Singularities of electromagnetic fields in polyhedral domains

Martin COSTABEL and Monique DAUGE

Abstract. *In this paper, we investigate the singular solutions of time harmonic Maxwell equations in a domain which has edges and polyhedral corners. It is now well known that in the presence of non-convex edges, the solution fields have no square integrable gradients in general and that the main singularities are the gradients of singular functions of the Laplace operator* [4, 5]. *We show how this type of result can be derived from the classical Mellin analysis, and how this analysis leads to sharper results concerning the singular parts which belong to* H^1 *: For the generating singular functions, we exhibit simple and explicit formulas based on (generalized) Dirichlet and Neumann singularities for the Laplace operator. These formulas are more explicit than the results announced in our note* [10].

Introduction

Solutions of time-harmonic three-dimensional Maxwell equations

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\,\mu\boldsymbol{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\,\varepsilon\boldsymbol{E} = \boldsymbol{J}$$

with electric or magnetic boundary conditions have singularities near corners and edges of the boundary of the domain. This well-known fact has, for example, important consequences for the construction of numerical approximation of the solution. Just as for other elliptic boundary value problems, the singularities can be analyzed by standard techniques [17, 21, 15, 12] that go back to Kondratev's technique of Mellin transformation.

The special form of Maxwell equations allows us to go further in the analysis of the regularity and the singularities. Some known results are:

- the H^1 regularity for convex domains, SARANEN [26],
- the $H^{1/2}$ regularity for Lipschitz domains, COSTABEL [8],
- a description of singular functions for cones with a smooth basis, SARANEN [25],
- corner singularities for the corresponding two-dimensional problem, MOUSSAOUI [23], see also section 3.

Further regularity results can be found in KŘIČEK - NEITTAANMÄKI [19], HAZARD - LENOIR [16], AMROUCHE - BERNARDI - DAUGE - GIRAULT [2]. In all these cases, the main singularity is the gradient of a singular function of the Dirichlet or Neumann problem for the Laplace operator, thus is reduced to a problem that can be considered as well known [15, 12]. This relation can be extended to a more general class of piecewise smooth domains that have "screen" or "crack" parts, BIRMAN - SOLOMYAK and FILONOV [4, 5, 13].

We will show that, in such a class of domains, not only the first, but *all singular functions* for the Maxwell boundary value problems can be obtained in simple ways from Dirichlet or Neumann singular functions of the Laplace operator.

More generally, we will assume that the domain Ω is polyhedral, that is, its boundary consists of plane faces, straight edges, and corner points: such a polyhedron needs not to be Lipschitz nor simply connected. Thus Ω can have screen parts, in which case there is only H^s regularity with $s < 1/2$. We find another type of non-Lipschitz domains with the same bad regularity: These domains are, as exemplified by a domain between two cones with the same vertex, not locally simply connected. Here the simple equation

$$\text{“}\textit{Maxwell regularity} = \textit{Dirichlet regularity } -1\text{ ”}$$

is violated: The Dirichlet problem can even have H^2 regularity in such a case. The singular functions at such a corner are generated by topological objects: the elements of the cohomology space of the base of the cone, see section 6.

For a better understanding of this new phenomenon, consider the case of a domain between two circular cones with the same vertex and the same axis. In spherical coordinates we have with $0 < \theta_0 < \theta_1 < \pi$ in a ball $B(0, \rho_0)$:

$$\Omega \cap B(0, \rho_0) = \{(\rho, \theta, \varphi) \mid \quad \rho \in (0, \rho_0),\ \theta_0 < \theta < \theta_1,\ \varphi \in [0, 2\pi)\}.$$

We consider the functions

$$\Phi(\rho, \theta, \varphi) = \log \tan \frac{\theta}{2} \quad \text{and} \quad \Psi(\rho, \theta, \varphi) = \varphi.$$

Both functions are harmonic, and with the cylindrical and cartesian variables

$$r = \sqrt{x^2 + y^2} = \rho \sin \theta \quad \text{and} \quad (x, y, z) = (\rho \sin \theta \cos \varphi, \rho \sin \theta \sin \varphi, \rho \cos \theta)$$

we have

$$\boldsymbol{u} := \mathbf{grad}\, \Phi = \frac{1}{\rho} \Big(\frac{xz}{r^2}\,, \ \frac{yz}{r^2}\,, \ -1 \Big)^{\mathrm{T}} \quad \text{and} \quad \boldsymbol{v} := \mathbf{grad}\, \Psi = \Big(-\frac{y}{r^2}\,, \ \frac{x}{r^2}\,, \ 0 \Big)^{\mathrm{T}}.$$

Both $\boldsymbol{u}$ and $\boldsymbol{v}$ are harmonic vector fields:

$$\mathbf{curl}\, \boldsymbol{u} = \mathbf{curl}\, \boldsymbol{v} = 0, \quad \operatorname{div} \boldsymbol{u} = \operatorname{div} \boldsymbol{v} = 0.$$

On $\partial\Omega$ near the vertex, $\boldsymbol{u}$ is normal and $\boldsymbol{v}$ is tangential. Thus $\boldsymbol{u}$ satisfies "electric" and $\boldsymbol{v}$ "magnetic" boundary conditions. Both functions are in H^s near the vertex for $s < 1/2$. They correspond to electrostatic and magnetostatic singularities, respectively.

Maxwell's equations are not an elliptic system. But the elimination of one of the two fields $\boldsymbol{E}$ or $\boldsymbol{H}$ yields a variational formulation in a special "physical" energy space which is $X := H(\mathrm{div}) \cap H(\mathbf{curl})$, the space of square integrable vector fields with square integrable curl and divergence. The underlying boundary value problem is then a second order elliptic system, but set in a non-standard space. Instead of this space X, also the Sobolev space H^1 could be used in a similar variational formulation. Both variational formulations are based on the same bilinear form and both have unique solutions which, for non-convex domains, will not coincide in general. The second one is the projection in X of the first one onto H^1. But the most important fact is that only the formulation in the space X gives back a solution of the original Maxwell equations.

This situation is important for numerical approximations: If one uses standard finite elements that are contained in H^1, then the "true" solution in X cannot be approximated, and mesh refinement at the corners and edges does not help. A possible solution is to augment the finite element space by the explicitly known singular functions, ASSOUS - CIARLET - SONNENDRÜCKER [3] and BONNET - HAZARD - LOHRENGEL [6]. In such a method, the approximation is determined by the regularity of the regular part, i.e. the solution minus the singular function. This regularity can be quite different for different choices of the singular function. In particular, if the singular function is not constructed by our explicit formulas, but from abstract principles, then this regularity can be quite low, typically H^s with $s < 4/3$, see sections 3 and 4.

Plan. The outline of our paper is as follows:

We begin with preliminaries where we define the class of polygonal and polyhedral domains in which we will work, and the basic functional spaces with L^2 curl and divergence. Then starting from the classical Maxwell equations for a homogeneous material we give equivalent variational formulations involving the form $\langle \mathbf{curl}, \mathbf{curl} \rangle + \langle \mathrm{div}, \mathrm{div} \rangle$ in spaces of X-type. The question of equivalent formulations will be discussed in more details and generality in section 7.

In section 1, we start with the *principal part* of the equations obtained in the preliminaries: these are our *Maxwell problems*; we discuss the alternative formulation in the H^1 subspaces, which we call *pseudo-Maxwell problems*, and the link with the singular solution spaces of the Laplace operator; we conclude this section by a result (Theorem 1.4) on the characterization of these problems by the regularity of the divergence.

In section 2, as a preparation for the description of all singularities of the solutions of the Maxwell and pseudo-Maxwell problems, we formulate some results from [12] on the edge and corner singularities of the Dirichlet and Neumann Laplace operator on a polyhedral domain. We present the main arguments of the proofs. We also obtain a precise description of complementary spaces in the X-spaces of the H^1 subspaces.

In section 3, we give a complete description of the Maxwell and pseudo-Maxwell singularities in plane polygonal domains, using the results of Lemma 3.1 whose proof is postponed to section 5. It turns out that, at each non-convex corner, one singular function is interchanged between the solutions of the Maxwell and pseudo-Maxwell problems.

In section 4, we state all our results about polyhedral domains, relying on Lemmas 4.1 and 4.4 which give explicit formulas for all the Maxwell corner and edge singularities respectively. We have got a classification of these singularities in three main types, for example concerning the electric field,

1. Gradients of Laplace Dirichlet singularities,
2. Divergence-free fields whose curls are gradients of Laplace Neumann singularities,
3. Fields whose divergences are Laplace Dirichlet singularities.

When the domain is not locally simply connected in the neighborhood of the corner, the first two types are enriched by topological singularities of similar structure (the notion of Dirichlet and Neumann Laplace singularities has to be extended in a suitable way), but this concerns only the singularity exponents -1 and 0, respectively. The other singularity exponents are those of the Laplace Neumann problem and those of the Laplace Dirichlet problem ± 1. Concerning the magnetic field, the roles of Neumann and Dirichlet conditions are interchanged. We conclude this section by results about the pseudo-Maxwell problems. While for the Maxwell problems the main singularities are gradients of Laplace singularities, for the pseudo-Maxwell problems they can only be described as sorts of Stokes singularities.

Lemma 4.4 is proved in section 5 devoted to singularities in plane sectors or wedges whereas Lemma 4.1 is proved in section 6 in which singularities in polyhedral cones are investigated. The results concerning polyhedral cones are summarized in Table 1. For solutions of the original time-harmonic Maxwell equations, the singularities of type 3 are absent, and there is a symmetry between the singularities of the electric and the magnetic field. The results are summarized in Table 2.

In section 7, we compare several different commonly used variational formulations for the time harmonic Maxwell equations. For this comparison, we can admit rather general conditions corresponding to anisotropic inhomogeneous materials. When the material coefficients are smooth, the principal singularities are those of problems with constant coefficients. This corresponds to the choice made in the first six sections.

Preliminaries

0.a Domains and Sobolev spaces

Here are first a few definitions about domains and spaces. We want to consider rather general piecewise smooth domains which need not to be Lipschitz, in general: such domains can easily appear in applications and some of them have already been studied as mentioned above. We denote by $\boldsymbol{x} = (x, y, z)$ the cartesian coordinates in $\mathbb{R}^3$ and by $\boldsymbol{x} = (x, y)$ the cartesian coordinates in $\mathbb{R}^2$.

As in [12], the definition of the classes of domains is recursive. Let $B(\boldsymbol{x}, r)$ denote the ball of center $\boldsymbol{x}$ and radius r. In $\mathbb{R}^2$ we define:

- The *2D corner domains* as bounded domains Ω in $\mathbb{R}^2$ or $\mathbb{S}^2$ such that in each point $\boldsymbol{x}$ of the boundary there exists $r_{\boldsymbol{x}} > 0$ small enough such that to each connected component $\Omega_{\boldsymbol{x},i}$, of $\Omega \cap B(\boldsymbol{x}, r_{\boldsymbol{x}})$ belongs a diffeomorphism $\chi_{\boldsymbol{x},i}$ transforming $\Omega_{\boldsymbol{x},i}$ into a neighborhood of the corner 0 of a plane sector of opening in $(0, 2\pi]$, $\boldsymbol{x}$ being sent into 0.
- The *polygonal domains* as the 2D corner domains with straight sides (indeed any bounded domain whose boundary is a finite union of segments).

In $\mathbb{R}^3$ we define:

- The *3D corner domains* as bounded domains Ω in $\mathbb{R}^3$ such that in each point $\boldsymbol{x}$ of the boundary there exists $r_{\boldsymbol{x}} > 0$ small enough such that to each connected component $\Omega_{\boldsymbol{x},i}$ of $\Omega \cap B(\boldsymbol{x}, r_{\boldsymbol{x}})$ belongs a diffeomorphism $\chi_{\boldsymbol{x},i}$ transforming $\Omega_{\boldsymbol{x},i}$ into a neighborhood of the corner 0 of a cone $\Gamma_{\boldsymbol{x},i}$ of the form $\{\boldsymbol{x} \in \mathbb{R}^3,\ \boldsymbol{x}/|\boldsymbol{x}| \in G_{\boldsymbol{x},i}\}$ with $G_{\boldsymbol{x},i}$ a 2D corner domain of $\mathbb{S}^2$, $\boldsymbol{x}$ being sent into 0.
- The *polyhedral domains* as the 3D corner domains with straight faces (indeed any bounded domain whose boundary is a finite union of polygons).

We say that Ω in one of these classes is *locally simply connected* if for any $\boldsymbol{x}$ in its boundary, $\Omega \cap B(\boldsymbol{x}, r_{\boldsymbol{x}})$ is simply connected.

The space $H^1(\Omega)$ is the space of complex-valued distributions $u \in \mathcal{D}'(\Omega)$ which belong to $L^2(\Omega)$ and such that each component of their gradients belongs to $L^2(\Omega)$. The space $H^{1/2}(\partial\Omega)$ is the space of traces of $H^1(\Omega)$ where it is understood that $\partial\Omega$ is the "unfolded" boundary of Ω, that is, in the neighborhood of each point $\boldsymbol{x} \in \overline{\Omega} \setminus \Omega$ in the topological boundary of Ω, $\partial\Omega$ is the disjoint union of the parts of the boundaries of $\Omega_{\boldsymbol{x},i}$ which are contained in $\overline{\Omega} \setminus \Omega$. The space $H^{-1/2}(\partial\Omega)$ is the dual space of $H^{1/2}(\partial\Omega)$.

Moreover, we introduce the spaces $H(\mathbf{curl}\,;\Omega)$ and $H(\operatorname{div}\,;\Omega)$:

$$H(\mathbf{curl}\,;\Omega) = \big\{\boldsymbol{u} \in \mathcal{D}'(\Omega)^3 \mid \quad \boldsymbol{u} \in L^2(\Omega)^3,\ \mathbf{curl}\,\boldsymbol{u} \in L^2(\Omega)^3\big\} \tag{0.1}$$

$$H(\operatorname{div}\,;\Omega) = \big\{\boldsymbol{u} \in \mathcal{D}'(\Omega)^3 \mid \quad \boldsymbol{u} \in L^2(\Omega)^3,\ \operatorname{div}\boldsymbol{u} \in L^2(\Omega)\big\}. \tag{0.2}$$

For any $\boldsymbol{u} \in H(\mathbf{curl}\,;\Omega)$, the tangential trace $\boldsymbol{u} \times \boldsymbol{n}$ is well defined in $H^{-1/2}(\partial\Omega)^3$ due to the Green formula:

$$\forall \boldsymbol{v} \in H^1(\Omega)^3 : \quad \int_\Omega \boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{v} = \langle \boldsymbol{u} \times \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial\Omega}. \tag{0.3}$$

Similarly, for any $\boldsymbol{u} \in H(\operatorname{div}\,;\Omega)$, the normal trace $\boldsymbol{u} \cdot \boldsymbol{n}$ is well defined in $H^{-1/2}(\partial\Omega)$ by the Green formula:

$$\forall \varphi \in H^1(\Omega) : \quad \int_\Omega \operatorname{div}\boldsymbol{u}\,\varphi + \boldsymbol{u} \cdot \mathbf{grad}\,\varphi = \langle \boldsymbol{u} \cdot \boldsymbol{n}, \varphi \rangle_{\partial\Omega}. \tag{0.4}$$

0.b Time harmonic Maxwell equations

The classical time harmonic Maxwell equations at the frequency ω in a homogeneous isotropic body occupying Ω, with permeability $\mu > 0$ and permittivity $\varepsilon > 0$ are

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\,\mu\boldsymbol{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\,\varepsilon\boldsymbol{E} = \boldsymbol{J} \quad \text{in} \quad \Omega. \tag{0.5a}$$

Here $\boldsymbol{E}$ is the electric part and $\boldsymbol{H}$ the magnetic part of the electromagnetic field. They are supposed to be square integrable fields. By a change of unknowns μ and ε can be set to 1. The right hand side $\boldsymbol{J}$ is the current density. The exterior boundary conditions on $\partial\Omega$ are those of the perfect conductor ($\boldsymbol{n}$ denotes the unit outer normal on $\partial\Omega$):

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{and} \quad \boldsymbol{H} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{0.5b}$$

We see that *if* ω *is not zero,* and if $\boldsymbol{J}$ belongs to $H(\operatorname{div};\Omega)$ we deduce from the above first order system (0.5a) that $\boldsymbol{E}$ and $\boldsymbol{H}$ belong to $H(\mathbf{curl};\Omega) \cap H(\operatorname{div};\Omega)$ and that, according to (0.3) and (0.4) the boundary conditions (0.5b) make sense. Thus there holds

$$\boldsymbol{E} \in X_N \quad \text{and} \quad \boldsymbol{H} \in X_T \tag{0.6}$$

where X_N and X_T are the two closed subspaces of $H(\mathbf{curl};\Omega) \cap H(\operatorname{div};\Omega)$ defined respectively as

$$X_N = \big\{\boldsymbol{u} \in H(\mathbf{curl};\Omega) \cap H(\operatorname{div};\Omega) \mid \quad \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\big\} \tag{0.7}$$

$$X_T = \big\{\boldsymbol{u} \in H(\mathbf{curl};\Omega) \cap H(\operatorname{div};\Omega) \mid \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega\big\}. \tag{0.8}$$

Integrating by parts with test functions in X_N or in X_T and taking into account the equations which can be obtained by calculating the divergence of both equations in (0.5a), we prove in section 7 (in a wider generality concerning the permeability μ and the permittivity ε) that there holds the following result.

Theorem 0.1 *We assume* $\omega \neq 0$. *Let* $\boldsymbol{J} \in H(\operatorname{div};\Omega)$ *and define the functionals on* $H(\mathbf{curl};\Omega) \cap H(\operatorname{div};\Omega)$

$$\boldsymbol{f}(\boldsymbol{v}) := i\omega \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} + \frac{1}{i\omega} \int_\Omega \operatorname{div} \boldsymbol{J} \operatorname{div} \boldsymbol{v} \qquad \textit{and} \qquad \boldsymbol{h}(\boldsymbol{v}) := \int_\Omega \boldsymbol{J} \cdot \mathbf{curl}\,\boldsymbol{v}.$$

(1) *If* $(\boldsymbol{E}, \boldsymbol{H})$ *solves* (0.5)*, then* $\boldsymbol{u} = \boldsymbol{E}$ *solves* (0.9)

$$\boldsymbol{u} \in X_N, \quad \forall \boldsymbol{v} \in X_N, \quad \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} - \omega^2\, \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{f}(\boldsymbol{v}), \tag{0.9}$$

and $\boldsymbol{u} = \boldsymbol{H}$ *solves* (0.10)

$$\boldsymbol{u} \in X_T, \quad \forall \boldsymbol{v} \in X_T, \quad \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} - \omega^2\, \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{h}(\boldsymbol{v}), \tag{0.10}$$

$(2a)$ *If* $\boldsymbol{u}$ *solves* (0.9) *and* ω^2 *is not an eigenvalue of the Laplace Dirichlet operator on* Ω*, then* $(\boldsymbol{E}, \boldsymbol{H}) = \big(\boldsymbol{u}, (i\omega)^{-1}\, \mathbf{curl}\,\boldsymbol{u}\big)$ *solves* (0.5)*.*

$(2b)$ *If* $\boldsymbol{u}$ *solves* (0.10) *and* ω^2 *is not an eigenvalue of the Laplace Neumann operator on* Ω*, then* $(\boldsymbol{E}, \boldsymbol{H}) = \big(i\omega^{-1}\,(\mathbf{curl}\,\boldsymbol{u} - \boldsymbol{J}), \boldsymbol{u}\big)$ *solves* (0.5)*.*

1 The Maxwell and pseudo-Maxwell problems

In this section, we assume that Ω is a simply connected polyhedral domain in $\mathbb{R}^3$.

1.a Variational formulations

We call *Maxwell problems* the principal parts of problems (0.9) and (0.10), i.e.

$$\boldsymbol{u} \in X_N, \quad \forall \boldsymbol{v} \in X_N, \quad \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \operatorname{div} \boldsymbol{u}\, \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{1.1}$$

and

$$\boldsymbol{u} \in X_T, \quad \forall \boldsymbol{v} \in X_T, \quad \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \operatorname{div} \boldsymbol{u}\, \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{1.2}$$

with a right-hand side $\boldsymbol{f} \in L^2(\Omega)^3$. Both problems are uniquely solvable since, as we assumed that Ω is simply connected, the form

$$a(\boldsymbol{u}, \boldsymbol{v}) := \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{v} + \operatorname{div} \boldsymbol{u}\, \operatorname{div} \boldsymbol{v}$$

is strongly coercive on X_N and X_T, i.e. satisfies

$$\exists c > 0, \quad \forall \boldsymbol{u} \in X_N, \quad a(\boldsymbol{u}, \boldsymbol{u}) \geq c\Big(\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\,\mathbf{curl}\,\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|^2_{L^2(\Omega)} \Big),$$

and the same in X_T, see [2] for example.

