PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 33 (1967), p. 21-57
<http://www.numdam.org/item?id=PMIHES_1967__33__21_0>

SINGULARITIES OF DIFFERENTIABLE MAPS

by J. M. BOARDMAN

CONTENTS

INTRODUCTION

Consider a smooth map $f : \mathrm{V} \to \mathrm{W}$ between smooth manifolds. Its local structure can be extremely complicated. As part of this structure we have the *singularity subsets* of f, which we now introduce briefly.

For each point $p \in \mathrm{V}$, we have the differential $f_p : \mathrm{T}_p \to \mathrm{T}_{fp}$, where T_p denotes the tangent space to V at p, and T_{fp} that to W at fp. One can classify the points of V according to the rank of f_p : $p \in \Sigma^i(f)$ if the kernel of f_p has dimension exactly i. (Our notation here is unusual; this set has been called $\mathrm{S}^i(f)$ if $v \leqslant w$, or $\mathrm{S}^{i-v+w}(f)$ if $v \geqslant w$, where v and w are the dimensions of V and W [7].)

Thom proved [7] that for "most" maps f, these sets $\Sigma^i(f)$ are in fact (non-compact) submanifolds of V. If this holds for f, we can consider the restriction $f | \Sigma^i(f) : \Sigma^i(f) \to \mathrm{W}$, which is again a smooth map of manifolds, and define $\Sigma^{i,j}(f) = \Sigma^j(f | \Sigma^i(f))$. One hopes that $\Sigma^{i,j}(f)$ is again a smooth manifold, so that $\Sigma^{i,j,k}(f)$ can be defined, etc. For examples, see [7], [9] and [5].

This direct approach is hardly satisfactory, and is certainly awkward technically; it is by no means trivial to see that $\Sigma^{i,j}(f)$ has any chance of being a manifold. We adopt a different approach, using *jet spaces*, which were introduced by Ehresmann [2]. We construct, by means of iterated jacobian extensions, subsets Σ^{I} of the jet space, independent of any map f. These induce subsets $\Sigma^{I}(f)$ of V, defined for any map f and any finite sequence I of integers. Our main theorem (§ 6) asserts that all the subsets Σ^{I} are *submanifolds* in the jet space, and we compute their codimensions. This implies that the sets $\Sigma^{I}(f)$ are submanifolds of V for " sufficiently good " maps f, with the same codimensions. Enough such maps exist, in the sense that they approximate to any given map. We also show that for these maps, our sets $\Sigma^{I}(f)$ coincide with those given by the naive geometric approach outlined above. Much use is made of the infinite jet space $J^{\infty}(V, W)$, and of a canonically defined vector bundle over it, which we call the *total tangent bundle*.

In § 7 we show that the inductive procedure due to Porteous [6], for constructing the sets $\Sigma^{I}(f)$ and the *intrinsic derivatives* of f, always works for " good " maps f.

As the proof is algebraic, analogous results are valid for the algebraic case, the complex case, etc.

Various conversations with B. Morin, I. Porteous, and Prof. R. Thom, have helped to clarify many points. Finally I should like to thank Prof. C. T. C. Wall for suggesting the problem to me originally.

§ 0. Notation.

R denotes the field of real numbers. It will usually be the groundfield. We write $A \otimes B$ and Hom(A, B) respectively for $A \otimes_{\mathbf{R}} B$ and $\mathrm{Hom}_{\mathbf{R}}(A, B)$. $\mathbf{R}^n$ denotes real n-space, regarded as a real vector space or as a smooth manifold.

All our manifolds are assumed to be smooth (in the C^{∞} sense), paracompact, without boundary, and finite-dimensional unless otherwise stated. They need not be connected. We write V^v for a manifold V having dimension v. Also all maps and bundles are tacitly assumed smooth. By a *submanifold* of a manifold, we mean a subset locally smoothly like $\mathbf{R}^m \times 0 \subset \mathbf{R}^m \times \mathbf{R}^n$ for suitable m and n, as in [3, p. 18, 19]; n is called the *codimension* of the submanifold. Submanifolds need not be closed subsets. The only infinite-dimensional manifolds we shall encounter are the infinite jet spaces and their submanifolds.

If A is a subspace of the base of the bundle ξ, $\xi|A$ denotes the part of the bundle over A. We call *any* sequence $\mathbf{E}_0 \supset \mathbf{E}_1 \supset \mathbf{E}_2 \supset \ldots \supset \mathbf{E}_k$ of vector bundles, or vector spaces, a *flag*. We shall usually, as here, denote vector bundles and maps by heavy type : **E**, **F**, $\mathbf{a} : \mathbf{E} \to \mathbf{F}$, etc. The vector bundle **Hom**(**E**, **F**) is defined when **E** and **F** have the same base space, and has fibre $\mathrm{Hom}(\mathbf{E}|p, \mathbf{F}|p)$ over p. It is occasionally convenient to write **R** also for any product vector bundle whose fibre is the field **R** of real numbers.

We find some of the ideas and notation of sheaf theory useful, although we use no results. If V is a manifold, and U is open in V, we write $\mathscr{F}_V(U)$, or simply $\mathscr{F}(U)$, for the **R**-algebra of (smooth) real-valued functions on U, and $\mathscr{F}(p)$ for the algebra of germs at $p \in V$ of (smooth) real-valued functions. We write $\mathfrak{m}_p$ for the ideal in $\mathscr{F}(p)$, or in $\mathscr{F}(U)$ for any open set U containing p, consisting of the functions that vanish at p.

We note that $\mathscr{F}(p)$ is a local ring (though not Noetherian), having $\mathfrak{m}_p$ as its unique maximal ideal. If, further, $\mathbf{E}$ is a smooth real vector bundle over V, we write similarly $\Gamma_{\mathbf{E}}(\mathrm{U})$ for the set of smooth sections of $\mathbf{E}$ defined on U, and $\Gamma_{\mathbf{E}}(p)$ for the germs of such at p. By means of scalar multiplication, $\Gamma_{\mathbf{E}}(\mathrm{U})$ is a $\mathscr{F}(\mathrm{U})$-module, and $\Gamma_{\mathbf{E}}(p)$ is a $\mathscr{F}(p)$-module. In particular the tangent bundle $\mathbf{T}=\mathbf{T}_{\mathrm{V}}$ of V is a smooth vector bundle, and we remark that $\Gamma_{\mathbf{T}}(\mathrm{U})$, the set of smooth vector fields on U, acts on the ring $\mathscr{F}(\mathrm{U})$ as the module consisting of all derivations of $\mathscr{F}(\mathrm{U})$.

Apart from sheaf notation, we reserve brackets mainly for bracketing. Functional notation is normally denoted merely by juxtaposition, and composition by $\circ$. We occasionally use . for multiplication and bracket combined: $\alpha x . \beta y$ means the product of αx and βy, and not $\alpha(x . \beta y)$. Some such convention is useful when dealing with multiplicative properties of operators.

References thus [ ] will be found listed at the end.

We shall use the symbol ■ to signal the end of a proof, or the absence of any further proof.

§ 1. Jet spaces and total vector fields.

In this section we recall (e.g. from [2], [8]) the jet spaces $\mathrm{J}^n(\mathrm{V}, \mathrm{W})$ and certain of their properties, some of them well known, some less well known. We shall be particularly interested in the infinite jet space $\mathrm{J}^\infty(\mathrm{V}, \mathrm{W})$, for reasons which will soon become clear. It is the only jet space over which the total tangent bundle $\mathbf{D}$ can be defined. This vector bundle, which is the natural setting for our theory, is in effect already well known in various disguises.

Since we are concerned only with local properties, we need study in detail only the case when $\mathrm{W}=\mathbf{R}^w$ and V is an open subset of $\mathbf{R}^v$ — provided we cast our definitions in a sufficiently invariant form to allow piecing together.

We take coordinate functions $(x_1, x_2, \ldots, x_v)$ on V or $\mathbf{R}^v$, and $(y_1, y_2, \ldots, y_w)$ on $\mathrm{W}=\mathbf{R}^w$. Given n $(0 \leqslant n \leqslant \infty)$, an equivalence relation $\sim_n$ is defined between germs of maps from V to W as follows: $f \sim_n g$ if and only if f and g are germs defined at the same point $p \in \mathrm{V}$, $fp=gp$, and their partial derivatives at p of orders $\leqslant n$ agree. The third condition may be expressed invariantly as

$$f^* = g^* : \mathscr{F}(q)/\mathfrak{m}_q^{n+1} \to \mathscr{F}(p)/\mathfrak{m}_p^{n+1},$$

where $q=fp=gp$. If $m \geqslant n$, $\sim_m$ is a smaller equivalence relation than $\sim_n$.

Definition (1.1). — The *jet space* $\mathrm{J}^n(\mathrm{V}, \mathrm{W})$ $(0 \leqslant n \leqslant \infty)$ is the set of equivalence classes of germs of maps from V to W, under the equivalence relation $\sim_n$. If $m \geqslant n$, we have the canonical *projection map* $\pi_n^m : \mathrm{J}^m(\mathrm{V}, \mathrm{W}) \to \mathrm{J}^n(\mathrm{V}, \mathrm{W})$. If f is a germ at p, then $f \rightsquigarrow p$ and $f \rightsquigarrow fp$ also induce projection maps $\pi_{\mathrm{V}}^n : \mathrm{J}^n(\mathrm{V}, \mathrm{W}) \to \mathrm{V}$ and $\pi_{\mathrm{W}}^n : \mathrm{J}^n(\mathrm{V}, \mathrm{W}) \to \mathrm{W}$. Given any map $f : \mathrm{U} \to \mathrm{W}$, where U is open in V, we have the associated *jet section* (called *flot* in [2]) $\mathrm{J}^n f : \mathrm{U} \to \mathrm{J}^n(\mathrm{V}, \mathrm{W})$, which takes each point $p \in \mathrm{U}$ to the class of the germ of f at p. The image $(\mathrm{J}^n f)p$ of p is called the *n-jet* of f at p.

We shall usually omit the suffix ∞.

Trivially $J^0(V, W) \cong V \times W$, by the projections π^0_V and π^0_W. There are obvious transitivity relations such as $J^n f = \pi^m_n \circ J^m f$ and $\pi^k_n = \pi^m_n \circ \pi^k_m$.

So far, $J^n(V, W)$ is merely a set. Let us take $W = \mathbf{R}^w$ and V open in $\mathbf{R}^v$, with the coordinate functions x_i on V and y_j on W as above. Corresponding to each coordinate function x_i on V, we have the coordinate vector field d_i on V. We shall always regard a vector field on V as a derivation of the ring $\mathscr{F}(V)$, as well as being a section of the tangent bundle $\mathbf{T}_V$. Then the vector field d_i may be defined by $d_i x_j = 0$ if $j \neq i$, and $d_i x_i = 1$; as such it is called partial differentiation with respect to x_i, and written $\partial/\partial x_i$. Any vector field on V has the form $\Sigma \alpha_i d_i$, where $\alpha_i \in \mathscr{F}(V)$ $(1 \leqslant i \leqslant v)$, and it is well known that these account for all the derivations of the $\mathbf{R}$-algebra $\mathscr{F}(V)$. It follows that the action of any vector field d on any function β on V is given by the well-known *chain rule*

$$d\beta = \sum_i d_i \beta . dx_i. \tag{1.2}$$

These vector fields d_i commute. Given any sequence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_v)$ of non-negative integers, we write formally d^σ for the iterated operator $d_1^{\sigma_1} d_2^{\sigma_2} \ldots d_v^{\sigma_v}$ on $\mathscr{F}(V)$, x^σ for $x_1^{\sigma_1} x_2^{\sigma_2} \ldots x_v^{\sigma_v}$, and $\sigma!$ for $\sigma_1! \, \sigma_2! \ldots \sigma_v!$ We call $|\sigma| = \sum_i \sigma_i$ the *order* of σ, or of d^σ. We can now write down an obvious set of coordinate functions on $J^n(V, W)$, at least when n is finite, namely the functions X_i, Y_j, and $Z_{j,\sigma}$, for $1 \leqslant i \leqslant v$, $1 \leqslant j \leqslant w$, and $1 \leqslant |\sigma| \leqslant n$; these are defined by

$$X_i = x_i \circ \pi_V; \quad Y_j = Z_{j,0} = y_j \circ \pi_W; \quad Z_{j,\sigma} f = (d^\sigma(y_j \circ f))p, \tag{1.3}$$

where f is the germ at p of any map from V to W. These functions define an isomorphism of sets $J^n(V, W) \cong V \times \mathbf{R}^k$, where the precise dimension k does not interest us. These isomorphisms may be used as local charts in an atlas to make $J^n(V, W)$ a smooth manifold, for general manifolds V and W. All the projection maps π^m_n, etc., become projection maps of smooth fibre bundles, and the jet sections $J^n f$ are smooth.

Remark. — If $n > 1$, $J^n(V, W)$ is *not* a *vector* bundle over $V \times W$ with projection π^n_0 (unless W is a vector space). This is because there is no canonical way of adding n-jets in the fibres. Nevertheless, its fibres are all diffeomorphic to euclidean space, and it has a canonical zero section (the jets of constant maps).

If $n = \infty$, it is still possible to make $J(V, W) = J^\infty(V, W)$ into a strange kind of infinite-dimensional manifold by this means. It is not necessary for us to do this, because all we require are the concepts of smooth function on $J(V, W)$, and certain vector fields *defined* algebraically as derivations. These structures are more easily introduced as follows. The projections π_n induce a map

$$J(V, W) \to \varprojlim J^n(V, W)$$

from $J(V, W)$ to the inverse limit of the finite jet spaces $J^n(V, W)$ which is trivially an injection. It is a well-known classical result, due to E. Borel, that this is actually an isomorphism, but we shall not make essential use of this fact. However, it does suggest what structures to give to J(V, W). We give J(V, W) the inverse limit topology, which has as base the sets $(\pi_n)^{-1}(U)$, where n is finite and U is open in $J^n(V, W)$. We give J(V, W) the limit differential structure, in the following sense, which is suggested by sheaf theory.

Definition (**1.4**). — A function $\Phi : U \to R$, where U is open in J(V, W), is called *smooth* if it is locally of the form $\Psi \circ \pi_n$, where Ψ is a smooth function on some open subset of $J^n(V, W)$. (The integer n may have to be unbounded on U.) We write $\mathscr{F}(U)$ for the ring of smooth functions on U, and $\mathscr{F}(s)$ for the ring of germs at $s \in J(V, W)$ of smooth functions. Thus $\mathscr{F}(s) = \varinjlim \mathscr{F}(\pi_n s)$.

Hence the composite $\Phi \circ Jf$ of any smooth function Φ on J(V, W) and any jet section $Jf : N \to J(V, W)$ is again smooth.

Transversality.

We shall need Thom's celebrated transversality lemma, which is the keystone of general position theory. The usual form [8] is not quite good enough for our purposes; we need a variant that applies to the infinite jet space J(V, W). The only " submanifolds " of J(V, W) we propose to consider are those of the form $(\pi_n)^{-1}(Q)$, where Q is a submanifold of $J^n(V, W)$ and n is finite; these submanifolds have finite codimension. It is obvious what transversality of a jet section to such a submanifold should mean, in view of the relation $J^n f = \pi_n \circ Jf$.

Theorem (**1.5**). — *Let* $Q_1, Q_2, \ldots$ *be countably many submanifolds of* $J^n(V, W)$ (where $n \leqslant \infty$). *The maps* $f : V \to W$ *whose jet section* $J^n f : V \to J^n(V, W)$ *is transverse to* Q_i *for all i form a dense subset of these space* L *of all maps from* V *to* W, *where* L *is equipped with the fine*-C^∞*-topology* (or any of the other usual topologies).

Proof. — If n is finite, this result is well known (see [8] or [1]). The space L is a Baire space, i.e. the intersection of countably many dense open subsets of L is still dense. The subset L_i of maps f such that $J^n f$ is transverse to Q_i is a countable intersection of dense open subsets, by the usual proof based on Sard's theorem; hence $\bigcap_i L_i$ is again of this type, and therefore dense. ∎

The significance of transversality is that if $J^n f$ is transverse to Q and Q has codimension k, then $(J^n f)^{-1}(Q)$ is a *submanifold* of V, also having codimension k. Here we use the convention that the empty set can be given the structure of a n-manifold for any integer n, positive or negative! (Clearly every point of it has a neighbourhood of the required form.) Thus if $v < k$, transversality asserts that $(J^n f)V$ does not meet Q.

Total vector fields.

We formally define total vector fields as certain operators acting on functions on the jet space J(V, W), by defining them locally on the coordinate system (1.3) and requiring the chain rule (1.2) to hold. This will succeed, because by our definition (1.4) any smooth function on J(V, W) is locally a smooth function of finitely many of these coordinate functions. The total tangent bundle will be defined as the vector bundle having total vector fields as its sections.

Assume now that V is open in $\mathbf{R}^v$, and that $W=\mathbf{R}^w$. We use the coordinate functions (1.3).

Definition (**1.6**). — The *total* vector field D_i $(1\leqslant i\leqslant v)$ on J(V, W) is defined by :

$$\begin{cases} D_iX_k=0 & \text{if } k\neq i,\\ D_iX_i=1, & \\ D_iZ_{j,\sigma}=Z_{j,\tau}, & \text{where } \tau=(\sigma_1,\ldots,\sigma_{i-1},\sigma_i+1,\sigma_{i+1},\ldots,\sigma_v).\end{cases}$$

Let U be open in J(V, W). We call a *total vector field* on U any linear combination of operators, $\sum_i \Phi_i D_i$, where $\Phi_i\in\mathscr{F}(U)$.

Hence $Z_{j,\sigma}=D^\sigma Y_j$, and we therefore have no further use for the symbols $Z_{j,\sigma}$. Operators with the above properties are sometimes called *total* differential operators; whence our terminology.

It is clear that we cannot define total vector fields on $J^n(V, W)$ unless $n=\infty$. This is why we are working with the infinite jet spaces.

We can now rewrite (1.3) in terms of (1.6) :

Lemma (**1.7**). — *The functions* X_i $(1\leqslant i\leqslant v)$ *and* $D^\sigma Y_j$ $(1\leqslant j\leqslant w;\ 0\leqslant|\sigma|\leqslant n)$ *factor through* $J^n(V, W)$, *to yield a set of coordinate functions on* $J^n(V, W)$. ∎

Before we can pass to global properties of total vector fields, or even define them globally, we must describe the total vector fields D_i invariantly. Let N be open in V, and $f: N\to W$ any map. By the definition (1.3) of $Z_{j,\sigma}$, the identity $D_i\Phi\circ Jf=d_i(\Phi\circ Jf)$ holds for the coordinate functions (1.3), and hence for any function $\Phi\in\mathscr{F}(U)$, where U is open in J(V, W), by the chain rule (1.2). More generally, by taking linear combinations, given any vector field d on N, there is a total vector field D on $(\pi_V)^{-1}(N)$, which is characterized as an operator by the equation

$$D\Phi\circ Jf=d(\Phi\circ Jf). \tag{1.8}$$

We have now eliminated the dependence of D on the particular coordinate systems; so that it can be defined for general open sets N with vector fields d on them.

Definition (**1.9**). — We define locally the *total tangent bundle* **D** over J(V, W) as the " smooth " vector bundle having the total vector fields $\{D_1, D_2, \ldots, D_v\}$ as a base of sections. Elements of the fibre over $s\in J(V, W)$ are called *total tangent vectors* at s. They induce linear functionals $\mathscr{F}(s)\to\mathbf{R}$ satisfying the usual derivation formula.

