

SINGULARITIES IN THE NILPOTENT SCHEME OF A CLASSICAL GROUP

BY

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ABSTRACT. If (X, x) is a pointed scheme over a ring k , we introduce a (generalized) partition $\text{ord}(x, X/k)$. If G is a reductive group scheme over k , the existence of a nilpotent subscheme $N(G)$ of $\text{Lie}(G)$ is discussed. We prove that $\text{ord}(x, N(G)/k)$ characterizes the orbits in $N(G)$, their codimension and their adjacency structure, provided that G is GL_n , or Sp_n , and $1/2 \in k$. For SO_n only partial results are obtained. We give presentations of some singularities of $N(G)$. Tables for its orbit structure are added.

Introduction. Let G be a reductive algebraic group over a field of characteristic p . Let \mathfrak{g} be its Lie-algebra and $N(G)$ the closed subset of the nilpotent elements of \mathfrak{g} , cf. [19]. The G -orbits in $N(G)$ are characterized by weighted Dynkin diagrams, cf. [20, III]. Consider the following question. *Is it possible to classify the orbits in $N(G)$ using only the local structure of the variety $N(G)$?* We prove in (4.3) that the answer is positive if G is GL_n or if G is Sp_n and $p \neq 2$.

To this end we introduce a local invariant "ord" for any pointed scheme in §1. We develop the theory of $N(G)$ over an arbitrary ground ring k in §2. In §3 we restrict our attention to the classical group schemes. Using a cross section we obtain information about the orbit structure of $N(G)$. Our main theorem (4.2) relates $\text{ord}(x, N(G)/k)$ to the Jordan normal form of the nilpotent endomorphism induced by x in the classical representation.

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Conventions and notations. The cardinality of a set V is denoted by $\# V$. Any infinite cardinal is represented by ∞ . If x is a real number then $[x]$ is the greatest integer in x . All rings are commutative with 1. Let M be a module over a ring A . If M is free the rank of M is denoted by $\text{rg}_A M$. An element $c \in A$ is called M -regular if $a: M \rightarrow M$ is injective. Let $a = (a_1, \dots, a_r)$ be a

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sequence in A . The ideal generated by \mathbf{a} is denoted by $\langle \mathbf{a} \rangle$. The sequence is called M -regular if a_i is $(M/\langle (a_j)_{j < i} \rangle M)$ -regular for all i , cf. [12, 0_{IV} 15.1].

Unless stated otherwise k is an arbitrary ground ring. General references for schemes and group schemes are [11], [12] and [8]. If we consider a k -scheme as a functor from k -algebras to sets, cf. [11, p. 17], then the letter R is used to denote an arbitrary k -algebra. If X is a k -scheme and R is a k -algebra then $X_{(R)}$ is the R -scheme $X \otimes_k R$. If X is an affine scheme then its coordinate ring is denoted by $A(X)$. If A is a local ring its maximal ideal is denoted by \mathfrak{m}_A and its residue field by $k(A)$. If X is a scheme and $x \in X$ then we write $\mathfrak{m}_x := \mathfrak{m}_A$ and $k(x) := k(A)$ where $A := \mathcal{O}_{X,x}$.

1. A near-partition for a local k -algebra.

(1.1) A *near-partition* λ is a subset of \mathbb{N}^2 such that if $(m, n) \in \lambda$ and $i \leq m$ and $j \leq n$ then $(i, j) \in \lambda$. The set of near-partitions is denoted by NP . The duality mapping $D: NP \rightarrow NP$ is induced by $(i, j) \mapsto (j, i)$. The set NP is ordered by $\lambda \leq \mu$ if and only if $\lambda \subset \mu$. We write $|\lambda| := \#\lambda$. A near-partition λ is called a *partition* if $|\lambda| < \infty$. The set of partitions is denoted by \mathcal{P} .

If $\lambda \in NP$, the nonincreasing sequences λ^* and λ_* in $\{0\} \cup \mathbb{N} \cup \{\infty\}$ are defined by

$$\lambda^n \geq i \Leftrightarrow (n, i) \in \lambda \Leftrightarrow \lambda_i \geq n.$$

Clearly $\lambda_i = (D\lambda)^i = \sup \{n \in \mathbb{N} \mid \lambda^n \geq i\}$, and dually. A near-partition λ is completely determined by its sequence λ^* (or λ_*). We write $\lambda_* = (\lambda_1, \dots, \lambda_r)$ if $\lambda_i = 0$ for $i > r$. If $\lambda, \mu \in NP$, we define $\lambda + \mu \in NP$ by $(\lambda + \mu)^n := \lambda^n + \mu^n$, where $x + \infty := \infty + x := \infty$ for all x . If $\lambda_* = (\lambda_1, \dots, \lambda_r)$ and $\mu_* = (\mu_1, \dots, \mu_s)$ then $(\lambda + \mu)_*$ is the sequence obtained by ordering $(\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s)$, see [9, Proposition 6].

(1.2) DEFINITION. A *linear extension* over a ring k is a surjective morphism $\epsilon: E \rightarrow A$ of local k -algebras such that $\mathfrak{m}_E \ker(\epsilon) = 0$. Its near-partition $\text{ord}(\epsilon)$ is defined by

$$\text{ord}^n(\epsilon) := \text{rg}_{k(E)}(\ker(\epsilon) \cap \mathfrak{m}_E^{n+1}).$$

A linear extension $\epsilon: E \rightarrow A$ is called *versal* over k if for any linear extension $\zeta: F \rightarrow B$ over k and any local k -morphism $\phi: A \rightarrow B$ there exists a (clearly local) k -morphism $\gamma: E \rightarrow F$ with $\zeta \circ \gamma = \phi \circ \epsilon$, see diagram (i).

(1.3) PROPOSITION. *Let diagram (i) be a commutative diagram of k -algebras such that ϵ and ζ are linear extensions, that ϕ is a flat local morphism and that $\mathfrak{m}_A B = \mathfrak{m}_B$. Then we have $\text{ord}(\epsilon) \geq \text{ord}(\zeta)$.*

$$(i) \quad \begin{array}{ccc} F & \xrightarrow{\zeta} & B \\ \gamma \uparrow & & \uparrow \phi \\ E & \xrightarrow{\epsilon} & A \end{array}$$

PROOF. Let $n \in \mathbb{N}$. We prove that $\text{ord}^n(\epsilon) \geq \text{ord}^n(\zeta)$. It suffices to prove that the ideal $\ker(\zeta) \cap m_F^{n+1}$ is generated by the image of $\ker(\epsilon) \cap m_E^{n+1}$. We may assume that $\ker(\epsilon) \cap m_E^{n+1} = 0$. Now the mapping $m_E^{n+1} \rightarrow A$ induced by ϵ is an injection of A -modules. Since B is flat over A , it follows that $m_E^{n+1} \otimes_E B \rightarrow B$ is injective and hence that $\text{Tor}^E(E/m_E^{n+1}, B) = 0$. This implies injectivity of

$$\ker(\zeta) \otimes_E (E/m_E^{n+1}) \rightarrow F \otimes_E (E/m_E^{n+1})$$

so that $\ker(\zeta) \cap m_E^{n+1}F = m_E^{n+1}\ker(\zeta) = 0$. On the other hand $m_A B = m_B$ implies that $m_E F + \ker(\zeta) = m_F$, so that $m_E^{n+1}F = m_F^{n+1}$. This proves $\ker(\zeta) \cap m_F^{n+1} = 0$.

(1.4) Let A be a local k -algebra. If $\epsilon: E \rightarrow A$ is a versal linear extension over k then (1.3) implies that $\text{ord}(\epsilon) \geq \text{ord}(\zeta)$ for any linear extension $\zeta: F \rightarrow A$ over k . On the other hand there exists a versal linear extension $\epsilon: E \rightarrow A$ over k . In fact, write $A = R/\mathfrak{A}$ where R is some polynomial k -algebra. Let M be the ideal in R such that $m_A = M/J$. Then $R/MJ \rightarrow A$ is a versal linear extension over k , compare [15, p. 37]. Now we can give the following:

DEFINITION. $\text{ord}(A/k) := \text{ord}(\epsilon)$ where $\epsilon: E \rightarrow A$ is some (or any) versal linear extension over k .

EXAMPLE. Let k be a field. Put $H := k[T_1, \dots, T_m]$. Let $\mathfrak{a} = (a_1, \dots, a_r)$ be a sequence in H . Let a_i be homogeneous of degree $1 + \lambda_i$ where λ is a partition with $\lambda_{r+1} = 0$. Assume that the ideal $\langle \mathfrak{a} \rangle$ is not generated by a strict subsequence of \mathfrak{a} . Consider the local ring $A := (H/\langle \mathfrak{a} \rangle)_{\mathfrak{p}}$ where $\mathfrak{p} = \langle T_1, \dots, T_m \rangle$. Then $\text{ord}(A/k) = \lambda$.

In fact $\epsilon: (H/\mathfrak{p}\langle \mathfrak{a} \rangle)_{\mathfrak{p}} \rightarrow A$ is a versal linear extension over k and $\text{ord}^n(\epsilon) = \#\{i | \lambda_i \geq n\} = \lambda^n$.

(1.5) PROPOSITION. Let A be a local k -algebra and R a k -algebra. Assume that A or R is flat over k . Let $\mathfrak{p} \in \text{Spec}(A \otimes_k R)$ contract to m_A . Then $\text{ord}(A/k) \leq \text{ord}((A \otimes_k R)_{\mathfrak{p}}/R)$.

PROOF. Let $\epsilon: E \rightarrow A$ be a versal linear extension over k . Put $I := \ker(\epsilon)$. Let $\mathfrak{q} \in \text{Spec}(E \otimes R)$ be the inverse image of \mathfrak{p} . Since A or R is flat over k , $(I \otimes R)_{\mathfrak{q}}$ is an ideal in $(E \otimes R)_{\mathfrak{q}}$. Put $F := (E \otimes R)_{\mathfrak{q}} / (I \otimes R)_{\mathfrak{q}}$, so that $\zeta: F \rightarrow (A \otimes R)_{\mathfrak{p}}$ is a linear extension over R . One verifies that $I \otimes_{k(E)} k(F) \rightarrow \ker(\zeta)$ is injective and hence that $\text{ord}(\epsilon) \leq \text{ord}(\zeta)$. This suffices.

(1.6) PROPOSITION. Let A be a local k -algebra, $\mathbf{x} = (x_1, \dots, x_m)$ an A -regular sequence in \mathfrak{m}_A and f a nonzero element of $\langle \mathbf{x} \rangle$. Put $B = A/\langle f \rangle$ and $C = A/\langle \mathbf{x} \rangle$. Let $r \in \mathbb{N}$ and let ρ be a partition with $\rho_* = (r - 1)$.

- (a) If $f \in \mathfrak{m}_A^r$ then $\rho + \text{ord}(A/k) \leq \text{ord}(B/k)$.
 (b) If $f \notin \mathfrak{m}_A^{r+1}$ then $\text{ord}(B/k) \leq \rho + \text{ord}(C/k)$.

PROOF. Let $\epsilon: E \rightarrow A$ be a versal linear extension over k . Put $I = \ker(\epsilon)$. Choose $y_i \in E$ with $\epsilon(y_i) = x_i$ and $g \in E$ with $\epsilon(g) = f$. Put $F := E/\mathfrak{g}m_E$ and $G := E/\mathfrak{m}_E\langle y \rangle$. The linear extensions $\zeta: F \rightarrow B$ and $\eta: G \rightarrow C$ are versal over k . Since \mathbf{x} is a regular sequence, we have $I \cap \langle y \rangle = 0$. So the induced mappings $I \rightarrow \ker(\zeta)$ and $I \rightarrow \ker(\eta)$ are injective. This implies that $\text{ord}(\epsilon) \leq \text{ord}(\zeta)$ and $\text{ord}(\epsilon) \leq \text{ord}(\eta)$.

(a) Now it suffices to prove:

(*) If $n < r$ then $1 + \text{ord}^n(\epsilon) = \text{ord}^n(\zeta)$.

We may assume that $g \in \mathfrak{m}_E^{n+1}$. The cokernel of the injection $I \cap \mathfrak{m}_E^{n+1} \rightarrow \ker(\zeta) \cap \mathfrak{m}_F^{n+1}$ is isomorphic to $\langle g \rangle/\mathfrak{g}m_E$; this proves (*).

(b) By (*) it suffices to prove: If $f \notin \mathfrak{m}_A^{n+1}$ then $\text{ord}^n(\zeta) \leq \text{ord}^n(\eta)$. We may assume $g \in \langle y \rangle$. Since $f \notin \mathfrak{m}_A^{n+1}$ we have $g \notin \mathfrak{m}_E^{n+1}$. Using that $I \cap \langle y \rangle = 0$, one shows that the mapping $\ker(\zeta) \cap \mathfrak{m}_F^{n+1} \rightarrow \ker(\eta) \cap \mathfrak{m}_G^{n+1}$ is injective.

REMARK. Usually (1.6) (a) is applied in the situation where f itself is A -regular, $m = 1$ and $x_1 = f$.

(1.7) If X is a k -scheme and $x \in X$ then (X, x) is called a *pointed k -scheme*. We define $\text{ord}(x, X/k) := \text{ord}(\mathcal{O}_{X, x}/k)$. Pointed k -schemes (X, x) and (Y, y) are called *smoothly equivalent* if there are smooth k -morphisms $f: Z \rightarrow X$, $g: Z \rightarrow Y$ and a point $z \in Z$ with $f(z) = x$, $g(z) = y$. This is an equivalence relation on the class of pointed k -schemes, to be denoted by $(X, x) \sim (Y, y)$. See [12, IV 17] for the definition and the basic properties of smooth morphisms.

