alpha}$.

In [2] an explicit formula for this singular vector is found. This solves the problem of finding singular vectors when A is a positive definite matrix. i.e. when $\mathfrak{g}_A$ is a finite-dimensional simple Lie algebra, since in this case all the roots are real.

The next in difficulty case is that of a matrix A with all eigenvalues positive except one which is 0. Such Lie algebras are called *affine* ones and they have a description independent of the general Kac-Moody algebra theory. In the simplest case such an algebra is a one-dimensional nontrivial central extension of the "current" algebra, i.e. the Lie algebras of polynomial functions on the circle with values in a simple finite-dimensional Lie algebra (sometimes, a larger algebra with a derivation added to this extension). A general affine algebra is a subalgebra of such an algebra (see 1.1 for details).

The imaginary (i.e. not real) roots of affine algebras are of the form $m \cdot \theta$, where m is a positive integer, θ the "minimal" imaginary root. The hyperplane $H_{n,m\theta}$ is determined by the equation (in λ)

$$\lambda(c) + g = 0 \tag{1}$$

where c is a fixed central element of our algebra and g a number. Since (1) does not depend on n and m we will hereafter write H_θ for brevity.

Received October 20, 1988; in revised form November 25, 1988

When the affine algebra is the extension of a current algebra a construction of certain singular vectors is given in [2] for the modules $M(\lambda)$ with imaginary λ. The recipe is as follows. Take the Casimir element $\sum_i g_i g^i \in U(\mathfrak{g})$, where $\{g_i\}$ is a basis of $\mathfrak{g}$ and $\{g^i\}$ of its dual, and make it into the infinite series

$$T_n = \tfrac{1}{2} \sum_{k+l=n} :(g_i \otimes t^k)\cdot(g^i \otimes t^l):$$

where $g \otimes t^m$ denotes the function $\varphi \mapsto g \cdot t^m$ on the circle (here $t = \exp(\sqrt{-1}\cdot\varphi)$ with φ the angle parameter) and

$$:(g_i \otimes t^k)\cdot(g^i \otimes t^l): = \begin{cases}(g_i \otimes t^k)\cdot(g^i \otimes t^l) \text{ if } k \leqq l\\ (g^i \otimes t^l)\cdot(g_i \otimes t^k) \text{ if } k > l\end{cases}$$

The series T_n is an element of the appropriately completed algebra $\tilde{U}(\mathfrak{g}_A)$. Such series, though infinite ones, determine operators in $M(\lambda)$ and provided (1) is satisfied they are $\mathfrak{g}_A$-endomorphisms. Therefore, applying these operators to the vacuum (highest weight) vector we find singular vectors in $M(\lambda)$.

The following conjectures are stated in [2]:

1) If $\mathfrak{g} = \mathrm{sl}(2)$ and λ is a generic point on H_θ then this construction gives all the singular vectors of $M(\lambda)$.

2) In general, it is possible to get all the singular vectors of $M(\lambda)$ if we apply the above construction not only to the Casimir element but to any element of the center of $U(\mathfrak{g})$.

In this paper I prove these conjectures not only for the current algebras but for all affine algebras. Moreover, the obtained information on singular vectors is sufficient to describe the structure of $M(\lambda)$ completely if λ is a generic point.

The contents of the paper is as follows. In §1 I give main definitions and formulate the main results. In §2 I prove the main technical result: the formula for the bracket of particular elements of the completed universal enveloping algebra $\tilde{U}(\mathfrak{g}_A)$ with elements from $\mathfrak{g}_A$ (Theorem 1). In §3 the results of §2 are applied to the study of the structure of modules $M(\lambda)$ over the extended current algebras (Theorem 2). In §4 I briefly explain how to generalize the obtained results for an arbitrary affine algebra. Finally, in §5 we study the structure of the algebra generated by the constructed elements of the completed enveloping algebra.

§1. Main definitions and formulation of main results.

1.1. *Main definitions.* Here we will give the main definitions concerning affine Lie algebras and Verma modules over them. The reader interested in the relations with the general theory of Kac-Moody algebras should refer to [3]–[5].

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, σ an automorphism of order

d of the corresponding Dynkin diagram. If we fix a Cartan subalgebra f then σ uniquely determines an automorphism of $\mathfrak{g}$ which we will also denote by σ, and $\mathfrak{g}$ decomposes into the direct sum of eigensubspaces

$$\mathfrak{g} = \bigoplus_{0 \leq j \leq d-1} \mathfrak{g}^{(j)},$$

where $\sigma|_{\mathfrak{g}^{(j)}} = \exp(2\pi\sqrt{-1}j/d) \cdot id$

Consider the current algebra $\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$. (A *current* is a polynomial on a manifold, in our case on the circle, with values in $\mathfrak{g}$ such that to an element $a \otimes t^k$ the function $\exp(k\sqrt{-1}\varphi) \cdot a$ corresponds.) In $\mathfrak{g} \otimes \mathbf{C}[t^{-1}, t]$, distinguish the subalgebra $\mathfrak{g}_\sigma = \bigoplus_{i \in Z} (\mathfrak{g}^{(\mathrm{res}_d i)} \otimes t^i)$, where $\mathrm{res}_d\, i$ is the residue of $i \bmod d$. The linear space of the affine algebra $\hat{\mathfrak{g}}_\sigma$ is $\mathfrak{g}_\sigma \oplus \mathbf{C} \cdot c \oplus \mathbf{C}d$ with the bracket given by the formula

(2)
$$[a \otimes t^n + \alpha \cdot c + \beta \cdot d, a' \otimes t^m + \alpha' \cdot c + \beta' \cdot d] =$$
$$[a, a'] \otimes t^{m+n} + \beta \cdot m \cdot a' \otimes t^m - \beta' \cdot n \cdot a \otimes t^n + \delta_{n,-m} \cdot n(a, a') \cdot c$$

where $(.,.)$ is the invariant scalar product in $\mathfrak{g}$.

COMMENTS. The vector field $d = t\dfrac{d}{dt}$ on the circle determines a natural derivation of the current algebra preserving $\mathfrak{g}_\sigma$. There is a one-dimensional central extension of $\mathfrak{g}_\sigma \oplus \mathbf{C} \cdot d$ with the help of the cocycle φ given by the formula

$$\varphi(a \otimes t^n, a' \otimes t^m) = \delta_{n,-m} \cdot n \cdot (a, a'), \ \varphi(d, a \otimes t^n) = 0$$

and the obtained algebra is $\hat{\mathfrak{g}}_\sigma$. Note that $\hat{\mathfrak{g}}_{\mathrm{id}}$ is the extension of the current algebra. In what follows we will abbreviate $\hat{\mathfrak{g}}_{\mathrm{id}}$ by $\hat{\mathfrak{g}}$.

In $\mathfrak{g}$, fix a Cartan subalgebra f and the corresponding root system $R = \Delta_+ \cup \Delta_-$, where $\Delta_+(\Delta_-)$ is the set of positive (negative) roots. Let $\mathfrak{g} = \mathfrak{n}_- \oplus f \oplus \mathfrak{n}_+$, where $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$ and $\mathfrak{g}_\alpha$ the root subspace corresponding to α, be the corresponding decomposition.

The case of $\hat{\mathfrak{g}}$ is similar. The subalgebra $\hat{f} = f \oplus \mathbf{C} \cdot c \oplus \mathbf{C}d$ is the Cartan subalgebra of $\hat{\mathfrak{g}}$.

In order to define the root system we fix an embedding $f^* \hookrightarrow \hat{f}^*$ extending the functionals on f onto $\mathbf{C} \cdot c \oplus \mathbf{C} \cdot d$ by zero. In $\hat{f}^*$, distinguish the functional θ: $f \oplus \mathbf{C} \cdot c \to 0$, $\mathbf{C}d \to 1$. Let the root system $\hat{\Delta}$ of $\hat{\mathfrak{g}}$ be the set $\hat{\Delta} = \hat{\Delta}_+ \cup \hat{\Delta}_-$, where

$$\hat{\Delta}_\pm = \Delta_\pm \cup \{\alpha \pm m\theta \colon \alpha \in R, m \in \mathbf{N}\} \cup \{\pm m \cdot \theta \colon m \in \mathbf{N}\}.$$

The roots of the form $R + m \cdot \theta$ are called *real*, those of the form $m \cdot \theta$ *imaginary*.

