Singular Vectors of the Virasoro Algebra in Terms of Jack Symmetric Polynomials

Katsuhisa Mimachi, Yasuhiko Yamada

Department of Mathematics, Kyushu University, Hakozaki, Fukuoka 812, Japan

Abstract: We present an explicit formula of the Virasoro singular vectors in terms of Jack symmetric polynomials. The parameter t in the Virasoro central charge $c = 13 - 6(t + 1/t)$ is just identified with the deformation parameter α of Jack symmetric polynomials $J_\lambda(\alpha)$. As a by-product, we obtain an integral representation of Jack symmetric polynomials indexed by the rectangular Young diagrams.

1. Introduction

The Virasoro algebra is an infinite dimensional Lie algebra defined by the generators $l_m (m \in \mathbb{Z})$ and c with the relations

$$[l_n, l_m] = (n - m)l_{m+n} + \frac{c}{12}(n^3 - n)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$ and $[c, l_m] = 0$ for $m \in \mathbb{Z}$. Among many mathematicians and physicists to investigate the representation theory of the Virasoro algebra (see [2, 3] for instance), Tsuchiya and Kanie classified the singular vectors of the Fock space representations in [8], where they constructed the integral representation of the singular vectors. But they missed obtaining explicit forms of such singular vectors. The present article will be devoted to this point: Namely, we express the singular vectors in terms of some symmetric polynomials, called Jack symmetric polynomials.

Jack symmetric polynomials were first introduced by Jack in 1970 to evaluate some definite integral. Subsequently, the systematic study was begun by Macdonald, who considered even more general cases: Macdonald's symmetric polynomials [6].

In this paper we show that the singular vectors of the Virasoro algebra are given by Jack symmetric polynomials indexed by the rectangular Young diagrams.

We believe that such realization of Jack symmetric polynomials would be helpful in future research both of the representation theory of infinite dimensional algebras and of hypergeometric type functions in several variables.

Some basic notations about the symmetric polynomials, with definition of Jack symmetric polynomials, are recalled in Sect. 2. We give the main result in Sect. 3,

after a short review on the Fock space representation. Sections 4 and 5 are devoted to its proof. The key ingredient of our proof is an expression of Jack symmetric polynomials by Euler–Selberg type integrals.

We remark that the present result can be viewed as a generalization of the result in [9]. They showed, by a direct computation, that $c = 1$ singular vectors are given by the Schur polynomials.

Among the related works we refer the reader to [1], where the singular vectors in Verma modules and associated differential equations were studied.

2. Jack Symmetric Polynomials

We recall the basic notion about the symmetric polynomials by following the standard reference [4].

A partition λ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non-negative integers in decreasing order; $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0$. The number of the non-zero λ_i's is called the length of λ, denoted by $l(\lambda)$; and the sum of λ_i's is the weight of λ, denoted by $|\lambda|$. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, we define the conjugate partition $\lambda' = (\lambda_1', \lambda_2', \ldots, \lambda_n')$ by $\lambda_i' = \text{Card}\{j; \lambda_j \geqq i\}$. The dominace (or natural) ordering on partitions is defined as $\lambda \geqq \mu \Leftrightarrow |\lambda| = |\mu|$ and $\lambda_1 + \cdots + \lambda_i \geqq \mu_1 + \cdots + \mu_i$ for all $i \geqq 1$.

Sometimes, given a partition λ we also use a notation which indicates the number of times each integer occurs: $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ with $m_i = m_i(\lambda) = \text{Card}\{j; \lambda_j = i\}$.

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$ we denote by x^β the monomial $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$. If λ is a partition of length $l(\lambda) \leqq n$, the monomial symmetric polynomial m_λ is defined by $m_\lambda(x_1, \ldots, x_n) = \sum x^\beta$, where the sum is taken over all distinct permutations β of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. For each integer $r \geqq 0$, the rth elementary symmetric polynomial e_r and the rth power sum p_r are defined by $e_r = \sum_{1 \leqq i_1 < i_2 < \cdots < i_r \leqq n} x_{i_1} x_{i_2} \cdots x_{i_r} = m_{(1^r)}$ and $p_r = \sum_{i=1}^n x_i^r = m_{(r)}$, respectively.