This vector bundle **D** is canonically identified by (1.8) and the correspondence $d\rightsquigarrow D$ with the induced bundle $(\pi_V)^*\mathbf{T}_V$. We write $D=\pi_V^*d$. Its sections are the total vector fields. Hence *any* jet section Jf gives us back the vector field $d=(Jf)^*D$. Since by definition any total vector field has the form $\sum_i \Phi_i D_i$ locally, we deduce from (1.8) that

$$D\Phi\circ Jf=((Jf)^*D)(\Phi\circ Jf). \tag{1.10}$$

In the language of differential equations, any map $f: V \to W$ yields an integral submanifold $(Jf)V$ of the subbundle $\mathbf{D}$ of the " tangent bundle " to $J(V, W)$. The situation here is obviously entirely different from the finite-dimensional case.

When U is open in $J(V, W)$, we have described the inclusion

$$(1.11) \qquad \Gamma_{\mathbf{T}}(\pi_V U) \subset \Gamma_{\mathbf{D}}(U),$$

where $\mathbf{T} = \mathbf{T}_V$ is the tangent bundle of V.

Lemma (**1.12**). — *Suppose* $D \in \Gamma_{\mathbf{D}}(U)$ *lies in the image of* (1.11). *Then* $D(\Phi \circ \pi_n)$ *factors through* π_{n+1} *for any function* $\Phi : \pi_n U \to \mathbf{R}$.

Proof. — Over a coordinate neighbourhood in V this is clear from (1.6), (1.7), and the chain rule (1.2). ∎

Remark (**1.13**). — For any open set U in $J(V, W)$, $\Gamma_{\mathbf{D}}(U)$ is generated, as a $\mathscr{F}(U)$-module, by $2v$ globally defined vector fields on V, included by (1.11). For the tangent bundle $\mathbf{T}_V$ of V is a direct summand of a product vector bundle of fibre dimension $2v$; the components in $\mathbf{T}_V$ of the base of sections of this product bundle are the required generators.

Lemma (**1.14**). — *The Lie product* $[D', D''] = D'D'' - D''D'$ *of any two total vector fields* D' *and* D'' *is again a total vector field.*

Proof. — This is clear from (1.6). Alternatively, from an invariant point of view, it is evident from (1.8) that the inclusion (1.11) respects Lie products. This, with bilinearity, is enough, for if Φ' and Φ'' are in $\mathscr{F}(U)$, and D' and D'' are total vector fields on U, we have

$$[\Phi'D', \Phi''D''] = \Phi'\Phi''[D', D''] + \Phi'D'\Phi''.D'' - \Phi''D''\Phi'.D'$$

(where . is used as in § 0). ∎

Rank and total rank.

Let us first define the rank and corank of a set of functions on any manifold V. Let $p \in N \subset V$, and let A be any subset of $\mathscr{F}_V(N)$. For each $\alpha \in A$ and each tangent vector $d \in \mathbf{T}_V | p$ to V at p, we have the real number $d\alpha \in \mathbf{R}$. This induces a linear map

$$(1.15) \qquad \mathbf{T}_V | p \to \mathbf{R}^A,$$

where $\mathbf{R}^A$ stands for the vector space consisting of all maps from A to $\mathbf{R}$.

Definition (**1.16**). — The *rank* of A at p, $\mathrm{rk}_p A$, is defined as the rank of the linear map (1.15). The *corank*, or *kernel rank*, of A at p, written $\mathrm{kr}_p A$, is defined as the dimension of the kernel of (1.15).

Evidently, $\mathrm{rk}_p A + \mathrm{kr}_p A = v$.

We can do the same for *total* tangent vectors. Given $s \in U \subset J(V, W)$, and a subset A of $\mathscr{F}(U)$, we construct the linear map

$$(1.17) \qquad \mathbf{D} | s \to \mathbf{R}^A.$$

Definition (**1.18**). — The *total rank* of A at s, written $\mathrm{trk}_s \mathrm{A}$, is defined as the rank of the linear map (1.17). The *total corank*, or *total kernel rank*, of A at s, written $\mathrm{tkr}_s \mathrm{A}$, is defined as the dimension of the kernel of (1.17).

Evidently, $\mathrm{trk}_s \mathrm{A} + \mathrm{tkr}_s \mathrm{A} = v$.

The sets N and U appearing in the above definitions are irrelevant, for if $d \in \mathbf{T}_V | p$ and $\alpha \in \mathrm{A}$, the number $d\alpha$ depends only on the germ of α at p.

The total rank is, in a sense, universal :

Lemma (**1.19**). — *Suppose* $s \in \mathrm{U} \subset \mathrm{J}(\mathrm{V}, \mathrm{W})$, $p = \pi_V s$, *and* $f : \mathrm{N} \to \mathrm{W}$ *is a map such that* $p \in \mathrm{N}$, N *is open, and* $(\mathrm{J}f)p = s$. *Let* A *be a subset of* $\mathscr{F}(\mathrm{U})$; *then we have also the set* $(\mathrm{J}f)^* \mathrm{A} \subset \mathscr{F}_V((\mathrm{J}f)^{-1}\mathrm{U})$. *Then*

$$\mathrm{trk}_s \mathrm{A} = \mathrm{rk}_p (\mathrm{J}f)^* \mathrm{A}, \qquad \mathrm{tkr}_s \mathrm{A} = \mathrm{kr}_p (\mathrm{J}f)^* \mathrm{A}.$$

Proof. — This is a trivial deduction from (1.8). ∎

Definition (**1.20**). — Let U be open in J(V, W). We call the set of functions $\{\Phi_1, \Phi_2, \ldots, \Phi_k\}$ in $\mathscr{F}(\mathrm{U})$ *totally independent at* $s \in \mathrm{U}$ if its total rank at s is exactly k. We say it is *totally independent on* U if it is totally independent at every point of U.

Clearly we must have $k \leqslant v$.

Lemma (**1.21**). — *Let* U *be open in* J(V, W), *and* $\{\Phi_1, \Phi_2, \ldots, \Phi_v\}$ *a totally independent set of functions on* U. *Then there exist total vector fields* D_i *on* U $(1 \leqslant i \leqslant v)$, *uniquely defined by the conditions*

$$\mathrm{D}_i \Phi_j = \delta_{ij} \qquad (1 \leqslant i, j \leqslant v),$$

where δ *denotes the Kronecker delta function. These form a base of sections over* U, i.e. *a* $\mathscr{F}(\mathrm{U})$*-base of the module* $\Gamma_{\mathbf{D}}(\mathrm{U})$. *Moreover,* $[\mathrm{D}_i, \mathrm{D}_j] = 0$ *for all* i, j.

Suppose in addition that $\Phi_1, \Phi_2, \ldots, \Phi_v$, *and an extra function* Ψ, *factor through functions on* $\pi_n \mathrm{U} \subset \mathrm{J}^n(\mathrm{V}, \mathrm{W})$. *Then for each* i, $\mathrm{D}_i \Psi$ *factors through* $\pi_{n+1}\mathrm{U}$.

Proof. — Since the total vector fields are to be uniquely specified, we may assume that U is so small that we already know $\mathbf{D}|\mathrm{U}$ is a product bundle. Let $\{\partial_1, \partial_2, \ldots, \partial_v\}$ be any base of sections of $\mathbf{D}|\mathrm{U}$. By definition, the determinant $\det(\partial_i \Phi_j) \neq 0$ everywhere on U, and the matrix $(\partial_i \Phi_j)$ therefore has an inverse, (α_{ij}) say. Put $\mathrm{D}_i = \sum_j \alpha_{ij} \partial_j$; these are the unique total vector fields having the desired property. They form a base of sections of $\mathbf{D}|\mathrm{U}$, because they are everywhere linearly independent.

For any i, j, k, $[\mathrm{D}_i, \mathrm{D}_j]\Phi_k = \mathrm{D}_i \mathrm{D}_j \Phi_k - \mathrm{D}_j \mathrm{D}_i \Phi_k = 0$. But by (1.14) $[\mathrm{D}_i, \mathrm{D}_j]$ is also a total vector field, from which fact we may conclude that it is zero.

If the functions Φ_i factor through $\pi_n \mathrm{U}$, and we choose a base of sections $\{\partial_1, \partial_2, \ldots, \partial_v\}$ induced from vector fields on V by (1.11), we see from (1.12) that the functions $\partial_i \Phi_j$ and hence α_{ij} factor through $\pi_{n+1}\mathrm{U}$. Thus $\mathrm{D}_i \Psi$ also factors through $\pi_{n+1}\mathrm{U}$. ∎

Let us give one consequence of transversality. We recall the ideal $\mathfrak{m}_q$ of functions vanishing at q.

Lemma **(1.22)**. — *Let* Q *be a submanifold of* $J(V, W)$, $p \in V$, *and* $f: V \to W$ *a map such that* Jf *is transverse to* Q *at* p, *and* $s = (Jf)p \in Q$. *Then near* p, $Z = (Jf)^{-1}(Q)$ *is a submanifold of* V. *Let* $\mathfrak{a} \subset \mathscr{F}(s)$ *be the ideal of all germs of functions vanishing on* Q. *Then*

$$\dim \operatorname{Ker}(f|Z)_* = \operatorname{tkr}_s(\mathfrak{a} + \pi_W^* \mathfrak{m}_q),$$

where $q = fp$ *and* $(f|Z)_*$ *is the differential* $\mathbf{T}_Z|p \to \mathbf{T}_W|q$ *of* $f|Z: Z \to W$.

Proof. — Near s, Q is the inverse image of a submanifold of $J^k(V, W)$, for some finite k. Transversality, with the aid of the implicit function theorem, implies that $(Jf)^* \mathfrak{a}$ generates the ideal of germs at p of functions vanishing on Z. By (1.19), $\operatorname{tkr}_s(\mathfrak{a} + \pi_W^* \mathfrak{m}_q) = \operatorname{kr}_p((Jf)^* \mathfrak{a} + f^* \mathfrak{m}_q)$. Now a tangent vector at p always kills $\mathfrak{m}_p^2$, kills $(Jf)^* \mathfrak{a}$ if and only if it is tangent to Z, and kills $f^* \mathfrak{m}_q$ if and only if it lies in $\operatorname{Ker} f_*$. ∎

This suggests that to define the singularity subsets $\Sigma^I(f)$ of a map f in a satisfactory manner we should work with the total coranks of certain ideals of functions on the jet space, and then use (1.22) to identify this approach with the naive geometric approach when suitable transversality conditions hold. This is precisely what we shall do in the later sections.

§ 2. The singularity subsets.

In this section we shall define the singularity subsets $\Sigma^I(f)$ of a map $f: V \to W$. The jacobian extensions of certain ideals of functions on V are clearly relevant. These ideals will depend on the map f. Since we shall need to know what happens as f is allowed to vary, we must say what sort of dependence this is. We avoid this difficulty by defining our jacobian extensions universally, on the jet space $J(V, W)$.

Let U be any open subset of $J(V, W)$, and $\mathscr{F}$ the sheaf of germs of smooth functions on $J(V, W)$, which we defined in (1.4).

Definition **(2.1)**. — Given any *subset* A of $\mathscr{F}(U)$, its *kth total jacobian extension* $\Delta^k A$ is defined as the *ideal* of $\mathscr{F}(U)$ generated by A and the set of all $n \times n$ minors $\det(D_i \alpha_j)$, where $D_i \in \Gamma_{\mathbf{D}}(U)$, $\alpha_j \in A$ $(1 \leqslant i, j \leqslant n)$, and $n = v - k + 1$. Conventionally we define $\Delta^{v+1} A = \mathscr{F}(U)$.

As always, v is the dimension of V, and $\mathbf{D}$ is the total tangent bundle over $J(V, W)$, defined in (1.9).

The curious index is chosen to make the statement of the following trivial lemma as simple as possible. We have the maximal ideal $\mathfrak{m}_s$ in $\mathscr{F}(U)$, of functions vanishing at s. In (1.18) we defined the total corank $\operatorname{tkr}_s A$ of A at s.

Lemma **(2.2)**. — *Suppose* $s \in U$ *and* $A \subset \mathfrak{m}_s$. *Then* $\operatorname{tkr}_s A \geqslant k$ *if and only if* $\Delta^k A \subset \mathfrak{m}_s$. ∎

The number of generators for the ideal $\Delta^k A$ in (2.1) is vast, and quite impractical for computation. We next show that most of them are redundant.

Lemma **(2.3)**. — *Suppose that the subset* A *of* $\mathscr{F}(U)$ *generates the ideal* $\mathfrak{a}$, *and that the subset* Ω *generates* $\Gamma_{\mathbf{D}}(U)$ *as a* $\mathscr{F}(U)$*-module. Then the ideal* $\Delta^k \mathfrak{a}$ *is generated by* A *together* *with all the* $n\times n$ *minors formed as in* (2.1), *but using only functions in* A *and total vector fields in* Ω, *where* $n=v-k+1$.

Proof. — We take a typical minor $\det(D_i\alpha_j)$, where $D_i\in\Gamma_{\mathbf{D}}(U)$ $(1\leqslant i\leqslant n)$, and $\alpha_j\in\mathfrak{a}$ $(1\leqslant j\leqslant n)$. By hypothesis, we can write $D_i=\sum_\lambda \varphi_{i\lambda}D_\lambda$ and $\alpha_j=\sum_\mu \psi_{j\mu}\alpha_\mu$ for suitable elements $\varphi_{i\lambda}$, $\psi_{j\mu}\in\mathscr{F}(U)$, $D_\lambda\in\Omega$, and $\alpha_\mu\in A$. Then

$$D_i\alpha_j=\sum_{\lambda,\mu}\varphi_{i\lambda}\psi_{j\mu}.D_\lambda\alpha_\mu+\sum_{\lambda,\mu}\varphi_{i,\lambda}D_\lambda\psi_{j,\mu}.\alpha_\mu,$$

with the bracketing convention mentioned in § 0. The minor $\det(D_i\alpha_j)$ therefore lies in the required ideal, by standard linear algebra. ∎

This simplifying result has many useful corollaries.

Corollary **(2.4)**. — $\Delta^k A=\Delta^k\mathfrak{a}$. ∎

Corollary **(2.5)**. — *If the ideal* $\mathfrak{a}$ *is generated by* $v-k$ *elements, then* $\Delta^k\mathfrak{a}=\mathfrak{a}$.

Proof. — All the $n\times n$ minors under consideration contain a repeated column, since $n=v-k+1$. ∎

Corollary **(2.6)**. — *If the ideal* $\mathfrak{a}$ *is generated by functions on a subset of* $J^r(V, W)$, *then* $\Delta^k\mathfrak{a}$ *is generated by functions on a subset of* $J^{r+1}(V, W)$.

Proof. — By (1.13), $\Gamma_{\mathbf{D}}(U)$ is generated by vector fields on $\pi_V U\subset V$. The result now follows from (1.12). ∎

Corollary **(2.7)**. — *If* $\mathfrak{a}$ *is a finitely generated ideal of* $\mathscr{F}(U)$, *then so is* $\Delta^k\mathfrak{a}$.

Proof. — By (1.13), $\Gamma_{\mathbf{D}}(U)$ is a finitely generated $\mathscr{F}(U)$-module. ∎

Corollary **(2.8)**. — Δ^k *is compatible with restriction. If* $U\supset U'$, $\mathfrak{a}$ *is a given ideal in* $\mathscr{F}(U)$, *and* $\mathfrak{a}'$ *is the ideal in* $\mathscr{F}(U')$ *generated by* $\mathfrak{a}|U'$, *then* $\Delta^k\mathfrak{a}'=\Delta^k(\mathfrak{a}|U')$ *is the ideal generated by* $(\Delta^k\mathfrak{a})|U'$.

Proof. — By (1.13), $\Gamma_{\mathbf{D}}(U')$ is generated as a $\mathscr{F}(U')$-module by globally defined total vector fields. ∎

Corollary **(2.9)**. — *For any subset* A *of* $\mathscr{F}(U)$, $\Delta^0 A$ *is the ideal generated by* A, *provided* U *is sufficiently small.*

Proof. — This is clear locally, for if U is small enough, $\Gamma_{\mathbf{D}}(U)$ is a free $\mathscr{F}(U)$-module on v generators, and then the minors under consideration all have a repeated row. Hence they contribute nothing to $\Delta^0 A$. ∎

The last corollary is a further reason for our indexing system.

Determinants tend to be unmanageable; we would much prefer to handle linear transformations. We shall show in due course that this can be arranged.

Before doing this, we show that there are results for subbundles of $\mathbf{D}|U$ parallel to (2.3), etc.

Lemma **(2.10)**. — *Let* $\mathbf{E}$ *be any vector subbundle of* $\mathbf{D}|U$, *and put* $\Gamma=\Gamma_{\mathbf{E}}(U)$. *Suppose that the ideal* $\mathfrak{a}$ *of* $\mathscr{F}(U)$ *is generated by the subset* A, *and that the* $\mathscr{F}(U)$-*module* Γ *is generated by the set* Ω *of total vector fields. Then the ideal* $\mathfrak{a}+\Gamma\mathfrak{a}$ *is generated by* A *and the set of functions on* U *of the form* $D\alpha$, *where* $D\in\Omega$, *and* $\alpha\in A$. (We write ΓA for the additive subset of $\mathscr{F}(U)$ generated by all elements of the form $D\alpha$, where $D\in\Gamma$ and $\alpha\in A$. It is already an ideal because Γ is a $\mathscr{F}(U)$-module.)

Proof. — Take a typical element $D=\sum_i \beta_i D_i$ of Γ, where $\beta_i \in \mathscr{F}(U)$ and $D_i \in \Omega$ $(1 \leqslant i \leqslant m)$, and a typical element $\alpha = \sum_j \gamma_j \alpha_j$, where $\gamma_j \in \mathscr{F}(U)$ and $\alpha_j \in A$ $(1 \leqslant j \leqslant n)$. Then

$$D\alpha = \sum_{i,j} \beta_i \gamma_j D_i \alpha_j + \sum_{i,j} \beta_i \alpha_j D_i \gamma$$

lies in the required ideal. ∎

Corollary (**2.11**). — $\mathfrak{a} + \Gamma\mathfrak{a} = \mathfrak{a} + \Gamma A$. ∎

Corollary (**2.12**). — *If* $\mathfrak{a}$ *is a finitely generated ideal of* $\mathscr{F}(U)$, *then so is* $\mathfrak{a} + \Gamma\mathfrak{a}$, *provided* U *is small enough.* ∎

Corollary (**2.13**). — *Let* A *be a subset of* $\mathscr{F}(U)$, *and* U′ *an open subset of* U. *Then the ideal* $\Gamma_{\mathbf{E}}(U')(A|U')$ *is generated by* $(\Gamma A)|U'$.

Proof. — From the proof of (2.10), and (1.13), we need only use sections of **E** defined over U in computing $\Gamma_{\mathbf{E}}(U')(A|U')$. ∎

Now we shall suppose that the subbundle of $\mathbf{D}|U$ is not merely any vector subbundle. Suppose given a subset C of $\mathscr{F}(U)$ containing exactly $v-k$ elements, which is totally independent everywhere on U (see § 1). Then the set of all total tangent vectors that annihilate C is a subbundle **K** of $\mathbf{D}|U$ having fibre dimension k. Such subbundles have special properties.

Lemma (**2.14**). — *The sections of* **K** *are closed under Lie product, i.e.*

$$[\Gamma_{\mathbf{K}}(U), \Gamma_{\mathbf{K}}(U)] \subset \Gamma_{\mathbf{K}}(U).$$

Proof. — Take any two sections D_1 and D_2 of **K**, and $\alpha \in C$. Then

$$[D_1, D_2]\alpha = D_1 D_2 \alpha - D_2 D_1 \alpha = 0.$$

But by (1.14) $[D_1, D_2]$ is again a total vector field, and therefore is a section of **K**. ∎

We come now to the main reduction lemma.