We will consider these problems rather than problems (0.9) and (0.10) in the major part of our paper because they are simpler and their edge and corner singularities have the same principal parts as problems (0.9) and (0.10).

The non-standard feature of these problems is the nature of the variational spaces. Both spaces are embedded in $H^1_{\mathrm{loc}}(\Omega)^3$ but not in $H^1(\Omega)^3$ in general: for a polyhedral domain Ω, it is known [9] that X_N is embedded in $H^1(\Omega)^3$ if and only if Ω is convex, and the same holds for X_T.

However, it turns out that the form a is also coercive on the subspaces of $H^1(\Omega)^3$

$$H_N = X_N \cap H^1(\Omega)^3, \quad H_T = X_T \cap H^1(\Omega)^3$$

which means that

$$\exists c > 0, \quad \forall \boldsymbol{u} \in H_N, \quad a(\boldsymbol{u}, \boldsymbol{u}) \geq c\, \|\boldsymbol{u}\|^2_{H^1(\Omega)},$$

and the same in H_T, see [9]. Therefore the variational *pseudo-Maxwell problems*

$$\tilde{\boldsymbol{u}} \in H_N, \quad \forall \boldsymbol{v} \in H_N, \quad \int_\Omega \mathbf{curl}\,\tilde{\boldsymbol{u}} \cdot \mathbf{curl}\,\boldsymbol{v} + \operatorname{div} \tilde{\boldsymbol{u}}\, \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{1.3}$$

and

$$\tilde{\boldsymbol{u}} \in H_T, \quad \forall \boldsymbol{v} \in H_T, \quad \int_\Omega \mathbf{curl}\, \tilde{\boldsymbol{u}} \cdot \mathbf{curl}\, \boldsymbol{v} + \operatorname{div} \tilde{\boldsymbol{u}} \, \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{1.4}$$

are uniquely solvable for any $\boldsymbol{f} \in L^2(\Omega)^3$.

In order to understand the nature of the non- H^1 solutions of the Maxwell problems (1.1) and (1.2), one first has to study complementary subspaces of the closed subspaces H_N of X_N and H_T of X_T. These complements are related to edge and corner singularities of the Dirichlet and Neumann problems for the Laplace operator, as we shall see. The Laplacian appears if we consider conservative fields, that is, gradients of potentials. The question is then the H^2 regularity of the potentials. Conversely, it has been known for some time that for the non- H^1 singularities, it is also sufficient to study gradients. To describe this result more precisely, we introduce the domains of the Laplace Dirichlet Δ^{Dir} and Neumann Δ^{Neu} operators as follows

$$\begin{aligned} D(\Delta^{\mathrm{Dir}}) &= \big\{\varphi \in \mathring{H}^1(\Omega) \mid \quad \Delta\varphi \in L^2(\Omega)\big\} \\ D(\Delta^{\mathrm{Neu}}) &= \big\{\varphi \in H^1(\Omega) \mid \quad \Delta\varphi \in L^2(\Omega) \quad \text{and} \quad \partial_n\varphi = 0 \text{ on } \partial\Omega\big\}. \end{aligned}$$

Let us recall from [15, 12] that for any polyhedron the space $H^2(\Omega) \cap \mathring{H}^1(\Omega)$ is closed in the domain $D(\Delta^{\mathrm{Dir}})$, but that for non-convex polyhedra, the elliptic regularity theorem does not hold between $L^2(\Omega)$ and $H^2(\Omega)$, and similarly for the Neumann problem. We can immediately see that for any function φ belonging to $D(\Delta^{\mathrm{Dir}})$ but not to $H^2(\Omega)$, the vector function $\mathbf{grad}\, \varphi$ belongs to X_N but not to H_N, and similarly with $D(\Delta^{\mathrm{Neu}})$ in relation with the spaces X_T and H_T. That the converse also holds has been shown by BIRMAN - SOLOMYAK [4, 5], *cf* also [6].

Theorem 1.1 *Let Ω be a locally simply connected polyhedral domain.*

(i) *For any closed complement K_{Dir} of $H^2(\Omega) \cap \mathring{H}^1(\Omega)$ in $D(\Delta^{\mathrm{Dir}})$, we have*

$$X_N = H_N \oplus \mathbf{grad}\, K_{\mathrm{Dir}}.$$

(ii) *For any closed complement K_{Neu} of $H^2(\Omega) \cap D(\Delta^{\mathrm{Neu}})$ in $D(\Delta^{\mathrm{Neu}})$, we have*

$$X_T = H_T \oplus \mathbf{grad}\, K_{\mathrm{Neu}}.$$

Precise descriptions of the spaces K_{Dir} and K_{Neu} will be given below in section 2. Our result is that in the presence of nonconvex edges, these spaces are infinite-dimensional (see Corollary 2.8), whereas for the corresponding two-dimensional problems, their dimension is equal to the number of nonconvex corners, see (2.7).

1.b The regularity of the divergence

The solutions of problems (1.1) and (1.2) have a divergence with optimal elliptic regularity, i.e. $\operatorname{div} \boldsymbol{u} \in H^1(\Omega)$, whereas nothing similar holds for the solutions of problems (1.3) and (1.4).

Theorem 1.2
(i) *The divergence* $q = \operatorname{div} \boldsymbol{u}$ *of the solution of problem* (1.1) *belongs to* $\mathring{H}^1(\Omega)$ *and is the solution of the Dirichlet problem*

$$q \in \mathring{H}^1(\Omega), \quad \Delta q = -\operatorname{div} \boldsymbol{f}. \tag{1.5}$$

(ii) *The divergence* $q = \operatorname{div} \boldsymbol{u}$ *of the solution of problem* (1.2) *belongs to* $H^1(\Omega)$ *and is the solution with zero mean value on* Ω *of the Neumann problem*

$$q \in H^1(\Omega), \quad \forall p \in H^1(\Omega), \quad \int_\Omega \mathbf{grad}\, q \cdot \mathbf{grad}\, p = -\int_\Omega \boldsymbol{f} \cdot \mathbf{grad}\, p.$$

(iii) *If* $\operatorname{div} \boldsymbol{f} = 0$*, then the solutions of both Maxwell problems* (1.1) *and* (1.2) *satisfy* $\operatorname{div} \boldsymbol{u} = 0$.

PROOF. *(i)* If $\boldsymbol{u}$ solves (1.1), then taking as test functions $\boldsymbol{v} = \mathbf{grad}\,\varphi$ with $\varphi \in D(\Delta^{\mathrm{Dir}})$ we obtain

$$\forall \varphi \in D(\Delta^{\mathrm{Dir}}), \qquad \big\langle \operatorname{div} \boldsymbol{u}\,,\, \Delta^{\mathrm{Dir}}\varphi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\varphi \big\rangle_\Omega\,, \tag{1.6}$$

with $\langle\ ,\ \rangle_\Omega$ the duality product in Ω . But the solution of (1.5) satisfies

$$\forall p \in \mathring{H}^1(\Omega), \qquad \big\langle \mathbf{grad}\, q\,,\, \mathbf{grad}\, p \big\rangle_\Omega = -\big\langle \boldsymbol{f}\,,\, \mathbf{grad}\, p \big\rangle_\Omega\,,$$

whence

$$\forall \varphi \in D(\Delta^{\mathrm{Dir}}), \qquad \big\langle q\,,\, \Delta^{\mathrm{Dir}}\varphi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\varphi \big\rangle_\Omega\,.$$

Thus $\operatorname{div} \boldsymbol{u} - q$ is orthogonal to the range of Δ^{Dir} , which is the whole $L^2(\Omega)$.

The proof of *(ii)* is similar and *(iii)* follows from uniqueness for the Dirichlet and Neumann problems. ■

For the divergence $\tilde{q}$ of the pseudo-Maxwell solution $\tilde{\boldsymbol{u}}$ of problem (1.3), there holds, instead of (1.6):

$$\forall \varphi \in H^2(\Omega) \cap \mathring{H}^1(\Omega), \qquad \big\langle \tilde{q}\,,\, \Delta^{\mathrm{Dir}}\varphi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\varphi \big\rangle_\Omega\,. \tag{1.7}$$

Now for this "very weak" Dirichlet problem, there is no uniqueness in general. We define

$$K^*_{\mathrm{Dir}} : \text{ orthogonal complement in } L^2(\Omega) \text{ of } \Delta^{\mathrm{Dir}}\big(H^2 \cap \mathring{H}^1(\Omega)\big)\,. \tag{1.8}$$

This space is isomorphic to K_{Dir}. Its elements are often called *dual singular functions* because of their duality with singularities. They are the solutions of the totally homogeneous problem in its "very weak" form (1.7), see [22, 15]. We obtain the following result.

Theorem 1.3 *For* $\boldsymbol{f} \in L^2(\Omega)^3$*, let* $\boldsymbol{u}$ *and* $\tilde{\boldsymbol{u}}$ *be the solutions of problems* (1.1) *and* (1.3) *respectively. Then* $\operatorname{div} \boldsymbol{u} - \operatorname{div} \tilde{\boldsymbol{u}} \in K^*_{\mathrm{Dir}}$. *A similar result holds for problems* (1.2) *and* (1.4).

1.c Boundary value formulations

Maxwell and pseudo-Maxwell solutions differ by their regularity, but they are, in fact, solutions of one and the same boundary value problem. To understand this somewhat unusual situation, we introduce the following non-symmetric weak formulations:

$$\boldsymbol{u} \in X_N, \quad \forall \boldsymbol{v} \in H_N, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{1.9}$$

$$\boldsymbol{u} \in X_T, \quad \forall \boldsymbol{v} \in H_T, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}. \tag{1.10}$$

Both Maxwell and pseudo-Maxwell solutions satisfy these problems. By using $\mathcal{C}^\infty$ functions in H_N or H_T as test functions, we see in the standard way that the "electric" problem (1.9) has the following strong form

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} - \mathbf{grad} \operatorname{div} \boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.11a}$$
$$\boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.11b}$$
$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.11c}$$

whereas the "magnetic" problems corresponds to the boundary value problem

$$\mathbf{curl}\,\mathbf{curl}\, \boldsymbol{u} - \mathbf{grad} \operatorname{div} \boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.12a}$$
$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.12b}$$
$$\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.12c}$$

Here the partial differential (vector Laplace) equation is understood in the distributional sense. The "stable" boundary conditions (1.11b) and (1.12b) correspond to the definition of the variational spaces. The "natural" boundary conditions (1.11c) and (1.12c) are obtained after integration by parts and have to be understood in a weak sense, (1.11c) for example in the sense of the "very weak" Dirichlet problem (1.7).

If one takes into account that $\mathcal{C}^\infty$ functions in H_N or H_T that vanish in a neighborhood of the singular parts of the boundary, are dense in H_N or H_T, see [11], it is easy to see that for $\boldsymbol{f} \in L^2(\Omega)^3$ and $\boldsymbol{u} \in X_N$ or $\boldsymbol{u} \in X_T$ respectively, the weak formulations (1.9), (1.10) and the boundary formulations (1.11), (1.12) are completely equivalent. For the rest of this section, we concentrate on the "electric" problem (1.9). The results for (1.10) are analogous.

Theorem 1.4
(i) *To any* $q \in K^*_{\mathrm{Dir}}$ *there exists a unique* $\boldsymbol{u} \in X_N$, *solution of* (1.9) *with* $\boldsymbol{f} = 0$ *such that*

$$\operatorname{div} \boldsymbol{u} = q.$$

(ii) *Let* $\boldsymbol{f} \in L^2(\Omega)^3$ *and* $\boldsymbol{u}$ *be a solution of* (1.9). *Then*

$\alpha)$ $\boldsymbol{u}$ *is a solution of the pseudo-Maxwell problem* (1.3) *if and only if* $\boldsymbol{u} \in H^1(\Omega)^3$.

$\beta)$ $\boldsymbol{u}$ *is a solution of the Maxwell problem* (1.1) *if and only if* $\operatorname{div} \boldsymbol{u} \in H^1(\Omega)$.

(iii) *If* $\operatorname{div} \boldsymbol{f} = 0$ *then* $\boldsymbol{u}$ *is a solution of the Maxwell problem* (1.1) *if and only if*

$$\operatorname{div} \boldsymbol{u} = 0.$$

PROOF. *(i)* Let $q \in K^*_{\mathrm{Dir}}$. The splitting $X_N = H_N \oplus \mathbf{grad}\, K_{\mathrm{Dir}}$ (Theorem 1.1) defines the bounded operator S

$$\begin{array}{rccll} S: & X_N & \longrightarrow & K_{\mathrm{Dir}} & \\ & \boldsymbol{u} & \longmapsto & S\boldsymbol{u}, & \text{such that} \quad \boldsymbol{u} - \mathbf{grad}\, S\boldsymbol{u} \in H_N. \end{array}$$

Let $\boldsymbol{u}$ be the solution of the problem

$$\boldsymbol{u} \in X_N, \quad \forall \boldsymbol{v} \in X_N, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + \operatorname{div} \boldsymbol{u}\ \operatorname{div} \boldsymbol{v} = \int_\Omega q\ \Delta(S\boldsymbol{v}).$$

As $S|_{H_N} = 0$, $\boldsymbol{u}$ satisfies (1.9) with $\boldsymbol{f} = 0$. With test functions $\boldsymbol{v} = \mathbf{grad}\, \psi$ for any $\psi \in H^2 \cap H^1_0(\Omega)$, we obtain that $\operatorname{div} \boldsymbol{u}$ is orthogonal to $\Delta^{\mathrm{Dir}}(H^2 \cap H^1_0(\Omega))$ whereas with test functions $\boldsymbol{v} = \mathbf{grad}\, \psi$, $\psi \in K_{\mathrm{Dir}}$, we have $S(\mathbf{grad}\, \psi) = \psi$ whence

$$\forall \psi \in K_{\mathrm{Dir}}, \quad \int_\Omega \operatorname{div} \boldsymbol{u}\ \Delta\psi = \int_\Omega q\ \Delta\psi.$$

As q is also orthogonal to $\Delta^{\mathrm{Dir}}(H^2 \cap H^1_0(\Omega))$, we deduce that $\operatorname{div} \boldsymbol{u} - q$ is orthogonal to $\Delta\psi$ for all $\psi \in D(\Delta^{\mathrm{Dir}})$, therefore $\operatorname{div} \boldsymbol{u} = q$.
If $\boldsymbol{f} = 0$ and $\operatorname{div} \boldsymbol{u} = 0$, (1.9) combined with the splitting $X_N = H_N \oplus \mathbf{grad}\, K_{\mathrm{Dir}}$ yields that

$$\forall \boldsymbol{v} \in X_N, \quad \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{v} + \operatorname{div} \boldsymbol{u}\ \operatorname{div} \boldsymbol{v} = 0.$$

Hence $\boldsymbol{u} = 0$, which proves the uniqueness.
(ii) $\alpha)$ was explained above.
(ii) $\beta)$ One direction is given by Theorem 1.2: if $\boldsymbol{u}$ solves (1.1), then $\operatorname{div} \boldsymbol{u} \in H^1(\Omega)$. Let conversely $\boldsymbol{u} \in X_N$ be a solution of (1.11) with $\operatorname{div} \boldsymbol{u} \in H^1(\Omega)$. By subtracting the solution of problem (1.1) with the same $\boldsymbol{f}$, we reduce to the case when $\boldsymbol{f} = 0$. Then

(1.11a) yields $\Delta \operatorname{div} \boldsymbol{u} = 0$, and (1.11c) with the assumption gives $\operatorname{div} \boldsymbol{u} \in \mathring{H}^1(\Omega)$. Therefore $\operatorname{div} \boldsymbol{u} = 0$. Whence $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} = 0$ and

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\boldsymbol{w} = 0, \quad \forall \boldsymbol{w} \in \mathcal{C}^\infty(\overline{\Omega})^3.$$

As $\mathcal{C}^\infty(\overline{\Omega})^3$ is dense in $H(\mathbf{curl}\,;\Omega)$, we find that $\mathbf{curl}\,\boldsymbol{u} = 0$. As Ω is simply connected, we can conclude that $\boldsymbol{u} = 0$. ∎

2 Singularities of the Laplace operator

In this section we recall and reformulate results from [15] and [12] relating to corner singularities in a polygonal domain and to edge and vertex singularities in a polyhedral domain. For the reader's convenience, we will sketch the main arguments of the proofs.

2.a Polygonal domains

The results of this paragraph are proved by GRISVARD, see for example [15], but we present them with the method of [12] which is inspired by the classical paper by KONDRAT'EV [17]. The presentation of [12] is based on the introduction for each corner of several spaces of pseudo-homogeneous functions, which allows an optimal treatment of the polynomial part in the data and the solution.

The notion of corner is obvious if Ω is a Lipschitz polygon. If Ω has crack part in its boundary, we have to use the notations introduced with the definition of a polygonal domain in the Introduction.

Let $\boldsymbol{a}$ be an element of the unfolded boundary of Ω, corresponding to a point $\boldsymbol{x} \in \overline{\Omega} \setminus \Omega$. Thus $\boldsymbol{a}$ belongs to the boundary of one of the $\Omega_{\boldsymbol{x},i}$ and we say that $\boldsymbol{a}$ is a corner of Ω if the corresponding sector $\Gamma_{\boldsymbol{x},i}$ is non-trivial (opening $\neq \pi$) and we denote $\Gamma_{\boldsymbol{x},i}$ by $\Gamma_{\boldsymbol{a}}$ and $\Omega_{\boldsymbol{x},i}$ by $\mathcal{V}_{\boldsymbol{a}}$. Let $\mathcal{A}$ be the set of the corners $\boldsymbol{a}$ of Ω. With each corner $\boldsymbol{a}$, we associate local polar coordinates such that

$$\Gamma_{\boldsymbol{a}} = \big\{(r_{\boldsymbol{a}}, \theta_{a}) \mid r_{\boldsymbol{a}} > 0,\ 0 < \theta_{\boldsymbol{a}} < \omega_{\boldsymbol{a}}\big\} \quad \text{where } \omega_{\boldsymbol{a}} \text{ is the opening of } \Gamma_{\boldsymbol{a}}.$$

Let φ be the solution of the problem $\Delta^{\mathrm{Dir}}\varphi = f$ with $f \in H^{s-1}(\Omega)$, $s > 0$. Away from any neighborhood of the corners of Ω, the solution φ has the optimal H^{s+1} regularity. Near each corner $\boldsymbol{a}$, φ has an asymptotics as $r_{\boldsymbol{a}} \to 0$, which contains in general other functions (the singularities) than the polynomials (the Taylor expansion).

Let us fix $\boldsymbol{a} \in \mathcal{A}$. The asymptotics and the regularity of the solution φ in a neighborhood of $\boldsymbol{a}$ only depend on special spaces of (pseudo) homogeneous functions $Y^\lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}})$ and $Z^\lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}})$ defined below. We drop the subscript $\boldsymbol{a}$ in the notations when no confusion is possible.

For any $\lambda \in \mathbb{C}$, let $S^\lambda_{\mathrm{Dir}}(\Gamma)$ be the space

$$S^\lambda_{\mathrm{Dir}}(\Gamma) = \big\{\Phi(r,\theta) = r^\lambda \sum_{q=0}^{Q} \log^q r \; \phi_q(\theta) \;|\; \phi_q \in \overset{\circ}{H}{}^1(0,\omega)\big\}. \tag{2.1}$$

Only certain subspaces $Y^\lambda_{\mathrm{Dir}}(\Gamma)$ contribute to the asymptotics of φ: For $\lambda \in \mathbb{C}$, $Y^\lambda_{\mathrm{Dir}}(\Gamma)$ is defined as the subspace of $S^\lambda_{\mathrm{Dir}}(\Gamma)$:

$$Y^\lambda_{\mathrm{Dir}}(\Gamma) = \big\{\Phi \in S^\lambda_{\mathrm{Dir}}(\Gamma) \;|\; \Delta\Phi \text{ is polynomial in } (x,y)\,\big\}. \tag{2.2}$$

- If λ is a positive integer, $Y^\lambda_{\mathrm{Dir}}(\Gamma)$ contains the space $P^\lambda_{\mathrm{Dir}}(\Gamma)$ of homogeneous polynomial (thus non-singular) functions Φ of degree λ satisfying the Dirichlet boundary conditions. The space of singularities $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ is defined as a complement of $P^\lambda_{\mathrm{Dir}}(\Gamma)$ in $Y^\lambda_{\mathrm{Dir}}(\Gamma)$

$$Y^\lambda_{\mathrm{Dir}}(\Gamma) = Z^\lambda_{\mathrm{Dir}}(\Gamma) \oplus P^\lambda_{\mathrm{Dir}}(\Gamma). \tag{2.3}$$

- If λ is not a positive integer, $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ is simply defined as

$$Z^\lambda_{\mathrm{Dir}}(\Gamma) = \big\{\Phi \in S^\lambda_{\mathrm{Dir}}(\Gamma) \;|\; \Delta\Phi = 0\big\}. \tag{2.4}$$

We denote by $\Lambda_{\mathrm{Dir}}(\Gamma)$ the set of $\lambda \in \mathbb{C}$ such that $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ is not reduced to $\{0\}$. This is a discrete set: this fact is a consequence of the ellipticity of the boundary value problem to which it corresponds, see AGRANOVICH - VISHIK [1]. We can refer to $\Lambda_{\mathrm{Dir}}(\Gamma)$ as the *set of singular exponents* because only its elements λ produce singularities, which are in general of the form $r^\lambda\phi(\theta)$.