To every root $\beta \in \hat{\Delta}$ the space $(\hat{\mathfrak{g}})_\beta \subset \hat{\mathfrak{g}}$

$$(\hat{\mathfrak{g}})_\beta = \begin{cases} \mathfrak{g}_\alpha \otimes t^m \text{ if } \beta = \alpha + m\theta \\ \mathfrak{f} \otimes t^m \quad \text{if } \beta = m \cdot \theta \end{cases}$$

corresponds. It follows directly from the definition that $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{f}} \oplus \hat{\mathfrak{n}}_+$, where $\hat{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in \hat{\Delta}_\pm} (\hat{\mathfrak{g}})_\alpha$ and $[h, x] = \beta(h)x$ for any $h \in \mathfrak{f}$ and $x \in (\hat{\mathfrak{g}})_\beta$.

Therefore, we have obtained the root decomposition of $\hat{\mathfrak{g}}$. Note that the dimension of the root subspace corresponding to a real root equals 1, that corresponding to an imaginary root equals $\dim \mathfrak{f} = \operatorname{rank} \mathfrak{g}$.

For an arbitrary affine algebra the definitions are a trifle more complicated. In the decomposition $\mathfrak{g} = \bigoplus_{0 \leq j \leq d-1} \mathfrak{g}^{(j)}$ the Lie algebra $\mathfrak{g}^{(0)}$ is simple and $\mathfrak{g}^{(1)}, \ldots, \mathfrak{g}^{(d-1)}$ are irreducible $\mathfrak{g}^{(0)}$-modules (cf. [3], [5]). Fix a Cartan subalgebra $\mathfrak{f}^{(0)} \subset \mathfrak{g}^{(0)}$ (if $\mathfrak{f}$ is the considered Cartan subalgebra of $\mathfrak{g}$ then $\mathfrak{f}^{(0)}$ coincides with the set of σ-invariant elements of $\mathfrak{f}$) and the corresponding sets $\Delta^j \supset \Delta^j_+ \cup \Delta^j_-$, the weight systems of the $\mathfrak{g}^{(0)}$-modules $\mathfrak{g}^{(j)}$ $(j = 1, \ldots, d-1)$. In these notations $\hat{\mathfrak{f}} = \mathfrak{f}^{(0)} \oplus \mathbf{C} \cdot c \oplus \mathbf{C} \cdot d$ is a Cartan subalgebra of $\hat{\mathfrak{g}}_\sigma$. In $\hat{\mathfrak{f}}^*$, the distinguished element θ is given by the formula

$$\theta \colon \mathfrak{f}^{(0)} \oplus \mathbf{C} \cdot c \to 0,\ \mathbf{C}d \to 1.$$

The root system of $\hat{\mathfrak{g}}$ is the set $\hat{\Delta} = \hat{\Delta}_+ \cup \hat{\Delta}_-$, where

$$\hat{\Delta}_\pm = \Delta^0_\pm \cup \{\alpha \pm m \cdot \theta \colon \alpha \in \Delta^{\operatorname{res}_d m}, m \in \mathbf{N}\}$$

The root $\alpha + m \cdot \theta$ is called *real* if $\alpha \neq 0$ and *imaginary* otherwise.

The root subspaces are

$$(\hat{\mathfrak{g}}_\sigma)_{\alpha + m \cdot \theta} = \begin{cases} (\mathfrak{g}^{(\operatorname{res}_d m)})_\alpha \otimes t^m \text{ if } \alpha \neq 0 \\ \mathfrak{f}^{(\operatorname{res}_d m)} \otimes t^m \quad \text{if } \alpha = 0 \end{cases}$$

where, as usual, $\mathfrak{f}^{(j)} \subset \mathfrak{f}$ is the eigenspace of σ corresponding to the eigenvalue $\exp(2\pi\sqrt{-1}\, j/d)$. Note that, as for the current algebras, the dimension of the root subspace corresponding to a real root equals 1, that corresponding to an imaginary root equals $\dim \mathfrak{f}^{(\operatorname{res}_d m)}$. We have the direct sum decomposition $\hat{\mathfrak{g}}_\sigma = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{f}} \oplus \hat{\mathfrak{n}}_+$, where $\hat{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in \hat{\Delta}_\pm} (\hat{\mathfrak{g}}_\sigma)_\alpha$.

The above decomposition enables us to define Verma modules over $\hat{\mathfrak{g}}_\sigma$. Let $\lambda \in \hat{\mathfrak{f}}^*$. Then the Verma module $M(\lambda)$ is generated by one vector v_λ, called the *vacuum* vector, such that

$$\hat{\mathfrak{n}}_+ \cdot v_\lambda = 0,\ \ h \cdot v_\lambda = \lambda(h)\, v_\lambda \text{ for } h \in \hat{\mathfrak{f}}.$$

In $M(\lambda)$, we introduce a graded $\hat{\mathfrak{g}}_\sigma$-module structure: Let $Q(Q_+)$ be the set of linear combinations of positive roots with (non-negative) integer coefficients. It follows from the definition that

$$M(\lambda) = \bigoplus_{\eta \in Q} M(\lambda)_\eta,$$

where $M(\lambda)_\eta = \{v \in M(\lambda)\colon h \cdot v = (\lambda - \eta)(h) \cdot v \text{ for } h \in \hat{\mathfrak{f}}\}$.

A non-zero element of $M(\lambda)$ is called a *singular vector of weight* μ if

$$\hat{\mathfrak{n}}_+ \cdot v = 0,\ h \cdot v = \mu(h) \cdot v \text{ for } h \in \hat{\mathfrak{f}}.$$

If v is a singular vector of weight $\mu \neq \lambda$ then the formula $\varphi(a \cdot v_\mu) = a \cdot v$ determines an embedding $\varphi\colon M(\mu) \to M(\lambda)$. Clearly, $v_\lambda \notin \varphi(M(\mu))$ since $v_\lambda \notin \bigoplus_{\beta \in Q_+} M(\lambda)_{\lambda - \mu + \beta}$ and $\varphi(M(\mu)) \in \bigoplus_{\beta \in Q_+} M(\lambda)_{\lambda - \mu + \beta}$ and therefore $M(\lambda)$ is reducible.

1.2. *The main construction.* The completion $\tilde{U}(\hat{\mathfrak{g}}_\sigma)$ of $U(\hat{\mathfrak{g}}_\sigma)$ formed by the series of the form $\sum_{x_1 + \ldots + x_k = n, x_1 \leqq \ldots \leqslant x_k} e_1 \otimes t^{x_1} \cdot \ldots \cdot e_k \otimes t^{x_k}$, where $e_i \otimes t^{x_i} \in \hat{\mathfrak{g}}_\sigma$, acts on Verma modules. Indeed, it is not difficult to see that for an arbitrary vector from $M(\lambda)$ only a finite number of terms of such series do not act by zero.

Denote $x'_1 \leqq \ldots \leqq x'_m$ the decomposition of n dual to a decomposition $x_1 \leqq \ldots \leqq x_k$. On the set of decompositions, introduce a function ε setting

$$\varepsilon(x_1, \ldots, x_k) = [(x'_m - x'_{m-1})! \cdot \ldots \cdot (x'_2 - x'_1)!\, x'_1!]^{-1}$$

Clearly, $k!\,\varepsilon(x_1, \ldots, x_k)$ is the number of different permutations of the set $\{x_1, \ldots, x_k\}$.

Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$. An automorphism σ of $\mathfrak{g}$ extends naturally onto $T(\mathfrak{g})$. Let $T(\mathfrak{g}) = \bigoplus_{0 \leqslant k \leqq d-1} T^{(k)}(\mathfrak{g})$ be the decomposition into eigenspaces with respect to σ.