THEOREM. If $(X, x) \sim (Y, y)$ then $\text{ord}(x, X/k) = \text{ord}(y, Y/k)$.

PROOF. We may assume that there is a smooth k -morphism $f: X \rightarrow Y$ with $f(x) = y$.

Using the regularity of the noetherian local ring $\mathcal{O}_{X, x}/\mathfrak{m}_y \mathcal{O}_{X, x}$ and the arguments of the proof of [12, IV 19.2.9], we construct a subscheme Z of X containing x such that $\mathcal{O}_{Z, x} = \mathcal{O}_{X, x}/\langle \mathbf{a} \rangle$ where \mathbf{a} is an $\mathcal{O}_{X, x}$ -regular sequence, that $\mathcal{O}_{Y, y} \rightarrow \mathcal{O}_{Z, x}$ is flat and that $\mathfrak{m}_y \mathcal{O}_{Z, x}$ is the maximal ideal of $\mathcal{O}_{Z, x}$. By (1.6) (a) we have $\text{ord}(x, X/k) \leq \text{ord}(x, Z/k)$. Using (1.3) one proves that $\text{ord}(x, Z/k) \leq \text{ord}(y, Y/k)$.

We may assume that $Y = \text{Spec } A$ and $y = \mathfrak{m}_A$ where A is a local k -algebra. Choose a versal linear extension $\epsilon: E \rightarrow A$ over k . By [12, IV 18.1.1] there is a smooth E -algebra R such that $\text{Spec}(A \otimes_E R)$ is isomorphic to an open neigh-

bourhood of x in X . So $\mathcal{O}_{X,x} \cong (A \otimes_E R)_{\mathfrak{p}}$ for some $\mathfrak{p} \in \text{Spec}(A \otimes_E R)$ contracting to \mathfrak{m}_A . By (1.5) we have $\text{ord}(A/E) \leq \text{ord}((A \otimes_E R)_{\mathfrak{p}}/R)$. It is easy to see that this implies $\text{ord}(\mathfrak{y}, Y/k) \leq \text{ord}(x, X/k)$.

(1.8) The following remark will not be used in the sequel. For proofs and details we refer to [13].

REMARK. Let A be a noetherian local k -algebra. Then $\text{ord}(A/k)$ is a partition. It is equal to $\text{ord}(\hat{A}/k)$ where \hat{A} is the completion of A . If k is noetherian regular and A is of essentially finite type over k , then $\text{ord}(A/k) = \text{ord}(A/\mathbb{Z})$. A is regular if and only if $\text{ord}(A/\mathbb{Z}) = 0$. If $A = R/J$ where J is an ideal in a noetherian regular local ring R , then $\text{ord}(A/\mathbb{Z})$ is determined by the sequence $\nu_*(J)$, cf. [14, p. 209].

2. The nilpotent scheme.

(2.1) Consider an action h of an affine group scheme G on an affine scheme X over k . We have the morphisms $h, \text{pr}_2: G \times_k X \rightrightarrows X$. The orbit Gx of $x \in X$ is defined as the subset $h(\text{pr}_2^{-1}(x))$ of X . Let V be a subscheme of X . Let U be the open set where the induced morphism $h^V: G \times_k V \rightarrow X$ is smooth. V is called a *cross section at x* if $x \in V$ and $e_{(V)}(x) \in U$. Here $e_{(V)}: V \rightarrow G \times_k V$ is induced by the unit $e \in G(k)$. The subscheme V is called a *global cross section* if $U \rightarrow \text{Spec}(k)$ is surjective. V is called an *invariant subscheme* if the morphism h^V factorizes over V .

Let $A(X)^G$ be the equalizer of the comorphisms $A(X) \rightrightarrows A(G) \otimes_k A(X)$. If Y is an affine k -scheme, a G -invariant k -morphism $f: X \rightarrow Y$ corresponds to a comorphism $A(Y) \rightarrow A(X)$ factorizing over $A(X)^G$. We define the *affine quotient* of the action by $[X/G] := \text{Spec}(A(X)^G)$. It is called *universal* if the induced morphism $[X_{(R)}/G_{(R)}] \rightarrow [X/G]_{(R)}$ is an isomorphism for any k -algebra R .

REMARKS. (a) Let G be smooth over k . Then pr_2 and h are smooth morphisms. If $x' \in Gx$ then $(X, x) \sim (X, x')$, cf. (1.7). If V is a cross section at x then $(X, x) \sim (V, x)$.

(b) The condition, that the affine quotient $[X/G]$ is universal, is a local condition on $\text{Spec}(k)$ for the topology $(f p q c)$, cf. [8, IV], see [13, p. 38]. If k is a field any affine quotient is universal.

(c) Other types of quotients are discussed in [17, p. 3].

(2.2) PROPOSITION. Assume in (2.1) that the morphism $X \rightarrow \text{Spec}(k)$ is smooth and irreducible cf. [12, IV 4.5.5], and that V is affine and a global cross section.

- (a) The morphism $A(X)^G \rightarrow A(V)$ is injective.
- (b) If $A(X)^G \rightarrow A(V)$ is bijective then $[X/G]$ is universal.

PROOF. (a) Consider a nonzero $f \in A(X)^G$. Assume that $f|_V = 0$. There is a commutative diagram (i), so we have $f \circ h^V = 0$.

$$(i) \quad \begin{array}{ccc} G \times_k V & \xrightarrow{\text{pr}_2} & V \\ h^V \downarrow & & \downarrow f|_V \\ X & \xrightarrow{f} & \text{Spec}(k[T]) \end{array}$$

The morphism h^V is flat on U , so $h^V(U)$ is an open subset of X with $f|_{h^V(U)} = 0$. Since $f \neq 0$, there is a generic point x of $\text{Supp}(f\mathcal{O}_X)$. Let $\mathfrak{p} \in \text{Spec}(k)$ be the image of x . Let ξ be the unique generic point of $X \otimes_k k(\mathfrak{p})$. As $h^V(U) \rightarrow \text{Spec}(k)$ is surjective we have $\xi \in h^V(U)$ and hence $x \neq \xi$. Since $\mathcal{O}_{X,x} \otimes_k k(\mathfrak{p})$ is regular there is an $\mathcal{O}_{X,x} \otimes_k k(\mathfrak{p})$ -regular element $t \in \mathfrak{m}_x$. By [12, IV 11.3.7], t is $\mathcal{O}_{X,x}$ -regular. It is easy to see that this contradicts the choice of x . The argument used here was suggested by P. Deligne.

(b) Let R be a k -algebra. We have to prove that $u: A(X)^G \otimes R \rightarrow A(X_{(R)})^{G(R)}$ is bijective. As the assumptions of (a) are stable under base-change, the morphism $v: A(X_{(R)})^{G(R)} \rightarrow A(V) \otimes R$ is injective by (a). So it suffices to observe that $v \circ u$ is bijective.

(2.3) Let G be a smooth affine group scheme over k . Recall that the Lie-algebra $\text{Lie}(G)$ is defined as the group functor such that $\text{Lie}(G)(R)$ is the (additively written) kernel of the morphism $G(R[\delta]/\langle \delta^2 \rangle) \rightarrow G(R)$ induced by $\delta \mapsto 0$ where R is a k -algebra. $\text{Lie}(G)$ is a smooth affine group scheme, in fact a vector bundle. There is a canonical action of G on $\text{Lie}(G)$. If R is a k -algebra then $\text{Lie}(G)_{(R)} = \text{Lie}(G_{(R)})$. See [8, II 4]. Usually we write $\mathfrak{g} := \text{Lie}(G)$.

If K is a field over k , a section $x \in \mathfrak{g}(K) = \text{Lie}(G_{(K)})(K)$ is nilpotent if and only if its image is a nilpotent endomorphism of F for some (or any) immersion of $G_{(K)}$ in a K -group $Gl(F)$, cf. [2, p. 151]. A point $x \in \mathfrak{g}$ is called *nilpotent* if the corresponding section $x \in \mathfrak{g}(k(x))$ is nilpotent.

(2.4) DEFINITION. Let G be a reductive group scheme over k , cf. [8, XIX 2.7]. If the affine quotient $[\mathfrak{g}/G]$ is universal, cf. (2.1), then we define the *nilpotent scheme* $N(G) := p^{-1}p(0)$ where $0 \in \mathfrak{g}(k)$ is the zero section and $p: \mathfrak{g} \rightarrow [\mathfrak{g}/G]$ is the quotient morphism.

PROPOSITION. Let $N(G)$ be defined.

- (a) $N(G)$ is a G -invariant closed subscheme of \mathfrak{g} .
- (b) If R is a k -algebra then $N(G_{(R)}) = N(G)_{(R)}$.
- (c) A point $x \in \mathfrak{g}$ is nilpotent if and only if $x \in N(G)$.

PROOF. (a) is trivial. (b) is a consequence of the assumption that $[\mathfrak{g}/G]$ is universal. (c) By (b) we may assume that k is a field and that $x \in \mathfrak{g}(k)$. Now it is well known. The “if-part” follows from Cayley-Hamilton by an embedding

of G in some $Gl(F)$. The “only-if-part” is a consequence of the following

LEMMA. *Let G be a reductive k -group over a field k . If $x \in \mathfrak{g}(k)$ has the additive Jordan decomposition $x = x_s + x_n$, then x_s is in the closure of the orbit Gx .*

PROOF. Adapt [22, (4.4)] or [21, p. 92].

(2.5) Let G be a reductive group scheme over k . By (2.1)(b) the existence of a nilpotent scheme is a local condition on $\text{Spec}(k)$ for the topology $(f p q c)$. So we assume that G is of constant type (cf. [8, XXII 2.7]) with specified root system $R = (M, R, \rho)$, i.e. a root system R in a given lattice M (cf. [7, p. 287]). Let t be the torsion index (cf. [7, p. 294]). Let f be the connection index (cf. [4, p. 167]). Consider the following conditions:

- (i) $t^{-1} \in k$ and if $R \cap 2M \neq \emptyset$ then $1/2 \in k$, cf. [7, p. 296].
- (ii) $t^{-1}f^{-1} \in k$.
- (iii) If R has a component of type A_l then $(l + 1)^{-1} \in k$, of type B_p, C_p, D_p, G_2 then $1/2 \in k$, of type E_6, E_7, F_4 then $1/6 \in k$, of type E_8 then $1/30 \in k$.

The conditions (ii) and (iii) are equivalent and imply (i).

(2.6) **THEOREM.** *Let G be as in (2.5) satisfying condition (i).*

(a) *The affine quotient $[\mathfrak{g}/G]$ is universal. The quotient morphisms $p: \mathfrak{g} \rightarrow [\mathfrak{g}/G]$ is flat. $N(G)$ is defined and flat over k .*

(b) *Let T be a maximal torus of G with Weyl group W , cf. [8, XXII 3]. Put $\mathfrak{t} := \text{Lie}(T)$. The affine quotient $[\mathfrak{t}/W]$ is universal. The canonical morphism $[\mathfrak{t}/W] \rightarrow [\mathfrak{g}/G]$ is an isomorphism.*

(c) *Assume that (2.5) (ii) holds. Let $\pi: G \rightarrow \text{ad}(G)$ be the projection onto the adjoint group, cf. [8, XXII 4.3]. Then $N(\text{ad}(G))$ is defined and equal to $N(G)$.*

PROOF. (1) We may assume that G is split with respect to a (resp. the) maximal torus T , cf. [8, XXII 2.3]. Now $T = D_S(M)$ and $A(\mathfrak{t}) = S(M) \otimes k$. The group scheme W is the constant group scheme associated to the abstract Weyl group of R . By (2.5) (i) and [7, pp. 295, 296] the affine quotient $[\mathfrak{t}/W]$ is universal and the quotient morphism $\mathfrak{t} \rightarrow [\mathfrak{t}/W]$ is flat.

(2) By [8, XIII 5.1] and [12, IV 17.8.3] the subscheme \mathfrak{t} is a global cross section for the action of G on \mathfrak{g} , cf. (2.1). By (2.2) this implies that $A(\mathfrak{g})^G \rightarrow A(\mathfrak{t})^W$ is injective.

(3) We may assume that $k = \mathbf{Z}[1/m]$, cf. [8, XXV 1]. It follows from [20, II 3.17'] and [22, p. 220] that $A(\mathfrak{g})^G \otimes_k \mathbf{Q} \rightarrow A(\mathfrak{t})^W \otimes_k \mathbf{Q}$ is bijective. Consider $a \in A(\mathfrak{t})^W$. There is $a_1 \in A(\mathfrak{g})^G$ and a nonzero $s \in k$ with $a_1 | \mathfrak{t} = sa$. Put $R = k/\langle s \rangle$. Now $a_1 \otimes 1_R |_{\mathfrak{t}(R)} = 0$, so by (2) we have $a_1 \otimes 1_R = 0$ in

$A(\mathfrak{g}) \otimes R$. So there is $a_2 \in A(\mathfrak{g})$ with $a_1 = sa_2$. Since s is $A(G) \otimes A(\mathfrak{g})$ -regular we have $a_2 \in A(\mathfrak{g})^G$. Since s is $A(\mathfrak{t})$ -regular we have $a = a_2|_{\mathfrak{t}}$. This proves that $A(\mathfrak{g})^G \rightarrow A(\mathfrak{t})^W$ is bijective. So we have proved (b).