In the case of Δ^{Dir}, $\Lambda_{\mathrm{Dir}}(\Gamma)$ is closely related to the spectrum of the one-dimensional Dirichlet problem on $(0,\omega)$

$$\overset{\circ}{H}{}^1(0,\omega) \ni \phi \quad \longmapsto \quad -\partial^2_\theta\phi \in H^{-1}(0,\omega). \tag{2.5}$$

Its eigenvalues are $(\frac{k\pi}{\omega})^2$ with eigenvectors $\sin\frac{k\pi}{\omega}\theta$ and for each $k \in \mathbb{Z}^*$ (set of non-zero integers) the functions $r^{k\pi/\omega}\sin\frac{k\pi}{\omega}\theta$ are non-zero elements of $Y^{k\pi/\omega}_{\mathrm{Dir}}(\Gamma)$. There holds

Lemma 2.1 *Let Γ be a plane sector of opening $\omega \neq \pi$. Then*

$$\Lambda_{\mathrm{Dir}}(\Gamma) = \begin{cases} \{\frac{k\pi}{\omega},\ k \in \mathbb{Z}^*\} & \text{if } \omega \neq 2\pi \\ \{\frac{k}{2},\ k < 0 \text{ or } k \text{ odd}\} & \text{if } \omega = 2\pi. \end{cases}$$

For any $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma)$, the singularity space $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ is generated by

$$\Phi^\lambda_{\mathrm{Dir}} := \begin{cases} r^\lambda \sin\lambda\theta & \text{if } \lambda \notin \mathbb{N} \\ r^\lambda(\log r \sin\lambda\theta + \theta\cos\lambda\theta) - \frac{1}{\omega}(-\frac{y}{\sin\omega})^\lambda & \text{if } \lambda \in \mathbb{N}, \end{cases}$$

where $(x,y) = (r\cos\theta, r\sin\theta)$ are Cartesian coordinates.

Concerning the Neumann problem, definitions are similar: we set

$$\begin{aligned} S^\lambda_{\mathrm{Neu}}(\Gamma) &= \big\{\Phi(x) = r^\lambda \textstyle\sum_{q=0}^{Q} \log^q r \ \phi_q(\theta) \mid \phi_q \in H^1(0,\omega)\big\}, \\ Y^\lambda_{\mathrm{Neu}}(\Gamma) &= \big\{\Phi \in S^\lambda_{\mathrm{Neu}}(\Gamma) \mid \Delta\Phi \ \text{ is polynomial} \quad \text{and} \quad \partial_n \Phi\big|_{\partial\Gamma} = 0\big\}, \end{aligned}$$

then the singularity spaces $Z^\lambda_{\mathrm{Neu}}(\Gamma)$ and the set of exponents $\Lambda_{\mathrm{Neu}}(\Gamma)$ are defined along the same lines as above. There holds

Lemma 2.2 *Let Γ be a plane sector of opening $\omega \neq \pi$. Then $\Lambda_{\mathrm{Neu}}(\Gamma) = \Lambda_{\mathrm{Dir}}(\Gamma)$ and for any $\lambda \in \Lambda_{\mathrm{Neu}}(\Gamma)$, the singularity space $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ is generated by*

$$\Phi^\lambda_{\mathrm{Neu}} := \begin{cases} r^\lambda \cos\lambda\theta & \text{if } \lambda \notin \mathbb{N} \\ r^\lambda(\log r \cos\lambda\theta - \theta\sin\lambda\theta) + \frac{1}{\omega}(-\frac{y}{\sin\omega})^\lambda & \text{if } \lambda \in \mathbb{N}. \end{cases}$$

Coming back to the polygonal domain Ω, we use a smooth cut-off function χ such that $\chi(r_{\boldsymbol{a}})$ is 1 in a neighborhood of $\boldsymbol{a}$ and is zero outside $\mathcal{V}_{\boldsymbol{a}}$, and we set

$$\varphi^\lambda_{\mathrm{Dir},\boldsymbol{a}}(x,y) = \chi(r_{\boldsymbol{a}})\,\Phi^\lambda_{\mathrm{Dir},\boldsymbol{a}}(r_{\boldsymbol{a}},\theta_{\boldsymbol{a}}),$$

where $\Phi^\lambda_{\mathrm{Dir},\boldsymbol{a}}$ is the generating function in Lemma 2.1 corresponding to the sector $\Gamma_{\boldsymbol{a}}$. Then one has the following theorem of splitting in regular and singular parts.

Theorem 2.3 *Let φ be such that $f = \Delta^{\mathrm{Dir}}\varphi$ belongs to $H^{s-1}(\Omega)$. If for all $\boldsymbol{a} \in \mathcal{A}$ the exponent s does not belong to $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}})$, then there exist coefficients $\gamma^\lambda_{\boldsymbol{a}}$ for each $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}}) \cap (0,s)$ such that*

$$\varphi - \sum_{\boldsymbol{a}\in\mathcal{A}} \sum_{0<\lambda<s} \gamma^\lambda_{\boldsymbol{a}}\,\varphi^\lambda_{\mathrm{Dir},\boldsymbol{a}} \ \in \ H^{s+1}(\Omega). \tag{2.6}$$

A similar statement holds for the Neumann problem.

After localization near a corner $\boldsymbol{a}$, the key of the proof is the Mellin transform

$$\mathcal{M}[\varphi](\lambda) = \frac{1}{2\pi}\int_0^\infty r^{-\lambda}\varphi(r\cos\theta, r\sin\theta)\,\frac{dr}{r}\,.$$

$\mathcal{M}[\varphi]$ is defined for $\operatorname{Re}\lambda \leq 0$ and holomorphic. Moreover, the inverse Mellin transform on the line $\operatorname{Re}\lambda = 0$ gives back φ:

$$\varphi = \int_{\operatorname{Re}\lambda=0} r^\lambda \mathcal{M}[\varphi](\lambda)\,d\lambda.$$

Let $h := r^2 f = r^2\Delta\varphi$. Then $\mathcal{M}[h]$ is defined for $\operatorname{Re}\lambda \leq s$, meromorphic with poles on $\lambda \in \mathbb{N}$ corresponding to the Taylor expansion of h in $\boldsymbol{a}$. Moreover, with

$$\mathcal{L}(\lambda) \ : \ \overset{\circ}{H}{}^1(0,\omega) \ni \phi \quad \longmapsto \quad \partial^2_\theta\phi + \lambda^2\phi \in H^{-1}(0,\omega),$$

for any $\lambda \in \mathbb{C}$ and $\phi \in \mathring{H}^1(0,\omega)$ there holds $r^2\Delta(r^\lambda\phi) = r^\lambda\mathcal{L}(\lambda)\phi$, whence

$$\forall\lambda \in \mathbb{C}, \quad \operatorname{Re}\lambda \le 0, \qquad \mathcal{L}(\lambda)\mathcal{M}[\varphi](\lambda) = \mathcal{M}[h](\lambda).$$

As $\mathcal{L}(\lambda)^{-1}$ is meromorphic in $\mathbb{C}$, the function $\mathcal{L}(\lambda)^{-1}\mathcal{M}[h](\lambda)$ is a meromorphic extension of $\mathcal{M}[\varphi]$ for $0 < \operatorname{Re}\lambda \le s$. The poles of $\mathcal{L}(\lambda)^{-1}$ are the square roots of the eigenvalues of the operator (2.5), i.e. the $\frac{k\pi}{\omega}$, $k \in \mathbb{Z}^*$. With φ_0 the inverse Mellin transform on the line $\operatorname{Re}\lambda = s$

$$\varphi_0 = \int_{\operatorname{Re}\lambda = s} r^\lambda \mathcal{L}(\lambda)^{-1}\mathcal{M}[h](\lambda)\, d\lambda$$

there holds by Cauchy's residue formula

$$\varphi_0 - \varphi = \sum_{0<\operatorname{Re}\lambda_0<s} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda\mathcal{L}(\lambda)^{-1}\mathcal{M}[h](\lambda).$$

Moreover $\chi\varphi_0$ is regular: it belongs to the subspace of $H^{s+1}(\Gamma)$ with zero Taylor part at the corner $\boldsymbol{a}$ of Γ. The residue in λ_0 belongs to the space $S^{\lambda_0}_{\mathrm{Dir}}(\Gamma)$ and satisfies

$$\begin{aligned} r^2\Delta\Big(\operatorname*{Res}_{\lambda=\lambda_0} r^\lambda\mathcal{L}(\lambda)^{-1}\mathcal{M}[h](\lambda)\Big) &= \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda\mathcal{M}[h](\lambda) \\ &= \sum_{|\alpha|=\lambda_0} \frac{x^{\alpha_1}y^{\alpha_2}}{\alpha_1!\,\alpha_2!}\partial^\alpha h(\boldsymbol{a}) \;=\; r^2 \sum_{|\alpha|=\lambda_0-2} \frac{x^{\alpha_1}y^{\alpha_2}}{\alpha_1!\,\alpha_2!}\partial^\alpha f(\boldsymbol{a}). \end{aligned}$$

Thus the residue in λ_0 of $r^\lambda\mathcal{L}(\lambda)^{-1}\mathcal{M}[h](\lambda)$ belongs to $Y^{\lambda_0}_{\mathrm{Dir}}(\Gamma)$. The separation from the polynomial part in $P^\lambda_{\mathrm{Dir}}(\Gamma)$, *cf* (2.3), yields the splitting of φ in the Theorem. ■

For $s = 1$ the only contribution to the singular part comes from non-convex angles with the first exponent $\frac{\pi}{\omega} < 1$. Thus spaces K_{Dir} and K_{Neu} as introduced in Theorem 1.1 can be defined as

$$K_{\mathrm{Dir}} = \operatorname{Span}\big\langle \varphi^{\pi/\omega_{\boldsymbol{a}}}_{\mathrm{Dir},\boldsymbol{a}} \;|\; \omega_{\boldsymbol{a}} > \pi\big\rangle \quad \text{and} \quad K_{\mathrm{Neu}} = \operatorname{Span}\big\langle \varphi^{\pi/\omega_{\boldsymbol{a}}}_{\mathrm{Neu},\boldsymbol{a}} \;|\; \omega_{\boldsymbol{a}} > \pi\big\rangle. \tag{2.7}$$

Moreover

$$K^*_{\mathrm{Dir}} = \operatorname{Span}\big\langle \varphi^{-\pi/\omega_{\boldsymbol{a}}}_{\mathrm{Dir},\boldsymbol{a}} - \psi_{\mathrm{Dir},\boldsymbol{a}} \;|\; \omega_{\boldsymbol{a}} > \pi\big\rangle \tag{2.8}$$

where $\psi_{\mathrm{Dir},\boldsymbol{a}}$ is the solution of the problem $\Delta^{\mathrm{Dir}}\psi = \Delta(\varphi^{-\pi/\omega_{\boldsymbol{a}}}_{\mathrm{Dir},\boldsymbol{a}})$ and similarly for the Neumann problem.

2.b Corner singularities in polyhedral domains

The results of this paragraph can be found in [15] and again we adopt the presentation of [12].

Let $\mathcal{C}$ be the set of the corners $\boldsymbol{c}$ of the polyhedral domain $\Omega \subset \mathbb{R}^3$, that we define similarly to the corners of a polygon, with the requirement that for any $\boldsymbol{c} \in \mathcal{C}$ the corresponding cone $\Gamma_{\boldsymbol{x},i}$, denoted by $\Gamma_{\boldsymbol{c}}$, is a non-trivial cone (i.e. it is neither a half space nor a dihedron). The corresponding neighborhood $\Omega_{\boldsymbol{x},i}$ is denoted by $\mathcal{V}_{\boldsymbol{c}}$. In local spherical coordinates $\rho_{\boldsymbol{c}} \in \mathbb{R}_+$, $\vartheta_{\boldsymbol{c}} \in \mathbb{S}^2$, the cone $\Gamma_{\boldsymbol{c}}$ is:

$$\Gamma_{\boldsymbol{c}} = \big\{(\rho_{\boldsymbol{c}}, \vartheta_{\boldsymbol{c}}) \mid \rho_{\boldsymbol{c}} > 0,\ \vartheta_{\boldsymbol{c}} \in G_{\boldsymbol{c}} \subset \mathbb{S}^2\big\} \quad \text{with a spherical polygonal domain } G_{\boldsymbol{c}}.$$

Let us fix $\boldsymbol{c} \in \mathcal{C}$. The asymptotics at $\boldsymbol{c}$ of the solution φ of a Dirichlet problem on Ω depends on the spaces $Y^\lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ and $Z^\lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ defined similarly to the case of dimension 2. We drop the subscript $\boldsymbol{c}$ in the notations when no confusion is possible.

Analogously to (2.1), we introduce

$$S^\lambda_{\mathrm{Dir}}(\Gamma) = \big\{\Phi(\rho,\vartheta) = \rho^\lambda \sum_{q=0}^{Q} \log^q\rho\ \phi_q(\vartheta) \mid\ \phi_q \in \mathring{H}^1(G)\big\}. \tag{2.9}$$

Then the spaces $Y^\lambda_{\mathrm{Dir}}(\Gamma)$ and $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ are still defined by (2.2) (where Δ is now the three-dimensional Laplacian) and (2.3) respectively. Moreover $\Lambda_{\mathrm{Dir}}(\Gamma)$ is still the set of $\lambda \in \mathbb{C}$ such that $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ is not reduced to $\{0\}$.

Let Δ_G^{Dir} be the positive Laplace-Beltrami operator with Dirichlet conditions on G. The operator Δ_G^{Dir} is self-adjoint with a compact inverse. Let $\mathfrak{S}(\Delta_G^{\mathrm{Dir}})$ be its spectrum. From the expression of Δ in polar coordinates

$$\rho^2\Delta = (\rho\partial_\rho)^2 + \rho\partial_\rho - \Delta_G$$

we obtain that the set of exponents $\Lambda_{\mathrm{Dir}}(\Gamma)$ contains the roots of the equations $\lambda(\lambda+1) = \mu$ with $\mu \in \mathfrak{S}(\Delta_G^{\mathrm{Dir}})$:

Lemma 2.4 *Let Γ be a polyhedral cone and let n_Γ be the number of its faces. Then*

$$\Lambda_{\mathrm{Dir}}(\Gamma) = \begin{cases} \big\{-\frac12 \pm \sqrt{\mu+\frac14},\ \mu \in \mathfrak{S}(\Delta_G^{\mathrm{Dir}})\big\} \setminus \mathbb{N} & \textit{if } n_\Gamma = 1 \\ \big\{-\frac12 \pm \sqrt{\mu+\frac14},\ \mu \in \mathfrak{S}(\Delta_G^{\mathrm{Dir}})\big\} & \textit{if } n_\Gamma = 2 \\ \big\{-\frac12 \pm \sqrt{\mu+\frac14},\ \mu \in \mathfrak{S}(\Delta_G^{\mathrm{Dir}})\big\} \cup \mathbb{N}_2 & \textit{if } n_\Gamma \geq 3, \end{cases}$$

where $\mathbb{N}_2$ is the set of integers ≥ 2. For any non integer $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma)$, the singularity space $Z^\lambda_{\mathrm{Dir}}(\Gamma)$ is the space of the functions $\Phi^\lambda_{\mathrm{Dir}} = \rho^\lambda\phi(\vartheta)$ where ϕ spans the eigenspace of Δ_G^{Dir} corresponding to the eigenvalue $\mu = \lambda(\lambda+1)$.

Coming back to the polyhedral domain Ω we set

$$\varphi^{\lambda,p}_{\mathrm{Dir},\boldsymbol{c}}(x,y,z) = \chi(\rho_{\boldsymbol{c}})\,\Phi^{\lambda,p}_{\mathrm{Dir},\boldsymbol{c}}(\rho_{\boldsymbol{c}},\vartheta_{\boldsymbol{c}}),$$

where $\Phi^{\lambda,p}_{\mathrm{Dir},\boldsymbol{c}}$, $p = 1,\dots,P^\lambda$, is a basis of $Z^\lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$. Then there holds

Theorem 2.5 *Let $\varphi \in \mathring{H}^1(\Omega)$ such that $\Delta^{\mathrm{Dir}}\varphi$ belongs to $H^{s-1}(\Omega)$ and let $\boldsymbol{c} \in \mathcal{C}$. If $s - \frac{1}{2}$ does not belong to $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ then there exist coefficients $\gamma_{\boldsymbol{c}}^{\lambda,p}$ for each $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) \cap (-\frac{1}{2}, s-\frac{1}{2})$ and each $p = 1, \ldots, P^\lambda$, such that*

$$\chi(\rho_{\boldsymbol{c}})\,\varphi - \sum_{-\frac{1}{2}<\lambda<s-\frac{1}{2}} \sum_p \gamma_{\boldsymbol{c}}^{\lambda,p}\,\varphi_{\mathrm{Dir},\boldsymbol{c}}^{\lambda,p} \;\in\; H^{s+1}\big(\mathbb{R}^+, \rho_{\boldsymbol{c}}^2\,d\rho_{\boldsymbol{c}}; L^2(G_{\boldsymbol{c}})\big), \tag{2.10}$$

where $H^{s+1}\big(\mathbb{R}^+, \rho^2\,d\rho; L^2(G)\big)$ denotes the H^{s+1} space on $\mathbb{R}_+$ with measure $\rho^2\,d\rho$ and values in $L^2(G)$.

The proof follows the same lines as that of Theorem 2.3 and is based on the Mellin transform of $\chi(\rho_{\boldsymbol{c}})\,\varphi$ with respect to the corner $\boldsymbol{c}$, i.e. (we drop the subscript $\boldsymbol{c}$)

$$\mathcal{M}[\varphi](\lambda) = \frac{1}{2\pi}\int_0^\infty \rho^{-\lambda}\varphi(\rho\vartheta)\,\frac{d\rho}{\rho}\,,$$

where $\vartheta \in G$. The "operator pencil" is now

$$\mathcal{L}(\lambda) \;:\; \mathring{H}^1(G) \ni \phi \quad\longmapsto\quad \lambda(\lambda+1)\phi - \Delta_G\phi \in H^{-1}(G),$$

and the "regular part" φ_0 is provided by the inverse Mellin transform on the line $\operatorname{Re}\lambda = s - \frac{1}{2}$ of $\mathcal{L}(\lambda)^{-1}\mathcal{M}[h](\lambda)$ with $h = \rho^2\Delta(\chi\varphi)$. But, due to the presence of the edges of Γ, corresponding to the corners of G, with φ_0 we obtain no improvement in angular regularity, but only in the radial direction. That is why the regular part in Theorem 2.5 belongs to $H^{s+1}\big(\mathbb{R}^+, \rho^2\,d\rho; L^2(G)\big)$. ■

Similar definitions and results hold for the Neumann boundary condition if we define the spectrum $\mathfrak{S}(\Delta_G^{\mathrm{Neu}})$ of Δ_G^{Neu} as the set of *non-zero* eigenvalues of Δ_G^{Neu}.

2.c Edge singularities in polyhedral domains

For the sake of brevity, we describe the following results only for Dirichlet boundary conditions. The results and the method are those of [12, §16]: see Theorem 16.9.

Let $\mathcal{E}$ be the set of the (open) edges $\boldsymbol{e}$ of Ω: for each point $\boldsymbol{x} \in \boldsymbol{e}$, there exists a neighborhood $\Omega_{\boldsymbol{x},i}$ in which Ω coincides with the wedge $W_{\boldsymbol{e}} = \Gamma_{\boldsymbol{e}} \times \mathbb{R}$, where $\Gamma_{\boldsymbol{e}}$ is a plane sector given in local polar coordinates by

$$\Gamma_{\boldsymbol{e}} = \big\{(r_{\boldsymbol{e}}, \theta_{\boldsymbol{e}}) \mid r_{\boldsymbol{e}} > 0,\ 0 < \theta_{\boldsymbol{e}} < \omega_{\boldsymbol{e}}\big\} \quad \text{with } \omega_{\boldsymbol{e}} \text{ the opening of } \Gamma_{\boldsymbol{e}}.$$

We obtain local cylindrical coordinates by the adjunction of a coordinate $z_{\boldsymbol{e}}$ along the edge $\boldsymbol{e}$. Let $\varphi \in \mathring{H}^1(\Omega)$ such that $\Delta^{\mathrm{Dir}}\varphi \in H^{s-1}(\Omega)$. Before giving in the next paragraph a description of the singularities of φ along the whole edge $\boldsymbol{e}$, we are going to discuss briefly the structure of its singularities along $\boldsymbol{e}$ away from the corners of Ω.