We proceed to define linear maps

$$\Psi_n\colon\ T^{(\mathrm{res}_d n)}(\mathfrak{g}) \to \tilde{U}(\hat{\mathfrak{g}}_\sigma),\ n \in \mathbb{Z}.$$

For this note that $T^{(\mathrm{res}_d n)}$ consists of linear combinations of monomials of the form $e_1 \otimes \ldots \otimes e_m$, where $e_j \in \mathfrak{g}^{(i_j)}$ and $i_1 + \ldots + i_m \equiv n \bmod d$. On the monomials of this form set

$$\Psi_n(e_1 \otimes \cdots \otimes e_m) = \tag{3}$$

$$= \sum_{x_1 \ldots + x_m = n, x_1 \leqq \ldots \leqslant x_m, x_j \equiv i_j} \varepsilon(x_1, \ldots, x_m)\, e_1 \otimes t^{x_1} \cdot \ldots \cdot e_m \otimes t^{x_m}$$

For the current algebras ($\sigma = \mathrm{id}$) the common domain of the operators Ψ_n coincides with the whole $T(\mathfrak{g})$. Notice that the restrictions of Ψ_n onto $S(\mathfrak{g})$ coincide with the operators $\tilde{\Psi}_n$ given with the help of the normal ordering

$$\Psi_n(e_1 \otimes \cdots \otimes e_m) = \frac{1}{m!} \sum_{x_1 + \cdots + x_m = n, x_j \equiv i_j} :e_1 \otimes t^{x_1} \cdot \ldots \cdot e_m \otimes t^{x_m}:$$

where

$$:e_1 \otimes t^{x_1} \cdot \dots \cdot e_m \otimes t^{x_m}: = e_{\tau(1)} \otimes t^{x_{\tau(1)}} \cdot \dots \cdot e_{\tau(m)} \otimes t^{x_{\tau(m)}}$$

and τ is the permutation of minimal length satisfying $x_{\tau(i)} \leqq x_{\tau(j)}$ for $i < j$. The equation $\Psi_n|_{S(\mathfrak{g})} = \tilde{\Psi}_n|_{S(\mathfrak{g})}$ follows immediately from the combinatorial interpretation of the funcion ε given above.

Since this construction is very important, we will give a third formula for the operators Ψ_n when $\hat{\mathfrak{g}}_\sigma/\mathbf{C}\cdot c$, $U(\hat{\mathfrak{g}}/\mathbf{C}\cdot c)$ and $\tilde{U}(\hat{\mathfrak{g}}/\mathbf{C}\cdot c)$ will be considered. (We will need this formula in §2.)

Set

$$\tilde{\tilde{\Psi}}_n(e_1 \otimes \cdots \otimes e_m) = \lim_{N\to\infty} \sum_{x_1+\cdots+x_m=n,\, x_i<N,\, x_j\equiv i_j} e_1 \otimes t^{x_1} \cdot \dots \cdot e_m \otimes t^{x_m}$$

As above, a simple corollary of the combinatorial interpretation of ε is the identity $\Psi_n|_{S(\mathfrak{g})} = \tilde{\tilde{\Psi}}_n|_{S(\mathfrak{g})}$

1.3. *Main results.* If Ω is the quadratic generator of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ then for $\sigma = \mathrm{id}$ we have ([4], [5] Exercise 12.11):

$$[\Psi_n(\Omega), e \otimes t^m] = m(c+g)\,\Psi_{n+m}\left(\frac{\partial \Omega}{\partial e}\right) \tag{4}$$

$$[\Psi_n(\Omega), \Psi_m(\Omega)] = (c+g)[(m-n)\Psi_{n+m}(\Omega) + \frac{\dim \mathfrak{g}}{12}\delta_{n,-m}(m^3 - m)c] \tag{5}$$

where $\dfrac{\partial f}{\partial e} = (e, f)$ and g is a number.

If $\sigma \neq \mathrm{id}$ then (4) still holds whereas (5) is replaced by

$$[\Psi_n(\Omega), \Psi_m(\Omega) = \tag{5'}$$

$$= (c+g)[(m-n)\Psi_{n+m}(\Omega) + \delta_{n,-m} \sum_{0<j<n-j} j(n-j)\dim \mathfrak{g}^{(\mathrm{res}_d(n-j))}c]$$

It follows from (5) and (5′) that the action of the operators $\Psi_n(\Omega)$ on $M(\lambda)$ determines a representation of the Virasoro algebra if $\lambda(c) \neq -g$ and from (4) that they constitute a commuting family of $[\hat{\mathfrak{g}}_\sigma, \hat{\mathfrak{g}}_\sigma]$ -endomorphism if $\lambda(c) = -g$.

It turns out that (4), and therefore the above statement, holds for an arbitrary element from $S(\mathfrak{g})^{\mathfrak{g}}$.

THEOREM 1. *Let* $p \in S(\mathfrak{g})^{\mathfrak{g}}$. *Then*

$$[d, \Psi_n(p)] = n\,\Psi_n(p),\ [\Psi_n(p), e \otimes t^m] = m(c+g)\,\Psi_{n+m}\left(\frac{\partial p}{\partial e}\right) \tag{6}$$

Theorem 1 being applied to $M(\lambda)$ with $\lambda(c) + g = 0$ gives the following construction of singular vectors. Let $p_1, \ldots, p_{\operatorname{rank}\mathfrak{g}}$ be algebraically independent generators of $S(\mathfrak{g})^{\mathfrak{g}}$ which can be chosen as eigenvectors with respect to σ, i.e. $\sigma(p_j) = \exp(2\pi \cdot \sqrt{-1}\, m_j/d)p_j$.

Denote $X_{ij} = \Psi_i(p_j)$, where $i < 0$ and i, j agree in a natural way: $i \equiv m_j (\operatorname{mod} d)$. If v_λ is the vacuum vector of $M(\lambda)$ then $X_{ij} \cdot v_\lambda$ is a singular vector of weight $\lambda + i \cdot \theta$ (as follows directly from definitions and (6)).

As we have shown in 1.2, the module $M(\lambda)$ is reducible and $\lambda \in H_\theta$, and therefore g from (6) coincides with g in equation (1) of the hyperplane H_θ (see Introduction).

This construction of singular vectors in the modules $M(\lambda)$ with $\lambda \in H_\theta$ can be generalized. Notice that if $\lambda \in H_\theta$ then $\lambda + i \cdot \theta \in H_\theta$. Indeed, $(\lambda + i \cdot \theta)(c) = \lambda(c) + i \cdot \theta(c) = -g$. Therefore, the vector $X_{rs} \cdot X_{ij} \cdot v_\lambda$ is also singular of weight $\lambda + (r + i)\theta$.

It follows from (6) that the polynomial ring in countably many variables $\mathbb{C}[\ldots, X_{ij}, \ldots]$, which hereafter will be denoted by $\mathbb{C}[X]$, acts in $M(\lambda)$ for $\lambda \in H_\theta$. Set $\deg X_{ij} = i$ and let $\mathbb{C}[X]^{(m)}$ be the space of homogeneous (with respect to this grading) polynomials of degree m. If $q \in \mathbb{C}[X]^{(m)}$ then $q \cdot v_\lambda$ is a singular vector of weight $\lambda + m \cdot \theta$.

It turns out that if λ is a generic point of H_θ then there are no more singular vectors in $M(\lambda)$. The exact formulation is as follows.

THEOREM 2. *Let $\lambda \in H_\theta$ such that $\lambda \notin \bigcup_{n \geq 1} \bigcup_{\alpha \in \hat{\Delta}_+ \setminus \mathbb{Z} \cdot \theta} H_n$, let v_λ be the vacuum vector of $M(\lambda)$, and let $N(\lambda)$ be the maximal proper submodule of $M(\lambda)$. Then*

1) $N(\lambda)$ is generated by the vectors $X_{ij} \cdot v_\lambda$, where $j \in \{1, 2, \ldots, \operatorname{rank}\mathfrak{g}\}$, $i \in \{m_j - d, m_j - 2d, \ldots\}$.