Lemma (**2.15**). — *Let* U *be open in* J(V, W), *and* C *and* **K** *as above. Put* $\Gamma = \Gamma_{\mathbf{K}}(U)$. *Then for any ideal* $\mathfrak{a}$ *of* $\mathscr{F}(U)$ *generated by a subset* A *containing* C *we have*

$$\Delta^k \mathfrak{a} = \mathfrak{a} + \Gamma\mathfrak{a}.$$

Moreover, the ideal ΓA *is independent of the choice of* C.

Proof. — Write $\mathbf{D}|U = \mathbf{K} \oplus \mathbf{L}$, and $C = \{\alpha_1, \alpha_2, \ldots, \alpha_{v-k}\}$. As in the proof of (1.21) we can define total vector fields D_i $(1 \leqslant i \leqslant v-k)$, *as sections of* **L**, by the equations $D_i \alpha_j = \delta_{ij}$ $(1 \leqslant i, j \leqslant v-k)$. These form a base of sections of **L**. We apply (2.3), using only the total vector fields D_i together with those in Γ, and using only those functions in A. By construction, $\Gamma C = 0$. Any non-zero $n \times n$ minor (where $n = v-k+1$) of the restricted jacobian matrix must contain a row of elements of the form $D\alpha$ with $D \in \Gamma$, and therefore lie in ΓA. On the other hand, we obtain the whole of ΓA, for the element $D\alpha$ $(D \in \Gamma, \alpha \in A)$ arises as the minor formed from the rows $\{D_1, D_2, \ldots, D_{v-k}, D\}$ and columns $\{\alpha_1, \alpha_2, \ldots, \alpha_{v-k}, \alpha\}$. Hence the ideal ΓA is generated by the $n \times n$ minors of the jacobian matrix $(D\alpha)$, where $\alpha \in A$ and D lies in Γ or is some D_i. As in the proof of (2.3), removing the restriction on the total vector field D does not enlarge this ideal. It is therefore independent of the choice of C.

Finally, $\Delta^k \mathfrak{a} = \mathfrak{a} + \Gamma A = \mathfrak{a} + \Gamma \mathfrak{a}$, by (2.11). ∎

Lemma **(2.16)**. — *Suppose further that in* (2.15) C *and* A *are sets of functions on* $\pi_n U \subset J^n(V, W)$. *Then the ideals* $\mathfrak{a} + \Gamma \mathfrak{a}$ *and* ΓA *are generated by functions on* $\pi_{n+1} U \subset J^{n+1}(V, W)$.

Proof. — We need consider only the case when $\pi_n U$ is small, so small that $C = \{\Phi_1, \Phi_2, \ldots, \Phi_{v-k}\}$ extends to a set $\{\Phi_1, \Phi_2, \ldots, \Phi_{v-k}, \ldots, \Phi_v\}$ of v functions on $\pi_n U$, totally independent everywhere on U. Then the base of total vector fields $\{D_1, D_2, \ldots, D_v\}$ on U constructed by (1.21) has the property that for any function Ψ on $\pi_n U$, $D_i \Psi$ is a function on $\pi_{n+1} U$. Since Γ has the $\mathscr{F}(U)$-base $\{D_{v-k+1}, D_{v-k+2}, \ldots, D_v\}$, we deduce that the ideal ΓA is generated by functions on $\pi_{n+1} U$.

Finally we observe that by (2.11) $\mathfrak{a} + \Gamma \mathfrak{a} = \mathfrak{a} + \Gamma A$. ∎

We are ready to define the singularity subsets and deduce some of their elementary properties.

Let $I = (i_1, i_2, \ldots, i_n)$ be any sequence of integers; we say I has *length n*. We propose to define a subset Σ^I of the jet space J(V, W) for each sequence I. Let N be a subset of J(V, W), $s \in N$, and put $q = \pi_W s \in W$. Then we have the ideal $\mathfrak{m}_q$ in $\mathscr{F}_W(q)$, or in $\mathscr{F}_W(\pi_W N)$, of real functions vanishing at q. We may include it by the projection π_W in $\mathscr{F}(s)$, or in $\mathscr{F}(N)$, as an additive subgroup. We recall from (1.18) the notion of total corank. Our definition is motivated by (2.2) and (1.22).

Definition **(2.17)**. — We work in $\mathscr{F}(s)$. We define the subset Σ^I of J(V, W) by

$$s \in \Sigma^I \text{ if and only if } \begin{cases} \operatorname{tkr}_s \mathfrak{m}_q = i_1, \\ \operatorname{tkr}_s \Delta^{i_1} \mathfrak{m}_q = i_2, \\ \operatorname{tkr}_s \Delta^{i_2} \Delta^{i_1} \mathfrak{m}_q = i_3, \\ \cdots\cdots\cdots \\ \operatorname{tkr}_s \Delta^{i_{n-1}} \ldots \Delta^{i_2} \Delta^{i_1} \mathfrak{m}_q = i_n. \end{cases}$$

Formally we put $\Sigma^{\emptyset} = J(V, W)$. We write $\Delta^I \mathfrak{m}_q$ for the ideal $\Delta^{i_n} \Delta^{i_{n-1}} \ldots \Delta^{i_2} \Delta^{i_1} \mathfrak{m}_q$.

This definition is clearly invariant, if not very useful. We could instead have worked with ideals in $\mathscr{F}(N)$ instead of $\mathscr{F}(s)$, and used the same definition as above. It follows from (2.4) and (2.8) that we obtain the same subset Σ^I.

Remark. — $\Sigma^I = \emptyset$ unless

$$\textbf{(2.18)} \qquad \begin{cases} \text{a)} \quad i_1 \geqslant i_2 \geqslant i_3 \geqslant \ldots \geqslant i_{n-1} \geqslant i_n \geqslant 0, \\ \text{b)} \quad v \geqslant i_1 \geqslant v - w, \\ \text{c)} \quad \text{if } i_1 = v - w, \text{ then } i_1 = i_2 = \ldots = i_n. \end{cases}$$

We have an increasing sequence of ideals, whose total coranks must therefore decrease. Also, $\mathfrak{m}_q$ is generated by w elements, which shows that $\operatorname{trk}_s \mathfrak{m}_q \leqslant w$ and hence

$\operatorname{tkr}_s \mathfrak{m}_q \geqslant v-w$. Further, $\Delta^{v-w}\mathfrak{m}_q = \mathscr{F}(\mathrm{N}).\mathfrak{m}_q$, by (2.4) and (2.5), which has the same total corank and jacobian extensions as $\mathfrak{m}_q$, by (2.4). We shall see that conversely $\Sigma^{\mathrm{I}} \neq \emptyset$ if (2.18) holds.

Remark (**2.19**). — If we keep $\mathrm{I}' = (i_1, i_2, \ldots, i_{n-1})$ fixed, and vary i_n, we partition the set $\Sigma^{\mathrm{I}'}$ into the sets Σ^{I}, according to the total corank of the ideal $\Delta^{\mathrm{I}'}\mathfrak{m}_q$. It follows that as I runs through all sequences of length n, $\mathrm{J}(\mathrm{V}, \mathrm{W})$ is partitioned into subsets Σ^{I}.

The ideal $\Delta^{\mathrm{I}}\mathfrak{m}_q$ is finitely generated, by (2.7), and indeed by functions on $\pi_n \mathrm{N} \subset \mathrm{J}^n(\mathrm{V}, \mathrm{W})$, by (2.6). Hence by (2.2) we have:

Lemma (**2.20**). — *Σ^{I} is the inverse image under $\pi_n : \mathrm{J}(\mathrm{V}, \mathrm{W}) \to \mathrm{J}^n(\mathrm{V}, \mathrm{W})$ of a subset of $\mathrm{J}^n(\mathrm{V}, \mathrm{W})$, if I has length n.* ■

We deduce from (2.17) our definition of the singularities of a map $f : \mathrm{V} \to \mathrm{W}$:

Definition (**2.21**). — Given a map $f : \mathrm{V} \to \mathrm{W}$, we define its singularity subset $\Sigma^{\mathrm{I}}(f)$ as $(\mathrm{J}f)^{-1}(\Sigma^{\mathrm{I}})$, for each sequence I, where $\mathrm{J}f : \mathrm{V} \to \mathrm{J}(\mathrm{V}, \mathrm{W})$ is the jet section.

Hence the sets $\Sigma^{\mathrm{I}}(f)$ in V possess the same partition properties (2.19) as the sets Σ^{I} in $\mathrm{J}(\mathrm{V}, \mathrm{W})$.

Lemma (**2.22**). — *Take $n = 1$. Then $p \in \Sigma^i(f)$ if and only if* $\dim \operatorname{Ker} f_* = i$, *where $f_* : \mathbf{T}_{\mathrm{V}}|p \to \mathbf{T}_{\mathrm{W}}|fp$ is the differential of f at p.*

Proof. — This result is immediate from (1.19) and (2.2). ■

This shows that our set $\Sigma^i(f)$ is as in [7], apart from the numbering system for i.

Remark. — The sets $\Sigma^{\mathrm{I}}(f)$ can be computed without reference to the jet space, and in a finite number of steps if, say, $f : \mathbf{R}^v \to \mathbf{R}^w$ is polynomial. For the total jacobian extension of an ideal of functions on the jet space induces the ordinary jacobian extension of the induced ideal of functions on $\mathbf{R}^v$. Thus the images of the ideals $\Delta^{\mathrm{I}}\mathfrak{m}_q$ may be computed as successive jacobian extensions of the subset $f^*\mathfrak{m}_q \subset \mathfrak{m}_p$, where $fp = q$. Moreover, by (2.3), we need take only carefully chosen coordinate functions and vector fields in the computations.

However, it is difficult to establish general properties of the sets Σ^{I} from the definition in terms of determinants. It is possible to use (2.15) to formulate the definition in terms of vector bundles, with no mention of determinants. This is what we shall do in § 3.

§ 3. Special flags of bundles.

It is clear that the definition (2.17) we have given of the singularity subsets Σ^{I} of $\mathrm{J}(\mathrm{V}, \mathrm{W})$, though invariant, is unmanageable. In this section we develop another approach, in terms of " special " flags of vector bundles, which is useful, but whose invariance requires proof.

Throughout this section let $\mathrm{I} = (i_1, i_2, \ldots, i_n)$ be a fixed sequence of integers of length $n > 0$, satisfying $v \geqslant i_1 \geqslant i_2 \geqslant \ldots \geqslant i_n \geqslant 0$, and put $\mathrm{I}' = (i_1, i_2, \ldots, i_{n-1})$, its curtailment. We set $i_0 = v$ to unify the definitions. We use much of the notation of § 1 without comment, and suppress the inclusions π_W^*, etc. In particular, we have the total tangent bundle $\mathbf{D}$ over $J(V, W)$, and the sheaf $\mathscr{F}$ of germs of functions on $J(V, W)$.

Definition (3.1). — A *special* I-*flag* $\mathfrak{K}$ over the open subset U of (J(V, W) is a flag of subbundles of $\mathbf{D}|U$,

$$\mathbf{D}|U = \mathbf{K}_0 \supset \mathbf{K}_1 \supset \mathbf{K}_2 \supset \ldots \supset \mathbf{K}_n,$$

where $\mathbf{K}_r$ has fibre dimension i_r $(0 \leqslant r \leqslant n)$, such that there exists a system of subsets C_r of $\mathscr{F}(U)$ $(1 \leqslant r \leqslant n)$ satisfying:

a) C_r contains exactly $i_{r-1} - i_r$ elements $(1 \leqslant r \leqslant n)$;

b) $C_1 \cup C_2 \cup \ldots \cup C_n$ is totally independent on U (see (1.20));

c) $\Gamma_r C_r = 0$, where Γ_r is the module of sections of $\mathbf{K}_r$ $(1 \leqslant r \leqslant n)$;

d) $C_1 \subset \mathscr{F}_W(\pi_W U) \subset \mathscr{F}(U)$, and

$$C_r \subset \Gamma_{r-1}\Gamma_{r-2}\ldots\Gamma_2\Gamma_1\mathscr{F}(\pi_W U) \subset \mathscr{F}(U) \qquad (1 < r \leqslant n),$$

(each Γ_r acts on $\mathscr{F}(U)$, and $\pi_W^* : \mathscr{F}_W(\pi_W U) \subset \mathscr{F}(U)$);

e) C_r is a set of functions on $\pi_{r-1}U \subset J^{r-1}(V, W)$ $(1 \leqslant r \leqslant n)$.

Axiom *e)* is a purely technical condition, designed to simplify one proof in § 5. We shall usually assume that some such system of subsets C_r has been selected.

Lemma (3.2). — Γ_r *is closed under Lie product*, $[\Gamma_r, \Gamma_r] \subset \Gamma_r$ $(1 \leqslant r \leqslant n)$.

Proof. — $\mathbf{K}_r$ is exactly the bundle of total tangent vectors killing $C_1 \cup C_2 \cup \ldots \cup C_r$, by *a)* and *b)*. Hence Γ_r is closed under Lie product, by (2.14). ∎

Remark (3.3). — We can curtail a special I-flag to obtain a special I′-flag, by forgetting $\mathbf{K}_n$. We call it $\mathfrak{K}'$. Thus assertions about special flags will all be proved by induction on the length of I. Conversely, given a special I′-flag, we can always extend it trivially to a special I-flag, if $i_n = i_{n-1}$, by taking $\mathbf{K}_n = \mathbf{K}_{n-1}$.

Definition (3.4). — The *ideal* $\mathfrak{z}_{\mathfrak{K}} \subset \mathscr{F}(U)$ of a special I-flag $\mathfrak{K}$ is defined as

$$\mathfrak{z}_{\mathfrak{K}} = \Gamma_1\mathscr{F}_W(N) + \Gamma_2\Gamma_1\mathscr{F}_W(N) + \ldots + \Gamma_n\Gamma_{n-1}\ldots\Gamma_2\Gamma_1\mathscr{F}_W(N),$$

where $N = \pi_W U \subset W$ and $\pi_W : \mathscr{F}_W(N) \subset \mathscr{F}(U)$. We say that $\mathfrak{K}$ is *null at* $s \in U$ if $\mathfrak{z}_{\mathfrak{K}} \subset \mathfrak{m}_s$, the ideal of functions vanishing at s.

We usually use this definition in the inductive form

$$(3.5) \qquad \begin{cases} \mathfrak{z}_{\mathfrak{K}} = \Gamma_1\mathscr{F}_W(\pi_W U) & \text{if } n = 1, \\ \mathfrak{z}_{\mathfrak{K}} = \mathfrak{z}_{\mathfrak{K}'} + \Gamma_n \mathfrak{z}_{\mathfrak{K}'} & \text{if } n > 1. \end{cases}$$

Lemma (3.6). — *The ideal* $\mathfrak{z}_{\mathfrak{K}}$ *in* $\mathscr{F}(U)$ *is independent of* $\mathfrak{K}$. *It is generated by functions on the subset* $\pi_n U$ *of* $J^n(V, W)$.

Proof. — By (2.15), the ideal $\Gamma_1\mathscr{F}_W(\pi_W U)$ is invariant. By induction on the length of I, suppose $\mathfrak{z}_{\mathfrak{K}'}$ is invariant. Then (2.15) shows that $\mathfrak{z}_{\mathfrak{K}} = \mathfrak{z}_{\mathfrak{K}'} + \Gamma_n \mathfrak{z}_{\mathfrak{K}'}$ is invariant. By n applications of (2.16) we see from (3.5) that $\mathfrak{z}_{\mathfrak{K}}$ is generated by functions on $\pi_n U$. ∎

Given a special I-flag $\mathfrak{K}$ over U, and an open subset $U' \subset U$, we can define in the obvious way the *restricted* special I-flag $\mathfrak{K}|U'$ over U'. We see from (2.13) and (1.13) that its ideal $\mathfrak{z}_{\mathfrak{K}|U'}$ is generated by $\mathfrak{z}_{\mathfrak{K}}|U'$.

Lemma (**3.7**). — *The ideal* $\mathfrak{z}_{\mathfrak{K}}$ *is finitely generated, if* U *is small enough.*

Proof. — If $\mathfrak{a}$ is a finitely generated ideal in $\mathscr{F}(U)$, so is $\mathfrak{a}+\Gamma_r\mathfrak{a}$, for any r, by (2.12). If U is small enough, e.g. $\pi_W U$ contained in a coordinate neighbourhood in W, the chain rule (1.2) shows that $\Gamma_1\mathscr{F}_W(\pi_W U)$ is a finitely generated ideal. Then from (3.5), by induction on length, $\mathfrak{z}_{\mathfrak{K}}$ is finitely generated. ∎

Actually the hypothesis on U is unnecessary.

Lemma (**3.8**). — *Suppose the real vector space* $\mathfrak{z}_{\mathfrak{K}}/(\mathfrak{z}_{\mathfrak{K}'}+\mathfrak{m}_s\mathfrak{z}_{\mathfrak{K}})=(\mathfrak{z}_{\mathfrak{K}}/\mathfrak{z}_{\mathfrak{K}'})\otimes_{\mathscr{F}(U)}\mathbf{R}$ *is spanned by the images of certain elements* $\alpha_i\in\mathfrak{z}_{\mathfrak{K}}$. *Then in* $\mathscr{F}(s)$ *the image of* $\mathfrak{z}_{\mathfrak{K}}$ *is generated as an ideal by the images of* $\mathfrak{z}_{\mathfrak{K}'}$ *and these elements* α_i.

Proof. — This is a particular case of a well-known lemma attributed to Nakayama. $\mathscr{F}(s)$ is a local ring (though not Noetherian) having $\mathfrak{m}_s$ as its unique maximal ideal. We have proved above, in (3.7), that the image of $\mathfrak{z}_{\mathfrak{K}}$ in $\mathscr{F}(s)$ is a finitely generated ideal $\mathfrak{a}$, say, since $\mathscr{F}(U)\to\mathscr{F}(s)$ is surjective. Let $\mathfrak{b}$ be the ideal in $\mathscr{F}(s)$generated by $\mathfrak{z}_{\mathfrak{K}'}$ and the elements α_i, and put $M=\mathfrak{a}/\mathfrak{b}$. Then we know M is a finitely generated $\mathscr{F}(s)$-module, and satisfies $M=\mathfrak{m}_s M$. We have to prove $M=0$.

Suppose that on the contrary $\{x_1, x_2, \ldots, x_r\}$ is a minimal set of generators of M, where $r>0$. We may write $x_r=\sum_{i=1}^{r}\beta_i x_i$, with $\beta_i\in\mathfrak{m}_s$ $(1\leqslant i\leqslant r)$. Since $1-\beta_r$ is invertible, we can deduce that $x_r=(1-\beta_r)^{-1}\sum_{i=1}^{r-1}\beta_i x_i$, which contradicts the minimality of r. ∎

We next establish the equivalence of this theory with that of § 2, in several steps.