(4) With (b) and (2) one proves that $[\mathfrak{g}/G]$ is universal in the same way as in (2.2) (b). Let U be the open subset of \mathfrak{g} where p is flat. Since $\mathfrak{t} \rightarrow [\mathfrak{g}/G]$ is flat by (1) and (b), and $\mathfrak{t} \subset \mathfrak{g}$ is a regular immersion, we have $\mathfrak{t} \subset U$ by [12, O_{IV} 15.1.16]. As U is G -invariant this implies $U = \mathfrak{g}$ by the lemma in (2.4). The other assertions of (a) follow immediately.

(5) In the notations of [8, XXII], condition (2.5) (ii) implies that the central isogenies $G \rightarrow \text{corad}(G) \otimes \text{ss}(G)$ and $\text{ss}(G) \rightarrow \text{ad}(G)$ are étale morphisms, by [8, VIII 2.1] and [8, XXI 6.5]. So we have an isomorphism

$$A(\text{Lie}(\text{ad}(G)))^{\text{ad}(G)} \otimes A(\text{Lie}(\text{corad}(G))) \cong A(\mathfrak{g})^G.$$

With this isomorphism one proves (c).

REMARKS. (i) Assume that the order of the Weyl group is invertible in k . By [22, (6.9)] the morphism p is normal cf. [12, IV 6.8.1]. (ii) If $l \geq 2$ there is a semisimple group scheme G of type D_l over Z such that $[\mathfrak{g}/G]$ is not universal.

(2.7) COROLLARY. *Let G be as in (2.6). Let d_1, \dots, d_r be the degrees of R . Consider the partition λ defined by $\lambda_i := d_{r+1-i} - 1$ if $i \leq r$ and $\lambda_{r+1} := 0$. Let x be a point of the zero section of \mathfrak{g} . Then $\text{ord}(x, N(G)/k) = \lambda$.*

PROOF. By (1.7) we may assume that G is split with maximal torus T . Let $A(\mathfrak{g})^G = k[a_1, \dots, a_r]$ where a_1, \dots, a_r are algebraically independent and a_i is homogeneous of degree $d_{r+1-i} = 1 + \lambda_i$, cf. [7, Theorem 3]. We have $\hat{O}_{N(G),x} = \hat{O}_{\mathfrak{g},x}/(a)$. Since $\hat{O}_{\mathfrak{g},x}$ is flat over $A(\mathfrak{g})^G$ the sequence a is $\hat{O}_{\mathfrak{g},x}$ -regular. By (1.6) (a) this implies that $\text{ord}(x, N(G)/k) \geq \lambda$. Let $\mathfrak{p} \in \text{Spec}(k)$ be the image of x . By (1.5) we may replace k by the residue field $k(\mathfrak{p})$. Now the assertion follows from the example in (1.4).

REMARK. If k is noetherian regular the multiplicity of the local ring $\hat{O}_{N(G),x}$ is equal to $\prod_{i=1}^r d_i$, i.e. the order of the Weyl group. This is proved in [13, p. 55] using the methods of [18]. Compare [16, p. 386].

3. In the classical Lie-algebras.

(3.1) We fix a free k -module F of rank n . The scheme $\text{End}(F)$ is defined by $\text{End}(F)(R) := \text{End}_R(F \otimes_k R)$, cf. [11, I 9]. The group scheme $GL(F)$ (resp. $SL(F)$) is the open (resp. closed) subscheme of $\text{End}(F)$ where the function $\det \in A(\text{End}(F))$ is invertible (resp. where $\det = 1$). $GL(F)$ and $SL(F)$ are reductive group schemes over k of type A_{n-1} , cf. [6] and [8]. $\text{End}(F)$ is identified with $\text{Lie}(GL(F))$ by $x \leftrightarrow 1 + \delta x$ where $x \in \text{End}(F)(R)$, see (2.3) or [8, II 4]. Now $\text{Lie}(SL(F))$ consists of the endomorphisms with zero trace.

Assume $1/2 \in k$. Let ϵ be 0 or 1. An ϵ -form ϕ on F is a nondegenerate bilinear form $\phi: F \times F \rightarrow k$ which is symmetric if $\epsilon = 0$, alternating if $\epsilon = 1$. By “nondegenerate” we mean that the mapping $F \rightarrow F^\vee$ defined by $f \mapsto \phi(f, -)$ is bijective. Let ϕ be an ϵ -form. The subgroup functor $G'(F, \phi)$ of $GL(F)$ is defined by $x \in G'(F, \phi)(R)$ if and only if

$$\phi(xf, xg) = \phi(f, g) \quad (f, g \in F \otimes R).$$

We define $G(F, \phi) := G'(F, \phi) \cap SI(F)$. If $\epsilon = 0$ then $G(F, \phi)$ is the special orthogonal group scheme. If $\epsilon = 1$ then $G(F, \phi) = G'(F, \phi)$; it is the symplectic group scheme. Put $l := [\frac{1}{2}n]$ and $\zeta := n - 2l$. So ζ is 0 or 1. If $\epsilon = 1$ then $\zeta = 0$. Now $G(F, \phi)$ is a semisimple group scheme of type B_l if $\epsilon = 0, \zeta = 1$, of type C_l if $\epsilon = 1, \zeta = 0$, of type D_l if $\epsilon = \zeta = 0$, cf. [6] and [8]. The common Lie-algebra of $G(F, \phi)$ and $G'(F, \phi)$ is denoted by $\mathfrak{g}(F, \phi)$. For $x \in \text{End}(F)(R)$ we have $x \in \mathfrak{g}(F, \phi)(R)$ if and only if

$$\phi(xf, g) + \phi(f, xg) = 0 \quad (f, g \in F \otimes R).$$

CONVENTION. In the rest of this paper we consider two cases.

Case I. $G := G' := GL(F), l := n$.

Case II. (ϵ, ζ) where $\epsilon, \zeta \in \{0, 1\}, \epsilon + \zeta \leq 1: 1/2 \in k, \phi$ is an ϵ -form on $F, G := G(F, \phi), G' := G'(F, \phi), n = 2l + \zeta$.

In both cases l is the reductive rank of G . We put $\mathfrak{g} := \text{Lie}(G)$. While considering Case II it is convenient to label concepts introduced for Case I with the index l , e.g. $\mathfrak{g} \subset \mathfrak{g}_l = \text{End}(F)$.

(3.2) LEMMA. Case II. Let ϕ_1 be another ϵ -form on F . Then there is a faithfully flat étale k -algebra R such that ϕ_1 and ϕ induce equivalent forms on $F \otimes R$.

PROOF. By [15, pp. 34, 35] the scheme $\text{Isom}(\phi_1, \phi)$ is smooth over k . If K is an algebraically closed field over k then $\text{Isom}(\phi_1, \phi)(K) \neq \emptyset$. Hence by [12, IV 17.16.3] there is a faithfully flat étale k -algebra R with $\text{Isom}(\phi_1, \phi)(R) \neq \emptyset$.

(3.3) DEFINITION. In Case I, $z \in \mathfrak{g}(R)$ is called a *standard nilpotent* with *base-data* (f, λ) if $f = (f_1, \dots, f_r)$ is a sequence in $F \otimes R$ and λ is a partition, such that $\lambda^1 = r$, that the set $\{z^a f_i\}$, where $1 \leq i \leq r$ and $0 \leq a < \lambda_i$, is a basis of $F \otimes R$ and that $z^a f_i = 0$ if $a \geq \lambda_i$.

In Case II, $z \in \mathfrak{g}(R)$ is called a *standard nilpotent* with *base-data* $(f, \lambda, \beta, \alpha)$ if $z \in \mathfrak{g}_l(R)$ is a standard nilpotent with base-data $(f, \lambda), \beta$ is a permutation of $\{1, \dots, r\}$ where $r = \lambda^1$, and $\alpha: \{1, \dots, r\} \rightarrow R$ is a mapping such that

$$(1) \quad \begin{cases} \phi(z^a f_i, z^b f_j) = (-1)^a \alpha(i) & \text{if } j = \beta i \text{ and } a + b + 1 = \lambda_i, \\ \phi(z^a f_i, z^b f_j) = 0 & \text{otherwise.} \end{cases}$$

REMARK. Clearly $|\lambda| = n$. In Case II the assumptions imply

$$(2) \quad \alpha(i)^{-1} \in R, \quad \beta^2 = \text{id}, \quad \lambda_{\beta i} = \lambda_i, \quad \alpha(\beta i) = (-1)^{\lambda_i^{-1} + \epsilon} \alpha(i).$$

(3.4) The set \mathcal{P}_ϵ is defined as the subset of \mathcal{P} consisting of the partitions λ such that for any $m \geq 1$ with $m \equiv \epsilon \pmod{2}$ the number of indices i with $\lambda_i = m$ is even. These partitions are called orthogonal, resp. symplectic; in [10, p. 556]. We define $\mathcal{P}(n)$ as the set of partitions λ with $|\lambda| = n$, and $\mathcal{P}_\epsilon(n) := \mathcal{P}_\epsilon \cap \mathcal{P}(n)$. We write $\mathcal{P}_{(\epsilon)}$ to denote \mathcal{P} in Case I and \mathcal{P}_ϵ in Case II. So in (3.3) we have $\lambda \in \mathcal{P}_{(\epsilon)}(n)$.

(3.5) If $x \in \mathfrak{g}$ is nilpotent, cf. (2.3), then the section $x \in \mathfrak{g}(\mathbf{k}(x))$ is a standard nilpotent by [20, IV]. Let $\lambda \in \mathcal{P}(n)$. We define $\mathfrak{D}(\lambda)$ as the set of points $x \in \mathfrak{g}$ such that the section x is a standard nilpotent with partition λ . In case II we have $\mathfrak{D}(\lambda) = \mathfrak{D}_f(\lambda) \cap \mathfrak{g}$, and $\mathfrak{D}(\lambda) \neq \emptyset$ if and only if $\lambda \in \mathcal{P}_\epsilon(n)$.

Let k be a field and $x \in \mathfrak{D}(\lambda)$. By [20, IV] we have $\mathfrak{D}(\lambda) = G'x$, and $\mathfrak{D}(\lambda) \neq Gx$ if and only if we are in the *very-even case*: Case II (0, 0) with λ_i even for all i .

(3.6) LEMMA. *Case I. If $\lambda \in \mathcal{P}(n)$, there is a standard nilpotent element $z \in \mathfrak{g}(k)$ with partition λ .*

Case II. If λ, β and α satisfy the conditions (3.3)(2), then there is an ϵ -form ϕ_1 on F and a standard nilpotent element $z \in \mathfrak{g}(F, \phi_1)(k)$ with base-data $(\mathbf{f}, \lambda, \beta, \alpha)$ for some sequence \mathbf{f} in F .

PROOF. Case I is trivial.

Case II. Choose a standard nilpotent $z \in \mathfrak{g}_f(k)$ with base-data (\mathbf{f}, λ) . Let $\phi_1: F \otimes F \rightarrow k$ be the bilinear form defined by (3.3)(1). One verifies that ϕ_1 is an ϵ -form on F with $z \in \mathfrak{g}(F, \phi_1)(k)$.

(3.7) *The standard cross section.* Let $z \in \mathfrak{g}(k)$ be a standard nilpotent element with base-data (\mathbf{f}, λ) , resp. $(\mathbf{f}, \lambda, \beta, \alpha)$. Below we construct a linear subscheme $L \subset \mathfrak{g}$ such that $\mathfrak{g}(R) = [\mathfrak{g}(R), z] \oplus L(R)$ for any k -algebra R . This implies that *the subscheme $z + L \subset \mathfrak{g}$ is a cross section for the adjoint action of G in all points of the section z , cf. (2.1).* In fact the tangent morphism of $\text{Ad}: G \times (z + L) \rightarrow \mathfrak{g}$ at the section (e, z) is the surjective morphism $\mathfrak{g} \oplus L \rightarrow \mathfrak{g}$ given by $(x, y) \mapsto [x, z] + y$. So smoothness of Ad at (e, z) follows from [12, IV 17.11.1].

Let Ψ be the set of pairs (i, a) with $0 \leq a < \lambda_i$. Put $f(i, a) := z^a f_i$. Then $\{f(\psi) \mid \psi \in \Psi\}$ is a basis of F . Let $\{u(\psi)\}$ be the dual basis of F^\vee . This means that $\{u(\psi)\}$ is the basis of $F^\vee = \text{Hom}(F, k)$ with

$$\langle u(\psi), f(\psi') \rangle = \delta_{\psi, \psi'} \quad (\text{Kronecker delta}).$$

The coordinates $\xi(\psi; \psi')$ on \mathfrak{g}_f are defined by $\xi(\psi; \psi')(x) = \langle u(\psi), xf(\psi') \rangle$.

Clearly $\{\xi(\psi; \psi') \mid \psi, \psi' \in \Psi\}$ is a basis of $\mathfrak{g}_r(k)$. Let $\{e(\psi; \psi')\}$ be the dual basis of $\mathfrak{g}_r(k)$. We have

$$e(\psi; \psi')f(\psi'') = \delta_{\psi', \psi''}f(\psi),$$

$$[e(i, a; j, b), z] = e(i, a; j, b - 1) - e(i, a + 1; j, b)$$

where $e(i, a; j, b) = 0$ if $a \geq \lambda_i$ or $b < 0$. In Case I let \mathfrak{g}_{ij} , L_{ij} and L be the linear subschemes of \mathfrak{g} defined by

$$\mathfrak{g}_{ij}(R) := \sum_{a,b} Re(i, a; j, b),$$

$$L_{ij}(R) := \sum Re(i, a; j, \lambda_j - 1), \quad 0 \leq a < \min(\lambda_i, \lambda_j),$$

$$L(R) := \sum_{i,j} L_{ij}(R).$$

Then we have $\mathfrak{g}_{ij} = [\mathfrak{g}_{ij}, z] \oplus L_{ij}$ and $\mathfrak{g} = [\mathfrak{g}, z] \oplus L$.