2) Any singular vector of $M(\lambda)$ of weight $\lambda + m - \theta$ is of the form $q \cdot v_\lambda$ for some $q \in \mathbb{C}[X]^{(m)}$.

COROLLARY 1. *Any submodule of $M(\lambda)$ is generated by singular vectors if λ satisfies the conditions of Theorem 2.*

Let $\Lambda^*[\ldots, \Xi_{ij}, \ldots]$ be the exterior algebra of the space $\bigoplus_{i,j} \mathbb{C} \cdot X_{ij}$. In $\Lambda^*[\ldots, \Xi_{ij}, \ldots]$, introduce a $\hat{\mathfrak{g}}_\sigma$-module structure setting

$$g \cdot \Xi_{i_1 j_1} \wedge \ldots \wedge \Xi_{i_k j_k} = \sum_{1 \leq m \leq k} \Xi_{i_1 j_1} \wedge \ldots \wedge g \cdot \Xi_{i_m j_m} \wedge \ldots \wedge \Xi_{i_k j_k}$$

where

$$g \cdot \Xi_{ij} = \begin{cases} 0 \text{ if } g \in \hat{\mathfrak{n}}_\pm \\ i \cdot \theta(g) \text{ if } g \in \hat{\mathfrak{f}} \end{cases}$$

COROLLARY 2. *For the irreducible module $L(\lambda) = M(\lambda)/N(\lambda)$, if λ satisfies the conditions of Theorem 2 then the following sequence of $\hat{\mathfrak{g}}_\sigma$-modules is exact:*

$$0 \leftarrow L(\lambda) \leftarrow M(\lambda) \overset{\partial_1}{\longleftarrow} \Lambda^1[\Xi] \otimes M(\lambda) \leftarrow \dots \overset{\partial_k}{\longleftarrow} \Lambda^k[\Xi] \otimes M(\lambda) \leftarrow \dots$$

where

$$\partial_k(\Xi_{i_1 j_1} \wedge \dots \wedge \Xi_{i_k j_k} \otimes v_\lambda) =$$

$$= \sum_{1 \leq m \leq k} (-1)^m \Xi_{i_1 j_1} \wedge \dots \wedge \hat{\Xi}_{i_m j_m} \wedge \dots \wedge \Xi_{i_k j_k} \otimes X_{i_m j_m} \cdot v_\lambda$$

Corollary 2 enables us to write easily the formal character of the module $L(\lambda)$ over an arbitrary affine algebra. For simplicity let us prove the corresponding formula for current algebras. Recall that if a module V is a direct sum of finite dimensional weight subspaces $V = \bigoplus_\alpha V_\alpha$ then its *formal character* is $\operatorname{ch} V = \sum_\alpha \dim V_\alpha \cdot e^\alpha$, where e^α is an element of the group algebra of $\mathfrak{h}^*$. For example,

$$\operatorname{ch} M(\lambda) = e^\lambda \prod_{\alpha \in \Delta_t} (1 - e^{-\alpha})^{-\dim(\hat{\mathfrak{g}})_\alpha}.$$

To write the formula for $\operatorname{ch} L(\lambda)$ explicitly denote by $p(n,k)^{(l)}$ the number of representations of a number n in the form of the sum of k different positive numbers of l "colours", where the numbers may differ either in value or in colour. It follows from Corollary 2 that

$$\text{(7)} \qquad \operatorname{ch} L(\lambda) = e^\lambda \sum_{m \geq 1} e^{-m \cdot \theta} \sum_{k \geq 1} (-1)^k p(m,k)^{(\operatorname{rank} \mathfrak{g})} \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{-\dim(\mathfrak{g})_\alpha}$$

From the identity

$$\sum_{1 \leq m,k < \infty} (-1)^k p(m,k)^{(l)} t^m = \prod_{i \geq 1} (1 - t^i)^l$$

which is subject to a straightforward verification we deduce

$$\operatorname{ch} L(\lambda) = e^\lambda \prod_{\alpha \in \Delta_+ \setminus \mathbb{N} \cdot \theta} (1 - e^{-\alpha})^{-\dim(\mathfrak{g})_\alpha}$$

which is the desired formula.

Similarly, Corollary 2 enables us to extend (7) for an arbitrary affine algebra which implies that (8) holds in general case.

Kac and Kazhdan conjectured this formula for an arbitrary Kac-Moody algebra for $\lambda = -\rho$, where ρ is a particular element of $\mathfrak{h}_A^*$ (see [1] and also formula (10) from §2 below which implies that $-\rho$ satisfies the conditions of Theorem 2).

§2. Proof of Theorem 1.

First let us derive some consequences of calculations. We are interested in an explicit form of $[\Psi_n, e \otimes t^m]$ for $p \in S(\mathfrak{g})^{\mathfrak{g}}$. The algebra $\tilde{U}(\hat{\mathfrak{g}}_\sigma)$ inherits the filtration from $U(\hat{\mathfrak{g}}_\sigma)$ thanks to Poincaré-Birkhoff-Witt theorem. Obviously, the highest term of this bracket with respect to this filtration equals $-\Psi_{n+m}(\operatorname{ad} e(p))$ and therefore vanishes if $p \in S(\mathfrak{g})^{\mathfrak{g}}$. Besides, we have $[\Psi_n(p), e \otimes 1] = 0$ for $m = 0$. Indeed, let $(x_1, \ldots, x_k)$ be the multidegree of $e_{i_1} \otimes t^{x_1} \cdot \cdots \cdot e_{i_k} \otimes t^{x_k}$. We can rewrite $\Psi_n(p)$ in the form $\Psi_n(p) = \sum_{(x_1,\ldots,x_k)} \Psi_n^{(x_1,\ldots,x_k)}(p)$, where $\Psi_n^{(x_1,\ldots,x_k)}$ is the sum of all the monomials of multidegree $(x_1, \ldots, x_k)$. Clearly, the image of $\Psi_n^{(x_1,\ldots,x_k)}: S(\mathfrak{g}) \to U(\hat{\mathfrak{g}}_\sigma)$ is closed with respect to the adjoint action of $\mathfrak{g} \subset \hat{\mathfrak{g}}_\sigma$ (but not the whole $\hat{\mathfrak{g}}_\sigma$) and the operator $\Psi_n^{(x_1,\ldots,x_k)}$ itself is a $\mathfrak{g}$-module morphism. Therefore,

$$[\Psi_n(p), e \otimes 1] = \sum_{(x_1,\ldots,x_k)} [\Psi_n^{(x_1,\ldots,x_k)}(p), e \otimes 1] =$$

$$= \sum_{(x_1,\ldots,x_k)} \Psi_n^{(x_1,\ldots,x_k)}(-\operatorname{ad} e(p)) = 0.$$

If $m \neq 0$ then the situation is more complicated since $\operatorname{ad} e \otimes t^m$ changes the multidegree. However, direct calculations establish the validity of the following lemma.

LEMMA 1. *Let $\pm\omega$ be the highest (lowest) weight of the $\mathfrak{g}$-module $\mathfrak{g}$ and $e_{\pm\omega}$ the corresponding vectors. Let $\{e_i\}$ be a basis of $\mathfrak{g}$ and $\{e^i\}$ the dual basis, e an arbitrary element of $\mathfrak{g}$, $p \in S(\mathfrak{g})^{\mathfrak{g}}$, $\hat{\mathfrak{g}}_\sigma = \hat{\mathfrak{g}}$.*
Then

1) $[\Psi_n(p), e \otimes t^m] = m\left(c\,\Psi_{n+m}\left(\dfrac{\partial p}{\partial e}\right) + \sum_{i \leq i \leq \deg p} \alpha_{n,m}^{(i)} \Psi_{n+m}(q_i)\right)$

where $q_i \in S^{\deg p - i - 1}(\mathfrak{g})$, $\alpha_{n,m}^{(i)} \in \mathbf{C}$;

2) $q_1 = \sum_i \operatorname{ad}[e_i, e]\left(\dfrac{\partial p}{\partial e^i}\right)$;

3) *if $e = e_{\pm\omega}$ then q_i is the highest (lowest) weight vector of $S^{\deg p - i - 1}(\mathfrak{g})$ of weight $\pm\omega$.*

It follows from Lemma 1 that $[\Psi_n(p), e \otimes t^m]$ is a degree 1 polynomial in e and the coefficient of e equals $\Psi_{n+m}\left(\dfrac{\partial p}{\partial e}\right)$.