Lemma (**3.9**). — *Let* $\mathfrak{K}$ *be a special* I*-flag defined over* $U\subset J(V, W)$. *Suppose* $s\in U$, *and put* $q=\pi_W s\in W$. *Then*

$$\mathfrak{z}_{\mathfrak{K}}+\mathscr{F}(U).\mathfrak{m}_q=\Delta^I\mathfrak{m}_q\equiv\Delta^{i_n}\ldots\Delta^{i_2}\Delta^{i_1}\mathfrak{m}_q.$$

Proof. — Let $\mathfrak{K}'$ be the special I'-flag obtained by curtailing $\mathfrak{K}$. Put $\mathfrak{a}=\Delta^{I'}\mathfrak{m}_q$. Then by induction on the length of I, suppose we have established $\mathfrak{a}=\mathfrak{z}_{\mathfrak{K}'}+\mathscr{F}(U).\mathfrak{m}_q$. We apply (2.15) to $C=C_1\cup C_2\cup\ldots\cup C_n$, $\mathbf{K}_n$, and $\mathfrak{a}$. We find

$$\Delta^I\mathfrak{m}_q=\Delta^{i_n}\mathfrak{a}=\mathfrak{a}+\Gamma_n\mathfrak{a}=\mathfrak{z}_{\mathfrak{K}'}+\mathscr{F}(U).\mathfrak{m}_q+\Gamma_n\mathfrak{z}_{\mathfrak{K}'}+\Gamma_n(\mathscr{F}(U).\mathfrak{m}_q).$$

Now by (3.5) $\mathfrak{z}_{\mathfrak{K}}=\mathfrak{z}_{\mathfrak{K}'}+\Gamma_n\mathfrak{z}_{\mathfrak{K}'}$, and by (2.11)

$$\mathscr{F}(U).\mathfrak{m}_q+\Gamma_n(\mathscr{F}(U).\mathfrak{m}_q)=\mathscr{F}(U).\mathfrak{m}_q+\Gamma_n\mathfrak{m}_q\subset\mathscr{F}(U).\mathfrak{m}_q+\mathfrak{z}_{\mathfrak{K}}.$$

Hence $\Delta^I\mathfrak{m}_q=\mathfrak{z}_{\mathfrak{K}}+\mathscr{F}(U).\mathfrak{m}_q$.

The initial induction step, when $n=1$, is slightly different. Application of (2.15) to C_1, $\mathbf{K}_1$, and $\mathfrak{m}_q$, yields $\Delta^{i_1}\mathfrak{m}_q=\mathscr{F}(U).\mathfrak{m}_q+\Gamma_1\mathfrak{m}_q=\mathscr{F}(U).\mathfrak{m}_q+\Gamma_1\mathscr{F}_W(\pi_W U)$, since Γ_1 kills constants. This last expression is $\mathscr{F}(U).\mathfrak{m}_q+\mathfrak{z}_{\mathfrak{K}}$, by (3.5). ∎

Lemma (**3.10**). — *Let* $\mathfrak{K}'$ *be a special* I'*-flag defined on an open set* $U'\ni s$, *and put* $q=\pi_W s$. *Suppose that* $\mathfrak{K}'$ *is null at* s, *and that* $\operatorname{tkr}_s(\mathfrak{z}_{\mathfrak{K}'}+\mathscr{F}(U').\mathfrak{m}_q)\leqslant i_n$. *Then there exists an open set* $U\subset U'$, $U\ni s$, *such that the restricted special* I'*-flag* $\mathfrak{K}'|U$ *extends to a special* I*-flag* $\mathfrak{K}$ *on* U.

Proof. — We have to construct U, C_n, and $\mathbf{K}_n$. This is easy if $n=1$. If $n \geqslant 2$ write $I''=(i_1, i_2, \ldots, i_{n-2})$, and $\mathfrak{K}''$ for the special I''-flag obtained by curtailing $\mathfrak{K}'$. ($\mathfrak{K}''=\emptyset$ and $\mathfrak{z}_{\mathfrak{K}''}=0$ if $n=2$.)

Since $\mathfrak{K}'$ is null at s, Γ_{n-1} annihilates the ideal $\mathfrak{z}_{\mathfrak{K}''}+\mathscr{F}(U').\mathfrak{m}_q$, and therefore $\operatorname{trk}_s(\mathfrak{z}_{\mathfrak{K}''}+\mathscr{F}(U').\mathfrak{m}_q) \leqslant v-i_{n-1}$. But we are given $\operatorname{trk}_s(\mathfrak{z}_{\mathfrak{K}'}+\mathscr{F}(U').\mathfrak{m}_q) \geqslant v-i_n$. If we compare these two ideals, we see that the only extra term in the second is $\Gamma_{n-1}\Gamma_{n-2}\ldots\Gamma_2\Gamma_1\mathscr{F}_W(\pi_W U')$. But $C_1 \cup C_2 \cup \ldots \cup C_{n-1} \subset \mathfrak{z}_{\mathfrak{K}''}$; hence there exists a subset C_n of this extra term having exactly $i_{n-1}-i_n$ elements, such that $C_1 \cup C_2 \cup \ldots \cup C_{n-1} \cup C_n$ is totally independent at s. We can further choose C_n as a set of functions on $\pi_{n-1}U' \subset J^{n-1}(V, W)$, because from (2.16) $\Gamma_{n-1}\ldots\Gamma_2\Gamma_1\mathscr{F}_W(\pi_W U')$ is generated by such functions. Then (3.1) *e)* holds. This set $C_1 \cup C_2 \cup \ldots \cup C_n$ is totally independent at s, and therefore totally independent in some neighbourhood U of s. We take $\mathbf{K}_n$ as the bundle of total tangent vectors on U that kill these functions. We have constructed a special I-flag over U. ∎

Theorem (**3.11**). — (Equivalence.)

a) *Let $\mathfrak{K}$ be a special* I*-flag over* U, *where* U *is open in* J(V, W). *Then $\mathfrak{K}$ is null at $s \in U$ if and only if $s \in \Sigma^I$.*

b) *If $s \in \Sigma^I$, there exists a special* I*-flag defined on some neighbourhood of s.*

Proof. — We may rewrite the definition (2.17) of Σ^I in the inductive form:

$s \in \Sigma^I$ if and only if $s \in \Sigma^{I'}$ and $\operatorname{tkr}_s \Delta^{I'}\mathfrak{m}_q = i_n$, where $q=\pi_W s$. We prove both parts of the theorem by induction on the length of I.

a) Assume $s \in \Sigma^{I'}$, and that $\mathfrak{K}'$ is null at s, where $\mathfrak{K}'$ is the curtailment of $\mathfrak{K}$. Then $s \in \Sigma^I$ if and only if $\operatorname{tkr}_s \Delta^{I'}\mathfrak{m}_q = i_n$. This condition may be written $\operatorname{tkr}_s \Delta^{I'}\mathfrak{m}_q \geqslant i_n$, since we know by (3.9) that $\Delta^{I'}\mathfrak{m}_q$ contains the totally independent set $C_1 \cup C_2 \cup \ldots \cup C_n$ of $v-i_n$ functions. By (2.2) this condition is equivalent to $\Delta^I\mathfrak{m}_q = \Delta^{i_n}\Delta^{I'}\mathfrak{m}_q \subset \mathfrak{m}_s$. Now by (3.9) $\Delta^I\mathfrak{m}_q = \mathfrak{z}_{\mathfrak{K}} + \mathscr{F}(U).\mathfrak{m}_q$, which is contained in $\mathfrak{m}_s$ if and only if $\mathfrak{K}$ is null at s.

b) Suppose $s \in \Sigma^I$; then certainly $s \in \Sigma^{I'}$. Suppose we already have a special I'-flag $\mathfrak{K}'$ defined on a neighbourhood U' of s. By *a)* above, $\mathfrak{K}'$ is null at s; and $\operatorname{tkr}_s(\mathfrak{z}_{\mathfrak{K}'}+\mathscr{F}(U').\mathfrak{m}_q)=i_n$ by (3.9) and the definition of Σ^I. Then (3.10) constructs the required special I-flag. ∎

We have in this theorem achieved our object of showing that the singularity subsets Σ^I can be defined by linear methods, without recourse to determinants.

§ 4. Multilinear forms on flags.

We show that a special flag, defined as in § 3, gives rise to various multilinear forms. In this and the following sections, we shall need many vector bundles and maps of vector bundles, which we denote by $\mathbf{E}$, $\mathbf{F}$, $\mathbf{a}: \mathbf{E} \to \mathbf{F}$, etc. We shall simplify the notation by using the same symbol $\mathbf{E}$, say, for any restricted vector bundle $\mathbf{E}|N$, preferring to specify separately the subset N concerned.

Take $I=(i_1, i_2, \ldots, i_n)$, $I'=(i_1, i_2, \ldots, i_{n-1})$, and a fixed special I-flag $\mathfrak{K}$ over

$U \subset J(V, W)$. We use all the notation of § 3. Take a fixed point $s \in U$, and put $q = \pi_W s \in W$. We shall write K_r for the fibre $\mathbf{K}_r | s$ of $\mathbf{K}_r$ over s; then $K_r \cong \Gamma_r / \mathfrak{m}_s \Gamma_r$, where Γ_r is as in § 3 the module of sections of $\mathbf{K}_r$. Thus we have a surjection $\Gamma_r \to K_r$ of vector spaces $(1 \leqslant r \leqslant n)$. From (3.4) we have

$$\mathfrak{z}_{\mathfrak{K}} = \mathfrak{z}_{\mathfrak{K}'} + \Gamma_n \Gamma_{n-1} \ldots \Gamma_2 \Gamma_1 \mathscr{F}_W(\pi_W U),$$

where $\mathfrak{K}'$ is the curtailed flag, a special I-flag. The iterated action on $\mathfrak{m}_q$ of the total vector fields may be expressed as a $\mathbf{R}$-linear map

$$\Gamma_n \otimes \Gamma_{n-1} \otimes \ldots \otimes \Gamma_2 \otimes \Gamma_1 \otimes \mathfrak{m}_q \to \mathfrak{z}_{\mathfrak{K}}.$$

Theorem (4.1). — *Let $\mathfrak{K}$ be a special* I*-flag over* U, *and $\mathfrak{K}'$ its curtailment. Then for each point $s \in U$ there exists a unique map $\varepsilon_{\mathfrak{K}}(s)$ such that the diagram*

$$\begin{array}{ccc} \Gamma_n \otimes \Gamma_{n-1} \otimes \ldots \otimes \Gamma_2 \otimes \Gamma_1 \otimes \mathfrak{m}_q & \longrightarrow & \mathfrak{z}_{\mathfrak{K}} \\ \downarrow & & \downarrow \\ K_n \otimes K_{n-1} \otimes \ldots \otimes K_2 \otimes K_1 \otimes (\mathfrak{m}_q/\mathfrak{m}_q^2) & \xrightarrow[\varepsilon_{\mathfrak{K}}(s)]{} & \mathfrak{z}_{\mathfrak{K}}/(\mathfrak{z}_{\mathfrak{K}'} + \mathfrak{m}_s \mathfrak{z}_{\mathfrak{K}}) \end{array}$$

commutes. Moreover, $\varepsilon_{\mathfrak{K}}(s)$ is symmetric, in the sense that it factors through the symmetric tensor product to yield

$$\bar{\varepsilon}_{\mathfrak{K}}(s) : (K_n \odot K_{n-1} \odot \ldots \odot K_2 \odot K_1) \otimes (\mathfrak{m}_q/\mathfrak{m}_q^2) \to \mathfrak{z}_{\mathfrak{K}}/(\mathfrak{z}_{\mathfrak{K}'} + \mathfrak{m}_s \mathfrak{z}_{\mathfrak{K}}).$$

(For the precise definition of the symmetric tensor product $K_n \odot K_{n-1} \odot \ldots \odot K_2 \odot K_1$, see (4.2) below.)

Proof. — Now $\mathrm{Ker}(\Gamma_r \to K_r)$ is generated additively by sections of the form αD, where $\alpha \in \mathfrak{m}_s$, and $D \in \Gamma_r$. Take $D_r \in \Gamma_r$ $(1 \leqslant r \leqslant n)$, $\alpha \in \mathfrak{m}_s$, and $\beta, \gamma \in \mathfrak{m}_q$.

a) $D_n D_{n-1} \ldots D_{r+1} \alpha D_r \ldots D_1 \beta = \Sigma D'\alpha . D''\beta$, for various differential operators D' and D'', products of the D_i (where . brackets and multiplies, as indicated in § 0). In this case, $D''\beta \in \mathfrak{z}_{\mathfrak{K}'}$ for every term except $\alpha . D_n D_{n-1} \ldots D_2 D_1 \beta$, which lies in $\mathfrak{m}_s \mathfrak{z}_{\mathfrak{K}}$.

b) $D_n D_{n-1} \ldots D_1(\beta . \gamma) = \Sigma D'\beta . D''\gamma$, where $D'\beta$ and $D''\gamma$ are in $\mathfrak{z}_{\mathfrak{K}'}$, except for the terms $D_n D_{n-1} \ldots D_1 \beta . \gamma$ and $\beta . D_n D_{n-1} \ldots D_1 \gamma$, which lie in $\mathfrak{m}_s \mathfrak{z}_{\mathfrak{K}}$.

c) If also $D_{r-1} \in \Gamma_r$ for some r $(1 < r \leqslant n)$,

$$D_n \ldots D_r D_{r-1} \ldots D_1 \beta - D_n \ldots D_{r-1} D_r \ldots D_1 \beta = D_n \ldots D_{r+1} [D_r, D_{r-1}] \ldots D_1 \beta,$$

which is in $\mathfrak{z}_{\mathfrak{K}'}$ because the Lie product $[D_r, D_{r-1}]$ lies in Γ_r, by (3.2).

Assertions *a)* and *b)* show that $\varepsilon_{\mathfrak{K}}(s)$ can be defined. According to (4.3) below, *c)* is enough to establish symmetry. ∎

Digression on symmetric tensor products.

Let A be a vector space. The permutation group G on n symbols acts on the n-fold tensor product $A \otimes A \otimes \ldots \otimes A$ by permuting the factors. The *symmetric tensor product* $A \odot A \odot \ldots \odot A$ is defined as the largest quotient of $A \otimes A \otimes \ldots \otimes A$ on which G acts trivially, i.e. the quotient of $A\otimes A\otimes\ldots\otimes A$ by the subspace spanned by all elements of the form $\sigma x - x$, where $\sigma\in G$ and $x\in A\otimes A\otimes\ldots\otimes A$. Since G is generated by transpositions, even by transpositions of two adjacent factors, we may restrict the choice of σ to such permutations.

Let Ω be a (unordered) base of A; then $\Omega\times\Omega\times\ldots\times\Omega$ is a base of $A\otimes A\otimes\ldots\otimes A$. G also acts on $\Omega\times\Omega\times\ldots\times\Omega$. We denote the orbit set by $\Omega\bigcirc\Omega\bigcirc\ldots\bigcirc\Omega$. Then $\Omega\bigcirc\Omega\bigcirc\ldots\bigcirc\Omega$ is a base of $A\bigcirc A\bigcirc\ldots\bigcirc A$.

For each i such that $1\leqslant i\leqslant n$, let B_i be a subspace of A. We wish to define the symmetric tensor product $B_1\bigcirc B_2\bigcirc\ldots\bigcirc B_n$. A certain amount of care is needed, because we have at least three candidates, in general all distinct.

Definition (**4.2**). — We define the *symmetric tensor product* $B_1\bigcirc B_2\bigcirc\ldots\bigcirc B_n$ as the image of the composite

$$B_1\otimes B_2\otimes\ldots\otimes B_n\subset A\otimes A\otimes\ldots\otimes A\rightarrow A\bigcirc A\bigcirc\ldots\bigcirc A.$$

However, there are common cases when the various choices coincide.

Lemma (**4.3**). — *Suppose the vector subspaces* B_i $(1\leqslant i\leqslant n)$ *of* A *satisfy*

a) $B_1\supset B_2\supset B_3\supset\ldots\supset B_{n-1}\supset B_n$,

or

b) $B_1\supset B_2\supset B_3\supset\ldots\supset B_{n-1}$, $B_{n-1}\subset B_n$, *and* $B_{n-1}=B_n\cap B_1$. *Then the kernel of the symmetrization map*

$$B\equiv B_1\otimes B_2\otimes\ldots\otimes B_n\rightarrow B_1\bigcirc B_2\bigcirc\ldots\bigcirc B_n$$

is spanned by elements of the form

$$b_1\otimes\ldots\otimes b_r\otimes b_{r+1}\otimes\ldots\otimes b_n - b_1\otimes\ldots\otimes b_{r+1}\otimes b_r\otimes\ldots\otimes b_n,$$

where $b_i\in B_i$ $(1\leqslant i\leqslant n)$, *and* $b_i\in B_r\cap B_{r+1}$ $(i=r, r+1)$, *for some* r $(1\leqslant r<n)$.

Proof. — Choose a base Ω of A such that $\Omega\cap B_i$ is a base of B_i for all i, which is possible in either case. It is clear that the exhibited element lies in the kernel Z of the symmetrization map; we have to prove that Z is spanned by such elements.

Take a general element $z\in Z$. Then

$$z=\sum_\lambda \alpha_\lambda\,\omega_{1,\lambda}\otimes\omega_{2,\lambda}\otimes\ldots\otimes\omega_{n,\lambda},$$

for various base elements $\omega_{i,\lambda}\in\Omega\cap B_i$ $(1\leqslant i\leqslant n)$, and real numbers α_λ, where λ runs through some indexing set. Now $(\Omega\bigcirc\Omega\bigcirc\ldots\bigcirc\Omega)\cap(B_1\bigcirc B_2\bigcirc\ldots\bigcirc B_n)$ is a base of $B_1\bigcirc B_2\bigcirc\ldots\bigcirc B_n$, by our choice of Ω. For each $\eta\in\Omega\bigcirc\Omega\bigcirc\ldots\bigcirc\Omega$, let z_η be the sum of the terms of z for which $\omega_{1,\lambda}\bigcirc\omega_{2,\lambda}\bigcirc\ldots\bigcirc\omega_{n,\lambda}=\eta$; then $z=\sum_\eta z_\eta$, and we must have again that $z_\eta\in Z$ for all η. We may rewrite z_η in the form

$$\sum_\sigma \beta_\sigma\,\sigma(\omega_1\otimes\omega_2\otimes\ldots\otimes\omega_n),$$

where the elements $\omega_i\in\Omega$ are not necessarily all distinct, and σ runs through certain permutations in G. With suitable numbering, the coefficient β_1 of the identity permutation will be non-zero, or else $z_\eta = 0$. Since $z_\eta \in Z$, we must have $\sum_\sigma \beta_\sigma = 0$. It follows that Z is spanned by elements of the form

$$\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_n - \sigma(\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_n),$$

subject to $\omega_i \in \Omega \cap B_i \cap B_{\sigma i}$ $(1 \leqslant i \leqslant n)$.

Let r be the maximum index such that $\sigma r \neq r$; then $\sigma r < r$, $\omega_r \in B_{\sigma r+1}$, and $\omega_k \in B_{\sigma r}$, where $\sigma k = \sigma r + 1$, in all cases. Let τ be the transposition $(\sigma r, \sigma r + 1)$. Let us write $y = \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_n$; then we observe that $\tau\sigma y \in B$, and that we may therefore write

$$y - \sigma y = (y - \tau\sigma y) + (\tau\sigma y - \sigma y).$$

The second term on the right, $\tau\sigma y - \sigma y$, has the required form. By construction, the disorder of $\tau\sigma$ is less than the disorder of σ, where we mean by the disorder of σ the number of pairs of integers (i, j) such that $i < j$ and $\sigma i > \sigma j$. Downward induction on disorder therefore yields the desired result. ∎

We can draw two corollaries from (4.1).

Corollary **(4.4)**. — *We may deduce from* $\varepsilon_\mathfrak{K}(s)$ *the bundle map*

$$(\mathbf{K}_n \bigcirc \mathbf{K}_{n-1} \bigcirc \ldots \bigcirc \mathbf{K}_2 \bigcirc \mathbf{K}_1) \otimes \mathbf{Hom}(\mathbf{P}, \mathbf{R}) \to \mathbf{R}$$

over $\Sigma^{I'}$, *where* $\mathbf{P} = \pi_W^* \mathbf{T}_W$, *and* $\mathbf{R}$ *denotes the product line bundle.*

Proof. For each $s \in \Sigma^{I'}$, $\mathfrak{K}'$ is null at s by (3.11), $\mathfrak{z}_{\mathfrak{K}'} \subset \mathfrak{m}_s$, and evaluation gives

$$\mathfrak{z}_\mathfrak{K} / (\mathfrak{z}_{\mathfrak{K}'} + \mathfrak{m}_s \mathfrak{z}_\mathfrak{K}) \to \mathscr{F}(s) / \mathfrak{m}_s \cong \mathbf{R}.$$

We identify $\mathbf{Hom}(\mathbf{P}, \mathbf{R}) | s$ with $\mathfrak{m}_q / \mathfrak{m}_q^2$. Then composition with $\varepsilon_\mathfrak{K}(s)$, as s varies, yields the required bundle map. We can deduce from (4.1) that it is continuous. ∎

Later on, when we have discovered that $\Sigma^{I'}$ is a submanifold of J(V, W), we shall be able to say that we have a smooth bundle map.