So, let us fix $\boldsymbol{e} \in \mathcal{E}$ and drop the subscript $\boldsymbol{e}$. We investigate $\varphi \in \mathring{H}^1(\Gamma\times\mathbb{R})$ with compact support, solution of the Dirichlet problem on the wedge $\Delta^{\mathrm{Dir}}\varphi \in H^{s-1}(\Gamma\times\mathbb{R})$.

Let ω be the opening of the sector Γ and $\big((x,y),z\big)$ be the coordinates in $\Gamma\times\mathbb{R}$. The partial Fourier transform with respect to the variable z, $\varphi(x,y,z)\mapsto\hat\varphi(x,y,\xi)$, transforms the equation $\Delta\varphi=f$ into

$$\forall\xi\in\mathbb{R},\quad \forall(x,y)\in\Gamma,\qquad (\partial_x^2+\partial_y^2-\xi^2)\hat\varphi(x,y,\xi)=\hat f(x,y,\xi).$$

The change of variables $(x,y,\xi)\mapsto(\tilde x,\tilde y,\xi)=(|\xi|x,|\xi|y,\xi)$ transforms the above problem for each non-zero ξ into

$$\forall(\tilde x,\tilde y)\in\Gamma,\qquad (\partial_{\tilde x}^2+\partial_{\tilde y}^2-1)\tilde\varphi(\tilde x,\tilde y,\xi)=\xi^2\tilde f(\tilde x,\tilde y,\xi). \tag{2.11}$$

Thus for each ξ, we have an equation on Γ involving the two-dimensional Laplacian as principal part. Writing the above equation (2.11) in the form

$$\Delta\psi=g+\psi\quad\text{in}\quad\Gamma,$$

we derive from Theorem 2.3 by a bootstrap argument that the singularities of ψ have themselves expansions as $r\to0$, starting with the singularities of Δ^{Dir} in Γ:

$$\psi(\tilde x,\tilde y)-\chi(\tilde r)\sum_{0<\lambda<s}\tilde\gamma^\lambda\Big(\Phi^\lambda_{\mathrm{Dir}}(\tilde r,\theta)+\sum_{1\le q<s-\lambda}\Phi^{\lambda;q}_{\mathrm{Dir}}(\tilde r,\theta)\Big)\ \in\ H^{s+1}(\Gamma), \tag{2.12}$$

where the sum extends over $\lambda\in\Lambda_{\mathrm{Dir}}(\Gamma)$ and integer q, and $\tilde r=\sqrt{\tilde x^2+\tilde y^2}$. The singular functions $\Phi^\lambda_{\mathrm{Dir}}$ are those defined in Lemma 2.1, while the supplementary ones:

$$\Phi^{\lambda;q}_{\mathrm{Dir}}\in S^{\lambda+q}_{\mathrm{Dir}}(\Gamma),$$

depend only on ω, λ and q (*). Coming back to the original variables (x,y,ξ) and using the homogeneity of the functions Φ, we obtain (if no λ belongs to $\mathbb{N}$)

$$\begin{aligned}\hat\varphi(x,y,\xi)-\sum_{0<\lambda<s}|\xi|^\lambda\,\check\gamma^\lambda(\xi)\Big(&\chi(r|\xi|)\,\Phi^\lambda_{\mathrm{Dir}}(r,\theta)\\&+\sum_{1\le q\le s-\lambda}|\xi|^q\chi(r|\xi|)\,\Phi^{\lambda;q}_{\mathrm{Dir}}(r,\theta)\Big)\ \in\ H^{s+1}(\Gamma).\end{aligned}$$

The inverse partial Fourier transform yields the splitting of φ into regular and singular parts

$$\begin{aligned}\varphi(x,y,z)-\sum_{0<\lambda<s}\Big(&\mathcal{K}_\Gamma[\gamma^\lambda](r,z)\ \Phi^\lambda_{\mathrm{Dir}}(r,\theta)\\&+\sum_{1\le q\le s-\lambda}\mathcal{K}_\Gamma[\partial_z^q\gamma^\lambda](r,z)\ \Phi^{\lambda;q}_{\mathrm{Dir}}(r,\theta)\Big)\ \in\ H^{s+1}(\Gamma\times\mathbb{R}),\end{aligned} \tag{2.13}$$

* The $\Phi^{\lambda;q}_{\mathrm{Dir}}$ are solutions of the recursive equations $\Delta^{\mathrm{Dir}}\Phi^{\lambda;q}_{\mathrm{Dir}}=\Phi^{\lambda;q-2}_{\mathrm{Dir}}$. If $\lambda\notin\mathbb{N}$, $\Phi^{\lambda;q}_{\mathrm{Dir}}$ is zero for odd q and of the form $c\,r^q\Phi^\lambda_{\mathrm{Dir}}$ for even q, with $c\in\mathbb{R}$ depending on ω, λ and q.

where the edge coefficient γ^λ is the inverse Fourier transform of $|\xi|^\lambda\,\check{\gamma}^\lambda(\xi)$. Thanks to the uniformity of the first splitting (2.12), we obtain that $|\xi|^s\check{\gamma}^\lambda(\xi)$ belongs to $L^2(\mathbb{R})$, and hence $\gamma^\lambda \in H^{s-\lambda}(\mathbb{R})$ (†). The operator $\mathcal{K}_\Gamma$ acts as a lifting of trace and is defined as

$$\widehat{\mathcal{K}_\Gamma[\delta]}(r,\xi) = \chi(r|\xi|)\,\hat{\delta}(\xi).$$

Finally, setting

$$\begin{aligned}\mathfrak{K}^\lambda_{\mathrm{Dir},\Gamma\,;\,s}[\gamma^\lambda](x,y,z) \;&=\; \mathcal{K}_\Gamma[\gamma^\lambda](r,z)\;\Phi^\lambda_{\mathrm{Dir}}(r,\theta) &\text{(2.14a)}\\ &\quad+ \sum_{1\,\le\, q\,\le\, s-\lambda} \mathcal{K}_\Gamma[\partial_z^q\gamma^\lambda](r,z)\;\Phi^{\lambda;q}_{\mathrm{Dir}}(r,\theta), &\text{(2.14b)}\end{aligned}$$

we can write the expansion (2.13) in a synthetic way as

$$\varphi - \sum_{0\,<\,\lambda\,<\,s} \mathfrak{K}^\lambda_{\mathrm{Dir},\Gamma\,;\,s}[\gamma^\lambda] \in H^{s+1}(\Gamma\times\mathbb{R}) \quad \text{with} \quad \gamma^\lambda \in H^{s-\lambda}(\mathbb{R}). \tag{2.15}$$

In the block $\mathfrak{K}^\lambda_{\mathrm{Dir},\Gamma\,;\,s}[\gamma^\lambda]$, the term $\mathcal{K}_\Gamma[\gamma^\lambda]\,\Phi^\lambda_{\mathrm{Dir}}$ in (2.14a) is the *leading term*.

2.d Combined corner and edge singularities in polyhedral domains

The results and the method are those of [12, §17]: see Theorem 17.13. In [12] the splitting of the solution in regular and singular parts is localized near each corner and in the remaining interior parts of each edge. Here we give a new formulation of the results which is global on the whole domain.

We are going to give a decomposition into regular and singular parts of the solution φ of the Dirichlet problem $\Delta^{\mathrm{Dir}}\varphi \in H^{s-1}(\Omega)$ on the polyhedral domain Ω. The corner contributions are already present in (2.10). As for the edge contributions, they have a similar structure to those of (2.15) and involve for each edge $\boldsymbol{e} \in \mathcal{E}$, coefficients $\gamma^\lambda_{\boldsymbol{e}}$ defined on $\boldsymbol{e}$ for λ in the set of exponents $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}})$, *cf* Lemma 2.1, associated with the bi-dimensional Laplace operator on the sector $\Gamma_{\boldsymbol{e}}$ generating the wedge which coincides with Ω in a neighborhood of $\boldsymbol{e}$.

In order to state the regularity of the coefficients $\gamma^\lambda_{\boldsymbol{e}}$ along the edge $\boldsymbol{e} \in \mathcal{E}$, we use a smooth function $d_{\boldsymbol{e}}$ on the closed edge $\overline{\boldsymbol{e}}$, which is equivalent to the distance to the ends of $\boldsymbol{e}$: if for example $\boldsymbol{e} = \{\boldsymbol{x} \mid r_{\boldsymbol{e}} = 0,\ z_{\boldsymbol{e}} \in (-1,+1)\}$, we can take $d_{\boldsymbol{e}}(z_{\boldsymbol{e}}) = 1 - z_{\boldsymbol{e}}^2$. The weighted Sobolev spaces $\mathbb{V}^m_\eta(\boldsymbol{e})$ which are the correct spaces for the coefficients $\gamma^\lambda_{\boldsymbol{e}}$ are defined for $m \in \mathbb{N}$ and $\eta \in \mathbb{R}$ by

$$\mathbb{V}^m_\eta(\boldsymbol{e}) = \big\{\gamma \in L^2(\boldsymbol{e}) \;\big|\quad (d_{\boldsymbol{e}})^{\eta+k}\,\partial^k_{z_{\boldsymbol{e}}}\gamma \in L^2(\boldsymbol{e}), \quad k = 0,1,\ldots,m\big\}$$

and by complex interpolation for non-integers m.

Combining the corner expansions (2.10) for each corner $\boldsymbol{c} \in \mathcal{C}$, with a blow up of each edge $\boldsymbol{e} \in \mathcal{E}$ at its ends (which are two corners of Ω) and an edge expansion like (2.15), we can prove

† Essential for the estimates is the invertibility of the operator $\Delta - 1$ from $\mathring{H}^1(\Gamma)$ onto $H^{-1}(\Gamma)$.

Theorem 2.6 (i) *Let* $\varphi \in \overset{\circ}{H}{}^1(\Omega)$ *such that* $\Delta^{\mathrm{Dir}}\varphi$ *belongs to* $H^{s-1}(\Omega)$. *If for all* $\boldsymbol{c} \in \mathcal{C}$, $s-\frac{1}{2}$ *does not belong to* $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ *and if for all* $\boldsymbol{e} \in \mathcal{E}$, s *does not belong to* $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}})$, *then there exist:*

- *Coefficients* $\gamma_{\boldsymbol{c}}^{\lambda,p}$ *for each* $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) \cap (-\frac{1}{2}, s-\frac{1}{2})$ *and* $p = 1, \ldots, P^\lambda$,
- *Functions* $\gamma_{\boldsymbol{e}}^{\lambda} \in \mathbb{V}_{-s}^{s-\lambda}(\boldsymbol{e})$ *for each* $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}}) \cap (0, s)$

such that

$$\varphi - \sum_{\boldsymbol{c}\in\mathcal{C}} \sum_{-\frac{1}{2} < \lambda < s-\frac{1}{2}} \sum_{p} \gamma_{\boldsymbol{c}}^{\lambda,p}\, \varphi_{\mathrm{Dir},\boldsymbol{c}}^{\lambda,p} - \sum_{\boldsymbol{e}\in\mathcal{E}} \sum_{0<\lambda<s} \mathfrak{K}_{\mathrm{Dir},\boldsymbol{e}\,;\,s}^{\lambda}[\gamma_{\boldsymbol{e}}^{\lambda}] \;\in\; H^{s+1}(\Omega). \tag{2.16}$$

(ii) *Conversely, for any coefficients* $\gamma_{\boldsymbol{c}}^{\lambda,p} \in \mathbb{R}$ *and any functions* $\gamma_{\boldsymbol{e}}^{\lambda} \in \mathbb{V}_{-s}^{s-\lambda}(\boldsymbol{e})$ *there exists a function* φ *admitting the expansion* (2.16) *and such that* $\Delta^{\mathrm{Dir}}\varphi \in H^{s-1}(\Omega)$.

To describe the operators $\mathfrak{K}_{\mathrm{Dir},\boldsymbol{e}\,;\,s}^{\lambda}$, we need to define the smoothing operator $\mathcal{K}_{\boldsymbol{e}}[\cdot]$ adapted to the edge $\boldsymbol{e}$, *cf* (2.14), and for this we introduce the stretched variable

$$\tilde{z}_{\boldsymbol{e}} = \int_0^{z_{\boldsymbol{e}}} \frac{1}{d_{\boldsymbol{e}}(z)}\, dz,$$

where $z = 0$ corresponds to an interior point of $\boldsymbol{e}$. The change of variable $z_{\boldsymbol{e}} \mapsto \tilde{z}_{\boldsymbol{e}}$ is one to one $\boldsymbol{e} \to \mathbb{R}$ and for any function γ defined on $\boldsymbol{e}$, we set $\tilde{\gamma}(\tilde{z}_{\boldsymbol{e}}) = \gamma(z_{\boldsymbol{e}})$. Then $\mathcal{K}_{\boldsymbol{e}}[\gamma](\rho_{\boldsymbol{e}}, z_{\boldsymbol{e}})$ is the convolution operator with respect to $\tilde{z}_{\boldsymbol{e}}$:

$$\mathcal{K}_{\boldsymbol{e}}[\gamma](\rho_{\boldsymbol{e}}, z_{\boldsymbol{e}}) = \int_{\mathbb{R}} \frac{1}{\rho_{\boldsymbol{e}}}\, \alpha\Big(\frac{t}{\rho_{\boldsymbol{e}}}\Big)\, \tilde{\gamma}(t - \tilde{z}_{\boldsymbol{e}})\, dt \quad \text{with} \quad \rho_{\boldsymbol{e}} = \frac{r_{\boldsymbol{e}}}{d_{\boldsymbol{e}}},$$

where α is a smooth function in $\mathcal{S}(\mathbb{R})$ such that $\int_{\mathbb{R}} \alpha = 1$. Then the block $\mathfrak{K}_{\mathrm{Dir},\boldsymbol{e}\,;\,s}^{\lambda}[\gamma_{\boldsymbol{e}}^{\lambda}]$ has the following structure

$$\begin{aligned} \mathfrak{K}_{\mathrm{Dir},\boldsymbol{e}\,;\,s}^{\lambda}[\gamma_{\boldsymbol{e}}^{\lambda}](x,y,z) \;&=\; \mathcal{K}_{\boldsymbol{e}}[\gamma_{\boldsymbol{e}}^{\lambda}](\rho_{\boldsymbol{e}}, z_{\boldsymbol{e}})\; \Phi_{\mathrm{Dir},\boldsymbol{e}}^{\lambda}(\rho_{\boldsymbol{e}}, \theta_{\boldsymbol{e}}) && \text{(2.17a)} \\ &\quad + \sum_{1\le|\boldsymbol{q}|\le s-\lambda} \mathcal{K}_{\boldsymbol{e}}[\gamma_{\boldsymbol{e}}^{\lambda;\boldsymbol{q}}](\rho_{\boldsymbol{e}}, z_{\boldsymbol{e}})\; \Phi_{\mathrm{Dir},\boldsymbol{e}}^{\lambda;\boldsymbol{q}}(\rho_{\boldsymbol{e}}, \theta_{\boldsymbol{e}}). && \text{(2.17b)} \end{aligned}$$

In the leading term (2.17a), $\Phi_{\mathrm{Dir},\boldsymbol{e}}^{\lambda}$ is the function defined in Lemma 2.1 for $\omega = \omega_{\boldsymbol{e}}$ and in (2.17b) $\boldsymbol{q}$ is a multi-index, $\gamma_{\boldsymbol{e}}^{\lambda;\boldsymbol{q}}$ is a derivative of $\gamma_{\boldsymbol{e}}^{\lambda}$ of order $\le |\boldsymbol{q}|$ and $\Phi_{\mathrm{Dir},\boldsymbol{e}}^{\lambda;\boldsymbol{q}}$ belongs to $S_{\mathrm{Dir}}^{\lambda+|\boldsymbol{q}|}(\Gamma_{\boldsymbol{e}})$.

The theorem of regularity is an obvious particular case of the previous one:

Theorem 2.7 *Let* φ *be such that* $\Delta^{\mathrm{Dir}}\varphi$ *belongs to* $H^{s-1}(\Omega)$. *If for all* $\boldsymbol{c} \in \mathcal{C}$, *the intersection* $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) \cap (-\frac{1}{2}, s-\frac{1}{2})$ *is empty and if for all* $\boldsymbol{e} \in \mathcal{E}$, *the intersection* $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}}) \cap (0, s)$ *is empty, then* $\varphi \in H^{s+1}(\Omega)$.

As a particular case of Theorem 2.6 for $s = 1$ we obtain a characterization of the space K_{Dir}, *cf* Theorem 1.1, namely that K_{Dir} is parametrized by certain discrete coefficients corresponding to corners and by edge coefficients from the weighted Sobolev spaces $\mathbb{V}^m_\eta(\boldsymbol{e})$ associated with nonconvex edges.

Corollary 2.8 *Let $\mathcal{C}_0 = \{c \in \mathcal{C},\ \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) \cap (-\frac{1}{2}, \frac{1}{2}) \neq \emptyset\}$ and $\mathcal{E}_0 = \{e \in \mathcal{E},\ \omega_{\boldsymbol{e}} > \pi\}$. For any $\boldsymbol{c} \in \mathcal{C}_0$ there is only one λ in $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) \cap (-\frac{1}{2}, \frac{1}{2})$ and we denote by $\Phi_{\mathrm{Dir},\boldsymbol{c}}$ the corresponding singularity. For any $\boldsymbol{e} \in \mathcal{E}_0$ there is only one λ in $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}}) \cap (0,1)$ and we denote by $\Phi_{\mathrm{Dir},\boldsymbol{e}} := r_{\boldsymbol{e}}^{\pi/\omega_e} \sin \pi\theta/\omega_{\boldsymbol{e}}$ the corresponding singularity. Then for any complementary space K_{Dir} of $H^2 \cap \mathring{H}^1(\Omega)$ in $D(\Delta^{\mathrm{Dir}})$ the following mapping is one to one:*

$$\begin{array}{ccc} K_{\mathrm{Dir}} & \longrightarrow & \prod_{\boldsymbol{c}\in\mathcal{C}_0} \mathbb{R} \times \prod_{\boldsymbol{e}\in\mathcal{E}_0} \mathbb{V}^{1-\pi/\omega_e}_{-1}(\boldsymbol{e}) \\ \varphi & \longmapsto & (\gamma_{\boldsymbol{c}},\ \ \gamma_{\boldsymbol{e}}), \end{array}$$

where γ_c and $\gamma_{\boldsymbol{e}}$ are the coefficients such that

$$\varphi - \sum_{\boldsymbol{c}\in\mathcal{C}_0} \gamma_{\boldsymbol{c}}\, \chi(\rho_{\boldsymbol{c}})\, \Phi_{\mathrm{Dir},\boldsymbol{c}}(\rho_{\boldsymbol{c}}, \vartheta_{\boldsymbol{c}}) - \sum_{\boldsymbol{e}\in\mathcal{E}_0} \mathcal{K}_{\boldsymbol{e}}[\gamma_{\boldsymbol{e}}](\rho_{\boldsymbol{e}}, z_{\boldsymbol{e}})\ \Phi_{\mathrm{Dir},\mathbf{e}}(\rho_{\boldsymbol{e}}, \theta_{\boldsymbol{e}})\ \in\ H^2(\Omega).$$

Remark 2.9 The only information which is not a straightforward consequence of Theorem 2.6 is the fact that for any $\boldsymbol{c} \in \mathcal{C}_0$ there is a single element λ in $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) \cap (-\frac{1}{2}, \frac{1}{2})$. This is a consequence of the monotonicity of Dirichlet eigenvalues. The first eigenvalue μ_1 of the Laplace-Beltrami operator on the unit sphere is simple and is 0. The second one μ_2 is triple and is equal to 2. Thus the first $\lambda \geq 0$ in $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ is > 0 and the second one is $> -\frac{1}{2} + \sqrt{\mu_2 + \frac{1}{4}} = 1$. ■

3 Singularities of Maxwell operators on polygonal domains

The previous analysis for the Laplace operator can be extended to any strongly elliptic boundary value problem, see [12]. We explain now how this analysis can be adapted to the Maxwell problems on polygonal domains in two dimensions. In this section, we describe the singular functions and the sets of singular exponents, and state our main results. The constructions leading to these results will be presented in section 5.