Case II. The coordinates $\eta(\psi; \psi')$ on \mathfrak{g} are defined by $\eta(\psi; \psi')(x) = \phi(f(\psi), xf(\psi'))$. Since $\eta(\psi; \psi') = (-1)^{1+\epsilon} \eta(\psi'; \psi)$ we have a basis of $\mathfrak{g}(k)^\sim$ consisting of the $\eta(i, a; j, b)$ with $i < j$, or $i = j$ and $a < b + \epsilon$. Let $y(\psi; \psi')$ be the dual basis of $\mathfrak{g}(k)$. One shows that

$$[y(i, a; j, b), z] = y(i, a; j, b - 1) + y(i, a - 1; j, b)$$

if $i < j$, or $i = j$ and $a < b - 1$,

$$[y(i, a; i, a + 1), z] = y(i, a - 1; i, a + 1) + 2\epsilon y(i, a; i, a),$$

$$[y(i, a; i, a), z] = y(i, a - 1; i, a) \quad \text{if } \epsilon = 1,$$

where $y(\psi; \psi') = 0$ if not yet defined. For $i \leq j$ let \mathfrak{g}_{ij} , L_{ij} and L be the linear subschemes of \mathfrak{g} defined by

$$\mathfrak{g}_{ij}(R) := \sum_{a,b} Ry(i, a; j, b),$$

$$L_{ij}(R) := \sum_b Ry(i, \lambda_i - 1; j, b) \quad \text{if } i < j,$$

$$L_{ii}(R) := \sum Ry(i, \lambda_i - 2 + \epsilon - a; i, \lambda_i - 1 - a), \quad 0 \leq a \leq \frac{1}{2}(\lambda_i - 2 + \epsilon),$$

$$L(R) := \sum_{i < j} L_{ij}(R).$$

Then we have $\mathfrak{g}_{ij} = [\mathfrak{g}_{ij}, z] \oplus L_{ij}$ and $\mathfrak{g} = [\mathfrak{g}, z] \oplus L$.

Let F^\sim be identified with F in such a way that $\langle u, f \rangle = \phi(u, f)$. Putting $|\lambda_a = (-1)^a \alpha(i)^{-1}$, we get the following glossary:

$$\begin{aligned}
 |i|_a f(i, a) &= u(\beta i, \lambda_i - 1 - a), \\
 |i|_a \eta(i, a; j, b) &= \xi(\beta i, \lambda_i - 1 - a; j, b) |g, \\
 y(i, a; j, b) &= |i|_a e(\beta i, \lambda_i - 1 - a; j, b) - (-1)^\epsilon |j|_b e(\beta j, \lambda_j - 1 - b; i, a) \\
 &\hspace{15em} \text{if } i < j, \text{ or } i = j \text{ and } a < b, \\
 y(i, a; i, a) &= |i|_a e(\beta i, \lambda_i - 1 - a; i, a) \quad \text{if } \epsilon = 1, \\
 y(i, a; j, b) &= 0 \quad \text{otherwise.}
 \end{aligned}$$

REMARK. In Case I our $z + L$ is one of the cross sections of Arnold [1].
 (3.8) An elementary calculation shows that $L(k)$ is a free k -module of rank $l + \gamma_{(\epsilon)}(\lambda)$ if we write $\gamma_{(\epsilon)}(\lambda) := \gamma(\lambda)$ in Case I and $\gamma_{(\epsilon)}(\lambda) := \gamma_\epsilon(\lambda)$ in Case II where

$$\begin{aligned}
 \gamma(\lambda) &:= 2 \sum (i - 1) \lambda_i \quad \text{if } \lambda \in P(n), \\
 \gamma_\epsilon(\lambda) &:= \sum (i - 1) \lambda_i + (2\epsilon - 1) [\frac{1}{2} \# \{i | \lambda_i \equiv 1 \pmod{2}\}] \quad \text{if } \lambda \in P_\epsilon(n).
 \end{aligned}$$

Now the centralizer of z in $\mathfrak{g}(k)$ is also a free k -module of rank $l + \gamma_{(\epsilon)}(\lambda)$. By [20, I 5.6] we have the following:

COROLLARY. Assume that k is a field.

- (a) If $x \in \mathfrak{D}(\lambda)$ then $\dim(Gx) = \dim(\mathfrak{g}) - l - \gamma_{(\epsilon)}(\lambda)$.
- (b) There is a unique nilpotent orbit C_{reg} of maximal dimension.
 $C_{\text{reg}} = \mathfrak{D}(v)$ where $v_* = (n)$ in the Cases I and II ($\epsilon, 1 - \epsilon$) and $v_* = (\eta - 1, 1)$ in Case II $(0, 0)$. We have $\dim(C_{\text{reg}}) = \dim(\mathfrak{g}) - l$. If C is another nilpotent orbit in \mathfrak{g} then $\dim(C) \leq \dim(\mathfrak{g}) - l - 2$.

See also [1], [20, IV 2.28] and [21, p. 136].

(3.9) The mapping $\Sigma: P(n) \rightarrow P$ is defined by $(\Sigma\lambda)^m := \sum_{i>m} \lambda_i$ ($m \in \mathbb{N}$). As the corresponding propositions in [10, p. 567] are false, we shall prove the following:

PROPOSITION. Let $\lambda, \mu \in P_{(\epsilon)}(n)$ be such that

$$\{\mu\} = \{v \in P_{(\epsilon)}(n) | \Sigma\lambda > \Sigma v \geq \Sigma\mu\}.$$

Then there are $\rho, \sigma, \tau \in P_{(\epsilon)}$ with $\lambda = \rho + \sigma$, $\mu = \rho + \tau$ and σ, τ as described in the following table.

Case	σ_*	τ_*	Res ^t ictions
I	(p, q)	$(p + 1, q - 1)$	$p \geq q \geq 1$
II	(a) (p, p)	$(p + 1, p - 1)$	$p \geq 1$ and $p \equiv \epsilon(2)$
	(b_1) (p, q)	$(p + 2, q - 2)$	$p \geq q \geq 2$
	(b_2) (p, p, q)	$(p + 1, p + 1, q - 2)$	$p \geq q \geq 2$
	(b_3) (p, q, q)	$(p + 2, q - 1, q - 1)$	$p \geq q \geq 1$
	(b_4) (p, p, q, q)	$(p + 1, p + 1, q - 1, q - 1)$	$p \geq q \geq 1$

} and $p \equiv q \not\equiv \epsilon(2)$

PROOF. See (1.1) for the addition of partitions. Case I may be left to the reader. Case II. It is easy to see that we may assume disjointness: if $\lambda_i = \mu_j$ then $\lambda_i = 0$. Now we have to prove $\lambda = \sigma, \mu = \tau$ as in the table.

(a) Assume that there is a minimal $l \in \mathbb{N}$ with $\lambda_l \neq 0$ and $\lambda_l \equiv \epsilon(2)$. There is a maximal $m \in \mathbb{N}$ with $\lambda_m = \lambda_l$. Define $\nu \in \mathcal{P}_\epsilon(n)$ by $\nu_l = \lambda_l + 1, \nu_m = \lambda_m - 1$ and $\nu_i = \lambda_i$ otherwise. Clearly $\Sigma\lambda > \Sigma\nu$. Using disjointness one proves $\Sigma\nu \geq \Sigma\mu$, so that $\nu = \mu$ and, again by disjointness, we are in case (a).

(b) Now $\lambda_i \not\equiv \epsilon(2)$ whenever $\lambda_i > 0$. By disjointness there is an $m \in \mathbb{N}$ with $\mu_m > \lambda_1 > \mu_{m+1}$. It is easy to see that we can define $\nu \in \mathcal{P}_\epsilon(n)$ satisfying $\Sigma\nu > \Sigma\mu$ as follows:

- If $\mu_m \not\equiv \epsilon(2)$, then $\nu_m = \mu_m - 2$ and $\nu_i = \mu_i$ if $i < m$;
- if $\mu_m \equiv \epsilon(2)$, then $\nu_{m-1} = \nu_m = \mu_m - 1$ and $\nu_i = \mu_i$ if $i < m - 1$;
- if $\mu_{m+1} \not\equiv \epsilon(2)$, then $\nu_{m+1} = \mu_{m+1} + 2$ and $\nu_i = \mu_i$ if $i > m + 1$;
- if $\mu_{m+1} \equiv \epsilon(2)$, then $\nu_{m+1} = \nu_{m+2} = \mu_{m+1} + 1$ and $\nu_i = \mu_i$ if $i > m + 2$.

One proves that $\Sigma\lambda \geq \Sigma\nu$, so that $\lambda = \nu$ and we are in one of the four cases (b).

(3.10) THEOREM. *Let k be a field. Consider $z \in \mathfrak{D}(\lambda)$ and $x \in \mathfrak{D}(\mu)$. We have $z \in \overline{Gx} - Gx$ if and only if $\Sigma\lambda > \Sigma\mu$.*

REMARK. This theorem is due to Gerstenhaber, see [9, p. 327] and [10, pp. 567–569]. His proof for Case II is incomplete, see (3.9). Our proof seems to be more explicit.

PROOF. We may assume that z and x are rational points. So z is a standard nilpotent in $\mathfrak{g}(k)$ with partition λ . If $i \in \mathbb{N}$ then the endomorphism z^i of F has rank $(\Sigma D\lambda)^i$, see (1.1) for the definition of D .

Assume that $z \in \overline{Gx} - Gx$. The rank of z^i is less than or equal to the rank of x^i . This implies $\Sigma D\lambda \leq \Sigma D\mu$ and hence $\Sigma\lambda \geq \Sigma\mu$ by [9, p. 327]. As $\lambda \neq \mu$ it is easy to see that $\Sigma\lambda > \Sigma\mu$.

Assume that $\Sigma\lambda > \Sigma\mu$. We have to prove that $z \in \overline{Gx}$. We may assume that λ and μ are as in (3.9). So λ and μ are not both very-even, cf. (3.5), and it suffices to prove that $z \in \overline{\mathfrak{D}(\mu)}$. Using the notations of (3.7) we shall construct $y \in \mathfrak{g}(k)$ and a sequence $f(t)$ ($t \in k$) in such a way that $z(t) = z + ty \in$

$\mathfrak{g}(k)$ is a standard nilpotent in $\mathfrak{g}_r(k)$ with base-data $(f(t), \mu)$ if $t \neq 0$. This will prove $z \in \overline{\mathfrak{D}(\mu)}$.

Using a direct sum decomposition we may assume $\rho = 0, \lambda = \sigma, \mu = \tau$; cf. (3.9).

Case I. We have $\lambda_* = (p, q)$ and $\mu_* = (p + 1, q - 1)$. Let $((f_1, f_2), \lambda)$ be base-data for z . Put $y := e(1, q - 1; 2, q - 1)$. Put $f_1(t) := f_2$ and, if $q > 1, f_2(t) := tf_1 - zf_2$. We have

$$\begin{aligned} 0 \leq a \leq q - 1 &\Rightarrow z(t)^a f_1(t) = z^a f_2, \\ q \leq a \leq p &\Rightarrow z(t)^a f_1(t) = tz^{a-1} f_1, \\ 0 \leq a \leq q - 2 &\Rightarrow z(t)^a f_2(t) = tz^a f_1 - z^{a+1} f_2, \\ z(t)^{p+1} f_1(t) &= 0 \quad \text{and} \quad z(t)^{q-1} f_2(t) = 0. \end{aligned}$$

This implies that $z(t) \in \mathfrak{D}(\mu)$ if $t \neq 0$.

Case II. Of the five possibilities, cf. (3.9), we only treat (b_3) and (b_4) with $q \geq 2$. The other cases are easier, see [13, (4.3.7)], and already settled in [10, pp. 568, 569]. We choose convenient base-data $((f_1, \dots, f_r), \lambda, \beta, \alpha)$ for z . The verifications are left to the reader.

(b_3) $\lambda_* = (p, q, q), p \equiv q \not\equiv \epsilon(2), r = 3, \beta = \text{id}, \mu_* = (p + 2, q - 1, q - 1)$. Choose

$$\begin{aligned} y &:= y(1, p - 1; 2, 0) + y(1, p - 1; 3, 0) \\ &= e(1, 0; 2, 0) + e(1, 0; 3, 0) + e(2, q - 1; 1, p - 1) - e(3, q - 1; 1, p - 1), \\ f_1(t) &:= f_2, f_2(t) := zf_2 \text{ and } f_3(t) := z^{p-q+1} f_1 - tf_2 + tf_3. \end{aligned}$$

(b_4) $\lambda_* = (p, p, q, q), p \equiv q \not\equiv \epsilon(2), r = 4, \beta = \text{id}, \mu_* = (p + 1, p + 1, q - 1, q - 1)$. Choose

$$\begin{aligned} y &:= y(1, p - 1; 3, 0) + y(1, p - 1; 4, 0) \\ &\quad + y(2, p - 1; 3, 0) + y(2, p - 1; 4, 0) \\ &= e(1, 0; 3, 0) + e(1, 0; 4, 0) + e(2, 0; 3, 0) + e(2, 0; 4, 0) \\ &\quad + e(3, q - 1; 1, p - 1) - e(4, q - 1; 1, p - 1) \\ &\quad - e(3, q - 1; 2, p - 1) + e(4, q - 1; 2, p - 1), \\ f_1(t) &:= f_1, f_2(t) := f_3, f_3(t) := zf_3 \text{ and } f_4(t) := z^{p-q+1} f_1 - tf_3 + tf_4. \end{aligned}$$

4. The classical nilpotent scheme, singularities.

(4.1) The *symmetrical polynomials* $\sigma_1, \dots, \sigma_n \in A(\text{End}(F))$ are defined by the equation

$$\det(x + T \cdot \text{id}) = T^n + \sum_{m=1}^n T^{n-m} \sigma_m(x)$$

in $R[T]$ where R is a k -algebra and $x \in \text{End}(F)(R)$. They are invariant under the adjoint action of $GL(F)$ on $\text{End}(F)$. Let $X = (x_{ij})$ be the matrix of x with respect to some basis f_1, \dots, f_n of F . Then

$$\sigma_m(x) = \sum \det(x_{ij})_{i,j \in I}$$

where the summation is over all subsets I of $\{1, \dots, n\}$ with $\# I = m$.