Corollary **(4.5)**. — *Evaluation induces a bundle map*

$$\mathbf{a}_\mathfrak{K} : (\mathbf{K}_n \bigcirc \mathbf{K}_n \bigcirc \mathbf{K}_{n-1} \bigcirc \ldots \bigcirc \mathbf{K}_2 \bigcirc \mathbf{K}_1) \otimes \mathbf{Hom}(\mathbf{P}, \mathbf{R}) \to \mathbf{R} \text{ over } \Sigma^I.$$

Proof. — We pointed out in (3.3) that we could trivially extend the special I-flag $\mathfrak{K}$ to a special I^+-flag $\mathfrak{K}^+$, where $I^+ = (i_1, i_2, \ldots, i_{n-1}, i_n, i_n)$. The assertion follows by applying (4.4) to $\mathfrak{K}^+$. ∎

N. B. — Even though the vector spaces K_i are invariant over Σ^I, $\varepsilon_\mathfrak{K}(s)$ and $\mathbf{a}_\mathfrak{K}$ depend on the choice of the special I-flag $\mathfrak{K}$. Later, in § 7, we shall compute their indeterminacy.

We shall need an upper bound for the number of generators of the ideal $\mathfrak{z}_\mathfrak{K}$, which, by (3.6), *is* independent of $\mathfrak{K}$. We find a bound by considering the multilinear forms we have just introduced. By construction,

$$\varepsilon_\mathfrak{K}(s) : K_n \otimes K_{n-1} \otimes \ldots \otimes K_2 \otimes K_1 \otimes (\mathfrak{m}_q / \mathfrak{m}_q^2) \to \mathfrak{z}_\mathfrak{K} / (\mathfrak{z}_{\mathfrak{K}'} + \mathfrak{m}_s \mathfrak{z}_\mathfrak{K})$$

is surjective. We deduce from Nakayama's lemma, in the form (3.8):

Lemma **(4.6)**. — *If the ideal* $\mathfrak{z}_{\mathfrak{K}'}$ *in* $\mathscr{F}(s)$ *can be generated by* ν' *elements, and the rank of the linear map* $\varepsilon_\mathfrak{K}(s)$ *is* κ, *then the ideal* $\mathfrak{z}_\mathfrak{K}$ *in* $\mathscr{F}(s)$ *can be generated by* $\nu' + \kappa$ *elements.* ∎

We have therefore to find a lower bound for the dimension of the kernel of $\varepsilon_{\mathfrak{K}}(s)$. Symmetrizing $\varepsilon_{\mathfrak{K}}(s)$ helped.

We have not yet exhausted our knowledge of Ker $\varepsilon_{\mathfrak{K}}(s)$. By the very definition (3.1) of special I-flag, Γ_r was chosen to annihilate a totally independent set $C_1 \cup C_2 \cup \ldots \cup C_r$ of functions on $U \subset J(V, W)$. Let us write this fact in bundle form.

We express the action of the Γ_i on $\mathscr{F}_W(\pi_W U)$ as a **R**-linear map

$$\Gamma_{r-1} \otimes \Gamma_{r-2} \otimes \ldots \otimes \Gamma_2 \otimes \Gamma_1 \otimes \mathscr{F}_W(\pi_W U) \to \mathscr{F}(U).$$

By definition (3.1) *d)*, C_r lies in the image of this map. We lift each of the $i_{r-1} - i_r$ elements of C_r. By means of the isomorphism $\mathbf{Hom}(\mathbf{P}, \mathbf{R}) | s \cong \mathscr{F}_W(\pi_W U)/(\mathfrak{m}_q^2 + \mathbf{R}.1)$, each element of $\Gamma_{r-1} \otimes \ldots \otimes \Gamma_1 \otimes \mathscr{F}_W(\pi_W U)$ yields a smooth section of the vector bundle

$$\mathbf{K}_{r-1} \otimes \mathbf{K}_{r-2} \otimes \ldots \otimes \mathbf{K}_2 \otimes \mathbf{K}_1 \otimes \mathbf{Hom}(\mathbf{P}, \mathbf{R}).$$

For each element of C_r, we have a section of this vector bundle.

Definition **(4.7)**. — Having made the preceding choices, we define the vector bundle $\mathbf{H}_r$ as the product vector bundle with fibre $\mathbf{R}^m$, where $m = i_{r-1} - i_r$, and the smooth bundle map $\mathbf{h}_r : \mathbf{H}_r \to \mathbf{K}_{r-1} \otimes \ldots \otimes \mathbf{K}_1 \otimes \mathbf{Hom}(\mathbf{P}, \mathbf{R})$ by sending the base sections of $\mathbf{H}_r$ to the sections corresponding to the elements of C_r.

Let H_r be the fibre of $\mathbf{H}_r$ over s, etc. Then we have, trivially, from (3.1) *c)* and *d)*, the result:

Lemma **(4.8)**. — *The composite map*

$$K_n \otimes K_{n-1} \otimes \ldots \otimes K_r \otimes H_r \xrightarrow[1 \otimes h_r]{} K_n \otimes \ldots \otimes K_1 \otimes (\mathfrak{m}_q/\mathfrak{m}_q^2) \xrightarrow[\varepsilon_{\mathfrak{K}}(s)]{} \mathfrak{z}_{\mathfrak{K}}/(\mathfrak{z}_{\mathfrak{K}'} + \mathfrak{m}_s \mathfrak{z}_{\mathfrak{K}})$$

is zero $(1 \leqslant r \leqslant n)$. ∎

Hence $\varepsilon_{\mathfrak{K}}(s)$ vanishes on the image of $1 \otimes h_r$ and on the kernel of the symmetrization map. We need to know how these various subspaces are related. The situation is somewhat complicated, and when dealing with symmetric tensor products we must, of course, be careful with brackets. In order to abbreviate, let us write for the moment:

$$\begin{aligned} \mathbf{F} &\equiv (\mathbf{K}_n \bigcirc \mathbf{K}_{n-1} \bigcirc \ldots \bigcirc \mathbf{K}_1) \otimes \mathbf{Hom}(\mathbf{P}, \mathbf{R}), \\ \mathbf{E}_r &\equiv (\mathbf{K}_n \bigcirc \mathbf{K}_{n-1} \bigcirc \ldots \bigcirc \mathbf{K}_r) \otimes \mathbf{H}_r, && (1 \leqslant r \leqslant n) \\ \mathbf{G}_r &\equiv (\mathbf{K}_n \bigcirc \mathbf{K}_{n-1} \bigcirc \ldots \bigcirc \mathbf{K}_r) \otimes \mathbf{Hom}(\mathbf{K}_{r-1}/\mathbf{K}_r, \mathbf{R}) && (1 \leqslant r \leqslant n), \\ \mathbf{K} &\equiv \mathbf{K}_n \otimes \mathbf{K}_{n-1} \otimes \ldots \otimes \mathbf{K}_1 \otimes \mathbf{Hom}(\mathbf{P}, \mathbf{R}). \end{aligned}$$

We again write F, E_r, etc., for the fibres over the typical point s.

The bundle map $\mathbf{h}_r : \mathbf{H}_r \to \mathbf{K}$ composed with the symmetrization map $\mathbf{K} \to \mathbf{F}$ yields a bundle map

$$\bar{\mathbf{h}}_r : \mathbf{E}_r \to \mathbf{F} \qquad (1 \leqslant r \leqslant n). \tag{4.9}$$

If $\mathfrak{K}$ is null at s, we have also the linear maps (4.5) associated to the subsequences $(i_1, i_2, \ldots, i_{r-1})$ of I,

$$a_{r-1} : K_{r-1} \otimes K_{r-1} \otimes K_{r-2} \otimes \ldots \otimes K_1 \otimes P' \to \mathbf{R} \qquad (1 < r \leqslant n),$$

and also $a_0 : K_0 \otimes P' \to \mathbf{R}$, where we put $P' = \mathrm{Hom}(P, \mathbf{R})$ and $\mathbf{K}_0 = \mathbf{D}|U$ (as we have already done in the definition of $\mathbf{G}_1$). We may partially dualize, and write this as

$$a'_{r-1} : K_{r-1} \otimes K_{r-2} \otimes \ldots \otimes K_1 \otimes P' \to \mathrm{Hom}(K_{r-1}, \mathbf{R}).$$

Since $\mathfrak{K}$ is null at s, this map factors through $\mathrm{Hom}(K_{r-1}/K_r, \mathbf{R})$. We deduce the map

$$k_r : K \to G_r \qquad (1 \leqslant r \leqslant n).$$

Lemma (**4.10**). — *Over* $s \in \Sigma^I$, *the above maps* $k_r : K \to G_r$ $(1 \leqslant r \leqslant n)$ *are symmetric, and yield maps* $\overline{k}_r : F \to G_r$. *The composite* $\overline{k}_r \overline{h}_t : E_t \to F \to G_r$ $(1 \leqslant r, t \leqslant n)$ *is zero if* $t < r$, *and an isomorphism if* $t = r$.

Proof. — Consider a'_{r-1}. Since it is symmetric, and $\mathfrak{K}$ is null at s, it factors through

$$\overline{a}'_{r-1} : \{(K_{r-1}/K_r) \odot (K_{r-2}/K_r) \odot \ldots \odot (K_1/K_r)\} \otimes P' \to \mathrm{Hom}(K_{r-1}/K_r, \mathbf{R}).$$

Since $\mathfrak{K}$ is null at s, and the set of $v - i_r$ functions $C_1 \cup C_2 \cup \ldots \cup C_r$ is totally independent at s, the sequence

$$(\mathbf{4.11}) \qquad 0 \to H_r \xrightarrow[a'_{r-1}h_r]{} \mathrm{Hom}(K_{r-1}, \mathbf{R}) \to \mathrm{Hom}(K_r, \mathbf{R}) \to 0$$

must be exact. It follows that $\overline{a}'_{r-1} h_r$ is an isomorphism. We also know that a'_{r-1} vanishes on the image of

$$K_{r-1} \otimes K_{r-2} \otimes \ldots \otimes K_t \otimes H_t \xrightarrow[1 \otimes h_t]{} K_{r-1} \otimes \ldots \otimes K_1 \otimes P'$$

whenever $t < r$, by (4.8) applied to a curtailment of $\mathfrak{K}$. When we have established the existence of $\overline{k}_r$, these facts will give us $\overline{k}_r \overline{h}_t = 0$ when $t \leqslant r$, and the required isomorphism when $t = r$.

We have established, by constructing $\overline{a}'_{r-1}$, that $k_r : K \to G_r$ factors through the quotient

$$A \equiv \{K_n \odot K_{n-1} \odot \ldots \odot K_r\} \otimes \{(K_{r-1}/K_r) \odot \ldots \odot (K_1/K_r)\} \otimes P'$$

of K. We assert that A is also a quotient of F, from which the rest of the lemma follows, by the diagram:

$$\begin{array}{ccccccc}
K_n \otimes \ldots \otimes K_r \otimes H_r & \longrightarrow & K & \longrightarrow & & & K_n \otimes \ldots \otimes K_r \otimes \mathrm{Hom}(K_{r-1}/K_r, \mathbf{R}) \\
\downarrow & & \downarrow & \searrow & & & \downarrow \\
E_r & \xrightarrow[\overline{h}_r]{} & F & \dashrightarrow & A & \longrightarrow & G_r
\end{array}$$

We must show that

$$B \equiv \{K_n \odot K_{n-1} \odot \ldots \odot K_r\} \otimes \{(K_{r-1}/K_r) \odot (K_{r-2}/K_r) \odot \ldots \odot (K_1/K_r)\}$$

is a quotient of $K_n \odot K_{n-1} \odot \ldots \odot K_1$. Now by (4.3) the kernel of the symmetrization map

$$K_n \otimes K_{n-1} \otimes \ldots \otimes K_2 \otimes K_1 \to K_n \odot K_{n-1} \odot \ldots \odot K_2 \odot K_1$$

is generated by elements of the form

$$d_n \otimes \ldots \otimes d_t \otimes d_{t-1} \otimes \ldots \otimes d_1 - d_n \otimes \ldots \otimes d_{t-1} \otimes d_t \otimes \ldots \otimes d_1,$$

where $d_i \in K_i$ $(1 \leqslant i \leqslant n)$ and also $d_{t-1} \in K_t$. It is clear that this element gives 0 in B unless $t=r$, in which case d_t and d_{t-1} lie in K_r, and therefore give 0 in K_{r-1}/K_r. Hence the kernel of the symmetrization map is contained in Ker k_r, and we can deduce the existence of $\bar{k}_r$. ∎

The map $\varepsilon_{\mathfrak{K}}(s)$ factors through F by (4.1), and vanishes on $\bar{h}_r E_r$ $(1 \leqslant r \leqslant n)$, by (4.8).

Corollary **(4.12)**. — *For any point* $s \in \Sigma^I$, *the rank of the linear map* $\varepsilon_{\mathfrak{K}}(s)$ *is at most*

$$\kappa \equiv w\kappa_1 - (i_0 - i_1)\kappa_1 - (i_1 - i_2)\kappa_2 - \ldots - (i_{n-1} - i_n)\kappa_n,$$

where $\kappa_r = \dim(K_n \bigcirc K_{n-1} \bigcirc \ldots \bigcirc K_r)(1 \leqslant r \leqslant n)$.

Proof. — From (4.10) we see that the maps $\bar{h}_r$ are all injective, and that their images in F form a direct sum of some subspace. The dimension of E_r is $(i_{r-1} - i_r)\kappa_r$, and the dimension of F is $w\kappa_1$, where w is the dimension of W. ∎

Hence we have by (4.6) an upper bound for the number of generators of the image of the ideal $\mathfrak{z}_{\mathfrak{K}}$ in $\mathscr{F}(s)$ whenever $s \in \Sigma^I$.

Remark. — Given $f: V \to W$, $p \in \Sigma^I(f)$, and a special I-flag $\mathfrak{K}$ on a neighbourhood of $s = (Jf)p$, Jf induces a flag in $\mathbf{T}_V$ having corresponding properties, including linear maps $(Jf)^* a_r$. By taking coordinates compatible with the flag, i.e. $(Jf)^* \mathbf{K}_r$ spanned by the first i_r coordinate vector fields (which is possible by (3.1) and (1.21)), it is easy to see that the linear maps $(Jf)^* a_r$ over p are arbitrary, subject to symmetry, (4.8), the nonsingularity condition (4.11), and the nullity condition

$$a_r | K_{r+1} \otimes K_r \otimes \ldots \otimes K_2 \otimes K_1 \otimes P' = 0$$

when $r < n$. In particular one can show that Σ^I is non-empty whenever I satisfies (2.18).

The best-known case of a multilinear form over a point $p \in V$ is the hessian quadratic form $\mathbf{T}_V | p \otimes \mathbf{T}_V | p \to \mathbf{R}$ of a map $f: V \to \mathbf{R}$, defined whenever p is a critical point of f. In this case the indeterminacy is zero.

§ 5. Special flags and tangent vectors.

We have established in § 4 an upper bound for the number of generators of the ideal $\mathfrak{z}_{\mathfrak{K}}$ of a special I-flag $\mathfrak{K}$. In this section we find a lower bound, by constructing formal tangent vectors to the jet space J(V, W). Our main theorem will hang on the fact that these two bounds coincide.

We first set up some machinery for constructing explicitly certain tangent vectors to J(V, W). We make much use of the notation of § 1.

Suppose given a fixed map $f: V^v \to W^w$, and a fixed smooth homotopy $f_t: V \to W$ passing through f, so that $f_0 = f$. Let U be open in J(V, W), and take $\Phi \in \mathscr{F}(U)$ and a total vector field $D \in \Gamma_{\mathbf{D}}(U)$. We have by (1.1) the jet section $Jf_t : V \to J(V, W)$, and we put

$$\begin{cases} \varphi = \Phi \circ Jf, \\ \varphi(t) = \Phi \circ Jf_t, \\ \varphi' = \varphi'(0), \end{cases} \qquad \begin{cases} d = (Jf)^* D, \\ d(t) = (Jf_t)^* D, \\ d' = d'(0), \end{cases}$$

where $\varphi'(t)$ and $d'(t)$ denote the t-derivatives of $\varphi(t)$ and $d(t)$. The derivative $d'(t)$ has a meaning if we interpret a vector field on V as a section of $\mathbf{T}_V$. Thus we have functions φ and φ', and vector fields d and d', all defined on $(Jf)^{-1}(U)$. We use corresponding notation for other functions and vector fields constructed in this way. If the total vector field D is induced from a vector field on V by (1.11), it is obvious that $d(t)$ is constant and $d' = 0$.

Lemma (5.1). — *Given* $p \in V$, $s = (Jf)p \in U$, *and a homotopy* f_t *passing through* f *as above, then* $\Phi \leadsto \varphi' p \in \mathbf{R}$ *is a tangent vector to* J(V, W) *at* s. *We have, for* $\Phi, \Psi \in \mathscr{F}(U)$ *and* $D \in \Gamma_{\mathbf{D}}(U)$, *the formulae*

$$(\varphi\psi)' = \varphi'\psi + \varphi\psi' \quad \textit{and} \quad (d\varphi)' = d'\varphi + d\varphi' \quad \textit{in} \quad \mathscr{F}_V((Jf)^{-1}(U)),$$

and

$$(\hat{\varphi} d)' = \varphi' d + d' \hat{\varphi} \quad \textit{in} \quad \Gamma_{\mathbf{T}}((Jf)^{-1}(U)),$$

where $\hat{\varphi}$ *denotes multiplication by* φ, *an operator on* $\mathscr{F}_V((Jf)^{-1}(U))$, *and* $\mathbf{T} = \mathbf{T}_V$ *is the tangent bundle to* V.

Proof. — $\Gamma_{\mathbf{D}}(U)$ is generated by $\Gamma_{\mathbf{T}}(V)$ as a $\mathscr{F}(U)$-module, by (1.13). The formulae follow. Evaluation of the first at p shows that we have constructed a tangent vector. ∎

Remark. — These tangent vectors are "tangents along the fibre" of $\pi_V : J(V, W) \to V$, i.e. in the kernel of $\mathbf{d}\pi_V$. It can be shown that all such tangent vectors can be obtained by this method.

Corollary (5.2). — *If* D_i $(1 \leqslant i \leqslant n)$ *is a total vector field on* U, *and* Φ *is a function on* U, *then*

$$(d_1 d_2 \ldots d_n \varphi)' = d_1' d_2 \ldots d_n \varphi + d_1 d_2' \ldots d_n \varphi + \ldots + d_1 d_2 \ldots d_n' \varphi + d_1 d_2 \ldots d_n \varphi'. \quad ∎$$

Lemma (5.3). — *Given* $p \in (Jf)^{-1}(U) \subset V$, *assume the homotopy* f_t *is such that* $\varphi' \in \mathfrak{m}_p^n$ *for all* $\Phi \in \mathscr{F}_W(\pi_W U)$, *included in* $\mathscr{F}(U)$ *by* $(\pi_W)^*$. *Then* $\psi' \in \mathfrak{m}_p^{n-r}$ *whenever* $\Psi \in \mathscr{F}(\pi_r U)$ *is a function on* $\pi_r U \subset J^r(V, W)$.