Jack symmetric polynomials $J_\lambda = J_\lambda(x_1, \ldots, x_n; \alpha)$, indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of length $l(\lambda) \leqq n$, are defined as the eigenfunctions for the operator

$$D(\alpha) = \frac{\alpha}{2} \sum_{i=1}^n x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{1 \leqq i \neq j \leqq n} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i} . \tag{1}$$

Namely, for a fixed partition λ, J_λ is a unique solution of

$$D(\alpha) J_\lambda = e_\lambda(\alpha) J_\lambda \tag{2}$$

and

$$e_\lambda(\alpha) = \frac{\alpha}{2} \sum_{i=1}^n \lambda_i(\lambda_i - 1) - \sum_{i=1}^n (i - 1)\lambda_i + (n - 1)|\lambda| \tag{3}$$

in the form

$$J_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$$

for $a_{\lambda\mu} \in \mathbb{C}(\alpha)$.

Jack symmetric polynomials J_λ are a common generalization of some well-known polynomials. Indeed, for particular cases of α, polynomials J_λ reduce to the following:

(i) When $\alpha = 1$, they are the Schur polynomials $s_\lambda(x_1, \ldots, x_n)$, i.e. the characters of the polynomial representations of $GL(n; \mathbb{C})$.

(ii) When $\alpha = 2$, they are the zonal polynomials $Z_\lambda(x_1, \ldots, x_n)$, i.e. the zonal spherical functions on $GL(n)/SO(n)$.

(iii) When $\alpha = 1/2$, they are the zonal spherical functions on $GL(2n)/Sp(n)$.

(iv) When $\alpha = 0$, we have $J_\lambda(x_1, \ldots, x_n; 0) = e_{\lambda'}(x_1, \ldots, x_n)$, the product of elementary symmetric polynomials corresponding to the conjugate λ' of λ.

(v) If α tends to ∞, suitably normalized $J_\lambda(x_1, \ldots, x_n; \alpha)$ reduce to the monomial symmetric polynomials $m_\lambda(x_1, \ldots, x_n)$.

For details of Jack symmetric polynomials, we refer the reader to [5 and 7].

3. Fock Representations and Main Theorem

To state our main result, we prepare the notion of the Fock space representations of the Virasoro algebra.

First set

$$\alpha_0 = \frac{\beta}{2} - \frac{1}{\beta}, \qquad \alpha_{r,s} = (1 + r)\frac{\beta}{2} - (1 + s)\frac{1}{\beta} \quad \text{and} \quad t = \frac{\beta^2}{2}$$

for $\beta \in \mathbb{C}$ and $r, s \in \mathbb{Z}$.

The Heisenberg algebra $\mathscr{A}$ is a $\mathbb{C}$-algebra generated by a_n ($n \in \mathbb{Z}$) with the relations

$$[a_n, a_m] = n\delta_{n+m,0} .$$

Denote by F_α a left $\mathscr{A}$-module defined by the generator $|\alpha\rangle$ and the relations

$$a_n|\alpha\rangle = 0 \ (n > 0), \qquad a_0|\alpha\rangle = \alpha|\alpha\rangle .$$

When we put

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_{n-m}a_m : -\alpha_0(n + 1)a_n$$

for $n \in \mathbb{Z}$, we obtain the relations

$$[L_n, a_m] = -ma_{n+m} - \alpha_0 n(n + 1)\delta_{n+m,0} \quad (m, n \in \mathbb{Z})$$

and

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \quad (m, n \in \mathbb{Z}) ,$$

the Virasoro algebra with the central charge $c = 1 - 12\alpha_0^2 = 13 - 6(t + \frac{1}{t})$. Here we use the normal products defined by

$$: a_n a_m := \begin{cases} a_n a_m & (n \leqq m) , \\ a_m a_n & (n > m) . \end{cases}$$

Moreover, on the module F_α for each $\alpha \in \mathbb{C}$, we have the relations

$$L_n|\alpha\rangle = 0 \ (n \in \mathbb{Z}_{>0}) \quad \text{and} \quad L_0|\alpha\rangle = h_\alpha|\alpha\rangle$$

with $h_\alpha = \frac{\alpha^2}{2} - \alpha_0\alpha$. These imply that F_α can be considered as the highest weight module of the Virasoro algebra with the highest weight vector $|\alpha\rangle$ and the highest weight h_α. We call this module F_α the Fock space representation of the Virasoro algebra.