3.a Bi-dimensional Maxwell equations

Maxwell equations in $\mathbb{R}^2$ are obtained from the three-dimensional ones by the elimination of one coordinate (say z) and of the corresponding component in the fields. The curl has now two (dual) forms, one scalar when applied to 2D fields $\operatorname{curl} \boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1$, and one vectorial when applied to scalar functions $\mathbf{curl}\, w = (\partial_2 w, -\partial_1 w)$. The definitions of the divergence and gradient are obvious: $\operatorname{div} \boldsymbol{v} = \partial_1 v_1 + \partial_2 v_2$, and $\mathbf{grad}\, w = (\partial_1 w, \partial_2 w)$. The space X_N is then the space of $\boldsymbol{v} \in L^2(\Omega)^2$ such that $\operatorname{curl} \boldsymbol{v}$ and $\operatorname{div} \boldsymbol{v}$ are $L^2(\Omega)$, and with tangential boundary conditions.

Thus the Maxwell "electric" problem under consideration is:

$$\boldsymbol{u} \in X_N, \quad \forall \boldsymbol{v} \in X_N, \quad \int_\Omega \operatorname{curl} \boldsymbol{u} \operatorname{curl} \boldsymbol{v} + \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}, \tag{3.1}$$

and the corresponding boundary value problem is for $\boldsymbol{f} \in L^2(\Omega)^2$:

$$\begin{cases} \mathbf{curl} \operatorname{curl} \boldsymbol{u} - \mathbf{grad} \operatorname{div} \boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 & \text{on } \partial\Omega, \\ \boldsymbol{u} \in L^2(\Omega)^2, \quad \operatorname{curl} \boldsymbol{u} \in L^2(\Omega) & \text{and} \quad \operatorname{div} \boldsymbol{u} \in H^1(\Omega). \end{cases} \tag{3.2}$$

Of course we have also the pseudo-Maxwell version of problem (3.1) by replacing the variational space X_N by $H_N = X_N \cap H^1(\Omega)^2$.

We are going to investigate the solutions $\boldsymbol{u}$ when the data $\boldsymbol{f}$ belong to the Sobolev space $H^{s-1}(\Omega)^2$, with $s \geq 1$. Since the boundary value problem (3.2) is an elliptic system, the solution $\boldsymbol{u}$ belongs to $H^{s+1}(\mathcal{V} \cap \Omega)^2$ for any open set $\mathcal{V}$ such that $\overline{\mathcal{V}}$ does not meet any corner of Ω . The singular behavior of $\boldsymbol{u}$ is attached to the corners $\boldsymbol{a}$ of Ω .

3.b Homogeneous function spaces

Like for Δ^{Dir} , we start with the introduction of the corresponding spaces of homogeneous fields $\boldsymbol{S}_N^\lambda$, $\boldsymbol{Y}_N^\lambda$ and $\boldsymbol{Z}_N^\lambda$ on a plane sector Γ .

Let us denote by $\overline{\Gamma}^* = \overline{\Gamma} \setminus \{0\}$ the closure of Γ without its vertex. Then $\mathcal{C}_0^\infty(\overline{\Gamma}^*)$ denotes the space of smooth functions with compact support contained in $\overline{\Gamma}^*$. We note that the space $S_{\mathrm{Dir}}^\lambda(\Gamma)$ introduced in (2.1) can be equivalently defined as

$$S_{\mathrm{Dir}}^\lambda(\Gamma) = \big\{ \Phi \in \overset{\circ}{H}{}^1_{\mathrm{loc}}(\overline{\Gamma}^*) \mid \Phi = r^\lambda \sum_{q=0}^{Q} \log^q r \; \phi_q(\theta) \big\},$$

where $\overset{\circ}{H}{}^1_{\mathrm{loc}}(\overline{\Gamma}^*)$ is the space of Φ such that for all $\chi \in \mathcal{C}_0^\infty(\overline{\Gamma}^*)$, the truncated function $\chi\Phi$ belongs to $\overset{\circ}{H}{}^1(\Gamma)$. We define similarly

$$\boldsymbol{S}_N^\lambda(\Gamma) = \big\{ \boldsymbol{U} \in X_N^{\mathrm{loc}}(\overline{\Gamma}^*) \mid \boldsymbol{U} = r^\lambda \sum_{q=0}^{Q} \log^q r \; \boldsymbol{v}_q(\theta) \big\}, \tag{3.3}$$

where $\boldsymbol{U} \in X_N^{\mathrm{loc}}(\Gamma^*)$ means that for all $\chi \in \mathcal{C}_0^\infty(\overline{\Gamma}^*)$, the truncated field $\chi\boldsymbol{U}$ belongs to X_N . Thus $\boldsymbol{U} \times \boldsymbol{n}$ is zero on $\partial\Gamma$. The space $\boldsymbol{Y}_N^\lambda(\Gamma)$ corresponding to $Y_{\mathrm{Dir}}^\lambda(\Gamma)$ in (2.2) is defined as:

$$\begin{aligned} \boldsymbol{Y}_N^\lambda(\Gamma) = \big\{ \boldsymbol{U} \in \boldsymbol{S}_N^\lambda(\Gamma) \mid \quad & \operatorname{div} \boldsymbol{U} \in S_{\mathrm{Dir}}^{\lambda-1}(\Gamma), \\ & \mathbf{curl} \operatorname{curl} \boldsymbol{U} - \mathbf{grad} \operatorname{div} \boldsymbol{U} \text{ is polynomial} \big\}. \end{aligned} \tag{3.4}$$

Note that the *natural boundary condition* $\operatorname{div} \boldsymbol{u} = 0$ on $\partial\Omega$ is present in the condition $\operatorname{div} \boldsymbol{U} \in S_{\mathrm{Dir}}^{\lambda-1}(\Gamma)$ which also takes into account the *regularity condition* $\operatorname{div} \boldsymbol{u} \in H^1(\Omega)$.

- If λ is a positive integer, $\boldsymbol{Y}_N^\lambda(\Gamma)$ contains the space $\boldsymbol{P}_N^\lambda(\Gamma)$ of homogeneous polynomial fields $\boldsymbol{U}$ of degree λ satisfying the boundary conditions $\boldsymbol{U}\times\boldsymbol{n}=0$ and $\operatorname{div}\boldsymbol{U}=0$ on $\partial\Gamma$. Like in (2.3), let $\boldsymbol{Z}_N^\lambda(\Gamma)$ be a complement of $\boldsymbol{P}_N^\lambda(\Gamma)$ in $\boldsymbol{Y}_N^\lambda(\Gamma)$

$$\boldsymbol{Y}_N^\lambda(\Gamma)=\boldsymbol{Z}_N^\lambda(\Gamma)\oplus\boldsymbol{P}_N^\lambda(\Gamma). \tag{3.5}$$

- If λ is not a positive integer, $\boldsymbol{Z}^\lambda(\Gamma)$ is simply defined as

$$\big\{\boldsymbol{U}\in\boldsymbol{S}_N^\lambda(\Gamma)\mid\ \operatorname{div}\boldsymbol{U}\in S_{\mathrm{Dir}}^{\lambda-1}(\Gamma),\ \ \mathbf{curl}\operatorname{curl}\boldsymbol{U}-\mathbf{grad}\operatorname{div}\boldsymbol{U}=0\big\}. \tag{3.6}$$

We denote by $\Lambda_N(\Gamma)$ the set of $\lambda\in\mathbb{C}$ such that $\boldsymbol{Z}_N^\lambda(\Gamma)$ is not reduced to $\{0\}$. We prove in section 5:

Lemma 3.1 *Let Γ be a plane sector of opening $\omega\neq\pi$. Then the set of electric Maxwell singular exponents is*

$$\Lambda_N(\Gamma)=\big\{\lambda\in\mathbb{R}\setminus\{1\}\ \mid\ \ \lambda+1\ \textit{or}\ \lambda-1\ \textit{belongs to}\ \Lambda_{\mathrm{Dir}}(\Gamma)\big\}.$$

The corresponding spaces $\mathbf{Z}_N^\lambda(\Gamma)$ of singular functions are generated:

(i) If $\lambda+1\in\Lambda_{\mathrm{Dir}}(\Gamma)$ by

$$\boldsymbol{U}_N^{\lambda,+}:=\begin{cases}\big(r^\lambda\sin\lambda\theta,\ r^\lambda\cos\lambda\theta\big) & \textit{if}\ \lambda\notin\mathbb{N}\\ \big(r^\lambda(\log r\sin\lambda\theta+\theta\cos\theta),\ r^\lambda(\log r\cos\lambda\theta-\theta\sin\theta)\big) & \textit{if}\ \lambda\in\mathbb{N},\end{cases}$$

These functions are the gradients of the Dirichlet singular functions of the Laplace operator Δ^{Dir}, thus have zero curls and regular divergences.

(ii) If $\lambda-1\in\Lambda_{\mathrm{Dir}}(\Gamma)$ by

$$\boldsymbol{U}_N^{\lambda,-}:=\begin{cases}\big(r^\lambda\sin\lambda\theta,\ -r^\lambda\cos\lambda\theta) & \textit{if}\ \lambda\notin\mathbb{N}\\ (r^\lambda(\log r\sin\lambda\theta+\theta\cos\theta),\ -r^\lambda(\log r\cos\lambda\theta-\theta\sin\theta)\big) & \textit{if}\ \lambda\in\mathbb{N},\end{cases}$$

The divergences of these functions are the singular functions of Δ^{Dir}.

3.c Regularity and singularities in X_N

With $\boldsymbol{U}_{N,\boldsymbol{a}}^{\lambda,p}$ for $p=\pm$ the generating functions in Lemma 3.1 corresponding to the sector $\Gamma_{\boldsymbol{a}}$, we set for each $\boldsymbol{a}\in\mathcal{A}$

$$\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,p}(x,y)=\chi(r_{\boldsymbol{a}})\,\boldsymbol{U}_{N,\boldsymbol{a}}^{\lambda,p}(r_{\boldsymbol{a}},\theta_{\boldsymbol{a}}).$$

Definition 3.2 Let $s\geq 0$ and $\boldsymbol{a}\in\mathcal{A}$. We call *admissible singular functions* the functions $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,p}\in X_N\setminus H^{s+1}(\Omega)^2$ such that $\operatorname{div}\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,p}\in H^1(\Omega)$. The set of corresponding exponents λ is called the set of *admissible exponents* and denoted by $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{a}})$. ■

Lemma 3.3 *For any $s \geq 0$ and $\boldsymbol{a} \in \mathcal{A}$ there holds*

$$\Lambda_{N;\,s}(\Gamma_{\boldsymbol{a}}) \;=\; \big\{\lambda \in (-1,s] \mid \; \lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}})\big\} \tag{3.7a}$$

$$\bigcup \; \big\{\lambda \in (1,s] \mid \; \lambda-1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}})\big\}. \tag{3.7b}$$

Indeed we deduce from Lemma 3.1 that $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,+}$ belongs to X_N if and only if $\lambda > -1$. The divergence of $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,+}$ is zero near $\boldsymbol{a}$, thus always belongs to $H^1(\Omega)$. However the divergence of $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,-}$ is non-zero and singular near $\boldsymbol{a}$ and belongs to $H^1(\Omega)$ if and only if $\lambda - 1 > 0$. Finally $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,\pm}$ belongs to $H^{s+1}(\Omega)^2$ if and only if $\lambda > s$. ■

Here follows the statement of regularity and singularity for problem (3.1).

Theorem 3.4 *Let $\boldsymbol{u}$ be the solution of problem* (3.1) *with $\boldsymbol{f} \in H^{s-1}(\Omega)^2$.*
(i) *If for all $\boldsymbol{a} \in \mathcal{A}$, the exponent s does not belong to $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{a}})$, then for each λ in $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{a}})$ there exist coefficients $\gamma_{\boldsymbol{a}}^{\lambda,+}$ if $\lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}})$ and $\gamma_{\boldsymbol{a}}^{\lambda,-}$ if $\lambda-1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{a}})$ such that*

$$\boldsymbol{u} \;-\; \sum_{\boldsymbol{a}\in\mathcal{A}} \; \sum_{\lambda\in\Lambda_{N;\,s}(\Gamma_{\boldsymbol{a}})} \gamma_{\boldsymbol{a}}^{\lambda,\pm}\, \boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,\pm} \;\in\; H^{s+1}(\Omega)^2. \tag{3.8}$$

(ii) *If for all $\boldsymbol{a} \in \mathcal{A}$ the set $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{a}})$ is empty, then $\boldsymbol{u} \in H^{s+1}(\Omega)^2$.*

The proof uses exactly the same tools as for Theorem 2.3. But now, the Mellin transform of $\boldsymbol{u}$ (localized) is defined for $\operatorname{Re}\lambda \leq -1$ and meromorphically extended up to $\operatorname{Re}\lambda \leq s$. The regular part is still the inverse Mellin transform on the line $\operatorname{Re}\lambda = s$. The residues belong to $\boldsymbol{Y}_N^{\lambda}(\Gamma)$ and we obtain a singular part which is a priori a linear combination of the $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,p}$ for all $\lambda \in \Lambda_N(\Gamma_{\boldsymbol{a}}) \cap (-1,s)$. But as $\operatorname{div}\boldsymbol{u}$ belongs to $H^1(\Omega)$, only admissible singularities subsist. ■

3.d Different choices of regular and singular parts

Theorem 3.4 shows the existence of a splitting into singular functions and a regular part that is as regular as desired. If the singular functions are not constructed according to our explicit formulas in Lemma 3.1, then additional singular terms can be exchanged between the "singular" and "regular" parts: we describe this phenomenon for the simple but important case of the first singularity in $X_N \setminus H_N$.

Let us consider the expansion (3.8) for $s = 1$: only one singular function for each reentrant corner does not belong to H^1: this is $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,+}$ for $\lambda = \pi/\omega_{\boldsymbol{a}} - 1$. If we put the other terms of the expansion (3.8) into the regular part, we obtain an expansion of the type

$$\boldsymbol{u} \;=\; \boldsymbol{u}_{(H)} \quad + \sum_{\boldsymbol{a}\in\mathcal{A},\;\boldsymbol{a}\ \text{non-convex}} \gamma_{\boldsymbol{a}}\, \boldsymbol{w}_{\boldsymbol{a}}, \tag{3.9}$$

with $\boldsymbol{w}_{\boldsymbol{a}} \in X_N \setminus H_N$ and $\boldsymbol{u}_{(H)} \in H_N$.

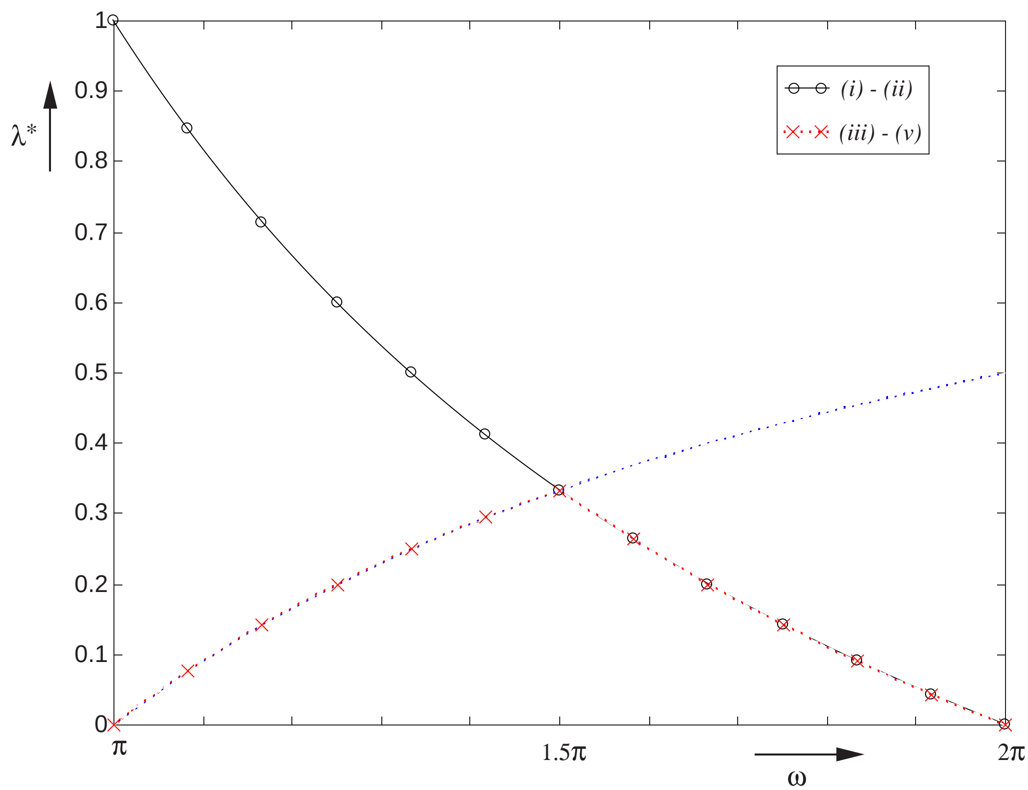
The question of the regularity of this pseudo regular part $\boldsymbol{u}_{(H)}$ is an important problem if one wants to use the splitting (3.9) for a numerical approximation of $\boldsymbol{u}$ by a singular function method, that is, by trial functions that are composed of the singular functions $\boldsymbol{w}_{\boldsymbol{a}}$ and regular functions (e.g. piecewise polynomials), see [20]. The convergence rate of the whole method is then determined by the regularity of $\boldsymbol{u}_{(H)}$.

We compare five constructions for $\boldsymbol{w}_{\boldsymbol{a}}$ and $\boldsymbol{u}_{(H)}$. In each case, $\boldsymbol{u}_{(H)}$ will have a decomposition itself, and its regularity is determined by its first singular function. For simplicity, we assume that there is just one reentrant corner of opening $\omega > \pi$ situated at the origin 0 and χ denotes a smooth cut-off function equal to 1 in a neighborhood of 0 .

In this case, $\boldsymbol{u}_{(H)} \in H^{\sigma+1}$ with $\sigma < \lambda^*$, where λ^* is the exponent of the first singular function in $\boldsymbol{u}_{(H)}$. Thus λ^* will directly yield the convergence rate of a singular function method for the approximation of $\boldsymbol{u}$.

(i) According to Lemma 3.1, the natural choice for $\boldsymbol{w}$ is $\mathbf{grad}(\chi\, r^{\pi/\omega} \sin\frac{\pi\theta}{\omega})$. In this case, the next exponent in $\Lambda_{N;\,s}(\Gamma)$ is $\lambda^* = \frac{2\pi}{\omega} - 1$ if $\omega \neq 2\pi$, and $\lambda^* = \frac{3\pi}{\omega} - 1 = \frac{1}{2}$ if $\omega = 2\pi$. Thus varying $\omega \in [\pi, 2\pi]$, the function λ^* covers the whole interval $(0, 1)$.

(ii) We can choose a divergence free form of $\boldsymbol{w}$: $\mathbf{curl}(\chi\, r^{\pi/\omega} \cos\frac{\pi\theta}{\omega})$. This expansion is, in fact, identical to *(i)*, so λ^* is the same.



(iii) A more abstract construction for $\boldsymbol{w}$ is described in [23] and [6]. Let φ be the solution of the Dirichlet problem

$$\Delta^{\mathrm{Dir}}\varphi = S_{\mathrm{Dir}}\,,$$

where S_{Dir} is the first dual singular function of the Dirichlet problem, i.e. a generator of the one-dimensional space K^*_{Dir}, see Theorem 1.3 and expression (2.8). Then the choice of $\boldsymbol{w} = \mathbf{grad}\,\varphi$ provides a splitting of $\boldsymbol{u}$ where $\boldsymbol{w}$ is orthogonal in X_N to the curl-free fields in H_N. Thus if $\mathbf{curl}\,\boldsymbol{f} = 0$, this is an orthogonal decomposition within the spaces of curl-free fields.

It is well known that S_{Dir} has a singular part $r^{-\pi/\omega}\sin\frac{\pi\theta}{\omega}$, see (2.8). Therefore, besides the main part $c_0\, r^{\pi/\omega}\sin\frac{\pi\theta}{\omega}$ with non-zero c_0, φ contains a singularity of exponent $2-\frac{\pi}{\omega}$. Thus $\boldsymbol{w}$ contains a term of exponent $1-\frac{\pi}{\omega}$. For $\omega \neq \frac{3\pi}{2},\ 2\pi$, this exponent does not belong to $\Lambda_{N;s}$. Thus $\boldsymbol{u}_{(H)}$ must contain this singularity, too, and we have

$$\lambda^* = \min\big\{\tfrac{2\pi}{\omega}-1,\ 1-\tfrac{\pi}{\omega}\big\} \in (0,\tfrac13].$$

This is less regular than the choice *(i)* if $\omega < \frac{3\pi}{2}$.