Proof. — As local coordinates on $J^r(V, W)$ at $(J^r f)p$, we may choose functions of the form $D_k D_{k-1} \ldots D_2 D_1 \Phi$, where Φ is defined on a subset of W, $k \leqslant r$, or Ψ, defined on a subset of V, by (1.7). For these functions the conclusion is immediate from (5.2). For general functions on $\pi_r U$, the result follows from this by the chain rule (1.2). ∎

Now let $\{\Phi_1, \Phi_2, \ldots, \Phi_v\}$ be a totally independent set of functions (see (1.20)) on U. Then (1.21) constructed canonically associated total vector fields D_i on U, defined by $D_i \Phi_j = \delta_{ij}$. As above, we assume given a homotopy f_t, and use it to define functions φ_i and φ_i', and vector fields d_i and d_i', defined on $(Jf)^{-1}(U)$, for $1 \leqslant i \leqslant v$. Total independence implies by (1.19) that the functions φ_i locally form a set of local coordinates on V, with the vector fields d_i as corresponding coordinate vector fields, since $d_i \varphi_j = \delta_{ij}$.

Then (5.1) applied to $d_i\varphi_j = \delta_{ij}$ yields $d'_i\varphi_j + d_i\varphi'_j = 0$. But any vector field d on $(Jf)^{-1}(U)$ has the form $\sum_i \alpha_i d_i$, where we must have $\alpha_i = d\varphi_i$. In this case we find

$$d'_i = -\sum_{j=1}^{v} d_i\varphi'_j \cdot d_j \qquad (1 \leqslant i \leqslant v). \tag{5.4}$$

Suppose now that $\mathfrak{K}$ is a special I-flag (see § 3) defined over U.

Lemma (5.5). — *Suppose the special* I-*flag* $\mathfrak{K}$ *is null at* $s \in U$. *Then we can find a set of* κ *tangent vectors* δ_k $(1 \leqslant k \leqslant \kappa)$ *to* J(V, W) *at* s *such that*:

a) $\delta_k(\Phi \circ \pi_{n-1}) = 0$ *for every function* Φ *on* $\pi_{n-1}U$,

b) *The induced maps* $\delta_k|\mathfrak{z}_{\mathfrak{K}} : \mathfrak{z}_{\mathfrak{K}} \to \mathbf{R}$ *are linearly independent in* $\mathrm{Hom}(\mathfrak{z}_{\mathfrak{K}}, \mathbf{R})$.

The number κ *is given by*

$$\kappa = w\kappa_1 - (i_0 - i_1)\kappa_1 - (i_1 - i_2)\kappa_2 - \ldots - (i_{n-1} - i_n)\kappa_n, \tag{5.6}$$

where $\kappa_r = \dim(K_n \bigcirc K_{n-1} \bigcirc \ldots \bigcirc K_r)$ *as in* (4.12).

Proof. — Take any map $f : V \to W$ such that $(Jf)p = s$, and put $q = \pi_W s = fp \in W$. We may as well work with germs at p and s. Take local coordinates $\{Y_1, Y_2, \ldots, Y_w\}$ on W at q, included as functions on J(V, W) by π_W. Now the set $C_1 \cup C_2 \cup \ldots \cup C_n$ of functions associated to the special I-flag $\mathfrak{K}$ in (3.1) is totally independent on U. If U is small enough, this set may be extended to a totally independent set of v functions $\{\Phi_1, \Phi_2, \ldots, \Phi_v\}$, numbered so that $C_r = \{\Phi_i : i_r < i \leqslant i_{r-1}\}$ for $1 \leqslant r \leqslant n$. Then as above we obtain the germs φ_i at p, which we may regard as a set of coordinates at p.

For each homotopy passing through f, we have defined new functions φ'_i and y'_i, and vector fields d'_i, near p. By (5.1), each such homotopy gives rise to a tangent vector $\mathscr{F}(s) \to \mathbf{R}$ given by $\Phi \rightsquigarrow \varphi' p$. Let Ξ be the set of all homotopies passing through f such that $y'_j \in \mathfrak{m}_p^n$. By (5.3), the corresponding tangent vectors annihilate all germs of functions on $J^{n-1}(V, W)$.

Let Ω be the set of all $(n+1)$-tuples $(G_n, G_{n-1}, \ldots, G_1, Y_i)$, where each G_r is one of the base total vector fields in Γ_r (the sections of $\mathbf{K}_r$; see (3.1)) for $1 \leqslant r \leqslant n$, and Y_i is one of the chosen coordinate functions on W. Then $\mathbf{R}^\Omega$, the set of all maps from Ω to $\mathbf{R}$, is a vector space. To each element $\omega \equiv (G_n, G_{n-1}, \ldots, G_1, Y_k)$ of Ω we associate the function $\Psi_\omega \equiv G_n G_{n-1} \ldots G_1 Y_k \in \mathfrak{z}_{\mathfrak{K}}$, the ideal (3.4) of the special flag $\mathfrak{K}$; we shall need only such functions. We consider the map $\theta : \Xi \to \mathbf{R}^\Omega$ defined by $(\theta\xi)\omega = \psi'_\omega p \in \mathbf{R}$ $(\omega \in \Omega; \xi \in \Xi)$. We define the *rank* of θ as the dimension of the subspace of $\mathbf{R}^\Omega$ spanned by its image. To complete the proof of (5.5), we have to show that $\mathrm{rk}\,\theta \geqslant \kappa$.

By (5.2) we have

$$\psi'_\omega = g'_n g_{n-1} \ldots g_1 y_k + g_n g'_{n-1} \ldots g_1 y_k + \ldots + g_n g_{n-1} \ldots g'_1 y_k + g_n g_{n-1} \ldots g_1 y'_k.$$

Thus we may express θ as the sum $\theta = \theta_0 + \theta_1 + \ldots + \theta_n$ of maps θ_r, by setting

$$\begin{cases} (\theta_0\xi)\omega = (g_n g_{n-1} \ldots g_2 g_1 y'_k)p, \\ (\theta_r\xi)\omega = (g_n g_{n-1} \ldots g'_r \ldots g_2 g_1 y_k)p \end{cases} \qquad (1 \leqslant r \leqslant n).$$

The triangle inequality gives

$$(5.7) \qquad \operatorname{rk}\theta \geqslant \operatorname{rk}\theta_0 - \operatorname{rk}\theta_1 - \operatorname{rk}\theta_2 - \ldots - \operatorname{rk}\theta_n.$$

The separate terms of this inequality will correspond to the terms of (5.6). We shall in fact prove

$$\begin{cases} \operatorname{rk}\theta_0 = w\kappa_1, \\ \operatorname{rk}\theta_r \leqslant (i_{r-1} - i_r)\kappa_r & (1 \leqslant r \leqslant n). \end{cases}$$

That $\operatorname{rk}\theta_0 = w\kappa_1$ is easy. The various g_r are coordinate vector fields on V, and we note that the y_j' may be chosen arbitrarily in $\mathfrak{m}_p^n$. Let $\overline{\Omega}$ denote Ω symmetrized as far as possible, so that $\overline{\Omega}$ has $w\kappa_1$ elements. Clearly θ_0 is symmetric, and factors through $\overline{\Omega}$. Each element $\omega \in \Omega$ corresponds to some differential operator $d^\sigma = g_n g_{n-1} \ldots g_1$ and coordinate y_k; if we choose the homotopy $\xi \in \Xi$ such that

$$\begin{cases} y_k' = \varphi^\sigma & \text{(abbreviated notation as in § 1)} \\ y_j' = 0 & \text{for } j \neq k, \end{cases}$$

we have $(\theta_0\xi)\omega = \sigma!$, and $\theta_0\xi$ vanishes on the rest of $\overline{\Omega}$. Hence $\operatorname{rk}\theta_0 = w\kappa_1$.

We still have to prove $\operatorname{rk}\theta_r \leqslant (i_{r-1} - i_r)\kappa_r$. We have the formula (5.4) $g_r' = \Sigma_j g_r \varphi_j' . d_j$. We substitute this in $\theta_r\xi$ and consider the j-th term, which yields on evaluation

$$(5.8) \qquad -\{g_n g_{n-1} \ldots g_{r+1}(g_r \varphi_j' . d_j) g_{r-1} \ldots g_1 y_k\} p.$$

There are three cases:

Case 1. — $j \leqslant i_r$. Then $D_j \in \Gamma_r$, and every term in the expansion of (5.8) vanishes by (3.4), since $\mathfrak{K}$ is null at s.

Case 2. — $j > i_{r-1}$. Then $\Phi_j \in C_1 \cup C_2 \cup \ldots \cup C_{r-1}$, and owing to the axiom (3.1) *e)* in the definition of special flag, Φ_j is a function on the subset $\pi_{r-2}U$ of $J^{r-2}(V, W)$. Hence $\varphi_j' \in \mathfrak{m}_p^{n-r+2}$, by (5.3). But φ_j' is preceded in (5.8) by only $n-r+1$ derivations, which shows that every term in the expansion of (5.8) is zero.

Case 3. — $i_r < j \leqslant i_{r-1}$. As in Case 2, $\varphi_j' \in \mathfrak{m}_p^{n-r+1}$. This time, only one term in the expansion of (5.8) can possibly be non-zero, namely

$$-g_n g_{n-1} \ldots g_{r+1} g_r \varphi_j' p . d_j g_{r-1} \ldots g_1 y_k p.$$

Only the first factor depends on the homotopy ξ, so there are at most κ_r independent functions on Ω here, since we have symmetry.

When we add up, Cases 1 and 2 contribute nothing to $\operatorname{rk}\theta_r$, and each of the $i_{r-1} - i_r$ indices j in Case 3 contributes at most κ_r. With (5.7), this completes the proof. ∎

Remark. — This proof is the only place where we use axiom *e)* of the definition (3.1) of a special I-flag. It would be desirable to dispense with the need for this axiom. Informed hindsight shows that all our results hold without it.

Corollary **(5.9)**. — *If the special* I*-flag* $\mathfrak{K}$ *is null at* $s\in J(V, W)$, *we have*

$$\mathrm{rk}_s\, \mathfrak{z}_{\mathfrak{K}} \geqslant \mathrm{rk}_s\, \mathfrak{z}_{\mathfrak{K}'} + \kappa,$$

where κ *is given by* (5.6), $\mathfrak{K}'$ *is the curtailed special* I'*-flag, and* $I'=(i_1, i_2, \ldots, i_{n-1})$.

Proof. — In (5.5) we produced κ tangent vectors. All we have yet to check is that these annihilate $\mathfrak{z}_{\mathfrak{K}'}$. This is obvious, because according to (3.6) $\mathfrak{z}_{\mathfrak{K}'}$ is generated by functions on $\pi_{n-1}U\subset J^{n-1}(V, W)$, and we assumed $\mathfrak{z}_{\mathfrak{K}'}\subset \mathfrak{m}_s$. ∎

§ 6. The singularity submanifolds.

In this section we collect together results from the previous sections, in order to state and deduce the main theorems.

We take fixed manifolds V^v and W^w, having dimensions v and w respectively.

Theorem **(6.1)**. — *For each sequence* $I=(i_1, i_2, \ldots, i_n)$ *of integers, the subset* Σ^I *defined in* (2.17) *of the jet space* J(V, W) *is a submanifold (not necessarily closed) having codimension* ν_I, *where the number* ν_I *is defined below* (6.5). *In fact,* Σ^I *is the inverse image of a submanifold of* $J^n(V, W)$ *having codimension* ν_I. *The set* Σ^I *is empty unless* I *satisfies* (2.18). *If* $s\in\Sigma^I$, *there exists a special* I*-flag* $\mathfrak{K}$ *(see* § 3*) defined on a neighbourhood* U *of* s, *whose ideal* $\mathfrak{z}_{\mathfrak{K}}$ (3.4) *in* $\mathcal{F}(U)$ *is locally the ideal of functions vanishing on* Σ^I. *If we set* $q=\pi_W s\in W$, *we have, over* U, $\mathfrak{z}_{\mathfrak{K}} + \mathfrak{m}_q.\mathcal{F}(s) = \Delta^I \mathfrak{m}_q$, *the iterated total jacobian extension (see* § 2*) of* $\mathfrak{m}_q$, *where* $\mathfrak{m}_q$ *is the ideal in* $\mathcal{F}_W(\pi_W U)$ *of functions vanishing at* q. *The ideal* $\mathfrak{z}_{\mathfrak{K}}$ *is independent of the choice of* $\mathfrak{K}$.

Theorem **(6.2)**. — *If* $f: V\to W$ *is a map whose jet section* $Jf: V\to J(V, W)$ *(see* § 1*) is transverse to* Σ^I, *then* $\Sigma^I(f)$, *which is defined as* $(Jf)^{-1}(\Sigma^I)$, *is a submanifold of* V *having codimension* ν_I. *If* I, j *denotes the extended sequence* $(i_1, i_2, \ldots, i_n, j)$, *we have*

$$\Sigma^{I,j}(f) = \Sigma^j(f\,|\,\Sigma^I(f)).$$

Also, when $I=\emptyset$, $\Sigma^j(f)=\{p\in V : \dim \operatorname{Ker} f_p = j\}$, *where* $f_p: T_p\to T_{fp}$ *is the differential of* f *at* p.

Theorem **(6.3)**. — *Any map* $f: V\to W$ *may be approximated in the fine-*C^∞ *sense by a map* $g: V\to W$ *whose jet section* $Jg: V\to J(V, W)$ *is transverse to all the submanifolds* Σ^I.

These are our main theorems.

Remark. — The principal assertion of (6.1), that Σ^I is a submanifold, was already known for $n=1$, or $i_2=i_3=\ldots=i_n=1$ [7] and for $n=2$ [4].

We have yet to define the number ν_I. Let $I=(i_1, i_2, \ldots, i_n)$ be any sequence of integers satisfying $i_1\geqslant i_2\geqslant \ldots \geqslant i_n \geqslant 0$. (We need consider only this case, by (2.18).)

Definition **(6.4)**. — We define λ_I as the number of sequences $(j_1, j_2, \ldots, j_n)$ of integers that satisfy

$$\begin{cases} \text{a)} & j_1\geqslant j_2\geqslant \ldots \geqslant j_n, \\ \text{b)} & i_r \geqslant j_r > 0 \qquad \text{for all } r\ (1\leqslant r\leqslant n); \end{cases}$$

μ_I as the number of sequences $(j_1, j_2, \ldots, j_n)$ of integers that satisfy

$$\begin{cases} \text{a)} \quad j_1 \geqslant j_2 \geqslant \ldots \geqslant j_n & \text{(as above),} \\ \text{b')} \quad i_r \geqslant j_r \geqslant 0 & \text{for all } r \ (1 \leqslant r \leqslant n), \text{ and } j_1 > 0; \end{cases}$$

and then define

$$(6.5) \quad \nu_I = (w - v + i_1)\mu_{i_1, i_2, \ldots, i_n} - (i_1 - i_2)\mu_{i_2, i_3, \ldots i_n} - (i_2 - i_3)\mu_{i_3, \ldots, i_n} - \ldots - (i_{n-1} - i_n)\mu_{i_n}.$$

For example, if $v = w$, we have $\nu_{2,2} = 10$, $\nu_{4,4,4} = 136$, $\nu_{5,5,5,5} = 625$, and $\nu_{8,7,6,5,4} = 7629$. The numbers soon become large.

If we take all the sequences counted in defining μ_I and omit any zeros from their ends, we obtain the identity

$$(6.6) \quad \mu_{i_1, i_2, \ldots, i_n} = \lambda_{i_1, i_2, \ldots, i_n} + \lambda_{i_1, \ldots, i_{n-1}} + \ldots + \lambda_{i_1, i_2} + \lambda_{i_1}.$$

Put $I' = (i_1, i_2, \ldots, i_{n-1})$ as usual, and also $i_0 = v$; then comparison of the formulae for ν_I and $\nu_{I'}$ gives, in conjunction with (6.6), $\nu_I = \nu_{I'} + \kappa$, where

$$(6.7) \quad \kappa = w\lambda_I - (i_0 - i_1)\lambda_I - (i_1 - i_2)\lambda_{i_2, i_3, \ldots, i_n} - \ldots - (i_{n-2} - i_{n-1})\lambda_{i_{n-1}, i_n} - (i_{n-1} - i_n)\lambda_{i_n}.$$

Proof of **(6.1)**. — Take $s \in \Sigma^I$. Then by (3.11), there exists a special I-flag $\mathfrak{K}$ over a neighbourhood U of s, and $\Sigma^I \cap U$ is exactly the set of zeros of its ideal $\mathfrak{z}_{\mathfrak{K}}$.

We see that the three formulae for κ in (6.7), (4.12), and (5.6) are identical, since $\kappa_r = \lambda_{i_r, i_{r+1}, \ldots, i_n}$ is the dimension of the symmetric tensor product $K_n \bigcirc K_{n-1} \bigcirc \ldots \bigcirc K_{r+1} \bigcirc K_r$. By induction on the length of I, (4.6) and (4.12) show that the image in $\mathscr{F}(s)$ of the ideal $\mathfrak{z}_{\mathfrak{K}}$ can be generated by ν_I elements. But (5.9) shows, again by induction on length, that $\mathrm{rk}_s \mathfrak{z}_{\mathfrak{K}} \geqslant \nu_I$. Thus the ideal $\mathfrak{z}_{\mathfrak{K}}$ in $\mathscr{F}(s)$ is generated by ν_I germs having linearly independent differentials.

Further, we know (and need to know!) from (3.7) that the ideal $\mathfrak{z}_{\mathfrak{K}}$ is finitely generated *locally*, i.e. that the ideal $\mathfrak{z}_{\mathfrak{K}|U'}$ in $\mathscr{F}(U')$ (which is generated by $\mathfrak{z}_{\mathfrak{K}}|U'$ by § 3) is finitely generated, for some small neighbourhood U′ of s; it follows that we can find a smaller neighbourhood U″ of s and ν_I functions on U″, whose differentials are linearly independent everywhere on U″, that generate the ideal $\mathfrak{z}_{\mathfrak{K}|U''}$. Their common zeros, $\Sigma^I \cap U''$, therefore form a smooth submanifold of U″ having codimension ν_I.

The other assertions in (6.1) are collected from (2.20), (2.18), (3.9), and (3.6). ∎

Proof of **(6.2)**. — Now $\Sigma^I(f)$ is a submanifold of V having codimension ν_I. By (2.22), $p \in \Sigma^j(f|\Sigma^I(f))$ if and only if $p \in \Sigma^I(f)$ and $\dim \operatorname{Ker} \mathbf{d}(f|\Sigma^I(f))|p = j$. By (1.22), (6.1), and the definition (2.17) of $\Sigma^{I,j}$, these conditions are equivalent to $p \in \Sigma^{I,j}(f)$. ∎

Proof of **(6.3)**. — This is immediate from the form of the transversality theorem given in (1.5). ∎

The formulae for the codimension of the singularity submanifolds Σ^I depend only on the difference $v - w$, not on v and w separately. This is no accident, for there is a " suspension " operation for singularities. Given any map $f: V \to W$, we may consider

$f\times 1 : V\times \mathbf{R} \to W\times \mathbf{R}$, and it is not difficult to prove that $\Sigma^I(f\times 1)=\Sigma^I(f)\times \mathbf{R}$, for all I.

More generally, consider a level-preserving map $f: V\times \mathbf{R}^k \to W\times \mathbf{R}^k$. For each $t\in\mathbf{R}^k$, we have $f_t : V\to W$. Then one can show that $\Sigma^I(f)\cap(V\times t)=\Sigma^I(f_t)$, for each I.