Since it remains to calculate the constant term, we will assume hereafter in this section that we consider the quotients of the considered algebras modulo the ideal generated by c. In particular, (4) implies that the series $T_n = \Psi_n(\Omega)$ constitute the algebra $\mathscr{L}$ of vector fields on the circle. This algebra naturally acts in the spaces F_λ of generalized λ-densities, i.e. tensor fields of the form $f(z)^{-\lambda}$ with $f \in \mathbf{C}[z^{-1}, z]$, $\lambda \in \mathbf{C}$ by the formula $T_m f_n = (n - \lambda - \lambda m) f_{n+m}$, where $f_n = z^n dz^{-\lambda}$.

LEMMA 2. *If $p \in S^k(\mathfrak{g})$ then the space spanned by $\Psi_n(p)$ $(n \in \mathbf{Z})$ is closed under the action of T_m $(m \in \mathbf{Z})$ and therefore is an $\mathscr{L}$-module. This module is isomorphic to $\mathscr{F}_{k-1}$.*

PROOF. Let $p = \sum p_{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}$ $(p_{i_1 \cdots i_k} \in \mathbf{C}, e_i \in \mathfrak{g})$.
The remark at the end of 1.2 implies that

$$[T_m, \Psi_n(p)] = [T_m, \tilde{\tilde{\Psi}}_n(p)] =$$

$$\lim_{N \to \infty} \sum_{x_1 + \cdots + x_k = n, x_i \leq N} \frac{1}{k!} \sum_{i_1, \ldots, i_k} [T_m, e_{i_1} \otimes t^{x_1} \cdot \cdots \cdot e_{i_k} \otimes t^{x_k}] =$$

$$\lim_{N \to \infty} \sum_{x_1 + \cdots + x_k = n, x_i \leq N} \frac{1}{k!} \sum_{i_1, \ldots, i_k} p_{i_1 \cdots i_k} \sum_{1 \leq j \leq k} x_j e_{i_1} \otimes t^{x_1} \cdot \cdots \cdot e_{i_j} \otimes t^{x_j + m} \cdot \cdots \cdot e_{i_k} \otimes t^{x_k} =$$

$$\lim_{N \to \infty} \frac{1}{k!} \sum_{y_1 + \cdots + y_k = n + m, y_i \leq N} \sum_{i_1, \ldots, i_k} \cdot (y_1 - m + y_2 - m + \cdots + y_k - m) e_{i_1} \otimes t^{y_1} \cdot \cdots \cdot e_{i_k} \otimes t^{y_k} =$$

$$(n - (k-1)m) \cdot \Psi_{n+m}(p).$$

Therefore the map $\Psi_n(p) \to z^{n+k-1} dz^{-k+1}$ determines an $\mathscr{L}$-module isomorphism.

Let us prove Theorem 1. Let us start with the extended current algebra. Lemma 2, 1) of Lemma 1 and the Jacobi identity imply that the maps φ_i: $\mathscr{F}_{k-1} \otimes \mathscr{F}_0 \to \mathscr{F}_{k-i-1}$, $\varphi_i(\Psi_n(p) \otimes (e \otimes t^m)) \mapsto \alpha_{n,m}^{(i)} \cdot \Psi_{n+m}(q_i)$ are $\mathscr{L}$-module homomorphisms . It is shown in [6] that any $\mathscr{L}$-module homomorphism $\mathscr{F}_{\lambda_1} \otimes \mathscr{F}_{\lambda_2} \to \mathscr{F}_{\lambda_3}$ is an invariant differential operator and the classification of such operators obtained by Grozman [9] implies that

$$\alpha_{n,m}^{(i)} = \begin{cases} 0 & \text{if } i \geq 2 \\ z \cdot m & \text{if } i = 1 \end{cases}$$

Therefore

$$[\Psi_n(p), e \otimes t^m] = m \left[c \, \Psi_{n+m}\left(\frac{\partial p}{\partial e}\right) + \Psi_{n+m}(q_1) \right]$$

It only remains to prove that

$$q_1 = z \frac{\partial p}{\partial e} \tag{9}$$

since applying $\Psi_n(p)$ to the vacuum vector of $M(\lambda)$, where $\lambda(c) = -z$, we get a singular vector which is only possible for $z = g$.

Let $e = e_{\pm\omega}$ (see Lemma 1). In this case to prove (9) let us recall that thanks to [7] we can take as generators of the algebra $S(\mathfrak{g})^{\mathfrak{g}}$ the polynomials $p_1, \dots, p_{\operatorname{rank}\mathfrak{g}}$ such that $\frac{\partial p_i}{\partial e} \in H$, where H is the space of harmonic polynomials. In particular, in the space of harmonic polynomials of any fixed degree there exists a unique up to a constant factor highest vector of weight $\pm\omega$, where ω is the highest weight of $\mathfrak{g}$. Thanks to heading 2 of Lemma 1, q_1 is harmonic and by heading 3 of Lemma 1 it is the highest vector of weight $\pm\omega$ and therefore proportional to $\frac{\partial p}{\partial e_{\pm\omega}}$. Therefore, the second of the formulas (6) is proved for $p \in \{p_1, \dots, p_{\operatorname{rank}\mathfrak{g}}\}$ if $e = e_{\pm\omega}$ and $\hat{\mathfrak{g}}_\sigma = \hat{\mathfrak{g}}$. To prove (6) for an arbitrary $p \in S(\mathfrak{g})^{\mathfrak{g}}$ and $e \in \mathfrak{g}$ for the extended current algebra we have to make use of the following obvious facts:

1) $\Psi_n(p_{i_1} \cdot \dots \cdot p_{i_k}) = \sum_{j_1 + \dots + j_k = n} \frac{1}{k!} :(\Psi_{j_1}(p_{i_1}) \cdot \dots \cdot \Psi_{j_k}(p_{i_k})): + \Psi_n(p)$

where $p \in S(\mathfrak{g})^{\mathfrak{g}}$ and $\deg p < \sum_{1 \le j \le k} \deg p_{i_j}$;

2) The space $\mathfrak{g} \oplus \mathbf{C} \cdot (e_\omega \otimes \mathbf{C}(e_{-\omega} \otimes t)$ generates $\hat{\mathfrak{g}}$.

Thus, Theorem 1 is proved for extended current algebras.

The second of the formulas (6) becomes obvious for an arbitrary affine algebra if we recall that $\hat{\mathfrak{g}}_\sigma \subset \hat{\mathfrak{g}}$. The first of the formulas (6) is an obvious corollary of definitions.

§3. Proof of Theorem 2. The case of extended current algebra.

Recall the main facts on the structure of Verma modules. If V is a submodule of $M(\lambda)$ then $V = \bigoplus_{\alpha \in Q_+} V_\alpha$, where $V_\alpha = M(\lambda)_\alpha \cap V$. Obviously, $V \neq M(\lambda)$ implies $V_0 = \{0\}$. Therefore, in $M(\lambda)$, there exists a maximal (perhaps zero) proper submodule $N(\lambda)$.