(iv) A similar construction in [23] and [3] is $\boldsymbol{w} = \mathbf{curl}\,\varphi$ with φ a solution of the Neumann problem

$$\Delta^{\mathrm{Neu}}\varphi = S_{\mathrm{Neu}}\,,$$

where S_{Neu} is the first dual singularity of the Neumann problem. This gives a splitting with $\boldsymbol{w}$ orthogonal in X_N to the divergence-free fields in H_N. Thus if $\operatorname{div}\boldsymbol{f} = 0$, this is an orthogonal decomposition within the spaces of divergence-free fields. This singular function $\boldsymbol{w}$ is linearly independent of the one in *(iii)*. But modulo $H^1(\Omega)$, these two functions are proportional, and their λ^* are also the same.

(v) Another natural construction is the orthogonal decomposition of $\boldsymbol{u}$ with respect to the inner product in X_N, with the part $\boldsymbol{u}_{(H)}$ in H_N and the residual $\gamma_0\boldsymbol{w}$ in $H_N^{\perp}$. Thus $\boldsymbol{u}_{(H)}$ is the solution $\tilde{\boldsymbol{u}}$ of the variational problem in H_N and $\gamma_0\boldsymbol{w}$ is the difference between the solutions of the Maxwell and pseudo-Maxwell problems with the same data, *cf* Theorem 1.3. In this case, $\boldsymbol{u}_{(H)}$ contains a singularity of exponent $1-\frac{\pi}{\omega}$, see (3.13) just below. Thus λ^* is the same as in *(iii)* and *(iv)*. Note that even if $\mathbf{curl}\,\boldsymbol{f} = 0$ or $\operatorname{div}\boldsymbol{f} = 0$, this decomposition is in general, *not the same* as the one in *(iii)* or *(iv)*, respectively.

3.e Singularities in H_N. Comparison

If instead of (3.1), we consider the variational pseudo-Maxwell problem in H_N, the corresponding boundary value problem is the same as (3.2) except the regularity requirements: we have now $\boldsymbol{u} \in H^1(\Omega)^2$ and no special regularity on $\operatorname{div}\boldsymbol{u}$, which is only $L^2(\Omega)$. The associated spaces of homogeneous functions are

$$\widetilde{\boldsymbol{S}}{}^{\lambda}_N(\Gamma) = \big\{\boldsymbol{U} \in H^{\mathrm{loc}}_N(\overline{\Gamma}^*)\ \big|\ \boldsymbol{U}(x) = r^{\lambda}\sum_{q=0}^{Q}\log^q r\ \boldsymbol{v}_q(\theta)\big\}, \tag{3.10}$$

and

$$\widetilde{\boldsymbol{Y}}_N^\lambda(\Gamma) = \big\{\boldsymbol{U} \in \widetilde{\boldsymbol{S}}_N^\lambda(\Gamma) \mid \quad \begin{array}{l} \operatorname{div} \boldsymbol{U} = 0 \quad \text{on} \quad \partial\Gamma, \\ \mathbf{curl}\operatorname{curl} \boldsymbol{U} - \mathbf{grad}\operatorname{div} \boldsymbol{U} \ \text{ is polynomial}\big\}. \end{array} \tag{3.11}$$

Then $\widetilde{\boldsymbol{Z}}_N^\lambda(\Gamma)$ and $\widetilde{\Lambda}_N(\Gamma)$ are defined similarly as their counterpart for X_N. As we are in dimension 2 with no other singularity than 0 in Γ, there holds

$$\widetilde{\Lambda}_N(\Gamma) = \Lambda_N(\Gamma). \tag{3.12}$$

But the set of *admissible exponents* is different. This is the set $\widetilde{\Lambda}_{N;\,s}(\Gamma_{\boldsymbol{a}})$ of the $\lambda \in \Lambda_N(\Gamma)$ such that $\boldsymbol{u}_{N,\boldsymbol{a}}^{\lambda,p} \in H_N \setminus H^{s+1}(\Omega)^2$. Then instead of (3.7), there holds

$$\widetilde{\Lambda}_{N;\,s}(\Gamma) = \Lambda_N(\Gamma) \cap (0, s]. \tag{3.13}$$

Thus we see that both sets $\Lambda_{N;\,s}(\Gamma)$ and $\widetilde{\Lambda}_{N;\,s}(\Gamma)$ are different if and only if $\Lambda_{\mathrm{Dir}}(\Gamma)$ has elements in the interval $[-1, +1]$, i.e. if $\omega > \pi$ (non-convex corner). Then

$$\Lambda_{N;\,s} = \big\{\tfrac{\pi}{\omega} - 1\big\} \bigcup \, (\Lambda_{N;\,s} \cap \widetilde{\Lambda}_{N;\,s}) \quad \text{and} \quad \widetilde{\Lambda}_{N;\,s} = \big\{1 - \tfrac{\pi}{\omega}\big\} \bigcup \, (\Lambda_{N;\,s} \cap \widetilde{\Lambda}_{N;\,s}).$$

4 Singularities of Maxwell operators on polyhedral domains

We continue the investigation of the Maxwell and pseudo-Maxwell problems, now on polyhedral domains in $\mathbb{R}^3$. Like for polygonal domains, we describe how the general theory of corner and edge singularities applies to the Maxwell problems and we present our main results. The detailed constructions follow in section 6.

If we assume that the right-hand side $\boldsymbol{f}$ belongs to $H^{s-1}(\Omega)^3$, with $s \geq 1$, the solutions of problems (1.1), (1.2), (1.3) or (1.4) are regular in any neighborhood which does not meet any corner or edge of Ω. The singular behavior of the solutions is attached to corners and edges.

4.a Corner singularities

Let us recall that $\mathcal{C}$ is the set of the corners of Ω and that in a neighborhood of each $\boldsymbol{c} \in \mathcal{C}$, Ω coincides locally with a polyhedral cone $\Gamma_{\boldsymbol{c}}$, to which correspond spherical coordinates $(\rho_{\boldsymbol{c}}, \vartheta_{\boldsymbol{c}})$.

As in dimension 2, we have to introduce the spaces of homogeneous fields $\boldsymbol{S}_N^\lambda$, $\boldsymbol{Y}_N^\lambda$ and $\boldsymbol{Z}_N^\lambda$ corresponding to the "electric" boundary conditions, on the cones $\Gamma_{\boldsymbol{c}}$ for any corner $\boldsymbol{c} \in \mathcal{C}$. The procedure for the "magnetic" conditions is strictly similar.

Let us fix a corner $\boldsymbol{c}$ and drop the subscript $\boldsymbol{c}$. The space $\boldsymbol{S}_N^\lambda(\Gamma)$ is defined as

$$\boldsymbol{S}_N^\lambda(\Gamma) = \big\{\boldsymbol{U} \in X_N^{\mathrm{loc}}(\overline{\Gamma}^*) \mid \ \boldsymbol{U} = \rho^\lambda \sum_{q=0}^{Q} \log^q \rho \, \boldsymbol{u}_q(\vartheta)\big\}. \tag{4.1}$$

We now define $\boldsymbol{Y}_N^\lambda(\Gamma)$ as the subspace of $\boldsymbol{S}_N^\lambda(\Gamma)$:

$$\begin{aligned}\boldsymbol{Y}_N^\lambda(\Gamma) = \big\{\boldsymbol{U} \in \boldsymbol{S}_N^\lambda(\Gamma) \mid \quad & \operatorname{div} \boldsymbol{U} \in S_{\mathrm{Dir}}^{\lambda-1}(\Gamma), \\ & \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U} - \mathbf{grad}\operatorname{div}\boldsymbol{U} \ \text{ is polynomial}\big\}.\end{aligned} \tag{4.2}$$

Then the singularity space $\boldsymbol{Z}_N^\lambda(\Gamma)$ is defined as in dimension 2, *cf* (3.5)-(3.6), and the set of electric Maxwell exponents $\Lambda_N(\Gamma)$ is the set of $\lambda \in \mathbb{C}$ such that $\boldsymbol{Z}_N^\lambda(\Gamma) \neq \{0\}$. We prove in section 6:

Lemma 4.1 *Let Γ be a polyhedral cone in $\mathbb{R}^3$ with vertex in 0 and characterized in spherical coordinates by $\vartheta \in G \subset \mathbb{S}^2$. Let $\boldsymbol{x}$ denote the vector of coordinates (x, y, z).*

a) The set of non-integer electric Maxwell singular exponents is given by

$$\Lambda_N(\Gamma) \setminus \mathbb{Z} = \big\{\lambda \in \mathbb{R} \setminus \mathbb{Z} \ \mid \quad \lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma) \ \text{ or } \ \lambda \in \Lambda_{\mathrm{Neu}}(\Gamma) \ \text{ or } \ \lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)\big\}.$$

The spaces $\boldsymbol{Z}_N^\lambda(\Gamma)$ have correspondingly the three types of generators:

Type 1. *If* $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$: $\boldsymbol{U}_N^{\lambda,1} = \mathbf{grad}\, \Phi_{\mathrm{Dir}}^{\lambda+1}$,

Type 2. *If* $\lambda \in \Lambda_{\mathrm{Neu}}(\Gamma)$: $\boldsymbol{U}_N^{\lambda,2} = \mathbf{grad}\, \Phi_{\mathrm{Neu}}^{\lambda} \times \boldsymbol{x}$,

Type 3. *If* $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$: $\boldsymbol{U}_N^{\lambda,3} = (2\lambda - 1)\Phi_{\mathrm{Dir}}^{\lambda-1}\boldsymbol{x} - \rho^2\, \mathbf{grad}\, \Phi_{\mathrm{Dir}}^{\lambda-1}$,

with $\Phi_{\mathrm{Dir}}^{\mu} \in Z_{\mathrm{Dir}}^{\mu}(\Gamma)$, cf *Lemma 2.4, and its Neumann analogue $\Phi_{\mathrm{Neu}}^{\lambda} \in Z_{\mathrm{Neu}}^{\lambda}(\Gamma)$.*

b) If G is simply connected, the values $\lambda = -1$ or 0 do not belong to $\Lambda_N(\Gamma)$.

c) If G is not simply connected, the values $\lambda = -1$ and 0 belong to $\Lambda_N(\Gamma)$ and the corresponding $\boldsymbol{U}_N^{\lambda,p}$ have zero curl and divergence.

Here are now the analogues of Definition 3.2 and Lemma 3.3. With a basis $\boldsymbol{U}_{N,\boldsymbol{c}}^{\lambda,p}$ of $\Lambda_N(\Gamma_{\boldsymbol{c}})$, we set for each $\boldsymbol{c} \in \mathcal{C}$

$$\boldsymbol{u}_{N,\boldsymbol{c}}^{\lambda,p}(x, y, z) = \chi(\rho_{\boldsymbol{c}})\, \boldsymbol{U}_{N,\boldsymbol{c}}^{\lambda,p}(\rho_{\boldsymbol{c}}, \vartheta_{\boldsymbol{c}}).$$

Definition 4.2 Let $s \geq 0$ and $\boldsymbol{c} \in \mathcal{C}$. We call *admissible singular functions* the functions $\boldsymbol{u}_{N,\boldsymbol{c}}^{\lambda,p} \in X_N \setminus H^{s+1}\big(\mathbb{R}^+, \rho_{\boldsymbol{c}}^2\, d\rho_{\boldsymbol{c}}; L^2(G_{\boldsymbol{c}})\big)^3$ such that $\operatorname{div} \boldsymbol{u}_{N,\boldsymbol{c}}^{\lambda,p} \in H^1(\Omega)$. The set of corresponding exponents λ is called the set of *admissible exponents* and denoted by $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})$. ∎

Lemma 4.3 *For any* $s \geq 0$ *and* $\boldsymbol{c} \in \mathcal{C}$ *there holds*

$$\Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}}) \;=\; \big\{\lambda \in (-\tfrac{3}{2}, s-\tfrac{1}{2}] \mid\; \lambda+1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})\big\} \tag{4.3a}$$

$$\bigcup\; \big\{\lambda \in (-\tfrac{1}{2}, s-\tfrac{1}{2}] \mid\; \lambda \in \Lambda_{\mathrm{Neu}}(\Gamma_{\boldsymbol{c}})\big\} \tag{4.3b}$$

$$\bigcup\; \big\{\lambda \in (\tfrac{1}{2}, s-\tfrac{1}{2}] \mid\; \lambda-1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})\big\} \tag{4.3c}$$

$$\bigcup\; \big\{-1, 0\big\} \cap (-\tfrac{3}{2}, s-\tfrac{1}{2}] \;\; \textit{if } G_c \textit{ is not simply connected.} \tag{4.3d}$$

Proof. Let us fix $\boldsymbol{c} \in \mathcal{C}$ and drop the subscript $\boldsymbol{c}$. If U is a non-zero homogeneous function of degree λ of the form $\rho^\lambda u(\vartheta)$ with $u \in L^2(G)$, there holds

$$\chi(\rho)\, U \in L^2(\Gamma) \iff \lambda > -\tfrac{3}{2} \tag{4.4a}$$

$$\chi(\rho)\, U \notin H^{s+1}\big(\mathbb{R}^+, \rho^2\, d\rho; L^2(G)\big) \iff \lambda \leq s - \tfrac{1}{2}. \tag{4.4b}$$

The fields $\boldsymbol{U}_N^{\lambda,1}$ have zero curl and divergence, like the $\boldsymbol{U}_N^{\lambda,p}$ for $\lambda = -1, 0$. Thus the corresponding $\chi(\rho)\, \boldsymbol{U}_N^{\lambda,p}$ belong to X_N if and only if they are in $L^2(\Gamma)$, whence (4.3a) and (4.3d).

The fields $\boldsymbol{U}_N^{\lambda,2}$ have zero divergence but non-zero curl, *cf* formulas (6.6). As their curl has the homogeneity $\lambda - 1$, $\chi(\rho)\, \boldsymbol{U}_N^{\lambda,2}$ belongs to X_N if and only if $\lambda - 1 > -\frac{3}{2}$, whence (4.3b).

The fields $\boldsymbol{U}_N^{\lambda,3}$ have non-zero curl and $\operatorname{div} \boldsymbol{U}_N^{\lambda,3} = (2\lambda^2 + \lambda)\Phi_{\mathrm{Dir}}^{\lambda-1}$. Thus $\operatorname{div}(\chi \boldsymbol{U}_N^{\lambda,3})$ belongs to $H^1(\Omega)$ if and only if $\lambda - 1 > -\frac{1}{2}$, whence (4.3c). ■

Similarly to the expansion (2.10) for solutions of Δ^{Dir}, we can prove by Mellin transform that the solution $\boldsymbol{u}$ of problem (1.1) with $\boldsymbol{f} \in H^{s-1}(\Omega)^3$ can be expanded in a neighborhood of each corner $\boldsymbol{c} \in \mathcal{C}$ so that there holds

$$\chi(\rho_{\boldsymbol{c}})\, \boldsymbol{u} - \sum_{\lambda \in \Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})} \sum_{p} \gamma_{\boldsymbol{c}}^{\lambda,p}\, \boldsymbol{u}_{N,\boldsymbol{c}}^{\lambda,p} \;\in\; H^{s+1}\big(\mathbb{R}^+, \rho_{\boldsymbol{c}}^2\, d\rho_{\boldsymbol{c}}; L^2(G_{\boldsymbol{c}})\big). \tag{4.5}$$

4.b Edge singularities

Let us recall that $\mathcal{E}$ is the set of the edges of Ω and that in a neighborhood of each $\boldsymbol{e} \in \mathcal{E}$, Ω coincides locally with a wedge $W_{\boldsymbol{e}} = \Gamma_{\boldsymbol{e}} \times \mathbb{R}$, to which correspond cylindrical coordinates $(r_{\boldsymbol{e}}, \theta_{\boldsymbol{e}}, z_{\boldsymbol{e}})$. Let us fix $\boldsymbol{e} \in \mathcal{E}$ and drop the subscript $\boldsymbol{e}$.

We have seen, see (2.13), that the edge singularities for Δ^{Dir} have a sort of tensor product structure whose polar parts (i.e. in (r, θ)) are the singularities at the corner of the sector Γ of the problems obtained by partial Fourier transform with respect to the tangential variable z. Moreover, the leading part of these singularities does not depend on the dual variable ξ of z. In other words, the leading singularities can be determined by the consideration of functions defined on the wedge $W = \Gamma \times \mathbb{R}$, but not depending on the variable z. This feature is common to any strongly elliptic problem. That is the reason for the introduction of the spaces

$$\boldsymbol{S}_N^\lambda(W) = \big\{ \boldsymbol{U} \in X_N^{\mathrm{loc}}(\overline{\Gamma}^* \times \mathbb{R}) \mid \ \boldsymbol{U} = r^\lambda \sum_{q=0}^{Q} \log^q r \, \boldsymbol{u}_q(\theta) \big\} \tag{4.6}$$

and

$$\begin{aligned} \boldsymbol{Y}_N^\lambda(W) = \big\{ \boldsymbol{U} \in \boldsymbol{S}_N^\lambda(W) \mid \quad & \operatorname{div} \boldsymbol{U} \in S_{\mathrm{Dir}}^{\lambda-1}(W), \\ & \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U} - \mathbf{grad}\operatorname{div}\boldsymbol{U} \ \text{ is polynomial} \big\}. \end{aligned} \tag{4.7}$$

Then the space of singular functions $\boldsymbol{Z}_N^\lambda(W)$ is defined along the same lines as for corners:

- If λ is a positive integer, $\boldsymbol{Y}_N^\lambda(W)$ contains the space $\boldsymbol{P}_N^\lambda(W)$ of homogeneous polynomial fields $\boldsymbol{U}$ of degree λ independent of z satisfying the boundary conditions $\boldsymbol{U} \times \boldsymbol{n} = 0$ and $\operatorname{div} \boldsymbol{U} = 0$ on ∂W. Let $\boldsymbol{Z}_N^\lambda(W)$ be such that
$$\boldsymbol{Y}_N^\lambda(W) = \boldsymbol{Z}_N^\lambda(W) \oplus \boldsymbol{P}_N^\lambda(W). \tag{4.8}$$
- If λ is not a positive integer, $\boldsymbol{Z}^\lambda(W)$ is simply defined as
$$\big\{ \boldsymbol{U} \in \boldsymbol{S}_N^\lambda(W) \mid \ \operatorname{div} \boldsymbol{U} \in S_{\mathrm{Dir}}^{\lambda-1}(W), \ \ \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U} - \mathbf{grad}\operatorname{div}\boldsymbol{U} = 0 \big\}. \tag{4.9}$$

We denote by $\Lambda_N(W)$ the set of $\lambda \in \mathbb{C}$ such that $\boldsymbol{Z}_N^\lambda(W)$ is not reduced to $\{0\}$. We prove in section 5:

Lemma 4.4 *The set of the exponents $\Lambda_N(W)$ attached to the wedge $W = \Gamma \times \mathbb{R}$ is*

$$\Lambda_N(W) = \{\lambda \in \mathbb{R} \setminus \{1\} \ \mid \ \ \lambda - 1, \ \lambda \ \textit{or}\ \lambda + 1 \ \textit{belongs to} \ \ \Lambda^{\mathrm{Dir}}(\Gamma)\}.$$

The spaces $\boldsymbol{Z}_N^\lambda(\Gamma)$ have correspondingly the three types of generators:

Type 1. *If* $\lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$: $\quad \boldsymbol{U}_N^{\lambda,1} = \big(\boldsymbol{U}_N^{\lambda,+}, \ 0\big) = \big(\mathbf{grad}\, \Phi_{\mathrm{Dir}}^{\lambda+1}, \ 0\big),$

Type 2. *If* $\lambda \in \Lambda_{\mathrm{Dir}}(\Gamma)$: $\quad \boldsymbol{U}_N^{\lambda,2} = \big(\mathbf{0}, \ \Phi_{\mathrm{Dir}}^\lambda\big),$

Type 3. *If* $\lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma)$: $\quad \boldsymbol{U}_N^{\lambda,3} = \big(\boldsymbol{U}_N^{\lambda,-}, \ 0\big),$

where $\Phi_{\mathrm{Dir}}^\mu \in Z_{\mathrm{Dir}}^\mu(\Gamma)$ are the Laplace Dirichlet plane singularities, cf *Lemma* 2.4 *and $\boldsymbol{U}_N^{\lambda,\pm}$ the electric Maxwell plane singularities,* cf *Lemma* 3.1.

Here are now the analogues of Definition 3.2 and Lemma 3.3.