Remark **(6.8)**. — The previous remarks give rise to a useful general procedure for making any map transverse by adding extra coordinates. Let $f: V\to W$ be any map. Then for sufficiently huge k we can take a " homotopy " $f_t : V\to W$ ($t\in\mathbf{R}^k$) passing through $f=f_0$, such that for some given r the associated map $V\times\mathbf{R}^k\to J^r(V, W)$ given by $(p, t)\rightsquigarrow(Jf_t)p$ is locally a projection. This map is certainly transverse to any submanifold of $J^r(V, W)$. The map we require is the associated level-preserving map $f' : V\times\mathbf{R}^k \to W\times\mathbf{R}^k$, which has corresponding transversality properties. In practice, one is interested only in transversality to particular submanifolds, and in this case the number k can be considerably reduced by judicious choice of the homotopy.

We have decomposed $J^r(V, W)$, and therefore for a given transverse map $f: V\to W$ the manifold V, as the disjoint union of finitely many submanifolds Σ^I or $\Sigma^I(f)$, where I runs through the sequences of length r. Such decompositions are called *manifold collections*, or *stratifications*. Their precise definition is still rather fluid. Unfortunately, it is a defect of the theory of singularities, in its present state, that this decomposition does not satisfy any of the sets of axioms proposed for manifold collections. For example, even the dimension axiom fails, an observation due essentially to Whitney [9].

To see this, take $V=W=\mathbf{R}^2$, with sets of coordinates $\{x_1, x_2\}$ on V and $\{y_1, y_2\}$ on W, and consider the map $f: V\to W$ given by

$$\begin{cases} y_1\circ f = x_1x_2 + x_1^6 \\ y_2\circ f = bx_2, \end{cases}$$

where b is a parameter. Then we find that $0\in\Sigma^{2,0}(f)$ if $b=0$, but that $0\in\Sigma^{1,1,1,1,1}(f)$ if $b\neq 0$. Thus $\Sigma^{2,0}$ contains points of the frontier of $\Sigma^{1,1,1,1,1}$, in spite of the fact that their respective codimensions are 4 and 5.

The criticism that the above map is not transverse is met by adding extra coordinates, as in (6.8). We could take the result as a map $f: \mathbf{R}^7\to\mathbf{R}^7$ given by

$$\begin{cases} y_1\circ f = x_1x_2 + x_1^6 + x_1^2x_3 + x_1^3x_4 + x_1^4x_5, \\ y_2\circ f = bx_2 + x_1x_6 + x_2x_7, \\ y_i\circ f = x_i \quad (3\leq i\leq 7). \end{cases}$$

The discussion remains essentially unchanged.

The remedy is in principle clear; we need to decompose the manifolds Σ^I still further, by taking account of more structure. We already have more structure available. Take $b=0$ in the above. We have a special 2-flag, whose form $\mathbf{a}|0$ is given in terms of the coordinate tangent vectors $\{\partial_1, \partial_2\}$ to V at 0 and $\{d_1, d_2\}$ to W at 0, by:

$$\partial_1\otimes\partial_1\rightsquigarrow 0, \quad \partial_1\otimes\partial_2\rightsquigarrow d_1, \quad \partial_2\otimes\partial_2\rightsquigarrow 0,$$

with zero indeterminacy. This is not a " general " symmetric linear map, and ought to be separated off from the ordinary points of $\Sigma^{2,0}$ in a submanifold of its own, with higher codimension.

The problem of finding an entirely satisfactory decomposition seems deep and difficult.

§ 7. Intrinsic derivatives.

Now that we know the subsets Σ^I of the jet space $J(V, W)$ are submanifolds, we return to the study of the bundle maps $\mathbf{a}_{\mathfrak{K}}$ introduced in § 4. These are now known to be smooth. We have pointed out already that they depend on $\mathfrak{K}$. In this section we look for their invariant properties.

Our principal tool is the concept of intrinsic derivative, introduced by Porteous [6]. Let $\mathbf{E}$ be a vector bundle over the manifold V. We consider the tangent bundle $\mathbf{T}_\mathbf{E}$ of its total space $|\mathbf{E}|$. The projection $\pi : |\mathbf{E}| \to V$ induces the short exact sequence

$$(7.1) \qquad 0 \to \mathbf{E}|p \to T_s \to T_p \to 0,$$

where $s \in |\mathbf{E}|$ and $p = \pi s$, and T_s and T_p are the tangent spaces at s and p. Any section χ of $\mathbf{E}$ such that $\chi p = s$ has a differential $\mathbf{d}\chi : \mathbf{T}_V \to \mathbf{T}_\mathbf{E}$ which splits (7.1). This is not in general canonical. However, if s lies on the zero section, we may take χ as the zero section itself, which certainly is canonical. In this case, (7.1) splits canonically,

$$(7.2) \qquad T_s \cong T_p \oplus \mathbf{E}|p.$$

Suppose now that χ is a general section of $\mathbf{E}$, and that $\alpha : V \to \mathbf{R}$ vanishes at p; then the section $\alpha\chi$ induces, with the aid of (7.2), a map $T_p \to \mathbf{E}|p$.

Lemma **(7.3)**. — *This map* $T_p \to \mathbf{E}|p$ *induced by the section* $\alpha\chi$ *of* $\mathbf{E}$ *as above, is given by* $d \rightsquigarrow d\alpha . \chi p$ *for each tangent vector* $d \in T_p$.

Proof. — We may work locally, and assume that $\mathbf{E}$ is a product bundle, having $\{\chi_1, \chi_2, \ldots, \chi_n\}$ as a base of sections. Suppose that $\chi = \sum_i \alpha_i \chi_i$ is a general section of $\mathbf{E}$, and that d is a tangent vector at p. If we compose χ with the product projection $|\mathbf{E}| \to \mathbf{E}|p$, and identify $\mathbf{E}|p$ with its own tangent space at any point, this composite induces $d \rightsquigarrow \sum_i d\alpha_i . \chi_i p$. Hence $\alpha\chi$ induces $d \rightsquigarrow \sum_i d(\alpha\alpha_i) . \chi_i p$. By assumption, $d(\alpha\alpha_i) = d\alpha . \alpha_i p + \alpha p . d\alpha_i = d\alpha . \alpha_i p$, so that the last map can be written

$$d \rightsquigarrow \sum_i d\alpha . \alpha_i p . \chi_i p = d\alpha . \chi p. \quad \blacksquare$$

Lemma **(7.4)** (Porteous). — *Let* $\mathbf{a} : \mathbf{E} \to \mathbf{F}$ *be a map of vector bundles over* V. *Then the map* $\mathbf{a}$ *induces the canonically defined intrinsic derivative of* $\mathbf{a}$ *at* $p \in V$,

$$(7.5) \qquad d(\mathbf{a}) : T_p \to \mathrm{Hom}(\mathrm{Ker}\ a, \mathrm{Coker}\ a),$$

where $a : E \to F$ *denotes the restriction of* $\mathbf{a}$ *to the fibres over* p.

Proof. — Let χ be any section of $\mathbf{E}$ such that $\mathbf{a}\chi p = 0$; then by the preceding discussion the image section $\mathbf{a}\chi$ induces a map $T_p \to F$. This map clearly depends

R-linearly on χ. We require a map that depends only on χp, not on χ. Now any section of **E** that vanishes at p is a **R**-linear combination of sections of the form $\alpha\chi$, where $\alpha p = 0$, $\alpha : V \to \mathbf{R}$, and χ is a general section of **E**. For such a section, (7.3) shows that the image of $T_p \to F$ lies in Im a. Dividing out by Im a yields the desired map, which we can write in the given form. ∎

The method suggested by Porteous for defining the singularity sets $\Sigma^I(f)$ of a map $f : V \to W$ is to apply (7.4) to the differential $\mathbf{d}f : \mathbf{T}_V \to f^*\mathbf{T}_W$ at p, define a submanifold of V by a rank condition, over which the maps (7.5) will assemble to form a new bundle map, apply (7.4) to this map, and continue as far as possible. We shall show that for sufficiently good maps, namely those whose jet sections are transverse to all the submanifolds Σ^I of J(V, W), this process always succeeds. Moreover, the necessary transversality condition at each step appears explicitly, and needs no mention of the jet space.

To prove all this, we shall nevertheless work universally, on the jet space J(V, W). Take a fixed sequence $I = (i_1, i_2, \ldots, i_n)$ of integers, and a special I-flag $\mathfrak{K}$ (see § 3) defined on the open subset U of J(V, W). As in (3.1) we may choose sets of functions C_r on U, and construct a bundle $\mathbf{H}_r$ having fibre dimension $i_{r-1} - i_r$, and a bundle map as in (4.7)

$$\mathbf{h}_r : \mathbf{H}_r \to \mathbf{K}_{r-1} \otimes \mathbf{K}_{r-2} \otimes \ldots \otimes \mathbf{K}_1 \otimes \mathbf{Hom}(\mathbf{P}, \mathbf{R})$$

over U for $1 \leqslant r \leqslant n$, where $\mathbf{P} = ((\pi_W)^* \mathbf{T}_W) \mid U$. We write Σ_r for Σ^J when $J = (i_1, i_2, \ldots, i_r)$ is obtained from I by curtailing, and Σ_0 for J(V, W).

As before, we write E, a, etc., for the fibre of a vector bundle **E**, etc., or a vector bundle map $\mathbf{a} : \mathbf{E} \to \mathbf{F}$, etc., over a typical point $s \in U$. We have the surjective symmetric linear map (4.1)

$$\varepsilon_{\mathfrak{K}}(s) : K_n \otimes K_{n-1} \otimes \ldots \otimes K_1 \otimes \mathrm{Hom}(P, \mathbf{R}) \to \mathfrak{z}_{\mathfrak{K}} / (\mathfrak{z}_{\mathfrak{K}'} + \mathfrak{m}_s \mathfrak{z}_{\mathfrak{K}})$$

where $\mathfrak{K}'$ is the curtailed special I′-flag, $I' = (i_1, i_2, \ldots, i_{n-1})$, and $\mathfrak{z}_{\mathfrak{K}}$ and $\mathfrak{z}_{\mathfrak{K}'}$ are their ideals (3.4). This was used in (4.5) to construct a symmetric bundle map, which we now write in the form

$$\mathbf{K}_n \otimes \mathbf{K}_n \otimes \mathbf{K}_{n-1} \otimes \ldots \otimes \mathbf{K}_1 \to \mathbf{P} \qquad \text{over } \Sigma_n.$$

(Here, and in future, we frequently partially dualize linear maps without comment, and use standard natural isomorphisms.) Actually, we can do better. Denote by $\mathbf{T}_r$ the tangent bundle to Σ_r $(0 \leqslant r \leqslant n)$, a subbundle of the restriction to Σ_r of the tangent bundle to J(V, W). (The fact that these vector bundles have infinite fibre dimension will cause no difficulty. Alternatively, we could work entirely over $J^n(V, W)$.) Then by (6.1), if $s \in \Sigma_{n-1}$, T_{n-1} is precisely the set of tangent vectors at s that annihilate $\mathfrak{z}_{\mathfrak{K}'}$; which shows that over Σ_n the above bundle map extends canonically to a bundle map

$$\mathbf{b}_n : \mathbf{T}_{n-1} \otimes \mathbf{K}_n \otimes \mathbf{K}_{n-1} \otimes \ldots \otimes \mathbf{K}_1 \to \mathbf{P} \qquad \text{over } \Sigma_n.$$

It is still induced by evaluation. Further, T_n is the set of tangent vectors at s that annihilate $\mathfrak{z}_{\mathfrak{K}}$, and is therefore by (6.1) the kernel of

$$b'_n : T_{n-1} \to \mathrm{Hom}(K_n \otimes K_{n-1} \otimes \ldots \otimes K_1, P).$$

Similarly for the curtailed flags. Let us write $\mathbf{b}_r$ for the corresponding bundle map over Σ_r $(1 \leqslant r \leqslant n)$, and set conventionally $\mathbf{b}_0 = \mathbf{d}\pi_W : \mathbf{T}_0 \to \mathbf{P}$. Then we have the exact sequence of bundles

$$(7.6) \qquad 0 \to \mathbf{T}_r \to \mathbf{T}_{r-1} \underset{\mathbf{b}'_r}{\to} \mathbf{Hom}(\mathbf{K}_r \otimes \mathbf{K}_{r-1} \otimes \ldots \otimes \mathbf{K}_1, \mathbf{P}) \qquad \text{over } \Sigma_r$$

for $1 \leqslant r \leqslant n$.

We deduce from (7.6) that

$$(7.7) \qquad \mathbf{K}_r = \mathbf{K}_1 \cap \mathbf{T}_{r-1} \qquad \text{over } \Sigma_r \qquad (1 \leqslant r \leqslant n).$$

For by induction on r assume $\mathbf{K}_r = \mathbf{K}_1 \cap \mathbf{T}_{r-1}$; then if $r < n$,

$$\mathbf{K}_1 \cap \mathbf{T}_r = (\mathbf{K}_1 \cap \mathbf{T}_{r-1}) \cap \mathbf{T}_r = \mathbf{K}_r \cap \mathbf{T}_r = \mathbf{Ker}(\mathbf{b}'_r | \mathbf{K}_r) = \mathbf{K}_{r+1} \qquad \text{over } \Sigma_{r+1},$$

by nullity and the short exact sequence (4.11). The induction starts trivially with $\mathbf{K}_1 = \mathbf{K}_1 \cap \mathbf{T}_0$.

We deduce also that

$$\mathbf{b}_r : \mathbf{T}_{r-1} \otimes \mathbf{K}_r \otimes \mathbf{K}_{r-1} \otimes \ldots \otimes \mathbf{K}_1 \to \mathbf{P} \qquad \text{over } \Sigma_r$$

is symmetric. According to (4.3) *b)* and (7.7), this assertion contains no new information.

Our next step is the construction of some bundles, which we shall need for expressing the intrinsic derivatives. For the moment, they are not genuinely bundles, but merely functions assigning to certain points $s \in U$ some vector space related to previously defined vector spaces. We work in the fibres over s.

We put $P_0 = P$. Suppose we have already defined a surjection

$$c_{r-1} : \mathrm{Hom}(K_{r-1} \bigcirc K_{r-2} \bigcirc \ldots \bigcirc K_1, P) \to P_{r-1} \qquad \text{over } \Sigma_{r-2},$$

where we use the symmetric tensor product as defined in (4.2). If $s \in \Sigma_{r-1}$ we define $e_r : P_{r-1} \to Q_r$ as the cokernel of the composite

$$K_{r-1} \xrightarrow[b'_{r-1} | K_{r-1}]{} \mathrm{Hom}(K_{r-1} \bigcirc K_{r-2} \bigcirc \ldots \bigcirc K_1, P) \xrightarrow[c_{r-1}]{} P_{r-1},$$

and then define $c_r : \mathrm{Hom}(K_r \bigcirc K_{r-1} \bigcirc \ldots \bigcirc K_1, P) \to P_r$ as the coimage of the composite u_r, which is

$$(7.8) \qquad \begin{array}{c} \mathrm{Hom}(K_r \bigcirc K_{r-1} \bigcirc \ldots \bigcirc K_1, P) \\ \cap \\ \mathrm{Hom}\{K_r \otimes (K_{r-1} \bigcirc \ldots \bigcirc K_1), P\} \cong \mathrm{Hom}\{K_r, \mathrm{Hom}(K_{r-1} \bigcirc \ldots \bigcirc K_1, P)\} \\ \downarrow \mathrm{Hom}(1, c_{r-1}) \\ \mathrm{Hom}(K_r, P_{r-1}) \\ \downarrow \mathrm{Hom}(1, e_r) \\ \mathrm{Hom}(K_r, Q_r). \end{array}$$

This inductive definition starts with $c_0 : \mathrm{P} = \mathrm{P}_0$ and continues for $1 \leqslant r \leqslant n$, ending with e_{n+1} and Q_{n+1}.

Thus P_r and Q_r are defined over s when $s \in \Sigma_{r-1}$.

In the fibres over s we consider the following linear maps:

$$c_r : \mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P}) \to \mathrm{P}_r \qquad (s \in \Sigma_{r-1};\ 0 \leqslant r \leqslant n),$$

$$e_{r+1} : \mathrm{P}_r \to \mathrm{Q}_{r-1} \qquad (s \in \Sigma_r;\ 0 \leqslant r \leqslant n),$$

$$\overline{h}'_t : \mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P}) \to \mathrm{Hom}\{(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_t) \otimes \mathrm{H}_t, \mathbf{R}\}$$

which is obtained from h_t (compare (4.9)), when $1 \leqslant t \leqslant r+1 \leqslant n+1$, and

$$\overline{k}''_t : \mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_t, \mathrm{K}_{t-1}/\mathrm{K}_t) \to \mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P})$$

which is obtained from $b'_{t-1} | \mathrm{K}_{t-1}$ with the help of (7.6) and (7.7), when $1 \leqslant t \leqslant r+1 \leqslant n+1$. (Some interpretation is required. When $t = r+1$, we interpret $\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_t$ as $\mathbf{R}$. We must also put $\mathbf{T}_{-1} = \mathbf{T}_0$.)

Lemma **(7.9)**. — a) *Over any point* $s \in \Sigma_r$ $(0 \leqslant r \leqslant n)$ *the maps* c_r *and* $\overline{h}'_t$ *(for* $1 \leqslant t \leqslant r$*) present* $\mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P})$ *as a product of vector spaces.*

b) *Over any point* $s \in \Sigma_{r+1}$ $(0 \leqslant r < n)$ *the maps* $e_{r+1}c_r$ *and* $\overline{h}'_t$ *(for* $1 \leqslant t \leqslant r+1$*) present* $\mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P})$ *as a product of vector spaces.*

c) *Over any point* $s \in \Sigma_r$ $(0 \leqslant r \leqslant n)$ *the maps* $\overline{k}''_t$ *(for* $1 \leqslant t \leqslant r$*) present* Ker c_r *as a sum of vector spaces.*

d) *Over any point* $s \in \Sigma_{r-1}$ $(0 \leqslant r < n)$ *the maps* $\overline{k}''_t$ *(for* $1 \leqslant t \leqslant r+1$*) present* $\mathrm{Ker}(e_{r+1}c_r)$ *as a sum of vector spaces.*

Proof. — We work by induction on r, and consider the commutative diagram over a point $s \in \Sigma_r$,

$$\begin{array}{ccccc}
\mathrm{K}_{r+1} \to \mathrm{K}_r \xrightarrow[b'_r | \mathrm{K}_r]{} \mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P}) & \xrightarrow[c_r]{} & \mathrm{P}_r & \xrightarrow[e_{r+1}]{} & \mathrm{Q}_{r+1} \\
\cap & & \cap & & \\
\mathrm{Hom}\{\mathrm{K}_r \otimes (\mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1), \mathrm{P}\} & \xrightarrow[v_r]{} & \mathrm{Hom}(\mathrm{K}_r, \mathrm{Q}_r), & &
\end{array}$$

where v_r is obtained from (7.8).

We assume *b)* and *d)* for $r-1$. Then $\mathrm{Ker}(e_r c_{r+1})$ is presented as a sum by injections

$$\mathrm{Hom}(\mathrm{K}_{r-1} \bigcirc \mathrm{K}_{r-2} \bigcirc \ldots \bigcirc \mathrm{K}_t, \mathrm{K}_{t-1}/\mathrm{K}_t) \to \mathrm{Hom}(\mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P})$$

for $1 \leqslant t \leqslant r$. Hence Ker v_r is presented as a sum by injections

$$\mathrm{Hom}\{\mathrm{K}_r \otimes (\mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_t), \mathrm{K}_{t-1}/\mathrm{K}_t\} \to \mathrm{Hom}\{\mathrm{K}_r \otimes (\mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1), \mathrm{P}\}.$$

By (4.10) these yield injections

$$\overline{k}''_t : \mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_t, \mathrm{K}_{t-1}/\mathrm{K}_t) \to \mathrm{Hom}(\mathrm{K}_r \bigcirc \mathrm{K}_{r-1} \bigcirc \ldots \bigcirc \mathrm{K}_1, \mathrm{P}),$$

whose images evidently lie in Ker c_r, and present as a sum some subspace of Ker c_r. Hence Ker c_r cannot be too small.