Let us define the Shapovalov form on $M(\lambda)$ (cf. [1]). Let $\omega: \mathfrak{g} \to \mathfrak{g}$ be the Cartan antiautomorphism of a simple finite-dimensional Lie algebra. Set

$$\hat{\omega}: \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}, \ \hat{\omega}|_{\mathfrak{f}} = \mathrm{id}, \ \hat{\omega}(e \otimes t^m) = \omega(e) \otimes t^{-m}.$$

Then $\hat{\omega}$ is also an antiautomorphism and $\hat{\omega}((\hat{\mathfrak{g}})_\alpha) = (\hat{\mathfrak{g}})_{-\alpha}$. The Shapovalov form is the bilinear functional $F(\lambda)(.,.)$ on $M(\lambda)$ given by the formulas

$$F(\lambda)(v_\lambda, v_\lambda) = 1, \ F(\lambda)(v_\lambda, M(\lambda)_\beta) = 0 \text{ if } \beta \neq 0,$$

$$F(\lambda)(a \cdot v_\lambda, b \cdot v_\lambda) = F(\lambda)(v_\lambda, \hat{\omega}(a) b v_\lambda)$$

$$F(\lambda)(M(\lambda)_\alpha, M(\lambda)_\beta) = 0 \text{ for } \alpha \neq \beta$$

and clearly the Shapovalov form is symmetric. Let $F_\eta(\lambda)$ be the restriction of the Shapovalov form onto $M(\lambda)_\eta$. Obviously, $N(\lambda)_\eta = \operatorname{Ker} F_\eta(\lambda)$. Kac and Kazhdan calculated the determinant $\det F_\eta(\lambda)$ of the Shapovalov form ([1]) and therefore found the conditions for reducibility of Verma modules.

THEOREM (Kac-Kazhdan).

$$\det F_\eta(\lambda) = \prod_{n=1}^{\infty} \prod_{\alpha \in \hat{\Delta}_+} \left(\lambda(h_\alpha) + \rho(h_\alpha) - n\frac{(\alpha,\alpha)}{2} \right)^{\dim \hat{\mathfrak{g}}_\alpha \cdot P(\eta - n\alpha)} \tag{10}$$

where P is the Kostant function ($P(\eta)$ equals the number of representations of a weight η as the linear combination of positive roots with non-negative integer coefficients).

We will not comment on formula (10) any more since in what follows we will only need the following statement. If $H_{n,\alpha}$ is the hyperplane given by the equation $\lambda(h_\alpha) + \rho(h_\alpha) - n^{(\alpha,\alpha)/2} = 0$ then the hyperplane $H_{n,m\theta}$ (θ an imaginary root) is given by equation (1), i.e. coincides with H_θ, and $H_\theta \setminus \bigcup_{n \geq 1} \bigcup_{\alpha \in \hat{\Delta}_+ \setminus \mathbf{N}\cdot\theta} H_{n,\alpha}$ is dense in H_θ.

Fix a functional $z \in \hat{f}^*$ so that $z(h_\alpha) \neq 0$ for all $\alpha \in \hat{\Delta}_+$ and draw the straight line $\pi = \{\lambda(\tau) : \lambda(\tau) = \lambda + \tau \cdot z, \tau \in \mathbf{C}\}$ through $\lambda \in \hat{f}^*$.

Consider the family of Verma modules $M(\lambda(\tau))$ with $\lambda(\tau) \in \pi$. Since all the Verma modules are isomorphic as linear spaces (indeed, $M(\lambda) \approx V(\hat{\mathfrak{n}}_-)$), there is a map

$$D: M(\lambda) \times M(\lambda) \to \mathbf{C}(\tau), \; D(a,b) = F(\lambda(\tau))(a,b)$$

In $M(\lambda)$, introduce the Jantzen filtration ([1]) $M(\lambda) = M^0 \supset M^1 \supset M^2 \supset \dots$ setting

$$M^i = \{v \in M(\lambda) : D(v,w) \in \tau^i \mathbf{C}[\tau] \text{ for all } w \in M(\lambda)\}$$

Obviously, $N(\lambda) = M^1$.

Let $\operatorname{ord}(Q(\tau))$ be the maximal s such that τ^s divides $Q(\tau)$. Then it is easy to see that ([1])

$$\operatorname{ord}(\det F_\eta(\lambda)) = \sum_{i \geq 1} \dim (M^i)_\eta \tag{11}$$

Fix $\lambda \in H_\theta$. As was shown in 1.3 the polynomial ring in countably many variables $\mathbf{C}[X]$ acts in $M(\lambda)$ which enables us to introduce one more filtration $M(\lambda) = {}^0M \supset {}^1M \supset {}^2M \supset \dots$

Let ${}^r\mathbf{C}[X]$ be the space of homogeneous polynomials of degree r, where $\deg' X_{ij} = 1$ (do not confuse with the grading introduced in 1.3). Set ${}^rM = \sum_{i \geq r} {}^i\mathbf{C}[X] \cdot M(\lambda)$.

We have

$$ {}^iM \subset M^i \tag{12} $$

Indeed, Theorem 1 implies ${}^1M \subset N(\lambda) = M^1$.
To prove (12) it remains to make use of the induction.
The following lemma is a key one in the proof of Theorem 2.

LEMMA 3. *Let* $\lambda \in H_\theta \setminus \bigcup_{n \geqq 1} \bigcup_{\alpha \in \hat{\Delta}_t \setminus \mathbb{N} \cdot \theta} H_{n,\alpha}$.
Then

1) $\operatorname{ord}(\det F_\eta(\lambda(\tau))) = \sum_{i,j \geqq 1} \operatorname{rank} \mathfrak{g} \cdot \dim M(\lambda)_{\eta - i \cdot j \cdot \theta}$

2) *If* $\operatorname{ord}(\det F_\eta(\lambda(\tau))) = \sum_{i \geqq 1} i \dim ({}^iM/{}^{i+1}M)_\eta$ *then* $({}^1M)_\eta = N(\lambda)_\eta$.

PROOF. 1) is an obvious corollary of formula (10) and the identity $\dim M(\lambda)_\eta = P(\eta)$.
2) Note that

$$ \sum_{i \geqq 1} i \cdot \dim ({}^iM/{}^{i+1}M)_\eta = \sum_{i \geqq 1} \dim ({}^iM)_\eta. $$

Therefore thanks to (11) and (12) we have

$$ \sum_{i \geqq 1} i \cdot \dim ({}^iM/{}^{i+1}M) \leqq \operatorname{ord}(\det F_\eta(\lambda(\tau))) $$

If this inequality turns into equality we get applying (11) again ${}^iM = M^i$, in particular ${}^1M = M^1$.

Fix an arbitrary homogeneous with respect to the weight decomposition subspace $H \subset M(\lambda)$ complementary to ${}^1M$. Let us identify H with a subspace of $U(\hat{\mathfrak{n}}_-)$ making use of the fact that $M(\lambda)$ is a free $U(\hat{\mathfrak{n}}_-)$-module with one generator. Introduce a partial ordering on the set of weights by the requirement that $\alpha < \eta$ if and only if $\eta - \alpha \in Q_+$.

LEMMA 4. Assume that $({}^1M)_\alpha = N(\lambda)_\alpha$ *for* $\alpha < \eta$. *Then*

$$ ({}^1M)_\alpha = (H \otimes \mathbb{C}[X] \cdot v_\lambda)_\alpha \text{ for } \alpha \leqq \eta \tag{13} $$

PROOF. 1) By induction in the degree of $e \in U(\hat{\mathfrak{n}}_-)$ let us prove that there is a decomposition of the form $e \cdot v_\lambda = \sum_i h_i \cdot f_i(X) \cdot v_\lambda$, where $h_i \in H$, $f_i(X) \in \mathbb{C}[X]$. If $\deg e_i = 1$ then $e \cdot v_\lambda \in H$. If $e \cdot v_\lambda \notin H$ then $e \cdot v_\lambda = e_H \cdot v_\lambda + e_N \cdot v_\lambda$ where $e_H \cdot v_\lambda \in H$, $e_N \cdot v_\lambda \in {}^1M$. The hypothesis of Lemma 4 implies that $e_N = e' \cdot f(X)$, where $f(X) \leftarrow \mathbb{C}[X]$ and $\deg e' < \deg e$, which enables us to apply the inductive hypothesis since $f(X)M(\lambda) = M(\lambda + (\deg f)\cdot\theta)$ and the functional $\lambda + m\cdot\theta$ satisfies the conditions imposed on λ.