Definition 4.5 Let $s \geq 0$ and $\boldsymbol{e} \in \mathcal{E}$. Let $\beta_{\boldsymbol{e}}$ be a smooth cut-off function with support away from the corners and the other edges of Ω and $\beta_{\boldsymbol{e}} \equiv 1$ in a neighborhood of a point in $\boldsymbol{e}$. We call *admissible singular functions* the functions $\boldsymbol{U}_{N,\boldsymbol{e}}^{\lambda,p}$ such that

$$\beta_{\boldsymbol{e}} \boldsymbol{U}_{N,\boldsymbol{e}}^{\lambda,p} \in X_N \setminus H^{s+1}(\Omega)^3 \quad \text{and} \quad \operatorname{div}\big(\beta_{\boldsymbol{e}} \boldsymbol{U}_{N,\boldsymbol{e}}^{\lambda,p}\big) \in H^1(\Omega).$$

The set of corresponding exponents λ is called the set of *admissible exponents* and denoted by $\Lambda_{N;\,s}(W_{\boldsymbol{e}})$. ■

Lemma 4.6 *For any* $s \geq 0$ *and* $\boldsymbol{e} \in \mathcal{E}$ *there holds*

$$\Lambda_{N;\,s}(W_{\boldsymbol{e}}) \;=\; \big\{\lambda \in (-1, s] \mid \lambda + 1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}})\big\} \tag{4.10a}$$
$$\bigcup \big\{\lambda \in (0, s] \mid \lambda \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}})\big\} \tag{4.10b}$$
$$\bigcup \big\{\lambda \in (1, s] \mid \lambda - 1 \in \Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}})\big\}. \tag{4.10c}$$

PROOF. Let us fix $\boldsymbol{e} \in \mathcal{E}$ and drop the subscript $\boldsymbol{e}$. If U is a non-zero homogeneous function of degree λ of the form $r^{\lambda}\mathrm{U}(\theta)$ with $\mathrm{U} \in \mathcal{C}^{\infty}([0,\omega])$, there holds

$$\beta(r,z)\,U(r,\theta) \in L^2(W) \iff \lambda > -1 \tag{4.11a}$$
$$\beta(r,z)\,U(r,\theta) \notin H^{s+1}(W) \iff \lambda \leq s. \tag{4.11b}$$

The fields $\boldsymbol{U}_N^{\lambda,1}$ have zero curl and divergence. Thus $\beta\boldsymbol{U}_N^{\lambda,1}$ belongs to X_N if and only if it is in $L^2(\Gamma)$, whence (4.10a).

The fields $\boldsymbol{U}_N^{\lambda,2}$ have zero divergence but non-zero curl. As their curl has the homogeneity $\lambda - 1$, $\beta\boldsymbol{U}_N^{\lambda,2}$ belongs to X_N if and only if $\lambda - 1 > -1$, whence (4.10b).

The fields $\boldsymbol{U}_N^{\lambda,3}$ have non-zero curl and $\operatorname{div}\boldsymbol{U}_N^{\lambda,3} = (2\lambda)\Phi_{\mathrm{Dir}}^{\lambda-1}$. Then $\operatorname{div}(\beta\boldsymbol{U}_N^{\lambda,3})$ belongs to $H^1(\Omega)$ if and only if $\lambda - 1 > -1$, whence (4.10b). ■

4.c Regularity and combined corner and edge singularities

We are now ready to give the main statement of splitting in regular and singular parts.

Theorem 4.7 *Let* $\boldsymbol{u}$ *be the solution of problem* (1.1) *with data* $\boldsymbol{f} \in H^{s-1}(\Omega)^3$ *with* $s \geq 1$. *We assume that:*

- *For all* $\boldsymbol{c} \in \mathcal{C}$, $s - \frac{1}{2}$ *does not belong to* $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})$,
- *For all* $\boldsymbol{e} \in \mathcal{E}$, s *does not belong to* $\Lambda_{N;\,s}(W_{\boldsymbol{e}})$,

Then there exist:

- *Coefficients* $\gamma_{\boldsymbol{c}}^{\lambda,p}$ *for each* $\lambda \in \Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})$ *and* p,
- *Functions* $\gamma_{\boldsymbol{e}}^{\lambda,p} \in \mathbb{V}_{-s}^{s-\lambda}(\boldsymbol{e})$ *for each* $\lambda \in \Lambda_{N;\,s}(W_{\boldsymbol{e}})$ *and* p

such that

$$\boldsymbol{u} - \sum_{\boldsymbol{c}\in\mathcal{C}} \sum_{\lambda\in\Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})} \sum_{p} \gamma_{\boldsymbol{c}}^{\lambda,p}\,\boldsymbol{u}_{\boldsymbol{c}}^{\lambda,p} - \sum_{\boldsymbol{e}\in\mathcal{E}} \sum_{\lambda\in\Lambda_{N;\,s}(W_{\boldsymbol{e}})} \sum_{p} \mathfrak{K}_{N,\boldsymbol{e};\,s}^{\lambda,p}[\gamma_{\boldsymbol{e}}^{\lambda,p}] \;\in\; H^{s+1}(\Omega). \tag{4.12}$$

In (4.12) *the block of singularities* $\mathfrak{K}^{\lambda,p}_{N,\boldsymbol{e}\,;\,s}[\gamma^{\lambda,p}_{\boldsymbol{e}}]$ *has a structure similar to the blocks appearing for the Laplace operator* Δ^{Dir} *in* (2.17)*, namely*

$$\mathfrak{K}^{\lambda,p}_{N,\boldsymbol{e}\,;\,s}[\gamma^{\lambda,p}_{\boldsymbol{e}}](x,y,z) = \mathcal{K}_{\boldsymbol{e}}[\gamma^{\lambda,p}_{\boldsymbol{e}}](\rho_{\boldsymbol{e}},z_{\boldsymbol{e}})\,\boldsymbol{U}^{\lambda,p}_{N,\boldsymbol{e}}(\rho_{\boldsymbol{e}},\theta_{\boldsymbol{e}}) \tag{4.13a}$$

$$+ \sum_{1\,\le\,|\boldsymbol{q}|\,\le\,s-\lambda} \mathcal{K}_{\boldsymbol{e}}[\gamma^{\lambda,p;\boldsymbol{q}}_{\boldsymbol{e}}](\rho_{\boldsymbol{e}},z_{\boldsymbol{e}})\,\boldsymbol{U}^{\lambda,p;\boldsymbol{q}}_{N,\boldsymbol{e}}(\rho_{\boldsymbol{e}},\theta_{\boldsymbol{e}}). \tag{4.13b}$$

In the leading term (4.13a)*,* $\boldsymbol{U}^{\lambda,p}_{N,\boldsymbol{e}}$ *is the function defined in Lemma* 4.4 *and in* (4.13b) $\boldsymbol{q}$ *is a multi-index,* $\gamma^{\lambda;\boldsymbol{q}}_{\boldsymbol{e}}$ *is a derivative of* $\gamma^{\lambda}_{\boldsymbol{e}}$ *of order* $\le |\boldsymbol{q}|$ *and* $\boldsymbol{U}^{\lambda,p;\boldsymbol{q}}_{N,\boldsymbol{e}}$ *belongs to* $S^{\lambda+|\boldsymbol{q}|}_N(W_{\boldsymbol{e}})$.

As a consequence, the corresponding regularity statement is

Theorem 4.8 *Let* $s \ge 1$. *If*
- *for all* $\boldsymbol{c}\in\mathcal{C}$, $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})$ *is empty,*
- *for all* $\boldsymbol{e}\in\mathcal{E}$, $\Lambda_{N;\,s}(W_{\boldsymbol{e}})$ *is empty,*

then for any data $\boldsymbol{f}\in H^{s-1}(\Omega)^3$ *the solution* $\boldsymbol{u}$ *of problem* (1.1) *belongs to* $H^{s+1}(\Omega)^3$.

Remark 4.9
(i) Let $\sigma\in(-1,s]$ such that for all $\boldsymbol{e}\in\mathcal{E}$ and all $\lambda\in\Lambda_{N;\,\sigma}(W_{\boldsymbol{e}})$ there holds $\sigma-\lambda<1$. Then we have the expansion of $\boldsymbol{u}$

$$\boldsymbol{u} - \sum_{\boldsymbol{c}\in\mathcal{C}}\ \sum_{\lambda\in\Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})}\ \sum_{p}\gamma^{\lambda,p}_{\boldsymbol{c}}\,\boldsymbol{u}^{\lambda,p}_{\boldsymbol{c}} - \sum_{\boldsymbol{e}\in\mathcal{E}}\ \sum_{\lambda\in\Lambda_{N;\,\sigma}(W_{\boldsymbol{e}})}\ \sum_{p}\mathcal{K}_{\boldsymbol{e}}[\gamma^{\lambda,p}_{\boldsymbol{e}}]\,\boldsymbol{U}^{\lambda,p}_{N,\boldsymbol{e}} \in H^{\sigma+1}(\Omega),$$

where the edge contributions are limited to the leading terms, without the complicated "shadow" terms (4.13b).

(ii) If moreover the sum over the corner singularities is restricted to $\lambda\in\Lambda_{N;\,\sigma}(\Gamma_{\boldsymbol{c}})$, then the coefficients $\gamma^{\lambda,p}_{\boldsymbol{e}}$ along the edges are not the same as in (4.12), and only belong to $\mathbb{V}^{s-\lambda}_{-\sigma}(\boldsymbol{e})$.

(iii) If, with $s\ge1$ and $\sigma\in(-1,s)$ there holds
- for all $\boldsymbol{c}\in\mathcal{C}$, $\Lambda_{N;\,\sigma}(\Gamma_{\boldsymbol{c}})$ is empty,
- for all $\boldsymbol{e}\in\mathcal{E}$, $\Lambda_{N;\,\sigma}(W_{\boldsymbol{e}})$ is empty,

then for any $\boldsymbol{f}\in H^{s-1}(\Omega)^3$ the solution $\boldsymbol{u}$ of problem (1.1) belongs to $H^{\sigma+1}(\Omega)^3$. ■

4.d Regularity of solutions

Taking advantage of the knowledge of the sets $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{c}})$ and $\Lambda_{N;\,\sigma}(W_{\boldsymbol{e}})$, *cf* Lemmas 4.1 and 4.4, we can give more explicit regularity statements than Theorem 4.8. Relying on this theorem, we only have to determine the minimal elements of the admissible sets of exponents for each corner and each edge.

(i) SCREENS AND OTHER NON-LIPSCHITZ DOMAINS. If Ω has screen parts, then it has edges $\boldsymbol{e}$ where $\omega_{\boldsymbol{e}} = 2\pi$. Then the least value in $\Lambda_{N;\infty}(W_{\boldsymbol{e}})$ is $-\frac{1}{2}$.

As $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ and $\Lambda_{\mathrm{Neu}}(\Gamma_{\boldsymbol{c}})$ have no element in $[-1,0]$, *cf* Lemma 2.4, the least value in $\Lambda_{N;\infty}(\Gamma_{\boldsymbol{c}})$ is -1 if Ω is not locally simply connected.

In both situations, the solution $\boldsymbol{u}$ of (1.1) satisfies

$$\boldsymbol{u} \in H^{\tau}(\Omega)^3, \quad \forall \tau < \tfrac{1}{2}, \qquad \text{and} \qquad \text{generically } \boldsymbol{u} \notin H^{\frac{1}{2}}(\Omega)^3 .$$

(ii) LOCALLY SIMPLY CONNECTED DOMAINS WITHOUT SCREENS. The least value in all the sets $\Lambda_{N;\infty}(W_{\boldsymbol{e}})$ for $\boldsymbol{e} \in \mathcal{E}$ is given by $\lambda_{\mathcal{E}} - 1$ where

$$\lambda_{\mathcal{E}} = \min_{\boldsymbol{e}\in\mathcal{E}} \left\{ \begin{array}{ll} \frac{\pi}{\omega_e} & \text{if } \omega_{\boldsymbol{e}} \neq \frac{\pi}{2} \\ 3 & \text{if } \omega_{\boldsymbol{e}} = \frac{\pi}{2} \end{array} \right\}. \tag{4.14}$$

The least value in all the sets $\Lambda_{N;\infty}(\Gamma_{\boldsymbol{c}})$ for $\boldsymbol{c} \in \mathcal{C}$ is given by $\min\{\lambda_{\mathcal{C}}^{\mathrm{Dir}} - 1, \lambda_{\mathcal{C}}^{\mathrm{Neu}}\}$ with

$$\lambda_{\mathcal{C}}^{\mathrm{Dir}} = \min_{\boldsymbol{c}\in\mathcal{C}} \ \lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) \quad \text{and} \quad \lambda_{\mathcal{C}}^{\mathrm{Neu}} = \min_{\boldsymbol{c}\in\mathcal{C}} \ \lambda_{\mathrm{Neu}}(\Gamma_{\boldsymbol{c}}), \tag{4.15}$$

where $\lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ and $\lambda_{\mathrm{Neu}}(\Gamma_{\boldsymbol{c}})$ are the least positive elements of $\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})$ and $\Lambda_{\mathrm{Neu}}(\Gamma_{\boldsymbol{c}})$: with $\mu_{\mathrm{Dir}}(G_{\boldsymbol{c}})$ and $\mu_{\mathrm{Neu}}(G_{\boldsymbol{c}})$ the least non-zero eigenvalue of the Laplace-Beltrami operators $\Delta_{G_{\boldsymbol{c}}}^{\mathrm{Dir}}$ and $\Delta_{G_{\boldsymbol{c}}}^{\mathrm{Neu}}$, there holds

$$\lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}}) = -\tfrac{1}{2} + \sqrt{\mu_{\mathrm{Dir}}(G_{\boldsymbol{c}}) + \tfrac{1}{4}} \quad \text{and} \quad \lambda_{\mathrm{Neu}}(\Gamma_{\boldsymbol{c}}) = -\tfrac{1}{2} + \sqrt{\mu_{\mathrm{Neu}}(G_{\boldsymbol{c}}) + \tfrac{1}{4}} .$$

The solution $\boldsymbol{u}$ of (1.1) satisfies

$$\boldsymbol{u} \in H^{1+\sigma}(\Omega)^3, \quad \forall \sigma, \ \big(\sigma \le s \text{ and } \sigma < \min\big\{\lambda_{\mathcal{E}} - 1, \ \lambda_{\mathcal{C}}^{\mathrm{Dir}} - \tfrac{1}{2}, \ \lambda_{\mathcal{C}}^{\mathrm{Neu}} + \tfrac{1}{2}\big\}\big). \tag{4.16}$$

Here are a few particular interesting situations

(a) If Ω is not convex, then $\boldsymbol{u} \in H^{\tau}(\Omega)^3$ for all $\tau < \min\{\lambda_{\mathcal{E}}, \ \lambda_{\mathcal{C}}^{\mathrm{Dir}} + \frac{1}{2}\}$. Thus $\boldsymbol{u}$ belongs to $H^{\frac{1}{2}}(\Omega)^3$ but not to $H^1(\Omega)^3$.

(b) If Ω is convex, the monotonicity of Dirichlet eigenvalues allows then to prove that $\lambda_{\mathcal{E}} \le \lambda_{\mathcal{C}}^{\mathrm{Dir}}$, thus $\boldsymbol{u} \in H^{1+\sigma}(\Omega)^3$ for all σ, $\sigma \le s$ and $\sigma < \min\{\lambda_{\mathcal{E}} - 1, \ \lambda_{\mathcal{C}}^{\mathrm{Neu}} + \frac{1}{2}\}$.

(c) If Ω is a parallelepiped, $\boldsymbol{u} \in H^{1+\sigma}(\Omega)^3$ for all σ, $\sigma \le s$ and $\sigma < 2$.

4.e Singularities of solutions

In the spirit of Remark 4.9, we are going to give simplified expressions for the singular part of $\boldsymbol{u}$ for σ small enough. We assume now that Ω is locally simply connected.

If we take, compare with (4.16),

$$\sigma < \min\big\{\lambda_{\mathcal{E}}, \ \lambda_{\mathcal{C}}^{\mathrm{Dir}} + \tfrac{1}{2}, \ \lambda_{\mathcal{C}}^{\mathrm{Neu}} + \tfrac{3}{2}\big\}, \tag{4.17}$$

then the singularities in the splitting (4.12) only involve gradients and we have

$$\boldsymbol{u} \;-\; \sum_{\boldsymbol{c}\in\mathcal{C}} \sum_{\lambda+1\in\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})} \gamma_{\boldsymbol{c}}^{\lambda}\,\chi(\rho_{\boldsymbol{c}})\;\mathbf{grad}\,\Phi_{\mathrm{Dir},\boldsymbol{c}}^{\lambda+1}(\rho_{\boldsymbol{c}},\vartheta_{\boldsymbol{c}}) \tag{4.18a}$$

$$-\; \sum_{\boldsymbol{e}\in\mathcal{E}} \sum_{\lambda+1\in\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}})} \mathcal{K}_{\boldsymbol{e}}[\gamma_{\boldsymbol{e}}^{\lambda}]\,\chi(\rho_{\boldsymbol{e}})\;\mathbf{grad}_{\boldsymbol{e}}\,\Phi_{\mathrm{Dir},\boldsymbol{e}}^{\lambda+1}(\rho_{\boldsymbol{e}},\theta_{\boldsymbol{e}}) \;\in\; H^{\sigma+1}(\Omega), \tag{4.18b}$$

where in (4.18a) $\lambda+1$ belongs to $(0,\sigma+\frac{1}{2})$, in (4.18b) $\lambda+1$ belongs to $(0,\sigma+1)$ and $\mathbf{grad}_{\boldsymbol{e}}$ is the gradient associated with the cartesian variables $x_{\boldsymbol{e}}=\rho_{\boldsymbol{e}}\cos\theta_{\boldsymbol{e}}$, $y_{\boldsymbol{e}}=\rho_{\boldsymbol{e}}\sin\theta_{\boldsymbol{e}}$ and $z_{\boldsymbol{e}}$.

For $\sigma=0$, we only have one contribution per non convex edge and at most one per corner: Denoting like in Corollary 2.8 $\mathcal{C}_0=\{c\in\mathcal{C},\ \lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})<\frac{1}{2}\}$ and $\mathcal{E}_0=\{e\in\mathcal{E},\ \omega_{\boldsymbol{e}}>\pi\}$, the splitting (4.18) takes the simplified form

$$\boldsymbol{u} \;-\; \sum_{\boldsymbol{c}\in\mathcal{C}_0} \gamma_{\boldsymbol{c}}\,\chi(\rho_{\boldsymbol{c}})\;\mathbf{grad}\,\Phi_{\mathrm{Dir},\boldsymbol{c}}(\rho_{\boldsymbol{c}},\vartheta_{\boldsymbol{c}}) \tag{4.19a}$$

$$-\; \sum_{\boldsymbol{e}\in\mathcal{E}_0} \mathcal{K}_{\boldsymbol{e}}[\gamma_{\boldsymbol{e}}]\,\chi(\rho_{\boldsymbol{e}})\;\mathbf{grad}_{\boldsymbol{e}}\,\Phi_{\mathrm{Dir},\boldsymbol{e}}(\rho_{\boldsymbol{e}},\theta_{\boldsymbol{e}}) \;\in\; H^{1}(\Omega), \tag{4.19b}$$

where $\Phi_{\mathrm{Dir},\boldsymbol{c}}$ and $\Phi_{\mathrm{Dir},\boldsymbol{e}}$ are the singularities of Δ^{Dir} associated with the smallest eigenvalue of the Dirichlet Laplace-Beltrami on $G_{\boldsymbol{c}}$ and $(0,\omega_{\boldsymbol{e}})$ respectively.

Remark 4.10 In the splittings (4.18) and (4.19), the singular generators can also be expressed as curls since for any harmonic and homogeneous function Φ of degree μ, *cf* (6.6b):

$$(\mu+1)\,\mathbf{grad}\,\Phi = \mathbf{curl}(\mathbf{grad}\,\Phi\times\boldsymbol{x})$$

and

$$\mathbf{grad}_{\boldsymbol{e}}(\rho_{\boldsymbol{e}}^{\mu}\sin\mu\theta_{\boldsymbol{e}}) = \mathbf{curl}_{\boldsymbol{e}}(\rho_{\boldsymbol{e}}^{\mu}\cos\mu\theta_{\boldsymbol{e}}),$$

where $\mathbf{curl}_{\boldsymbol{e}}$ denotes the two-dimensional vectorial curl in the $(x_{\boldsymbol{e}},y_{\boldsymbol{e}})$ plane, completed by a zero tangential component along the edge. ■

Another interesting question in the framework of the splittings (4.18) and (4.19), is to know whether it is possible to write the singular parts as gradients in a global way.

Lemma 4.11 *Let* $\sigma\in[0,\lambda_{\mathcal{E}})$ *and* $\mathbf{c}\in\mathcal{C}$. *Let* $\Phi=\Phi_{\mathrm{Dir},\boldsymbol{c}}^{\mu}$ *be a singularity of the Laplace Dirichlet problem on* $\Gamma_{\boldsymbol{c}}$. *Then*

$$\chi(\rho_{\boldsymbol{c}})\;\mathbf{grad}\,\Phi - \mathbf{grad}\,\big(\chi(\rho_{\boldsymbol{c}})\Phi\big) \in H^{\sigma+1}(\Omega) \tag{4.20}$$

and $\varphi:=\chi(\rho_{\boldsymbol{c}})\Phi$ *belongs to* $\mathring{H}^1(\Omega)$ *and is such that* $\Delta\varphi$ *is in* $H^{\sigma}(\Omega)$.