An element $|\chi\rangle \in F_\alpha$ satisfying the relations

$$L_n|\chi\rangle = 0 \ (n \in \mathbb{Z}_{>0}) \quad \text{and} \quad L_0|\chi\rangle = (h_\alpha + N)|\chi\rangle$$

is called the *singular vector* of degree $N \in \mathbb{Z}$. To analyze the structure of the Fock module F_α, it plays an important role. Denote by $SF_\alpha(N)$ the set of singular vectors of degree N in F_α. If $\alpha \notin \{\alpha_{r,s}; r, s \in \mathbb{Z}_{>0} \text{ or } r, s \in \mathbb{Z}_{<0}\}$, F_α is irreducible and $\dim SF_\alpha(N) = 0$ for any N. If $\alpha = \alpha_{r,s}$ for $r, s \in \mathbb{Z}_{>0}$, $\dim SF_\alpha(rs) = 1$, i.e. there exists a unique singular vector $|\chi_{r,s}\rangle \in F_{\alpha_{r,s}}$ of degree $N = rs$ up to a constant factor. Furthermore, there exists a polynomial $P_{r,s} \in \mathbb{C}[a_{-1}, a_{-2}, a_{-3}, \ldots, a_{-rs}]$ whose coefficients are rational functions in β such that $SF_{\alpha_{r,s}}(rs) = \mathbb{C}P_{r,s}|\alpha_{r,s}\rangle$. Since we are not interested in the constant factor for defining the polynomial $P_{r,s}$, we fix it in what follows by the formula (4) below.

Here we reach the stage to give our main result:

Theorem 1. *We have*

$$P_{r,s} = \rho_n(J_{(r^s)}(x_1, \ldots, x_n; t)),$$

where $n \geq rs$ and

$$\rho_n : \mathbb{C}[x_1, x_2, \ldots, x_n]^{S_n} \to \mathbb{C}[a_{-1}, a_{-2}, a_{-3}, \ldots, a_{-n}]$$

is an algebra homomorphism determined by

$$\rho_n(p_m(x)) = (-1)^{m-1}\beta a_{-m}$$

for $1 \leq m \leq n$.

4. Proof of Theorem

On the manifold $M_r = \{(z_1, \ldots, z_r) \in (\mathbb{C}^*)^r; \ z_i \neq z_j (1 \leq i < j \leq r)\}$, set

$$\Phi = \prod_{1 \leq i < j \leq r}(z_i - z_j)^{2t} \prod_{1 \leq i \leq r} z_i^{(1-r)t-1}$$

and $d\zeta = dz_1 \ldots dz_r$. Let $H_r(M_r; S)$ be the twisted de Rham homology with coefficient S, local system associated with Φ. This homology is studied in [8]. They construct a cycle $\mathscr{C} \in H_r(M_r; S)$ which gives the following.

Lemma 2. *There exists a cycle $\mathscr{C} \in H_r(M_r; S)$ such that*
   (a) *unless $m_1 + \cdots + m_r = 0$,*

$$\int_{\mathscr{C}} \Phi \prod_{1 \leq i \leq r} z_i^{m_i} d\zeta = 0 \quad \text{for} \quad m_i \in \mathbb{Z} ,$$

(b) *if* $d(d+1)t \notin \mathbb{Z}$ *and* $d(r-d)t \notin \mathbb{Z}$ *for* $1 \leqq d \leqq r-1$,

$$\int_{\mathscr{C}} \Phi d\zeta = \frac{1}{(r-1)!} \prod_{j=1}^{r-1} \frac{\Gamma((j-r)t)\Gamma((j+1)t+1))}{\Gamma(t+1)} \;.$$

See Proposition 4.2 in [8].