2) Let us prove the uniqueness of the decomposition (13). Let $\sum_{i\geq 0} h_i\cdot f_i(X)\cdot v_\lambda = 0$ Without loss of generality we may assume that $\deg f_0 \leq \deg f_i$ and the polynomials $f_0, f_1, \dots$ are linearly independent. Then

$$h_0\cdot f_0(X)\cdot v_\lambda = -\sum_{i\geq 1} h_i\cdot f_i(X)\cdot v_\lambda$$

which implies that $h_0\cdot f_0(X)\cdot v_\lambda$ belongs to a proper submodule of $f_0(X)M(\lambda)$. Since $f_0(X)\cdot M(\lambda) = M(\lambda + (\deg' f)\cdot\theta)$, the conditions of the lemma imply that $h_0 = h'\cdot f$, $f\in \mathbf{C}[X]$ which contradicts $h_0\cdot v_\lambda \in H$.

Let us pass to the proof of Theorem 2. To prove heading 1) by induction in $\alpha\in Q_+$ let us show that $({}^1M)_\alpha = N(\lambda)_\alpha$. We have $({}^1M)_\alpha = N(\lambda)$ for $\alpha < \theta$ since in this case $\det F_\alpha(\lambda) \neq 0$. We have $\operatorname{ord}(\det F_\theta(\lambda)) = \operatorname{rank}\mathfrak{g}$ for $\alpha = \theta$ and

$$\dim({}^iM)_\alpha = \begin{cases}\operatorname{rank}\mathfrak{g} \text{ if } i = 1\\ \quad 0 \quad \text{ if } i > 1\end{cases}$$

Therefore, $\operatorname{ord}(\det F_\theta(\lambda)) = \sum_{i\geq 1} i\cdot\dim({}^iM/{}^{i+1}M)_\theta$. Thanks to heading 2) of Lemma 3 the initial statement of induction is proved.

Let $({}^1M)_\alpha = N(\lambda)_\alpha$ for $\alpha < \eta$. Making use of Lemma 4 we select a basis B in $({}^1M)_\eta$ of the form $B = \{b_{ij} = h_i\cdot X_j\cdot h_i\in H, X_j\in\mathbf{C}[X]\}$, where j is a multiindex of the form $j = \{i_1, j_1, k_1; i_2, j_2, k_2; \dots\}$ and $X_j = X^{k_1}_{i_1 j_1}\cdot X^{k_2}_{i_2 j_2}\cdot\cdots$.
We have

$$\text{(14)}\qquad \operatorname{ord}(\det F_\eta(\lambda)) = \sum_{l,k\geq 1}\operatorname{rank}\mathfrak{g}\cdot\dim M(\lambda - k\cdot l\cdot\theta)_{\eta - l\cdot k\cdot\theta} =$$

$$\sum_{l,k\geq 1}\ \sum_{1\leq m\leq \operatorname{rank}\mathfrak{g}} \#\{b_{ij}\in B\colon X_j\in X^l_{km}\cdot\mathbf{C}[X]\} =$$

$$\sum_{b_{ij}\in B}\deg' X_j = \sum_{i\geq 1} i\cdot\dim({}^iM/{}^{i+1}M)_\eta$$

Here # denotes the cardinality of a set. The first of the equations (14) follows from 1) of Lemma 3, the second one from Lemma 4, the third one is obvious, and the fourth one follows from Lemma 4. As the result we get

$$\operatorname{ord}(\det F_\eta(\lambda(\tau))) = \sum_{i\geq 1} i\cdot\dim({}^iM/{}^{i+1}M)_\eta$$

By 2) of Lemma 3 this proves the identity $({}^1M)_\eta = N(\lambda)_\eta$ and therefore 1) of Theorem 2.

To prove 2) notice that 1) and Lemma 4 imply $M(\lambda) = H\otimes\mathbf{C}[X]\cdot v_\lambda$. Let w be a singular vector of $M(\lambda)$ different from v_λ. Then

$$w = \sum_{0 \leq i \leq n} h_i \cdot x_i,\ h_i \in H,\ x \in \mathbf{C}[X] \cdot v_\lambda$$

where the vectors $x_0, \ldots, x_n$ are linearly independent. Since $\hat{\mathfrak{n}}_+ \cdot w = 0$, then under the action of an appropriate element from $U(\hat{\mathfrak{n}}_+)$ on w we get $x_0 = \sum_{1 \leq i \leq n} h'_i \cdot x_i$ which contradicts to the fact of the decomposition $M(\lambda) = H \otimes \mathbf{C}[X] \cdot v_\lambda$.

§4. Proof of Theorem 2. The case of an arbitrary affine algebra.

In the above proof of Theorem 2 the characteristic properties of current algebras had been used twice: in the proof of 1) of Lemma 3 we have made use of the identity

$$\dim (\hat{\mathfrak{g}})_{j \cdot \theta} = \operatorname{rank} \mathfrak{g},\ j \in \mathbf{Z}$$

and in formula (14) we have assumed in addition to the reference to Lemma 3 that

(15) $$\# \{X_{ij}: i = m\} = \operatorname{rank} \mathfrak{g}$$

In the general case $\dim (\hat{\mathfrak{g}}_\sigma)_{j \cdot \theta} = \dim \mathfrak{f}^{\operatorname{res}_d j}$, where d is the order of σ; and it is not difficult to see that (14) and therefore the proof of Theorem 2 holds also for an arbitrary affine algebra if we make use of the following generalization of formula (15).

LEMMA 5. *There is a system of generators of the ring* $S(\mathfrak{g})$ *such that*

(15′) $$\# \{X_{ij}: |_i = m\} = \dim \mathfrak{f}^{(\operatorname{res}_d m)}$$

PROOF. It follows from the definition of the elements X_{ij} (see 1.3) that we have to establish the existence of a family of generators $p_1, \ldots, p_{\operatorname{rank} \mathfrak{g}} \in S(\mathfrak{g})^{\mathfrak{g}}$ such that $\sigma(p_i) = \exp(2\pi\sqrt{-1}\, m_i/d) \cdot p_i$ and

$$\# \{m_i,\ i = 1, \ldots, \operatorname{rank} \mathfrak{g}: m_i \equiv m\} = \dim \mathfrak{f}^{(\operatorname{res}_d m)}$$

The nontrivial automorphism σ only exists for the diagrams of type A_n, D_n and E_6. It is convenient to distinguish the following two cases.

1) The case of A_n and D_4. In this case direct verifications show that the standard (Chevalley) systems of generators (see, e.g. [8], Ch. VIII) will do.

2) The case of E_6 and D_n where $n > 4$. In this case it is possible to do without tiresome calculations. The algebras of this type have a unique nontrivial automorphism which is a reflection ([8]) and therefore

$$\dim \mathfrak{h}^{(j)} = \begin{cases} \operatorname{rank} \mathfrak{g} - 1 & \text{if } j = 0 \\ 1 & \text{if } j = 1 \end{cases}$$

The group $\tilde{W} = W \rtimes \{e, \sigma\}$ is generated by reflections (here W is the Weyl group of $\mathfrak{g}$). This group acts in the polynomial ring $S(f)^w \cong S(\mathfrak{g})^{\mathfrak{g}}$. Clearly, the generators of this ring can be chosen to be eigenvectors with respect to σ with eigenvalues ± 1. Since $(S(f)^w)^{\tilde{W}}$ is also a polynomial ring, the number of generators which changes sign under σ does not exceed 1. The fact that there exists at least one generator which changes sign is proved by the standard methods of the theory of groups generated by reflections (see, e.g. [8] Exercise 1 to §8, Ch. VIII).

§5. The brackets of the elements $\Psi_i(p_j)$ and the Zamolodchikov algebra.

In this section we will give the simplest results on the structure of the algebra generated by the elements $X_{ij} = \Psi_i(p_j)$, where $i \in \mathbb{Z}$, $p_1, \dots, p_{\mathrm{rank}\,\mathfrak{g}}$ are the generators of $S(\mathfrak{g})^{\mathfrak{g}}$. Hereafter we will replace $\hat{\mathfrak{g}}$ by $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}]$ (exclude d) and denote the derived algebra also by $\hat{\mathfrak{g}}$.