Case II. Clearly $\sigma_m | \mathfrak{g} \in A(\mathfrak{g})^G$. Let Φ be the matrix $\phi(f_i, f_j)$. We have $x \in \mathfrak{g}(R)$ if and only if ${}^t X = -\Phi X \Phi^{-1}$. This implies that $\sigma_m | \mathfrak{g} = 0$ if m is odd. Assume $\epsilon = \zeta = 0$. We define $\tau_l \in A(\mathfrak{g})$ by $\tau_l(x) := \text{Pf}(\Phi X)$, where Pf denotes the Pfaffian, cf. [3, §5, no. 2]. Using loc. cit. one proves that $\tau_l^2 = \det(\Phi) \sigma_n$ and that $\tau_l \in A(\mathfrak{g})^G$.

We define the sequence $\mathfrak{a} = (a_1, \dots, a_l)$ in $A(\mathfrak{g})$ as follows. In Case I we put $a_i := \sigma_i$. In Case II ($\epsilon, 1 - \epsilon$) we put $a_i := \sigma_{2i}$. In Case II (0, 0) we put $a_i := \sigma_{2i}$ if $i < l$, and $a_l := \tau_l$.

- THEOREM.** (a) $A(\mathfrak{g})^G$ is the free polynomial ring $k[a_1, \dots, a_l]$.
 (b) The sequence \mathfrak{a} is $A(\mathfrak{g})$ -regular (in any order).
 (c) $N(G) = \text{Spec}(A(\mathfrak{g})/\langle \mathfrak{a} \rangle)$, it is flat over k .
 (d) $N(G)$ is smooth over k in the points of $\mathfrak{D}(\nu)$ where ν is, cf. (3.8)(b).
 (e) If k is a normal ring then $N(G)$ is a normal scheme.

PROOF. (a) Let $u: k[T_1, \dots, T_l] \rightarrow A(\mathfrak{g})^G$ be defined by $T_i \mapsto a_i$. We have to prove that u is bijective. Replacing k by a faithfully flat k -algebra (cf. (3.6) and (3.2)), we may assume the existence of a standard nilpotent $z \in \mathfrak{g}(k)$ with partition ν , cf. (3.8)(b). Let $z + L$ be the cross section of (3.7). By (2.2) the morphism $v: A(\mathfrak{g})^G \rightarrow A(z + L)$ is injective. Case by case one shows that $v \circ u$ is bijective, so that u is bijective.

(b) and (c). By [7], Theorem (2.6) applies. So $A(\mathfrak{g})$ is flat over $A(\mathfrak{g})^G$. So we have (b) and (c).

(d) We may use the cross section of (a). Now $(z + L) \cap N(G)$ is a cross section at z for the action of G on $N(G)$, and the assertion follows from $(z + L) \cap N(G) \cong \text{Spec}(k)$.

(e) By (c) and [12, IV 6.14.1] we may assume that k is a field. Now $N(G)$ is nonsingular in codimension one, by (d) and (3.8)(b). So $N(G)$ is normal by Serre's criterion [12, IV 5.8.6].

REMARKS. (i) There are other ways to prove the theorem, either avoiding (3.7) or avoiding (2.6) and [7]. (ii) It can be shown that $N(SI(F))$ exists and is equal to $N(GL(F))$, where k is arbitrary. Here (2.6) does not apply.

(4.2) If $\lambda \in \mathcal{P}(n)$, the partition $\Sigma \lambda$ is defined in (3.9). Case II ($\epsilon, 1 - \epsilon$). If $\lambda \in \mathcal{P}_\epsilon(n)$ where $n = 2l + 1 - \epsilon$, then we define the partition $\Sigma_\epsilon \lambda$ by $(\Sigma_\epsilon \lambda)_i := (\Sigma \lambda)_{2i - \epsilon}$. Case II (0, 0). If $\lambda \in \mathcal{P}_0(n)$ where $n = 2l$, then we define $\Sigma_0 \lambda$

$:= \theta + \nu$ where $\theta, \nu \in \mathcal{P}$ are given by $\theta_i := (\Sigma\lambda)_{2i+1}$ and $\nu_* := (\frac{1}{2}\lambda^1 - 1)$. Note: in the last case λ^1 is even and $(\Sigma\lambda)_1 = \lambda^1 - 1$. We write $\Sigma_{(\epsilon)}$ to denote Σ in Case I and Σ_ϵ in Case II. $\Sigma_{(1)}$ means that $\epsilon = 0$ is excluded in Case II.

THEOREM. Consider $x \in \mathfrak{D}(\lambda)$. Then $\text{ord}(x, N(G)/k) = \Sigma_{(1)}\lambda$ in Cases I and II (1, 0), and $\text{ord}(x, N(G)/k) \geq \Sigma_0\lambda$ in Case II (0, ζ).

PROOF. (1) By (1.7) we may replace $(N(G), x)$ by a smoothly equivalent pointed scheme. So by (3.6) and (3.2) we may assume the existence of a standard nilpotent $z \in \mathfrak{g}(k)$ with partition λ . By (2.1) (a) we may assume that $x = z(\mathfrak{p})$ for some $\mathfrak{p} \in \text{Spec}(k)$. Put $A := \hat{O}_{\mathfrak{g}, x}$ and $B := \hat{O}_{N(G), x}$. We have $B = A/\langle a \rangle$ where a is the A -regular sequence of (4.1), or rather its image in A .

(2) Let J be the ideal in $A(\mathfrak{g})$ corresponding to the section z . So x corresponds to the prime ideal $J + \mathfrak{p}A(\mathfrak{g})$. We claim

- (a) If $1 \leq i \leq n$ and $m := 1 + (\Sigma\lambda)_{n+1-i}$ then $\sigma_i \in J^m$.
- (b) In Case II (0, 0) we have $\tau_i \in J^m$ where $m := \frac{1}{2}\lambda^1$.

PROOF OF (a). It suffices to consider Case I. Let (\mathfrak{f}, λ) be base-data for z . Using the notation of (3.7) we define

$$\sigma_P := \det \xi(\psi; \psi')_{\psi, \psi' \in P}$$

if $\emptyset \neq P \subset \Psi$. So $\sigma_i = \Sigma \sigma_P$ where the summation is over all P with $\#P = i$. If $\xi(\psi; \psi') \notin J$ then we have $\psi' = (j, a)$, $\psi = (j, a + 1)$ for some j and a . Consider P with $\#P = i$. If π is a permutation of P then one verifies that

$$\#\{(j, a) \in P \mid \pi(j, a) \neq (j, a + 1)\} \geq 1 + (\Sigma\lambda)_{n+1-i} = m.$$

This implies $\sigma_P \in J^m$, proving (a).

PROOF OF (b). We may assume that k is reduced. Now the assertion follows from $\tau_i^2 = \det(\Phi)\sigma_n \in J^{2m}$, cf. (a).

(3) By (1.6)(a) it follows from (2)(a), (b) that $\text{ord}(B/k) \geq \Sigma_{(\epsilon)}\lambda$. This proves the theorem in Case II (0, ζ). In the rest of the proof Case II (0, ζ) is excluded. It suffices to prove

$$(*) \quad \text{ord}(B/k) \leq \Sigma_{(1)}\lambda.$$

By (1.5) and (4.1)(c) we may replace k by an algebraic closure of the field $k(\mathfrak{p})$. So henceforth k is an algebraically closed field. Now x and z may be identified. Let (\mathfrak{f}, λ) , resp. $(\mathfrak{f}, \lambda, \beta, \alpha)$ be its base-data.

(4) We prove (3) (*) by induction on $n = |\lambda|$. The cases with $n \leq 1$ are trivial. So assume $n \geq 2$. Put $r := \lambda^1$. Let ρ be the partition with $\rho_* = (r - 1)$. The partition μ is defined as follows.

Case I. $\mu_r := \lambda_r - 1$, $\mu_i := \lambda_i$ if $i \neq r$.

Case II. If λ_r is even then $\mu_r := \lambda_r - 2$ and $\mu_i := \lambda_i$ otherwise. If λ_r is

odd so that $\lambda_{r-1} = \lambda_r$, then $\mu_{r-1} := \mu_r := \lambda_r - 1$ and $\mu_i := \lambda_i$ otherwise.

One verifies that $\mu \in \mathcal{P}_{(1)}$ and that $\sum_{(1)} \lambda = \rho + \sum_{(1)} \mu$.

We have $F = \sum kf(\psi)$, $\psi \in \Psi$, cf. (3.7). Let P be the subset of Ψ containing $(r, 0)$ and in Case II also $(\beta r, \lambda_r - 1)$. Put $F' := \sum kf(\psi)$, $\psi \notin P$, and $F'' := \sum kf(\psi)$, $\psi \in P$. Clearly $F = F' \oplus F''$. In Case II the form $\phi' := \phi|_{F'}$ is nondegenerate and hence a 1-form on F' . We put $G' := Gl(F')$ in Case I and $G' := G(F', \phi')$ in Case II. So the convention (3.1) concerning G' is not applied here. We put $\mathfrak{g}' := Lie(G')$, etc.

Let

$$\begin{pmatrix} x' & x_2 \\ x_1 & x_3 \end{pmatrix}$$

be the matrix of x with respect to the decomposition $F = F' \oplus F''$. Now x' is a standard nilpotent in $\mathfrak{g}'(k)$ with partition μ . Consider the ring

$$B' := \mathcal{O}_{N(G'), x'} = \mathcal{O}_{\mathfrak{g}', x'} / \langle (\sigma'_i)_{i < n} \rangle.$$

By induction we have $\text{ord}(B'/k) \leq \sum_{(1)} \mu$. One verifies that

$$w \mapsto \begin{pmatrix} w & x_2 \\ x_1 & x_3 \end{pmatrix}$$

defines a regular immersion $u: \mathfrak{g}' \rightarrow \mathfrak{g}$ such that $u(x') = x$, $u^0(\sigma_n) = 0$ and $u^0(\sigma_i) = \sigma'_i$ if $i < n$, where $u^0: A(\mathfrak{g}) \rightarrow A(\mathfrak{g}')$ is the comorphism. Put $R := A / \langle (\sigma_i)_{i < n} \rangle$ so that $B = R / \langle f \rangle$ where f is the image of σ_n in R . Now there is an R -regular sequence x in R such that $B' \cong R / \langle x \rangle$ and $f \in \langle x \rangle$. By (1.6)(b) this implies

$$\text{ord}(B/k) \leq \rho + \text{ord}(B'/k) \leq \rho + \sum_{(1)} \mu = \sum_{(1)} \lambda$$

provided that $f \notin m_R^{r+1}$. So in order to prove the theorem it suffices to prove that

$$(*) \quad \sigma_\Psi \notin \langle (\sigma_P)_{P \neq \Psi} \rangle + m_A^{r+1}$$

where we have used the notation of (2).

(5) In Case II we normalize the base-data of x as follows: $\beta i \neq i$ if and only if λ_i is odd; $|\beta i - i| \leq 1$ for all i ; if $i \geq \beta i$ then $\alpha(i) = 1$. Now $i \leq \beta i$ implies $\alpha(i) = (-1)^{\lambda_i}$. With the notation of (3.7) we define a linear subvariety M of \mathfrak{g} .

Case I. $M := \sum ke(i, 0; j, \lambda_j - 1)$ ($1 \leq i, j \leq r$).

Case II. $M := \sum ky(i, \lambda_i - 1; j, \lambda_j - 1)$ where the summation is over all pairs (i, j) such that $i = j$ or $i \leq \beta i < j \leq \beta j$. So in this case $M \subset M_r$.

The ring $A(x + M)$ is considered as a graded k -algebra such that x corresponds to the augmentation ideal. The functions $\sigma_P|_{x + M}$ are homogeneous,

$\sigma_\Psi |x + M$ is homogeneous of degree r . So it suffices to prove

$$(*) \quad \sigma_\Psi |x + M \notin \langle (\sigma_P |x + M)_{P \neq \Psi} \rangle.$$

(6) *Case I.* It is easy to see that $x + M$ has a subvariety $x_1 + M_1$ such that $\sigma_P |x_1 + M_1 \neq 0$ if and only if $P = \Psi$. This proves (5)(*) and the theorem.