It is known that a polynomial given by the integral

$$P_{r,s} = \int_{\mathscr{C}} \prod_{1 \leqq i < j \leqq r} (z_i - z_j)^{2t} \prod_{i=1}^{r} z_i^{(1-r)t-1-s} \widehat{E}(z_i) d\zeta \tag{4}$$

on the above cycle $\mathscr{C}$ determines the singular vector $|\chi_{r,s}\rangle = P_{r,s}|\alpha_{r,s}\rangle$.

Here $\widehat{E}(z)$ is defined by

$$\widehat{E}(z) = \exp\left(\beta \sum_{m=1}^{n} \frac{a_{-m}}{m} z^m\right)$$

with $n \geqq rs$. Since $\widehat{E}(z) = \rho_n(E(z))$ with $E(z) = \sum_{i=0}^{n} e_i(x)z^i = \prod_{k=1}^{n}(1 + x_k z)$, the following is enough to prove our theorem:

Proposition 3. *Jack symmetric polynomial* $J_{(r^s)}(x_1,\ldots,x_n;t)$ *for* $r \leqq n$ *is expressed by*

$$N_r(t) \int_{\mathscr{C}} \Phi \prod_{1 \leqq i \leqq r} z_i^{-s} E(z_i) \, d\zeta \,,$$

where

$$N_r(t) = (r-1)! \prod_{j=1}^{r-1} \frac{\Gamma(t+1)}{\Gamma((j-r)t)\Gamma((j+1)t+1))} \;.$$

Proof. By Lemma 2, we see that the integral expresses a homogeneous polynomial of degree rs and that $m_{(r^s)}$ is its leading term. Hence we need only to prove that the integral

$$\int_{\mathscr{C}} \Phi \prod_{1 \leqq i \leqq r} z_i^{-s} E(z_i) \, d\zeta$$

satisfies the differential equation (2) with $\alpha = t$, which will be given in the next section[1]

5. The Differential Equation

In this section, we denote by

$$\phi_1 \equiv \phi_2$$

the equality

$$\int_{\mathscr{C}} \phi_1 \tilde{\Phi} d\zeta = \int_{\mathscr{C}} \phi_2 \tilde{\Phi} d\zeta$$

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[1] During our preparation of the present paper, we received the preprint "Collective Field Theory, Calogero–Sutherland Model and Generalized Matrix Models" by H. Awata et al. (hep-th/9411053). Although this kind of integral representation for the Jack polynomials is also considered there, the argument on the cycles is missing.

with

$$\tilde{\Phi} = \Phi \prod_{1 \le i \le r} z_i^{-s} E(z_i) = \prod_{1 \le i < j \le r} (z_i - z_j)^{2t} \prod_{1 \le i \le r} z_i^{(1-r)t-1-s} \prod_{\substack{1 \le i \le r \\ 1 \le k \le n}} (1 + x_k z_i)$$

for our fixed cycle $\mathscr{C}$.

Lemma 4. *We get*

$$\sum_{\substack{1 \le k \le n \\ 1 \le i \le r}} \frac{1}{1 + x_k z_i} \equiv r(n-s), \tag{5}$$

and

$$\{(1-r)t + n - s - 1\} \sum_{\substack{1 \le k \le n \\ 1 \le i \le r}} \frac{1}{1 + x_k z_i} - \sum_{\substack{1 \le i \le r \\ 1 \le k_1 < k_2 \le n}} \frac{2}{(1 + x_{k_1} z_i)(1 + x_{k_2} z_i)}$$

$$+ 2t \sum_{\substack{1 \le k \le n \\ 1 \le i_1 < i_2 \le r}} \frac{1}{(1 + z_{i_1} x_k)(1 + z_{i_2} x_k)} \equiv 0. \tag{6}$$