In the simplest case $\mathfrak{g} = \mathrm{sl}(2)$, $\mathrm{rank}\,\mathfrak{g} = 1, p_1$ is the Casimir element, and as follows from (4) the corresponding algebra is the unversal enveloping algebra for the Virasoro algebra. In the presence of the generators p_i of degree greater than 2 the situation gets more complicated.

LEMMA 6. *Let $p, q \in S(\mathfrak{g})^{\mathfrak{g}}$, Z be the centre of $\tilde{U}(\mathfrak{g})/(c + g)\tilde{U}(\hat{\mathfrak{g}})$ and π: $\tilde{U}(\mathfrak{g}) \to \tilde{U}(\hat{\mathfrak{g}})/(c + g)\tilde{U}(\hat{\mathfrak{g}})$ the natural projection. Then*

$$[\Psi_m(p), \Psi_n(q)] = (c + g)Y \text{ and } \pi Y \in Z \tag{16}$$

PROOF. It follows from Theorem 1 that (16) holds for some $Y \in \tilde{U}(\hat{\mathfrak{g}})$. Taking the bracket of (16) with an arbitrary $e \in \hat{\mathfrak{g}}$ we get

$$(c + g)^2\, U(\hat{\mathfrak{g}}) \in [e, [\Psi_n(p), \Psi_m(q)] = (c + g)[e, Y]$$

Therefore $[e, Y] \in (c + g)\, \tilde{U}(\hat{\mathfrak{g}})$ as required.

Clearly, any element of Z determines a singular vector in $M(\lambda)$ if $\lambda(c) + g = 0$, Therefore, Lemma 6 and Theorem 1 impose some restrictions onto the right-hand side of (16). For $\mathfrak{g} = \mathrm{sl}(3)$ these restrictions suffice to calculate Y.

Let us give the results of the calculations. Let f_1, f_2 be invariant polynomials on sl(3) given by the formula $f_i(e) = \mathrm{tr}\,\rho_i(e)$, where $\rho_i = S^{i+1}\rho$ and ρ is the identity (standard) representation of sl(3). The corresponding polarizations p_1 and p_2 of f_1 and f_2 respectively are the generators of $S(\mathfrak{g})^{\mathfrak{g}}$.

The algebra we are interested in is generated by the elements $T_n = \Psi_n(p_1)$ and $W_n = \Psi_n(p_2)$. Let us calculate the brackets between these elements. The formula (4) takes in this case the form

$$[T_m, T_n] = (c + 3)[(n - m)\, T_{n+m} + \delta_{n,-m} \tfrac{2}{3}(m^3 - m)c \tag{17}$$

Lemmas 2 and 6 imply

$$[T_m, W_n] = (c + 3)(n - 2m)\, W_{n+m} \tag{18}$$

Lemma 6 and Theorem 2 imply that

$$(19)\quad [W_m, W_n] = (c+3)\Big[\sum_{i+j=m+n} \alpha_{ij}^{mn} :T_iT_j: + x_{mn}^{(1)} T_{m+n} C + x_{mn}^{(2)} W_{m+n} + $$
$$+ x_{mn}^{(3)} T_{m+n} + \delta_{m,-n}(x_{mn}^{(4)} \cdot c^2 + x_{mn}^{(5)} c)\Big]$$

The simple calculations making use of an explicit formula for the elements p_1, p_2 show that

$$(20)\qquad \alpha_{ij}^{mn} = \tfrac{1}{3}(m-n)$$

Note also that the elements

$$(21)\qquad W_m, T_n, :T_iT_j:, T_i \cdot c^j, W_m \cdot T_n, (i, j, m, n \in \mathbf{Z})$$

are linearly independent.

LEMMA 7. *There exists a unique algebra with generators T_m, W_n, C $(m, n \in \mathbf{Z})$ and defining relations (17)–(20) such that the elements (21) are linearly independent. The following formulas are valid.*

$$x_{mn}^{(3)} = \begin{cases} \dfrac{n-m}{48}(2n^2 + 2m^2 - 7m \cdot n - 8) & \text{if } m+n \in 2\mathbf{Z} \\ \dfrac{n-m}{48}(2n^2 + 2m^3 - 7m \cdot n + 4) & \text{if } m+n \in 2\mathbf{Z}+1 \end{cases}$$

$$x_{mn}^{(1)} = \tfrac{1}{18}(n-m)(n^2 + m^2 - \tfrac{1}{2}m \cdot n - 4) + \tfrac{1}{3}x_{mn}^{(3)}$$

$$x_{mn}^{(2)} = 0;\ x_{mn}^{(4)} = \tfrac{43}{27}\binom{n+2}{5};\ x_{mn}^{(5)} = \tfrac{11}{3}\binom{n+2}{5}.$$

Lemma 7 suggests that a similar description in terms of defining relations is also possible for the algebras $\hat{\mathfrak{g}}$ constructed for $\mathfrak{g} = \mathrm{sl}(n)$, $n > 3$.

Note that the passage to the algebra $\tilde{U}(\hat{\mathfrak{g}}) \times \tilde{Q}(c)$, where $\tilde{Q}[c]$ is the field of algebraic functions, we get with the help of the above construction a realization of the Zamolodchikov algebra which appeared in conformal field theory ([10]). For this in notations of [10] set $L_n = -\dfrac{T_n}{(c+3)}$,

$$C = \frac{8c}{c+3},\quad V_n = \sqrt{3} \cdot 4 \cdot (c+3)^{-3/2} \cdot (22 + 5 \cdot c)^{-1/2} W_n.$$

ACKNOWLEDGEMENTS. I am deeply grateful to D. B. Fuchs for the invaluable help and B. L. Feigin for the useful discussions.

ADDED NOTE. After this work was completed I received a preprint of [11], where similar results are obtained for the algebras $A_l^{(1)}, B_l^{(1)}, C_l^{(1)}$.

This paper is a version of my talks at the Levkovo Winter School on representation theory and at the seminar on supermanifolds in December 1987.

BIBLIOGRAPHY

1. V. G. Kac and D. A. Kazhdan, *Structure of representations with highest weight of infinite-dimensional Lie algebras.* Adv. Math. 1979 v. 34, 97–108.
2. B. L. Feigin, D. B. Fuchs and F. G. Malikov, *Singular vectors in Verma modules over Kac-Moody algebras.* Funct. Anal Appl. 1986 v. 20, n2, 25–37 (Russian).
3. V. G. Kac, *Simple irreducible Lie algebras of finite growth.* Math USSR Izvestia, ser. math 1968, v. 32 1923–1967 (Russian).
4. V. G. Kac and D. H. Peterson, *Infinitedimensional Lie algebras, theta functions and modular forms*, Adv. Math. 1983, v. 50.
5. V. G. Kac, *Infinitedimensional Lie algebras Birkhäuser*, 1983.
6. B. L. Feigin and D. B. Fuchs, *Skewsymmetric invariant differential operators on the line and Verma modules over Virasoro algebra.* Funct. Anal. Appl. 1982, v. 16, n2, 47–63 (Russian; In: D. Leites (ed). Seminar on supermanifolds n5, Reports of Dept. Math. Stockholm Univ. n25, 1986.
7. B. Kostant, *Lie group representations on polynomial rings.* Bull. Amer. Math. Soc. 1967, v1, n4.
8. N. Bourbaki, *Groups et algèbres de Lie.* Ch. VII-VIII. Hermann, Paris.
9. A. A. Kirillov, *Invariant differential operators over geometric quantities*, JOSMAR, v30, 1984
10. A. B. Zamolodchikov, *Additional infinite symmetries in twodimensional quantum field theory*, Soviet theor math. physics, 1985, v. 65, n3.
11. T. Hayashi, *Sugaware operators and Kac-Kazhdan conjecture*, Inv. Math. 1988, v. 94, 13–52.

SEMINAR ON SUPERMANIFOLDS
c/o D. LEITES
DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
BOX 6701
11385 STOCKHOLM
SWEDEN