Case II. Consider the subvariety $x_1 + M_1$ of $x + M$ where

$$x_1 := x + \sum_{i > \beta_i} y(i, \lambda_i - 1; i, \lambda_i - 1),$$

$$M_1 := \sum k y(i, \lambda_i - 1; j, \lambda_j - 1) \quad (i \leq \beta_i, i \leq j \leq \beta_j).$$

Now x_1 is a standard nilpotent in $\mathfrak{g}_r(k)$ with base-data (f', λ') such that

$$M_1 = \sum_{i < j} k(e'(i, 0; j, \lambda'_j - 1) + e'(j, 0; i, \lambda'_i - 1))$$

with respect to the new base-data. In order to prove (5)(*) and hence the theorem, it suffices to show that

$$\sigma_\Psi |x_1 + M_1 \notin \langle (\sigma_P |x_1 + M_1)_{P \neq \Psi} \rangle.$$

This is a consequence of the following:

LEMMA. *Assume $\text{char}(k) \neq 2$. Let $r \in \mathbb{N}$. Consider the ring $k[T_{ij}]$ where $1 \leq i \leq j \leq r$. Put $T_{ij} := T_{ji}$ if $i > j$. Put $Q := \{1, \dots, r\}$. If $\emptyset \neq P \subset Q$ define $\sigma_P := \det(T_{ij})_{i,j \in P}$. Then $\sigma_Q \notin \langle (\sigma_P)_{P \neq Q} \rangle$.*

PROOF. We may assume $r \geq 3$. Let I be the ideal generated by all T_{ij} such that $1 \neq |i - j| \neq r - 1$, and all T_{ij}^2 . It is easy to see that $\sigma_P \notin I$ if and only if $P = Q$.

(4.3) The following facts are not proved here, see [13, pp. 11–13].

- (i) The mapping $\Sigma_{(1)}: P_{(1)}(n) \rightarrow \mathcal{P}$ is injective.
- (ii) If $\Sigma_1 \lambda \leq \Sigma_1 \mu$ where $\lambda, \mu \in P_1(n)$ then $\Sigma \lambda \leq \Sigma \mu$.
- (iii) If $\lambda \in P_{(1)}(n)$ then $\gamma_{(1)}(\lambda) = 2|\Sigma_{(1)} \lambda|$.

Using (1.7), (2.1)(a), (3.5), (3.8), (3.10), (4.2) we get the following

COROLLARY. *Case I and II (1, 0). Let $x \in \mathfrak{D}(\lambda)$ and $y \in N(G)$.*

- (a) $y \in \mathfrak{D}(\lambda)$ if and only if $\text{ord}(y, N(G)/k) = \text{ord}(x, N(G)/k)$.
- (b) $y \in Gx$ if and only if $(N(G), y) \sim (N(G), x)$, cf. (1.7).

Assume that k is a field.

- (c) $\text{codim}(Gx, N(G)) = 2|\text{ord}(x, N(G)/k)|$.
- (d) $y \in \overline{Gx}$ if and only if $\text{ord}(y, N(G)/k) \geq \text{ord}(x, N(G)/k)$.

(4.4) **REMARK.** In (4.2) Case II (0, ζ), inequality occurs if λ_1 is even and also if $\lambda_* = (3, 3, 2, 2)$, but we have equality if $\lambda_* = (3, 3, 2, 2, 1)$. In the last case we have

$$\text{codim}(Gx, N(G)) = \gamma_0(\lambda) > 2|\Sigma_0\lambda| = 2|\text{ord}(x, N(G)/k)|$$

if k is a field, compare (4.3)(c) and (4.9) table B_5 .

(4.5) The polynomials f_a are defined by $f_a := 0$ if $a < 0$, $f_0 := 1$ and $f_a := \sum_{i \geq 1} X_i f_{a-i}$ if $a > 0$. They are determined by the generating function

$$\sum_{a=0}^{\infty} T^a f_a = \left(1 - \sum_{i \geq 1} X_i T^i\right)^{-1}.$$

Clearly $f_a(X_1) = X_1^a$ if $a \geq 0$. One can prove that

$$f_a(X_1, X_2) = \sum \binom{a-i}{i} X_1^{a-2i} X_2^i \quad (0 \leq i \leq \lfloor a/2 \rfloor).$$

Let A^m denote the affine space over k of rank m , say with coordinate ring $k[X_1, \dots, X_m]$. It is pointed in some point of the origin section. The *Kleinian singularities* A_l and D_l are the pointed subschemes of A^3 given by one equation:

$$\begin{aligned} A_l, l \geq 1, & \text{ by } X_1^{l+1} + X_2 X_3 = 0, \\ D_l, l \geq 3, & \text{ by } X_1^{l-1} - X_1 X_2^2 + X_3^2 = 0, \text{ if } 1/2 \in k. \end{aligned}$$

We define the following *singularities*.

If $l \geq 3$, AA_l in $A^2 \times A^4$ by

$$\begin{cases} f_l(X_1, X_2) + Y_1 Y_3 + Y_2 Y_4 = 0, \\ X_2 f_{l-1}(X_1, X_2) - Y_4(X_1 Y_2 - X_2 Y_1) + Y_2 Y_3 = 0. \end{cases}$$

If $1/2 \in k$ and $l \geq 3$, BB_l in $A^2 \times A^4$ by

$$\begin{cases} f_{l-1}(2X_1, -X_2^2) - 2Y_1 Y_3 + Y_2^2 - Y_4^2 = 0, \\ X_2^2 f_{l-2}(2X_1, -X_2^2) + (Y_3 - X_1 Y_1)^2 - X_2^2 Y_1^2 - 2Y_4(X_1 Y_4 - X_2 Y_2) = 0. \end{cases}$$

If $1/2 \in k$ and $l \geq 2$, CC_l in $A^3 \times A^2$ by

$$(X_3^2 - X_1 X_2)^l + X_1 Y_1^2 + 2X_3 Y_1 Y_2 + X_2 Y_2^2 = 0.$$

If $1/2 \in k$ and $l \geq 5$, CD_l in $A^2 \times A^4$ by

$$\begin{cases} f_{l-2}(X_1, X_2) + X_1 Y_2^2 - X_2 Y_1^2 - Y_3^2 + 2Y_2 Y_4 = 0, \\ X_2 f_{l-3}(X_1, X_2) + X_2(X_1 Y_1^2 + Y_2^2 - 2Y_1 Y_3) + Y_4^2 = 0. \end{cases}$$

If $1/2 \in k$ and $l \geq 3$, DD_l in $A^3 \times A^3$ by

$$\begin{cases} (x_1^2 + x_2^2 + x_3^2)^{l-1} + y_1^2 + y_2^2 + y_3^2 = 0, \\ x_1 y_1 + x_2 y_2 + x_3 y_3 = 0. \end{cases}$$

(4.6) PROPOSITION. Assume in Case II that $l + \epsilon + \zeta \geq 3$. Consider $\lambda \in P_{(\epsilon)}(n)$ with $0 < \gamma_{(\epsilon)}(\lambda) < 6$, cf. (3.8). If $x \in \mathfrak{D}(\lambda)$ then $(N(G), x)$ is smoothly equivalent (cf. (1.7)) to the singularity (cf. (4.5)) given in the following table.

G	λ_*	Dynkin diagram	$\gamma_{(\epsilon)}(\lambda)$	singularity
$Gl_n, n \geq 2$	$(n-1, 1)$	$\left\{ \begin{array}{c} \leftarrow \overset{0}{\text{---}} \text{---} \text{---} \rightarrow \\ \leftarrow \overset{1}{\text{---}} \text{---} \overset{1}{\text{---}} \text{---} \rightarrow \end{array} \right\}$	2	A_{n-1}
$SO_{2l+1}, l \geq 2$	$(2l-1, 1, 1)$	$\leftarrow \overset{2}{\text{---}} \overset{0}{\text{---}} \rightarrow$	2	A_{2l-1}
$Sp_{2l}, l \geq 2$	$(2l-2, 2)$	$\leftarrow \overset{0}{\text{---}} \overset{2}{\text{---}} \rightarrow$	2	D_{l+1}
$SO_{2l}, l \geq 3$	$(2l-3, 3)$	$\leftarrow \overset{0}{\text{---}} \overset{2}{\text{---}} \overset{2}{\text{---}} \overset{2}{\text{---}} \rightarrow$	2	D_l
$Gl_n, n \geq 4$	$(n-2, 2)$	$\left\{ \begin{array}{c} \leftarrow \overset{0}{\text{---}} \overset{2}{\text{---}} \overset{2}{\text{---}} \overset{0}{\text{---}} \text{---} \text{---} \rightarrow \\ \leftarrow \overset{1}{\text{---}} \overset{1}{\text{---}} \overset{1}{\text{---}} \overset{1}{\text{---}} \text{---} \text{---} \rightarrow \end{array} \right\}$	4	AA_{n-1}
$SO_{2l+1}, l \geq 3$	$(2l-3, 3, 1)$	$\leftarrow \overset{0}{\text{---}} \overset{2}{\text{---}} \overset{0}{\text{---}} \rightarrow$	4	BB_l
$Sp_{2l}, l \geq 2$	$l = 2, (2, 2, 1)$	$\left\{ \begin{array}{c} \leftarrow \overset{1}{\text{---}} \overset{0}{\text{---}} \rightarrow \\ \overset{0}{\text{---}} \overset{2}{\text{---}} \overset{0}{\text{---}} \rightarrow \\ \leftarrow \overset{0}{\text{---}} \overset{2}{\text{---}} \overset{0}{\text{---}} \overset{2}{\text{---}} \rightarrow \end{array} \right\}$	4	CC_l
	$l = 3, (3, 3)$		4	DD_n
	$l \geq 4, (2l-4, 4)$		4	CD_{l+1}
$SO_{2l}, l \geq 3$	$(2l-3, 1, 1, 1, 1)$	$\left\{ \begin{array}{c} \leftarrow \overset{2}{\text{---}} \overset{0}{\text{---}} \overset{0}{\text{---}} \overset{2}{\text{---}} \rightarrow \\ \leftarrow \overset{0}{\text{---}} \overset{2}{\text{---}} \overset{0}{\text{---}} \overset{2}{\text{---}} \rightarrow \end{array} \right\}$	4	DD_l
	$l = 4, (4, 4)$		4	
	$l \geq 5, (2l-5, 5)$		4	CD_l

REMARK. We have $\gamma_{(\epsilon)}(\lambda) = \text{codim}(Gx, N(G)_{(k(x))})$. For the singularities with $\gamma_{(\epsilon)}(\lambda) = 2$, compare [5] and [21, pp. 140–158]. In the table we have added the Dynkin diagram of the section $x \in \mathfrak{g}(k(x))$, cf. [20, III, IV], where $\leftarrow \text{---}$ means a string with numbers 2 attached to the nodes.

PROOF. The classification of all possibilities for λ is easy. By the sequence of reductions used in (4.2)(1) we may assume that $x = z(\mathfrak{p})$ where z is a standard nilpotent with partition λ and $\mathfrak{p} \in \text{Spec}(k)$. In Case II the base-data for z may be prescribed within the bounds set by (3.3)(2). Let $z + L$ be the cross section of (3.7). Then $(z + L) \cap N(G)$ is a cross section at z for the action of G on $N(G)$. So $(N(G), x)$ is smoothly equivalent to $((z + L) \cap N(G), z(\mathfrak{p}))$ by (2.1) (a). The two singularities to be determined for Gl_n will be examples in (4.7) and (4.8). We do not give the tedious calculations needed to settle Case II, see [13, p. 79] for some indications.

(4.7) Case I with $\lambda_* = (p, 1^q)$, i.e. $(p, 1, \dots, 1)$ with q times 1. We have $n = p + q$ and $r := \lambda^1 = q + 1$. On $z + L$ we define the coordinate functions ξ_a, ξ_{ij} as follows: if R is a k -algebra and $x \in (z + L)(R)$, then

$$(1) \quad x = z - \sum_{a=1}^p \xi_a(x)e(1, p-a; 1, p-1) - \sum_{(i,j) \neq (1,1)} \xi_{ij}(x)e(i, 0; j, \lambda_j - 1).$$

So $A(z + L) = k[\xi_a, \xi_{ji}]$. Put $\xi_{11} = 0$.

If $a \geq 1$, let $s_a, h_a \in k[\xi_{ij}]$ be defined by

$$(2) \quad \begin{cases} s_a = \sum \det(\xi_{ij})_{i,j \in I} \\ h_a = \sum \det(\xi_{ij})_{i,j \in \{1\} \cup I} \end{cases}$$

where in both cases the summation is over the subsets I of $\{2, \dots, r\}$ with $\#I = a$.

Clearly, if $a \geq r$ then $s_a = h_a = 0$. The subscheme $(z + L) \cap N(G)$ of $z + L$ is defined by the equations $\sigma_m|(z + L) = 0$ ($1 \leq m \leq n$). One verifies that

$$(3) \quad \begin{cases} (-1)^m \sigma_m|(z + L) = \xi_m + s_m + \sum_{a=1}^{m-1} \xi_a s_{m-a} & \text{if } 1 \leq m \leq p, \\ (-1)^m \sigma_m|(z + L) = h_{m-p} + s_m + \sum_{a=1}^p \xi_a s_{m-a} & \text{if } p < m \leq n. \end{cases}$$

The first p equations can be solved inductively. With the notations of (4.5) we obtain $\xi_m = f_m(-s_1, -s_2, \dots, -s_q)$ ($1 \leq m \leq p$). So $(z + L) \cap N(G)$ is isomorphic to the subscheme of $\text{Spec } k[\xi_{ij}]$ defined by the equations

$$(4) \quad h_{m-p} + \sum_{a=0}^p s_{m-a} f_a(-s_1, \dots, -s_q) = 0 \quad (p < m \leq n).$$

EXAMPLES. (a) $\lambda_* = (n - 1, 1)$. Putting $X_1 = -\xi_{22}, X_2 = \xi_{12}, X_3 = \xi_{21}$, we get the singularity A_{n-1} , cf. (4.5).