Proof. Let d_z stand for the total derivative with respect to $z = (z_1, \ldots, z_r)$. The equality

$$d_z(z_i(-)^{i-1} \tilde{\Phi} dz_1 \ldots \hat{dz_i} \ldots dz_r)$$

$$= \left[n + (1-r)t - s - \sum_{1 \le k \le n} \frac{1}{1 + x_k z_i} + 2t \sum_{\substack{1 \le j \le r \\ j \ne i}} \frac{z_i}{z_i - z_j} \right] \tilde{\Phi} d\zeta$$

gives

$$d_z \sum_{i=1}^{r} ((-)^{i-1} z_i \tilde{\Phi} dz_1 \ldots \hat{dz_i} \ldots dz_r) = \left[r(n-s) - \sum_{\substack{1 \le k \le n \\ 1 \le i \le r}} \frac{1}{1 + x_k z_i} \right] \tilde{\Phi} d\zeta,$$

since

$$\sum_{i=1}^{r} \sum_{\substack{1 \le j \le r \\ j \ne i}} \frac{z_i}{z_i - z_j} = \binom{r}{2}.$$

It completes the proof of (5).

Subsequently we get

$$d_z \left(\frac{z_i}{(1 + x_k z_i)} (-)^{i-1} \tilde{\Phi} dz_1 \ldots \hat{dz_i} \ldots dz_r \right) = \left[\frac{(1-r)t - s + n - 1}{1 + x_k z_i} \right.$$

$$\left. - \sum_{\substack{1 < \ell \le n \\ \ell \ne k}} \frac{1}{(1 + x_\ell z_i)(1 + x_k z_i)} + 2t \frac{z_i}{1 + x_k z_i} \sum_{\substack{1 < j \le r \\ j \ne i}} \frac{1}{z_i - z_j} \right] \tilde{\Phi} d\zeta$$

for $1 \leqq i \leqq r$ and $1 \leqq k \leqq n$, since

$$\frac{z_l x_\ell}{(1 + x_\ell z_i)(1 + x_k z_l)} = \frac{1}{1 + x_k z_i} - \frac{1}{(1 + x_\ell z_i)(1 + x_k z_l)} \;.$$

Thus we obtain

$$d_z \left(\sum_{l=1}^{r} \frac{z_l}{1 + x_k z_i} (-)^{i-1} \tilde{\Phi} dz_1 \ldots \hat{dz_i} \ldots dz_r \right) = \left[\{(1-r)t - s + n - 1\} \sum_{1 \leqq i \leqq r} \frac{1}{1 + x_k z_i} \right.$$

$$\left. - \sum_{1 \leqq i \leqq r} \sum_{\substack{1 \leqq \ell \leqq n \\ \ell \neq k}} \frac{1}{(1 + x_\ell z_i)(1 + x_k z_l)} + 2t \sum_{1 \leqq i < j \leqq r} \frac{1}{(1 + x_k z_i)(1 + x_k z_j)} \right] \tilde{\Phi} d\zeta \;.$$

Here we use the identity

$$\sum_{i=1}^{r} \sum_{\substack{1 \leqq j \leqq r \\ j \neq i}} \frac{z_l}{(1 + x_k z_i)(z_i - z_j)} = \sum_{1 \leqq i < j \leqq r} \frac{z_i}{(1 + x_k z_i)(z_l - z_j)}$$

$$+ \sum_{1 \leqq i < j \leqq r} \frac{z_j}{(1 + x_k z_j)(z_j - z_i)} = \sum_{1 \leqq i < j \leqq r} \frac{1}{(1 + x_k z_l)(1 + x_k z_j)}$$

Summing up for k, we finally obtain

$$d_z \left(\sum_{k=1}^{n} \sum_{l=1}^{r} \frac{z_i}{1 + x_k z_i} (-)^{i-1} \tilde{\Phi} dz_1 \ldots \hat{dz_i} \ldots dz_r \right)$$

$$= \left[\{(1-r)t - s + n - 1\} \sum_{1 \leqq k \leqq n} \sum_{1 \leqq i \leqq r} \frac{1}{1 + x_k z_i} \right.$$

$$- \sum_{1 \leqq i \leqq r} \sum_{1 \leqq k < \ell \leqq n} \frac{2}{(1 + x_\ell z_i)(1 + x_k z_i)}$$

$$\left. + 2t \sum_{1 \leqq k \leqq n} \sum_{1 \leqq i_1 < i_2 \leqq r} \frac{1}{(1 + x_k z_{i_1})(1 + x_k z_{i_2})} \right] \tilde{\Phi} d\zeta \;,$$

which leads to the equality (6).