On the other hand, $\mathrm{Hom}(K_{r-1}\bigcirc\ldots\bigcirc K_1, P)$ is presented as a product by projections

$$\begin{cases} e_r c_{r-1} : \mathrm{Hom}(K_{r-1}\bigcirc\ldots\bigcirc K_1, P) \to Q_r, \\ \mathrm{Hom}(K_{r-1}\bigcirc\ldots\bigcirc K_1, P) \to \mathrm{Hom}\{(K_{r-1}\bigcirc\ldots\bigcirc K_t)\otimes H_t, \mathbf{R}\}, \end{cases}$$

for $1 \leqslant t \leqslant r$. These induce a product presentation of $\mathrm{Hom}\{K_r\otimes(K_{r-1}\bigcirc\ldots\bigcirc K_1), P\}$, whose restriction to $\mathrm{Hom}(K_r\bigcirc K_{r-1}\bigcirc\ldots\bigcirc K_1, P)$ yields projections

$$\begin{cases} c_r : \mathrm{Hom}(K_r\bigcirc K_{r-1}\bigcirc\ldots\bigcirc K_1, P) \to P_r, \\ \overline{h}'_t : \mathrm{Hom}(K_r\bigcirc K_{r-1}\bigcirc\ldots\bigcirc K_1, P) \to \mathrm{Hom}\{(K_r\bigcirc\ldots\bigcirc K_t)\otimes H_t, \mathbf{R}\}, \end{cases}$$

again using (4.10). It is not immediately apparent that these maps present $\mathrm{Hom}(K_r\bigcirc\ldots\bigcirc K_1, P)$ as a product; all we know from this is that the intersection of their kernels is zero. Hence $\mathrm{Ker}\, c_r$ cannot be too large.

However, $\overline{h}'_t$ and $\overline{k}''_t$ have the same rank, so that our two estimates for the dimension of $\mathrm{Ker}\, c_r$ agree. This shows that we must after all have sum and product presentations of $\mathrm{Ker}\, c_r$, as required. This proves *a)* and *c)* for r.

Finally we must compare $\mathrm{Ker}\, c_r$ with $\mathrm{Ker}(e_{r+1}c_r)$, when $s\in\Sigma_{r+1}$. We must divide out by $\mathrm{Im}(b'_r|K_r)$, which is isomorphic to K_r/K_{r+1}, and projects isomorphically to $\mathrm{Hom}(H_r, \mathbf{R})$. These last two spaces are the required extra factors in *b)* and *d)*. The induction starts trivially with $r=0$. ∎

Corollary **(7.10)**. — *The fibres* P_r *and* Q_r *form smooth vector bundles* $\mathbf{P}_r$ *and* $\mathbf{Q}_r$ *over* Σ_r $(1\leqslant r\leqslant n)$, *and we have smooth bundle maps*

$$\mathbf{c}_r : \mathbf{Hom}(\mathbf{K}_r\bigcirc\mathbf{K}_{r-1}\bigcirc\ldots\bigcirc\mathbf{K}_1, \mathbf{P}) \to \mathbf{P}_r \qquad \textit{over } \Sigma_r,$$

$$\mathbf{e}_r : \mathbf{P}_{r-1} \to \mathbf{Q}_r \qquad \textit{over } \Sigma_r,$$

$$\mathbf{u}_r : \mathbf{Hom}(\mathbf{K}_r\bigcirc\mathbf{K}_{r-1}\bigcirc\ldots\bigcirc\mathbf{K}_1, \mathbf{P}) \to \mathbf{Hom}(\mathbf{K}_r, \mathbf{Q}_r) \qquad \textit{over } \Sigma_r.$$

Proof. — We deduce from (7.9) that the dimensions of P_r and Q_r are constant over Σ_r. Fibre cokernels and images of smooth bundle maps are again smooth vector bundles, provided only that their fibre dimensions are locally constant. ∎

Lemma **(7.11)**. — *We have the bundle map*

$$\mathbf{c}_{n-1}\mathbf{b}'_{n-1}|\mathbf{K}_{n-1} : \mathbf{K}_{n-1} \to \mathbf{Hom}(\mathbf{K}_{n-1}\bigcirc\mathbf{K}_{n-2}\bigcirc\ldots\bigcirc\mathbf{K}_1, \mathbf{P}) \to \mathbf{P}_{n-1} \qquad \textit{over } \Sigma_{n-1},$$

Over Σ_n, *its kernel is* $\mathbf{K}_n$ *and its cokernel is* $\mathbf{Q}_n$. *Then* (7.4) *gives rise to the intrinsic derivative bundle map*

$$\mathbf{d}(\mathbf{c}_{n-1}\mathbf{b}'_{n-1}|\mathbf{K}_{n-1}) : \mathbf{T}_{n-1} \to \mathbf{P}_n \subset \mathbf{Hom}(\mathbf{K}_n, \mathbf{Q}_n) \qquad \textit{over } \Sigma_n,$$

which coincides with the composite

$$\mathbf{T}_{n-1} \underset{\mathbf{b}'_n}{\to} \mathrm{Hom}(\mathbf{K}_n\bigcirc\mathbf{K}_{n-1}\bigcirc\ldots\bigcirc\mathbf{K}_1, \mathbf{P}) \underset{\mathbf{u}_n}{\to} \mathbf{Hom}(\mathbf{K}_n, \mathbf{Q}_n).$$

Proof. — We work in the fibres over $s\in\Sigma_n$. Since $b_{n-1}(1\otimes h_t)$ vanishes on

$$K_{n-1}\otimes K_{n-1}\otimes K_{n-2}\otimes\ldots\otimes K_t\otimes H_t$$

for $t \leqslant n-1$ by (4.8), and we have the product presentation (7.9) *a)*, we know

$$\mathrm{Ker}(c_{n-1}b'_{n-1}|\mathrm{K}_{n-1}) = \mathrm{Ker}(b'_{n-1}|\mathrm{K}_{n-1}) = \mathrm{K}_n.$$

The cokernel of $c_{n-1}b'_{n-1}|\mathrm{K}_{n-1}$ is Q_n by definition.

The rest of the lemma is trivial. Both bundle maps were defined, in effect, by taking vector fields passing through given tangent vectors, applying their product to a function on W, and evaluating. The only difficulty was to arrange for the result to be well defined. In the case of b'_n this was done by taking only those vector fields lying in a special I-flag $\mathfrak{R}$; $\mathbf{K}_r$ is intrinsically defined only over Σ_r, and a special I-flag is a way of artificially extending the bundle $\mathbf{K}_r$ locally, in a carefully controlled way. In that case, dividing out by certain subspaces, as in the definition of intrinsic derivative, was unnecessary.

It follows that $\mathbf{d}(\mathbf{c}_{n-1}\mathbf{b}'_{n-1}|\mathbf{K}_{n-1})$ factors through the image of $\mathbf{u}_n$, which is $\mathbf{P}_n$. ∎

Definition **(7.12)**. — Given the sequence $\mathrm{I} = (i_1, i_2, \ldots, i_n)$ as before, we define the $(n+1)$th *total intrinsic derivative* as

$$\mathbf{d}_{n+1} = \mathbf{d}(\mathbf{c}_{n-1}\mathbf{b}'_{n-1}|\mathbf{K}_{n-1}) : \mathbf{T}_{n-1} \to \mathbf{P}_n \qquad \text{over } \Sigma_n$$

for $n > 0$, and $\mathbf{d}_1 = \mathbf{d}\pi_{\mathrm{W}} : \mathbf{T}_{-1} = \mathbf{T}_0 \to \mathbf{P}_0$.

Lemma **(7.13)**. — *The intrinsic derivative $\mathbf{d}_{n+1}$ yields the short exact sequence*

$$0 \to \mathbf{T}_n \to \mathbf{T}_{n-1} \xrightarrow[\mathbf{d}_{n+1}]{} \mathbf{P}_n \to 0 \qquad \textit{over } \Sigma_n.$$

Proof. — We again work in the fibres over $s \in \Sigma_n$. As before, from (4.8),

$$b_n : \mathrm{T}_{n-1} \otimes \mathrm{K}_n \otimes \mathrm{K}_{n-1} \otimes \ldots \otimes \mathrm{K}_1 \otimes \mathrm{Hom}(\mathrm{P}, \mathbf{R}) \to \mathbf{R}$$

vanishes on $(1 \otimes h_t)(\mathrm{T}_{n-1} \otimes \mathrm{K}_n \otimes \mathrm{K}_{n-1} \otimes \ldots \otimes \mathrm{K}_t \otimes \mathrm{H}_t)$ for $t \leqslant n$. This fact, with the product presentation (7.9) *a)*, shows that

$$\mathrm{Ker}\ d_{n+1} = \mathrm{Ker}(u_n b'_n) = \mathrm{Ker}\ b'_n = \mathrm{T}_n,$$

by (7.6). If we compute the dimensions of $\mathrm{T}_{n-1}/\mathrm{T}_n$ from (6.1) and (6.7), and of P_n from (7.9), we find that they agree, term for term. Hence the sequence must be exact. ∎

Theorem **(7.14)**. — *Over Σ_n, the bundles, $\mathbf{P}_n$, $\mathbf{Q}_n$, and $\mathbf{K}_n$, and the bundle maps*

$$\mathbf{c}_n : \mathbf{Hom}(\mathbf{K}_n \bigcirc \mathbf{K}_{n-1} \bigcirc \ldots \bigcirc \mathbf{K}_1, \mathbf{P}) \to \mathbf{P}_{n-1},$$

$$\mathbf{e}_n : \mathbf{P}_{n-1} \to \mathbf{Q}_n,$$

$$\mathbf{u}_n : \mathbf{Hom}(\mathbf{K}_n \bigcirc \mathbf{K}_{n-1} \bigcirc \ldots \bigcirc \mathbf{K}_1, \mathbf{P}) \to \mathbf{Hom}(\mathbf{K}_n, \mathbf{Q}_n),$$

and

$$\mathbf{d}_{n+1} : \mathbf{T}_{n-1} \to \mathbf{P}_n,$$

are invariant, in the sense that they do not depend on the choice of the special I-*flag* $\mathfrak{R}$.

Proof. — This is evident from (7.11) and the inductive definitions, granted that $\mathbf{c}_0 : \mathbf{P} = \mathbf{P}_0$ and $\mathbf{d}\pi_W : \mathbf{T}_0 \to \mathbf{P}$ are invariant. ∎

Hence we are fully justified in calling $\mathbf{d}_{n+1}$ the intrinsic derivative; it depends only on the sequence **I**.

We are now ready to assemble our results, and interpret them in the case of the singularities of a map. Now $\mathbf{K}_r$ is contained in **D**, the total tangent bundle, by definition, but $\mathbf{T}_r$ is not. We put $S_r = T_r \cap D$ over the point $s \in J(V, W)$. These spaces do not form a bundle over any of the sets Σ_r. Given a map $f : V \to W$, we may use its jet section $Jf : V \to J(V, W)$ to induce bundles $\mathbf{K}_r$, $\mathbf{P}_r$, etc., and bundle maps $\mathbf{c}_r$, etc., over subsets of V, which we denote by the same letters. We again use K_r, P_r, c_r, etc., for their fibres over a typical point $p \in V$. Our conclusion will be that in suitable cases these bundles and maps can be constructed on V without reference to special flags or to the jet space. On the other hand, the direct description fails to elicit certain of their properties, notably symmetry.

Given a map $f : V \to W$ and a sequence $I = (i_1, i_2, \ldots, i_n)$ of integers such that $v \geqslant i_1 \geqslant i_2 \geqslant \ldots \geqslant i_n \geqslant 0$, we consider the following inductive hypothetical construction:

We start from the data (α_1):

(α_1) Take $\Sigma_0(f) = V$, $\mathbf{S}_{-1} = \mathbf{T}_V$, $\mathbf{K}_0 = \mathbf{T}_V$, $\mathbf{c}_0 : \mathbf{P} = \mathbf{P}_0 = f^*\mathbf{T}_W$, and

$$\mathbf{d}_1(f) = \mathbf{d}f : \mathbf{S}_{-1} = \mathbf{T}_V \to f^*\mathbf{T}_W = \mathbf{P}_0.$$

More generally, suppose that for some r $(0 < r \leqslant n)$ we have the data:

(α_r) *a)* A submanifold $\Sigma_{r-1}(f)$ of V.

b) A subbundle $\mathbf{S}_{r-2} \subset \mathbf{T}_V$ defined over $\Sigma_{r-1}(f)$.

c) Subbundles $\mathbf{K}_{r-1} \subset \mathbf{K}_{r-2} \subset \ldots \subset \mathbf{K}_1 \subset \mathbf{K}_0 = \mathbf{T}_V$ defined over $\Sigma_{r-1}(f)$, such that $\mathbf{K}_{r-1} \subset \mathbf{S}_{r-2}$.

d) A bundle map

$$\mathbf{c}_{r-1} : \mathbf{Hom}(\mathbf{K}_{r-1} \bigcirc \mathbf{K}_{r-2} \bigcirc \ldots \bigcirc \mathbf{K}_1, \mathbf{P}) \to \mathbf{P}_{r-1} \qquad \text{over } \Sigma_{r-1}(f).$$

e) A bundle map $\mathbf{d}_r(f) : \mathbf{S}_{r-2} \to \mathbf{P}_{r-1}$ over $\Sigma_{r-1}(f)$.

Then we continue the construction with:

(β_r) We define $\Sigma_r(f)$ as the set of points $p \in \Sigma_{r-1}(f) \subset V$ over which $d_r(f) | K_{r-1} : K_{r-1} \to P_{r-1}$ has kernel rank i_r.

(γ_r) Over $\Sigma_r(f)$, we may define $\mathbf{K}_r \subset \mathbf{K}_{r-1}$ as the kernel bundle and $\mathbf{e}_r : \mathbf{P}_{r-1} \to \mathbf{Q}_r$ as the cokernel of the bundle map $\mathbf{d}_r(f) | \mathbf{K}_{r-1} : \mathbf{K}_{r-1} \to \mathbf{P}_{r-1}$.

(δ_r) We define the bundle map

$$\mathbf{u}_r : \mathbf{Hom}(\mathbf{K}_r \bigcirc \mathbf{K}_{r-1} \bigcirc \ldots \bigcirc \mathbf{K}_1, \mathbf{P}) \to \mathbf{Hom}(\mathbf{K}_r, \mathbf{Q}_r) \qquad \text{over } \Sigma_r(f)$$

in terms of $\mathbf{c}_{r-1}$ and $\mathbf{e}_r$ as in (7.8).

(ε_r) Assume the bundle map $\mathbf{u}_r$ in (δ_r) has constant rank over $\Sigma_r(f)$, so that we can define the coimage bundle map

$$\mathbf{c}_r : \mathbf{Hom}(\mathbf{K}_r \bigcirc \mathbf{K}_{r-1} \bigcirc \ldots \bigcirc \mathbf{K}_1, \mathbf{P}) \to \mathbf{P}_r \qquad \text{over } \Sigma_r(f).$$

We call $\mathbf{P}_r$ the *symmetric subbundle* of $\mathbf{Hom}(\mathbf{K}_r, \mathbf{Q}_r)$.

(ζ_r) Assume that the intrinsic derivative bundle map (7.5) $\mathbf{S}_{r-1} \to \mathbf{Hom}(\mathbf{K}_r, \mathbf{Q}_r)$ over $\Sigma_r(f)$ of $\mathbf{d}_r(f)|\mathbf{K}_{r-1}$ factors through $\mathbf{P}_r$ to yield

$$\mathbf{d}_{r+1}(f) : \mathbf{S}_{r-1} \to \mathbf{P}_r \qquad \text{over } \Sigma_r(f),$$

where $\mathbf{S}_{r-1}$ is the tangent bundle of $\Sigma_{r-1}(f)$.

(η_r) Assume that $\mathbf{K}_r \subset \mathbf{S}_{r-1}$ over $\Sigma_r(f)$.

(θ_r) Assume that $\Sigma_r(f)$ is a submanifold of V.

Then we have (α_{r+1}), by restricting domains of definition as necessary.

We need to know when the assumptions hold, so that the construction can proceed. Their validity will be expressed in terms of the following condition:

(ι_t) $d_{t+1}(f) : \mathrm{S}_{t-1} \to \mathbf{P}_t$ is surjective in the fibres over points $p \in \Sigma_t(f)$.

Theorem **(7.15)**. — *If* (ι_t) *holds for* $0 < t < r$, *the assumptions* (ε_r), (ζ_r), *and* (η_r), *are valid, and* $\Sigma_r(f)$ *is exactly* $\Sigma^{\mathrm{J}}(f)$, *the subset defined in* (2.21), *where* $\mathrm{J} = (i_1, i_2, \ldots, i_r)$. *If also* ($\iota_r$) *is true, then the assumption* (θ_r) *is valid. The conditions* (ι_t) *hold for all t if and only if the jet section* $\mathrm{J}f : \mathrm{V} \to \mathrm{J}(\mathrm{V}, \mathrm{W})$ *is transverse to* Σ_t *for all t, which is true for a fine-*C^∞*-dense subset of the maps* $f : \mathrm{V} \to \mathrm{W}$.

Proof. — We see by induction that locally all the above bundles, etc., are induced from the corresponding bundles with the same names obtained from special flags over open sets in J(V, W). For these the required inductive assumptions hold, if we use $\mathbf{T}_r$ instead of $\mathbf{S}_r$, and all there is left to check is that the conditions (ι_t) are equivalent to transversality, so that the $\Sigma_t(f)$ are submanifolds and have the bundles $\mathbf{S}_t$ as their tangent bundles.

We have to compare (ι_t) with the condition:

(κ_t) $\qquad \mathrm{J}f : \mathrm{V} \to \mathrm{J}(\mathrm{V}, \mathrm{W})$ is transverse to Σ_t.

Now by the short exact sequence of (7.13), (ι_t) is equivalent to $\mathbf{T}_t + (\mathbf{T}_{t-1} \cap \mathbf{D}) = \mathbf{T}_{t-1}$ over $\Sigma_t \cap (\mathrm{J}f)\mathrm{V}$, or alternatively $\mathbf{T}_t + \mathbf{D} = \mathbf{T}_{t-1} + \mathbf{D}$. Also ($\kappa_t$) may be written $\mathbf{T}_t + \mathbf{D} = \mathbf{T}_0$. Hence given ($\kappa_{t-1}$), the conditions ($\iota_t$) and ($\kappa_t$) are equivalent. This is all we need. ∎

In particular, the intrinsinc derivatives $\mathbf{d}_{r+1}$ induce bundle maps $\mathbf{d}_{r+1}(f)$.

Definition **(7.16)**. — Given a suitable map $f : \mathrm{V} \to \mathrm{W}$, and a sequence I, the *r-th intrinsic derivative of f* $(1 \leqslant r \leqslant n+1)$

$$\mathbf{d}_r(f) : \mathbf{S}_{r-2} \to \mathbf{P}_{r-1} \qquad \text{over } \Sigma_{r-1}(f)$$

is induced from $\mathbf{d}_r|(\mathbf{T}_{r-2} \cap \mathbf{D})$ by $\mathrm{J}f$.

It is therefore symmetric.

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Manuscrit reçu le 10 juin 1966.
Révisé le 20 février 1967.