(b) $\lambda_* = (n - 2, 1, 1)$. The scheme $(z + L) \cap N(G)$ is isomorphic to the subscheme of $A^8 = \text{Spec}(k[\xi_{ij}])$, where $1 \leq i, j \leq 3 \leq i + j$, defined by the equations

$$(5) \quad \begin{cases} f_{n-1}(-s_1, -s_2) - h_1 = 0, \\ s_2 f_{n-2}(-s_1, -s_2) + h_2 = 0, \end{cases}$$

where

$$\begin{aligned} s_1 &= \xi_{22} + \xi_{33}, \\ s_2 &= \xi_{22}\xi_{33} - \xi_{23}\xi_{32}, \quad \text{and} \quad h_2 = \begin{vmatrix} 0 & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}, \\ h_1 &= \xi_{12}\xi_{21} + \xi_{13}\xi_{31}, \end{aligned}$$

(4.8) *Case I for arbitrary λ .* We use a different cross section, viz. $z + L''$ defined by $L'' := \sum_{i,j} L''_{ij}$ where $L''_{ij} := L_{ij}$ if $i \neq 1$ or $j = 1$ and $L''_{1,j}(R) := \sum_{0 < b < \lambda_j} \text{Re}(1, 0; j, b)$ if $j \neq 1$, see (3.7). Again we have

$$(N(G), x) \sim ((z + L'') \cap N(G), z(\mathfrak{p})).$$

Put $p := \lambda_1, q := n - \lambda_1$ and $\mu_* := (p, 1^q)$. Put $z' := \sum_{a=0}^{p-2} e(1, a + 1; 1, a)$, so that z' is a standard nilpotent element in $\mathfrak{g}(k)$ with partition μ . In the obvious way we define base-data (f', μ) for z' . The cross section $z' + L'$ at z' used in (4.7) contains $z + L''$. So we can use the elimination in (4.7) of $\xi_a, 1 \leq a \leq p$, substituting into the matrix (ξ_{ij}) at some places the constant functions 0 or -1 , cf. (4.7)(1).




EXAMPLE. If $\lambda_* = (n - 2, 2), n \geq 4$, we use the matrix

$$(\xi_{ij}) = \begin{pmatrix} 0 & Y_3 & Y_4 \\ Y_1 & -X_1 & -1 \\ Y_2 & -X_2 & 0 \end{pmatrix}$$

and we obtain the equations (4.7)(5) where $s_1 = -X_1, s_2 = -X_2, h_1 = Y_1 Y_3 + Y_2 Y_4$ and $h_2 = Y_4(X_1 Y_2 - X_2 Y_1) - Y_2 Y_3$. So $(z + L'') \cap N(G)$ is isomorphic to the singularity AA_{n-1} , cf. (4.5).

(4.9) Tables for the orbits in $N(G)$. We give the adjacency structure (cf. (3.10)), the Dynkin diagram (cf. [20, IV]), the codimension of the orbits $\gamma_{(\epsilon)}(\lambda)$ (cf. (3.8)), and the partition $\text{ord} = \text{ord}(x, N(G)/k)$ (cf. (4.2)). The number of orbits is denoted by $\#$. In the cases SO_{2l} with even l , the partition λ may represent two orbits, cf. (3.5). We give the Dynkin diagram of one of them and indicate how to get the other one by the symbol \curvearrowright .

For SO_n we give $\Sigma_0 \lambda$, which is a lower bound of ord , cf. (4.2). Whenever there are reasons to assume $\text{ord} \neq \Sigma_0 \lambda$, we give a conjectured value of ord or a question mark. As $D_2 = A_1 + A_1, B_2 = C_2$ and $D_3 = A_3$, the values of ord for the cases SO_4, SO_5 and SO_6 are not conjectural.

A_1	Gl_2	λ_*	Dy \circ	$\gamma(\lambda)$	ord_*	
		2	2	0	0	
$\# = 2$		1 1	0	2	1	
A_2	Gl_3	λ_*	Dy $\circ \text{---} \circ$	$\gamma(\lambda)$	ord_*	
		3	2 2	0	0	
		2 1	1 1	2	1	
$\# = 3$		1 1 1	0 0	6	2 1	
$D_2 = A_1 + A_1$	SO_4	λ_*	Dy $\circ \text{---} \circ$	$\gamma_0(\lambda)$	$(\Sigma_0 \lambda)_*$	ord_*
		3 1	2 2	0	0	
		2 2	\curvearrowright 2	2	0	1
$\# = 4$		1^4	0 0	4	1 1	



$B_2 = C_2$	SO_5	λ_*	Dy \rightleftarrows	$\gamma_0(\lambda)$	$(\Sigma_0 \lambda)_*$	ord.*
		5	2 2	0	0	0
		3 1 1	2 0	2	1	1
		2 2 1	0 1	4	1	2
$\neq = 4$		1^5	0 0	8	3 1	3 1

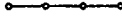

$C_2 = B_2$	Sp_4	λ_*	Dy \rightleftarrows	$\gamma_1(\lambda)$	ord.*
		4	2 2	0	0
		2 2	0 2	2	1
		2 1 1	1 0	4	2
$\neq = 4$		1^4	0 0	8	2 1

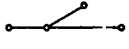
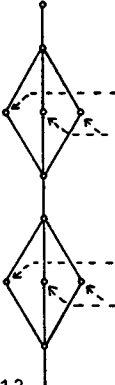
$A_3 = D_3$	Gl_4	λ_*		$\gamma(\lambda)$	ord.*
		4	2 2 2	0	0
		3 1	2 0 2	2	1
		2 2	0 2 0	4	1 1
		2 1 1	1 0 1	6	2 1
$\neq = 5$		1^4	0 0 0	12	3 2 1

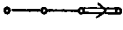
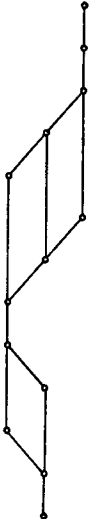
$D_3 = A_3$	SO_6	λ_*		$\gamma_0(\lambda)$	$(\Sigma_0 \lambda)_*$	ord.*
		5 1	2 2 2	0	0	
		3 3	0 2 2	2	1	
		3 1 1 1	2 0 0	4	1 1	
		2 2 1 1	0 1 1	6	1 1	2 1
$\neq = 5$		1^6	0 0 0	12	3 2 1	

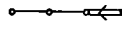
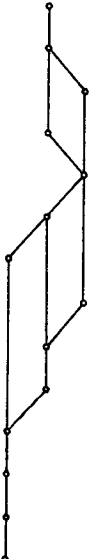
B_3	SO_7	λ_*		$\gamma_0(\lambda)$	$(\Sigma_0 \lambda)_*$	ord.*?
		7	2 2 2	0	0	
		5 1 1	2 2 0	2	1	
		3 3 1	0 2 0	4	1 1	
		3 2 2	1 0 1	6	2 1	
		3 1 ⁴	2 0 0	8	3 1	
		2 2 1 ³	0 1 0	10	3 1	3 2
$\neq = 7$		1^7	0 0 0	18	5 3 1	

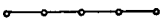

C_3	Sp_6	λ_*		$\gamma_1(\lambda)$	ord.
		6	2 2 2	0	0
		4 2	2 0 2	2	1
		4 1 1	2 1 0	4	2
		3 3	0 2 0	4	1 1
		2 2 2	0 0 2	6	2 1
		2 2 1 1	0 1 0	8	3 1
		2 1 ⁴	1 0 0	12	4 2
$\#\# = 8$		1 ⁶	0 0 0	18	5 3 1

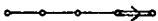
A_4	Gl_5	λ_*		$\gamma(\lambda)$	ord.
		5	2 2 2 2	0	0
		4 1	2 1 1 2	2	1
		3 2	1 1 1 1	4	1 1
		3 1 1	2 0 0 2	6	2 1
		2 2 1	0 1 1 0	8	2 1 1
		2 1 ³	1 0 0 1	12	3 2 1
$\#\# = 7$		1 ⁵	0 0 0 0	20	4 3 2 1


D_4	SO_8	λ_*		$\gamma_0(\lambda)$	$(\sum_0 \lambda)_*$	ord.?
		7 1	2 2 2 2	0	0	
		5 3	2 0 2 2	2	1	
		5 1 1 1	2 2 0 0	4	1 1	
		4 4	0 2 0 [↔] 2	4	1	1 1
		3 3 1 1	0 2 0 0	6	1 1 1	
		3 2 2 1	1 0 1 1	8	2 1 1	
		3 1 ⁵	2 0 0 0	12	3 2 1	
		2 ⁴	0 0 0 [↔] 2	12	2 1 1	3 2 1
		2 2 1 ⁴	0 1 0 0	14	3 2 1	3 2 2
$\#\# = 12$		1 ⁸	0 0 0 0	24	5 3 3 1	

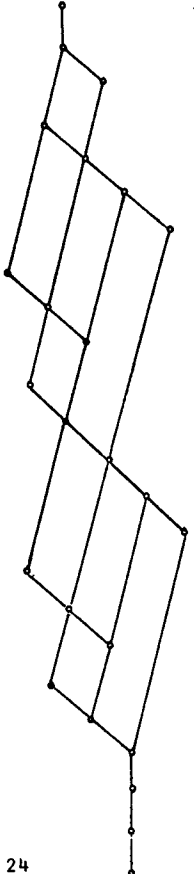
B_4	SO_9	λ_*		$\gamma_0(\lambda)$	$(\Sigma_0 \lambda)_*$	ord_* ?	
		9		2 2 2 2	0	0	
		7 1 1		2 2 2 0	2	1	
		5 3 1		2 0 2 0	4	1 1	
		5 2 2		2 1 0 1	6	2 1	
		5 1 ⁴		2 2 0 0	8	3 1	
		4 4 1		0 2 0 1	6	1 1	2 1
		3 3 3		0 0 2 0	8	2 1 1	
		3 3 1 ³		0 2 0 0	10	3 1 1	
		3 2 2 1 1		1 0 1 0	12	3 2 1	
		3 1 ⁶		2 0 0 0	18	5 3 1	
		2 ⁴ 1		0 0 0 1	16	3 2 1	3 3 2
		2 2 1 ³		0 1 0 0	20	5 3 1	5 3 2
$\# = 13$		1 ⁹		0 0 0 0	32	7 5 3 1	

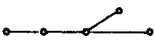
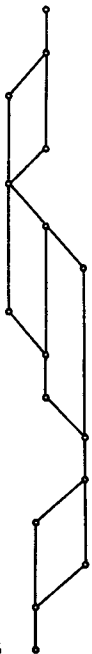
C_4	Sp_8	λ_*		$\gamma_1(\lambda)$	ord_*	
		8		2 2 2 2	0	0
		6 2		2 2 0 2	2	1
		6 1 1		2 2 1 0	4	2
		4 4		0 2 0 2	4	1 1
		4 2 2		2 0 0 2	6	2 1
		4 2 1 1		2 0 1 0	8	3 1
		4 1 ⁴		2 1 0 0	12	4 2
		3 3 2		0 1 1 0	8	2 1 1
		3 3 1 1		0 2 0 0	10	3 1 1
		2 2 2 2		0 0 0 2	12	3 2 1
		2 2 2 1 1		0 0 1 0	14	4 2 1
		2 2 1 ⁴		0 1 0 0	18	5 3 1
		2 1 ⁶		1 0 0 0	24	6 4 2
$\# = 14$		1 ⁸		0 0 0 0	32	7 5 3 1



A_5	Gl_6	λ_*		$\gamma(\lambda)$	ord_*
		6	2 2 2 2 2	0	0
		5 1	2 2 0 2 2	2	1
		4 2	2 0 2 0 2	4	1 1
		4 1 1	2 1 0 1 2	6	2 1
		3 3	0 2 0 2 0	6	1 1 1
		3 2 1	1 1 0 1 1	8	2 1 1
		3 1 1 1	2 0 0 0 2	12	3 2 1
		2 2 2	0 0 2 0 0	12	2 2 1 1
		2 2 1 1	0 1 0 1 0	14	3 2 1 1
		2 1 ⁴	1 0 0 0 1	20	4 3 2 1
$\#\# = 11$		1 ⁶	0 0 0 0 0	30	5 4 3 2 1

B_5	SO_{11}	λ_*		$\gamma_0(\lambda)$	$(\Sigma_0 \lambda)_*$	ord_* ?
		11	2 2 2 2 2	0	0	
		9 1 1	2 2 2 2 0	2	1	
		7 3 1	2 2 0 2 0	4	1 1	
		7 2 2	2 2 1 0 1	6	2 1	
		7 1 1 1 1	2 2 2 0 0	8	3 1	
		5 5 1	0 2 0 2 0	6	1 1 1	
		5 3 3	2 0 0 2 0	8	2 1 1	
		5 3 1 1 1	2 0 2 0 0	10	3 1 1	
		5 2 2 1 1	2 1 0 1 0	12	3 2 1	
		5 1 ⁶	2 2 0 0 0	18	5 3 1	
		4 4 3	0 1 1 0 1	10	2 1 1	2 2 1
		4 4 1 1 1	0 2 0 1 0	12	3 1 1	3 2 1
		3 3 3 1 1	0 0 2 0 0	14	3 2 1 1	
		3 3 2 2 1	0 1 0 1 0	16	3 2 1 1	= ord_* , cf. (4.4)
		3 3 1 ⁵	0 2 0 0 0	20	5 3 1 1	
		3 2 2 2 2	1 0 0 0 1	20	4 3 2 1	
		3 2 2 1 ⁴	1 0 1 0 0	22	5 3 2 1	
		3 1 ⁸	2 0 0 0 0	32	7 5 3 1	
		2 2 2 2 1 ³	0 0 0 1 0	26	5 3 2 1	5 3 3 2
		2 2 1 ⁷	0 1 0 0 0	34	7 5 3 1	7 5 3 2
$\#\# = 21$		1 ¹¹	0 0 0 0 0	50	9 7 5 3 1	