Lemma 5. *We have*

$$D(t)\tilde{\Phi} = \left[t \left\{ n \binom{r}{2} - (r-1) \sum_{\substack{1 \leqq k \leqq n \\ 1 \leqq i \leqq r}} \frac{1}{1 + x_k z_i} + \sum_{1 \leqq k \leqq n} \sum_{1 \leqq i < j \leqq r} \frac{1}{(1 + x_k z_l)(1 + x_k z_j)} \right\} \right.$$

$$\left. + r \binom{n}{2} - \sum_{1 \leqq k \leqq r} \sum_{1 \leqq l < j \leqq n} \frac{1}{(1 + x_i z_k)(1 + x_j z_k)} \right] \tilde{\Phi} \;. \tag{7}$$

Proof. Note that

$$x_i \partial_{x_i} \tilde{\Phi} = x_i \left\{ \frac{z_1}{1 + x_i z_1} + \cdots + \frac{z_r}{1 + x_i z_r} \right\} \tilde{\Phi}$$

$$= \left\{ r - \frac{1}{1 + x_i z_1} - \cdots - \frac{1}{1 + x_i z_r} \right\} \tilde{\Phi}$$

and

$$x_i^2 \partial_{x_i}^2 \tilde{\Phi} = 2 \sum_{1 \leq j_1 < j_2 \leq r} \frac{x_i^2 z_{j_1} z_{j_2}}{(1 + x_i z_{j_1})(1 + x_i z_{j_2})} \tilde{\Phi}$$

$$= 2 \sum_{1 \leq j_1 < j_2 \leq r} \left(1 - \frac{1}{1 + x_i z_{j_1}} \right) \left(1 - \frac{1}{1 + x_i z_{j_2}} \right) \tilde{\Phi}$$

$$= 2 \left[\binom{r}{2} - \sum_{1 \leq j_1 < j_2 \leq r} \left(\frac{1}{1 + x_i z_{j_1}} + \frac{1}{1 + x_i z_{j_2}} \right) \right.$$

$$\left. + \sum_{1 \leq j_1 < j_2 \leq r} \frac{1}{(1 + x_i z_{j_1})(1 + x_i z_{j_2})} \right] \tilde{\Phi}$$

$$= 2 \left[\binom{r}{2} - (r - 1) \sum_{1 \leq j \leq r} \frac{1}{1 + x_i z_j} \right.$$

$$\left. + \sum_{1 \leq j_1 < j_2 \leq r} \frac{1}{(1 + x_i z_{j_1})(1 + x_i z_{j_2})} \right] \tilde{\Phi} .$$

Hence, the equality

$$\sum_{1 \leq j_1 \neq j_2 \leq n} \frac{x_{j_1}}{(x_{j_1} - x_{j_2})(1 - x_{j_1} z_i)} = \sum_{1 \leq j_1 < j_2 \leq n} \frac{x_{j_1}}{(x_{j_1} - x_{j_2})(1 - x_{j_1} z_i)}$$

$$+ \sum_{1 \leq j_1 < j_2 \leq n} \frac{x_{j_2}}{(x_{j_2} - x_{j_1})(1 - x_{j_2} z_i)}$$

$$= \sum_{1 \leq j_1 < j_2 \leq n} \frac{1}{(1 + z_i x_{j_1})(1 + z_i x_{j_2})}$$

implies (7).

At this stage, combining Lemmas 4–5, we reach the desired assertion:

$$[D(t) - e_{(r^s)}(t)] \int_{\mathscr{C}} \tilde{\Phi} d\zeta = 0 . \tag{8}$$

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