C_5	Sp_{10}	λ		$\gamma_1(\lambda)$	ord.
		10	2 2 2 2 2	0	0
		8 2	2 2 2 0 2	2	1
		8 1 1	2 2 2 1 0	4	2
		6 4	2 0 2 0 2	4	1 1
		6 2 2	2 2 0 0 2	6	2 1
		6 2 1 1	2 2 0 1 0	8	3 1
		6 1 1 1 1	2 2 1 0 0	12	4 2
		5 5	0 2 0 2 0	6	1 1 1
		4 4 2	0 2 0 0 2	8	2 1 1
		4 4 1 1	0 2 0 1 0	10	3 1 1
		4 3 3	1 0 1 1 0	10	2 2 1
		4 2 2 2	2 0 0 0 2	12	3 2 1
		4 2 2 1 1	2 0 0 1 0	14	4 2 1
		4 2 1 ⁴	2 0 1 0 0	18	5 3 1
		4 1 ⁶	2 1 0 0 0	24	6 4 2
		3 3 2 2	0 1 0 1 0	14	3 2 1 1
		3 3 2 1 1	0 1 1 0 0	16	4 2 1 1
		3 3 1 ⁴	0 2 0 0 0	20	5 3 1 1
		2 2 2 2 2	0 0 0 0 2	20	4 3 2 1
		2 ⁴ 1 1	0 0 0 1 0	22	5 3 2 1
		2 ³ 1 ⁴	0 0 1 0 0	26	6 4 2 1
		2 2 1 ⁶	0 1 0 0 0	32	7 5 3 1
		2 1 ⁸	1 0 0 0 0	40	8 6 4 2
$\neq = 24$		1 ¹⁰	0 0 0 0 0	50	9 7 5 3 1



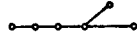
D_5	SO_{10}	λ_*		$\gamma_0(\lambda)$	$(\Sigma_0 \lambda)_*$	ord. _* ?
		9 1		2 2 2 2 2 0	0	
		7 3		2 2 0 2 2 2	1	
		7 1 1 1		2 2 2 0 0 4	1 1	
		5 5		0 2 0 2 2 4	1 1	
		5 3 1 1		2 0 2 0 0 6	1 1 1	
		5 2 2 1		2 1 0 1 1 8	2 1 1	
		5 1 ⁵		2 2 0 0 0 12	3 2 1	
		4 4 1 1		0 2 0 1 1 8	1 1 1	2 1 1
		3 3 3 1		0 0 2 0 0 10	2 1 1 1	
		3 3 2 2		0 1 0 1 1 12	2 1 1 1	3 1 1 1
		3 3 1 ⁴		0 2 0 0 0 14	3 2 1 1	
		3 2 2 1 ³		1 0 1 0 0 16	3 2 2 1	
		3 1 ⁷		2 0 0 0 0 24	5 3 3 1	
		2 ⁴ 1 1		0 0 0 1 1 20	3 2 2 1	3 3 2 2
		2 2 1 ⁶		0 1 0 0 0 26	5 3 3 1	5 3 3 2
$\# = 16$		1 ¹⁰		0 0 0 0 0 40	7 5 4 3 1	

A_6	GL_7	λ_*		$\gamma(\lambda)$	ord. _*
		7		2 2 2 2 2 2	0 0
		6 1		2 2 1 1 2 2	2 1
		5 2		2 1 1 1 1 2	4 1 1
		5 1 1		2 2 0 0 2 2	6 2 1
		4 3		1 1 1 1 1 1	6 1 1 1
		4 2 1		2 0 1 1 0 2	8 2 1 1
		4 1 ³		2 1 0 0 1 2	12 3 2 1
		3 3 1		0 2 0 0 2 0	10 2 1 1 1
		3 2 2		1 0 1 1 0 1	12 2 2 1 1
		3 2 1 1		1 1 0 0 1 1	14 3 2 1 1
		3 1 ⁴		2 0 0 0 0 2	20 4 3 2 1
		2 2 2 1		0 0 1 1 0 0	18 3 2 1 1 1
		2 2 1 ³		0 1 0 0 1 0	20 4 3 2 1 1
		2 1 ⁵		1 0 0 0 0 1	30 5 4 3 2 1
$\# = 15$		1 ⁷		0 0 0 0 0 0	42 6 5 4 3 2 1

D_6

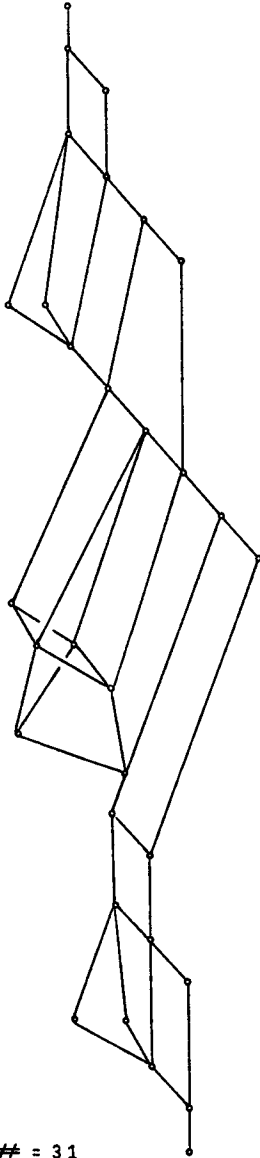
SO_{12}

$\lambda \cdot$



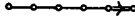
$\gamma_0(\lambda) (\Sigma_0 \lambda)_0$

ord.?

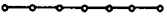


$\# = 31$

11,1	2 2 2 2 2 2	0	0	
9 3	2 2 2 0 2 2	2	1	
9 1 1 1	2 2 2 2 0 0	4	1 1	
7 5	2 0 2 0 2 2	4	1 1	
7 3 1 1	2 2 0 2 0 0	6	1 1 1	
7 2 2 1	2 2 1 0 1 1	8	2 1 1	
7 1 ⁵	2 2 2 0 0 0	12	3 2 1	
6 6	0 2 0 2 0 2	6	1 1	1 1 1
5 5 1 1	0 2 0 2 0 0	8	1 1 1 1	
5 3 3 1	2 0 0 2 0 0	10	2 1 1 1	
5 3 2 2	2 0 1 0 1 1	12	2 1 1 1	?
5 3 1 ⁴	2 0 2 0 0 0	14	3 2 1 1	
5 2 2 1 1 1	2 1 0 1 0 0	16	3 2 2 1	
5 1 ⁷	2 2 0 0 0 0	24	5 3 3 1	
4 4 3 1	0 1 1 0 1 1	12	2 1 1 1	?
4 4 2 2	0 2 0 0 0 2	14	2 1 1 1	?
4 4 1 ⁴	0 2 0 1 0 0	16	3 2 1 1	?
3 3 3 3	0 0 0 2 0 0	16	3 2 1 1 1	
3 3 3 1 ³	0 0 2 0 0 0	18	3 2 2 1 1	
3 3 2 2 1 1	0 1 0 1 0 0	20	3 2 2 1 1	?
3 3 1 ⁶	0 2 0 0 0 0	26	5 3 3 1 1	
3 2 ⁴ 1	1 0 0 0 1 1	24	4 3 2 2 1	
3 2 2 1 ⁵	1 0 1 0 0 0	28	5 3 3 2 1	
3 1 ⁹	2 0 0 0 0 0	40	7 5 4 3 1	
2 2 2 2 2 2	0 0 0 0 0 2	30	4 3 2 2 1	?
2 2 2 2 1 ⁴	0 0 0 1 0 0	32	5 3 3 2 1	?
2 2 1 ⁸	0 1 0 0 0 0	42	7 5 4 3 1	?
1 ¹²	0 0 0 0 0 0	60	9 7 5 5 3 1	

B_6	SO_{13}	λ_*		$\gamma_0(\lambda)$	$(\sum_0 \lambda)_*$	ord_*?
		13	2 2 2 2 2 2	0	0	
		11,1,1	2 2 2 2 2 0	2	1	
		9 3 1	2 2 2 0 2 0	4	1 1	
		9 2 2	2 2 2 1 0 1	6	2 1	
		9 1 1 1 1	2 2 2 2 0 0	8	3 1	
		7 5 1	2 0 2 0 2 0	6	1 1 1	
		7 3 3	2 2 0 0 2 0	8	2 1 1	
		7 3 1 1 1	2 2 0 2 0 0	10	3 1 1	
		7 2 2 1 1	2 2 1 0 1 0	12	3 2 1	
		7 1 ⁶	2 2 2 0 0 0	18	5 3 1	
		6 6 1	0 2 0 2 0 1	8	1 1 1	2 1 1
		5 5 3	0 2 0 0 2 0	10	2 1 1 1	
		5 5 1 1 1	0 2 0 2 0 0	12	3 1 1 1	
		5 4 4	1 0 1 1 0 1	12	2 2 1 1	
		5 3 3 1 1	2 0 0 2 0 0	14	3 2 1 1	
		5 3 2 2 1	2 0 1 0 1 0	16	3 2 1 1	?
		5 3 1 ⁵	2 0 2 0 0 0	20	5 3 1 1	
		5 2 2 2 2	2 1 0 0 0 1	20	4 3 2 1	
		5 2 2 1 ⁴	2 1 0 1 0 0	22	5 3 2 1	
		5 1 ⁸	2 2 0 0 0 0	32	7 5 3 1	
		4 4 3 1 1	0 1 1 0 1 0	16	3 2 1 1	3 2 2 1
		4 4 2 2 1	0 2 0 0 0 1	18	3 2 1 1	3 3 2 1
		4 4 1 ⁵	0 2 0 1 0 0	22	5 3 1 1	5 3 2 1
		3 3 3 3 1	0 0 0 2 0 0	20	3 3 2 1 1	
		3 3 3 2 2	0 0 1 0 1 0	22	4 3 2 1 1	
		3 3 3 1 ⁴	0 0 2 0 0 0	24	5 3 2 1 1	
		3 3 2 2 1 ³	0 1 0 1 0 0	26	5 3 2 1 1	?
		3 3 1 ⁷	0 2 0 0 0 0	34	7 5 3 1 1	
		3 2 ⁴ 1 1	1 0 0 0 1 0	30	5 4 3 2 1	
		3 2 2 1 ⁶	1 0 1 0 0 0	36	7 5 3 2 1	
		3 1 ¹⁰	2 0 0 0 0 0	50	9 7 5 3 1	
		2 ⁶ 1	0 0 0 0 0 1	36	5 4 3 2 1	?
		2 ⁴ 1 ⁵	0 0 0 1 0 0	40	7 5 3 2 1	?
		2 2 1 ⁹	0 1 0 0 0 0	52	9 7 5 3 1	?
		1 ¹³	0 0 0 0 0 0	72	11,9,7,5,3,1	

= 35

A_7	Gl_8	λ		$\gamma(\lambda)$	ord.
		8	2 2 2 2 2 2 2 2	0	0
		7 1	2 2 2 0 2 2 2 2	2	1
		6 2	2 2 0 2 0 2 2 2	4	1 1
		6 1 1	2 2 1 0 1 2 2 2	6	2 1
		5 3	2 0 2 0 2 0 2 2	6	1 1 1
		5 2 1	2 1 1 0 1 1 2 2	8	2 1 1
		5 1 1 1	2 2 0 0 0 2 2 2	12	3 2 1
		4 4	0 2 0 2 0 2 0 2	8	1 1 1 1
		4 3 1	1 1 1 0 1 1 1 1	10	2 1 1 1
		4 2 2	2 0 0 2 0 0 2 2	12	2 2 1 1
		4 2 1 1	2 0 1 0 1 0 2 2	14	3 2 1 1
		4 1 ⁴	2 1 0 0 0 1 2 2	20	4 3 2 1
		3 3 2	0 1 1 0 1 1 0 2	14	2 2 1 1 1
		3 3 1 1	0 2 0 0 0 2 0 2	16	3 2 1 1 1
		3 2 2 1	1 0 1 0 1 0 1 2	18	3 2 2 1 1
		3 2 1 ³	1 1 0 0 0 1 1 2	22	4 3 2 1 1
		3 1 ⁵	2 0 0 0 0 0 2 2	30	5 4 3 2 1
		2 2 2 2	0 0 0 2 0 0 0 2	24	3 3 2 2 1 1
		2 2 2 1 1	0 0 1 0 1 0 0 2	26	4 3 2 2 1 1
		2 2 1 ⁴	0 1 0 0 0 1 0 2	32	5 4 3 2 1 1
		2 1 ⁶	1 0 0 0 0 0 1 2	42	6 5 4 3 2 1
$\# = 22$		1 ⁸	0 0 0 0 0 0 0 2	56	7 6 5 4 3 2 1

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