Remark 4.12 The limitation of the regularity in (4.20) and for $\Delta\Phi$ comes only from the brutal cut-off of the edge asymptotics of φ away from the corner. A refined cut-off procedure [12, §16.C], would yield a similar statement without any limitation on σ. ■

Lemma 4.13 *Let* $\sigma \in [0, \lambda_{\mathcal{E}})$ *and* $\boldsymbol{e} \in \mathcal{E}$. *Let* $\Phi = \Phi^{\lambda+1}_{\mathrm{Dir},\boldsymbol{e}}$ *be a singularity of the Laplace Dirichlet problem on* $\Gamma_{\boldsymbol{e}}$, *with* $\lambda < \sigma$. *Let for* $s \geq \sigma$, $\gamma \in \mathbb{V}^{s-\lambda}_{-\sigma}(\boldsymbol{e})$. *Then,*

$$\mathcal{K}_{\boldsymbol{e}}[\gamma]\,\chi(\rho_{\boldsymbol{e}})\ \mathbf{grad}_{\boldsymbol{e}}\,\Phi(\rho_{\boldsymbol{e}},\theta_{\boldsymbol{e}}) - \mathbf{grad}\big(\mathcal{K}_{\boldsymbol{e}}[d_{\boldsymbol{e}}\,\gamma]\,\chi(\rho_{\boldsymbol{e}})\,\Phi(\rho_{\boldsymbol{e}},\theta_{\boldsymbol{e}})\big) \in H^{\sigma+1}(\Omega) \tag{4.21}$$

and $\varphi := \mathcal{K}_{\boldsymbol{e}}[d_{\boldsymbol{e}}\,\gamma]\,\chi(\rho_{\boldsymbol{e}})\Phi$ *belongs to* $\overset{\circ}{H}{}^1(\Omega)$ *and is such that* $\Delta\varphi$ *is in* $H^\sigma(\Omega)$.

Remark 4.14 Beyond what could be done by the introduction of correct "shadow" terms, it is impossible to avoid the limitation of the regularity by the weight $-\sigma$ in the space containing the edge coefficient γ. This implies that, if we apply such a refined statement to the edge terms in (4.18b), we have a sharp limitation by the smallest corner exponents which does not correspond to gradients ($\lambda^{\mathrm{Neu}}_{\mathcal{C}} + \frac12$ and $\lambda^{\mathrm{Dir}}_{\mathcal{C}} + \frac32$). ■

As a consequence of the expansion (4.18) and of the two previous lemmas, we obtain

Theorem 4.15 *Let* Ω *be locally simply connected,* $\sigma < \min\big\{\lambda_{\mathcal{E}},\ \lambda^{\mathrm{Neu}}_{\mathcal{C}} + \frac12,\ \lambda^{\mathrm{Dir}}_{\mathcal{C}} + \frac32\big\}$ *and* $s \geq \sigma$. *Then for any data* $\boldsymbol{f} \in H^{s-1}(\Omega)^3$ *the solution* $\boldsymbol{u}$ *of problem* (1.1) *can be split in the following way*

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{reg},\sigma} + \mathbf{grad}\,\varphi \tag{4.22}$$

where $\boldsymbol{u}_{\mathrm{reg},\sigma} \in H^{\sigma+1}(\Omega)^3$ *and* φ *can be written as*

$$\varphi = \sum_{\boldsymbol{c}\in\mathcal{C}}\ \sum_{\lambda\in\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{c}})} \gamma^\lambda_{\boldsymbol{c}}\,\chi(\rho_{\boldsymbol{c}})\,\Phi^\lambda_{\mathrm{Dir},\boldsymbol{c}}(\rho_{\boldsymbol{c}},\vartheta_{\boldsymbol{c}}) \tag{4.23a}$$

$$+ \sum_{\boldsymbol{e}\in\mathcal{E}}\ \sum_{\lambda\in\Lambda_{\mathrm{Dir}}(\Gamma_{\boldsymbol{e}})} \mathcal{K}_{\boldsymbol{e}}[d_{\boldsymbol{e}}\gamma^\lambda_{\boldsymbol{e}}]\,\chi(\rho_{\boldsymbol{e}})\,\Phi^\lambda_{\mathrm{Dir},\boldsymbol{e}}(\rho_{\boldsymbol{e}},\theta_{\boldsymbol{e}}). \tag{4.23b}$$

Here $\varphi \in \overset{\circ}{H}{}^1(\Omega)$ *satisfies* $\Delta\varphi \in H^\sigma(\Omega)$.

When applied with $\sigma = 0$, the above statement can be compared with Theorem 1.1 which gives the splitting of any element of X_N in the sum of an element of H_N and of a term $\mathbf{grad}\,\varphi$ with $\varphi \in \overset{\circ}{H}{}^1(\Omega)$ such that $\Delta\varphi \in L^2(\Omega)$, *cf* [4, 5, 13].

4.f Singularities in H_N

We conclude this section by some remarks on the regularity of the variational problem (1.3) posed in $H_N \subset H^1(\Omega)^3$. In section 3.e, this question was discussed for polygons. We saw that the set of exponents $\widetilde{\Lambda}_{N;\,s}(\Gamma_{\boldsymbol{a}})$ associated with the pseudo-Maxwell problem was, in general, different from $\Lambda_{N;\,s}(\Gamma_{\boldsymbol{a}})$ associated with the Maxwell problem. Still these exponents were related to those of the Dirichlet problem for the Laplace operator in a simple way.

Consider now the case of a three-dimensional polyhedral corner $\boldsymbol{c}$. If the cone $\Gamma_{\boldsymbol{c}}$ is *convex*, then the X_N-singular functions belong to H^1, and for the H_N-singular functions, the divergence has H^1 regularity. Thus the two problems have the same singularities near the corner $\boldsymbol{c}$.

We suppose therefore that the cone $\Gamma_{\boldsymbol{c}}$ is *not convex*. The set of admissible exponents $\widetilde{\Lambda}_{N;\,s}(\Gamma_{\boldsymbol{c}})$ associated with problem (1.3) is defined by the usual procedure like in dimension 2, by setting instead of (4.1) and (4.2):

$$\widetilde{\boldsymbol{S}}_N^{\lambda}(\Gamma_{\boldsymbol{c}}) = \big\{\boldsymbol{U} \in H_N^{\mathrm{loc}}(\overline{\Gamma}_{\boldsymbol{c}}^*) \mid \ \boldsymbol{U} = \rho^{\lambda} \sum_{q=0}^{Q} \log^q \rho\, \boldsymbol{u}_q(\vartheta)\big\} \tag{4.24}$$

$$\begin{aligned} \widetilde{\boldsymbol{Y}}_N^{\lambda}(\Gamma_{\boldsymbol{c}}) = \big\{\boldsymbol{U} \in \widetilde{\boldsymbol{S}}_N^{\lambda}(\Gamma_{\boldsymbol{c}}) \mid \quad & \operatorname{div} \boldsymbol{U} = 0 \ \text{ on } \partial\Gamma_{\boldsymbol{c}}, \\ & \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U} - \mathbf{grad}\operatorname{div}\boldsymbol{U} \ \text{ is polynomial}\big\}, \end{aligned} \tag{4.25}$$

and defining the singularity space $\widetilde{\boldsymbol{Z}}_N^{\lambda}(\Gamma)$ and the set of exponents $\widetilde{\Lambda}_N(\Gamma_{\boldsymbol{c}})$ correspondingly.

The set of admissible exponents for a right hand side in $H^{s-1}(\Omega)^3$ is then simply (compare with Lemma 4.3)

$$\widetilde{\Lambda}_{N;\,s}(\Gamma_{\boldsymbol{c}}) = \big\{\lambda \in \mathbb{C} \mid \ \lambda \in \widetilde{\Lambda}_N(\Gamma_{\boldsymbol{c}}) \quad \text{and} \quad \operatorname{Re}\lambda \in (-\tfrac{1}{2}, s - \tfrac{1}{2}]\big\}.$$

There exist general results on the exponents of the singular functions for this problem in the case of a Lipschitz cone (KOZLOV - MAZYA - ROSSMANN [18]). For instance, there is a strip $-1 \le \operatorname{Re}\lambda \le 0$ that does not contain such exponents. Note that this does not imply $H^{3/2}(\Omega)$ regularity, as it would for a cone with a regular base, because we have strong edge singularities here. The lowest edge exponent is $\lambda^* \le 1/3$, see section 3.d *(iii)*, and this corresponds to $H^{1+\sigma}$ regularity for $\boldsymbol{u}$ and H^{σ} regularity for $\operatorname{div}\boldsymbol{u}$ for all $\sigma < \lambda^*$.

5 Maxwell edge singularities

In this section, we are going to prove Lemmas 3.1 and 4.4 characterizing the singularities attached to the corner of a plane sector Γ and to the edge of a wedge $\Gamma \times \mathbb{R}$.

5.a 3D Maxwell singularities in a wedge

The wedge is equal to $\Gamma \times \mathbb{R}$ with a plane sector Γ of opening $\omega \in (0, 2\pi]$, $\omega \neq \pi$; the polar coordinates are denoted by (r, θ), the cartesian coordinates in the plane of Γ are denoted by (x, y), and z is a perpendicular coordinate.

Let $\lambda \in \mathbb{C}$. We look for non-polynomial solutions $\boldsymbol{U}$ of the system

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{U} = \boldsymbol{F} & \text{in } \Gamma\times\mathbb{R}, \quad \boldsymbol{F} \text{ polynomial,} \\ \boldsymbol{U}\times\boldsymbol{n} = 0, \quad \mathrm{div}\,\boldsymbol{U} = 0 & \text{on } \partial\Gamma\times\mathbb{R}, \\ \boldsymbol{U} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma\times\mathbb{R}), \quad \mathrm{div}\,\boldsymbol{U} \in S^{\lambda-1}_{\mathrm{Dir}}(\Gamma\times\mathbb{R}), \end{cases} \tag{5.1}$$

where $\boldsymbol{S}^{\lambda}_{N}(\Gamma\times\mathbb{R})$ is the space (4.6) of pseudo-homogeneous fields of degree λ with 3 components but not depending on z. Let us note that, *when* $\lambda \notin \mathbb{N}$*, the above problem reduces to find the non-zero solutions* $\boldsymbol{U} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma\times\mathbb{R})$ *of* $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{U} = 0$ *with the same boundary conditions.*

Let now $(\boldsymbol{V}, W)$ be the decomposition of the field $\boldsymbol{U}$ in the system of cartesian coordinates $((x,y),z)$. As $\boldsymbol{U}$ does not depend on the variable z, we obtain that system (5.1) splits into 2 independent problems:

$$\begin{cases} \mathbf{curl}\,\mathrm{curl}\,\boldsymbol{V} - \mathbf{grad}\,\mathrm{div}\,\boldsymbol{V} = \boldsymbol{f} & \text{in } \Gamma, \quad \boldsymbol{f} \text{ polynomial,} \\ \boldsymbol{V}\times\boldsymbol{n} = 0, \quad \mathrm{div}\,\boldsymbol{V} = 0 & \text{on } \partial\Gamma, \\ \boldsymbol{V} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma), \quad \mathrm{div}\,\boldsymbol{U} \in S^{\lambda-1}_{\mathrm{Dir}}(\Gamma), \end{cases} \tag{5.2}$$

and

$$\begin{cases} -\Delta W = f & \text{in } \Gamma, \quad f \text{ polynomial,} \\ W = 0 & \text{on } \partial\Gamma, \\ W \in S^{\lambda}_{\mathrm{Dir}}(\Gamma). \end{cases} \tag{5.3}$$

Indeed problem (5.2) is exactly the problem of finding the space $\boldsymbol{Y}^{\lambda}_{N}(\Gamma)$ (3.4) associated with two-dimensional Maxwell equations in the sector Γ and problem (5.3) is the problem of finding the space $Y^{\lambda}_{\mathrm{Dir}}(\Gamma)$ (2.2) associated with the two-dimensional Laplacian in Γ. For this latter problem, see Lemma 2.1.

Let us now consider the two-dimensional "Maxwell-type" problem (5.2). We introduce two auxiliary scalar variables

$$\Psi = \mathrm{curl}\,\boldsymbol{V} \quad \text{and} \quad q = \mathrm{div}\,\boldsymbol{V}. \tag{5.4}$$

Taking the divergence of the first line of (5.2) yields equation (5.5a) below. Equations (5.5b) and (5.5c) are straightforward ($S^{\lambda}_{\mathrm{Dir}}$ and $S^{\lambda}_{\mathrm{Neu}}$ are defined in §2.a)

$$-\Delta q = \mathrm{div}\,\boldsymbol{f} \ \text{ in } \Gamma, \qquad q = 0 \ \text{ on } \partial\Gamma, \qquad \text{with} \quad q \in S^{\lambda-1}_{\mathrm{Dir}}(\Gamma). \tag{5.5a}$$

$$\mathbf{curl}\,\Psi = \mathbf{grad}\,q + \boldsymbol{f} \ \text{ in } \Gamma, \qquad \text{with} \quad \Psi \in S^{\lambda-1}_{\mathrm{Neu}}(\Gamma). \tag{5.5b}$$

$$\mathrm{curl}\,\boldsymbol{V} = \Psi, \ \ \mathrm{div}\,\boldsymbol{V} = q \ \text{ in } \Gamma, \quad \boldsymbol{V}\times\boldsymbol{n} = 0 \ \text{ on } \partial\Gamma, \qquad \text{with} \quad \boldsymbol{V} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma). \tag{5.5c}$$

We easily see that the system of equations (5.5) is equivalent to (5.2).

5.b Non-integral exponents

In order to solve system (5.5), we begin with the simpler situation when λ *is not a positive integer.* Then the above system of equations reduces to

$$-\Delta q = 0 \ \text{ in } \Gamma, \qquad q = 0 \ \text{ on } \partial\Gamma, \qquad \text{with} \quad q \in S^{\lambda-1}_{\mathrm{Dir}}(\Gamma). \tag{5.6a}$$

$$\mathbf{curl}\,\Psi = \mathbf{grad}\,q \ \text{ in } \Gamma, \qquad \text{with} \quad \Psi \in S^{\lambda-1}_{\mathrm{Neu}}(\Gamma). \tag{5.6b}$$

$$\operatorname{curl} \boldsymbol{V} = \Psi, \ \operatorname{div} \boldsymbol{V} = q \ \text{ in } \Gamma, \quad \boldsymbol{V} \times \boldsymbol{n} = 0 \ \text{ on } \partial\Gamma, \quad \text{with} \quad \boldsymbol{V} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma). \tag{5.6c}$$

We can split the solutions of system (5.6) into three natural types:

1. $q = 0$, $\Psi = 0$ and $\boldsymbol{V}$ general non-zero solution of (5.6c).
2. $q = 0$, Ψ general non-zero solution of (5.6b) and $\boldsymbol{V}$ particular solution of (5.6c).
3. q general non-zero solution of (5.6a), Ψ particular solution of (5.6b) and $\boldsymbol{V}$ particular solution of (5.6c).

Let us study successively these three types.

Type 1.
Since $\operatorname{curl} \boldsymbol{V} = 0$ on the simply connected domain Γ, $\boldsymbol{V} = (V_1, V_2)$ is the gradient of a function Φ. Thus we have:

$$\begin{cases} V_r := \cos\theta\, V_1 + \sin\theta\, V_2 = \partial_r \Phi \\ V_\theta := -\sin\theta\, V_1 + \cos\theta\, V_2 = \frac{1}{r}\partial_\theta \Phi, \end{cases}$$

whence (we denote by $\tilde{V}$ the function $\tilde{V}(r,\theta) = V(x,y)$)

$$\tilde{\Phi}(r,\theta) - \tilde{\Phi}(1,0) = \int_1^r \tilde{V}_r(r',0)\, dr' \ + \ r\int_0^\theta \tilde{V}_\theta(r,\theta')\, d\theta', \tag{5.7}$$

which proves that since $\boldsymbol{V}$ belongs to $\boldsymbol{S}^{\lambda}_{N}(\Gamma)$, Φ is the sum of a function in $S^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$ and a constant. Therefore, Φ can be found in $S^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$. Then (5.5c) is equivalent to

$$\Delta\Phi = 0 \ \text{ in } \Gamma \quad \text{and} \quad \Phi = 0 \ \text{ on } \partial\Gamma, \quad \text{with} \quad \Phi \in S^{\lambda+1}_{\mathrm{Dir}}(\Gamma). \tag{5.8}$$

Hence, $\lambda + 1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ and Φ belongs to the space $Z^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$, *cf* Lemma 2.1: with the complex writing $\zeta = re^{i\theta}$ of the coordinates, a generator of $Z^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$ is given by $\operatorname{Im}\zeta^{\lambda+1}$.

Type 2.
We easily see that Ψ is zero and a particular solution of (5.6c) is $\boldsymbol{V} = 0$.

Type 3.
From equation (5.6a), we obtain that $\lambda - 1$ belongs to $\Lambda^{\mathrm{Dir}}(\Gamma)$ and that q belongs to $Z^{\lambda-1}_{\mathrm{Dir}}(\Gamma)$: thus q is proportional to $\operatorname{Im}\zeta^{\lambda-1}$. Then it is easy to see that $\Psi = -\operatorname{Re}\zeta^{\lambda-1}$ is a particular solution of (5.6b), and that $\boldsymbol{V} = \frac{1}{2\lambda}(\operatorname{Im}\zeta^{\lambda}, -\operatorname{Re}\zeta^{\lambda})$ is a particular solution of (5.6c).

5.c Integral exponents

When λ is a positive integer, we are searching for non-polynomial solutions of system (5.5). Similarly to the case when λ is not an integer, we split the solutions of the system (5.5) into the three types:

1. q and Ψ are polynomial and $\boldsymbol{V}$ is a non-polynomial solution of (5.5c).
2. q is polynomial, Ψ is a non-polynomial solution of (5.5b) and $\boldsymbol{V}$ a particular solution of (5.5c).
3. q is a non-polynomial solution of (5.5a), Ψ a particular solution of (5.5b) and $\boldsymbol{V}$ a particular solution of (5.5c).

Now the arguments are based on the evaluation of dimensions of polynomial spaces. Let Q^λ be the space of homogeneous polynomials of degree λ. We recall that $P^\lambda_{\mathrm{Dir}}(\Gamma)$ the subspace of $q \in Q^\lambda$ with zero traces on $\partial\Gamma$. We divide our study into three subcases:

(i) $\omega \neq 2\pi$ and $\lambda - 1$ does not belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$: In equation (5.5a) the r.h.s. $\operatorname{div} \boldsymbol{f}$ is any polynomial in $Q^{\lambda-3}$, thus the dimension of the range is $(\lambda-2)_+$. The dimension of $P^{\lambda-1}_{\mathrm{Dir}}(\Gamma)$ is $(\lambda-2)_+$ too. Moreover equation (5.5a) defines an operator from $P^{\lambda-1}_{\mathrm{Dir}}(\Gamma)$ into $Q^{\lambda-3}$ which is one to one due to the assumption that $\lambda-1$ does not belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$. Therefore this operator is onto.

• The r.h.s. of equation (5.5b) is any field in $Q^{\lambda-2} \times Q^{\lambda-2}$ which is divergence free. Thus the dimension of its range is $2(\lambda-1) - (\lambda-2)_+$ which is equal to λ if $\lambda \geq 2$ and 0 if $\lambda = 1$. The dimension of $Q^{\lambda-1}$ is equal to λ and equation (5.5b) defines an operator from $Q^{\lambda-1}$ into $\{\boldsymbol{g} \in Q^{\lambda-2} \times Q^{\lambda-2},\ \operatorname{div} \boldsymbol{g} = 0\}$, which is one to one for any $\lambda \geq 2$, thus onto.

• The r.h.s. (Ψ, q) of (5.5c) is any element of $Q^{\lambda-1} \times P^{\lambda-1}_{\mathrm{Dir}}(\Gamma)$. Thus the dimension of its range is $\lambda + (\lambda-2)_+ = 2(\lambda-1)$ if $\lambda \geq 2$ and 1 if $\lambda = 1$. The space of polynomial solutions of (5.6c) is

$$\{\boldsymbol{V} \in Q^\lambda \times Q^\lambda \mid \quad \boldsymbol{V} \times \boldsymbol{n} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{V} = 0 \quad \text{on} \quad \partial\Gamma\}. \tag{5.9}$$

Its dimension is $2(\lambda+1) - 4 = 2(\lambda-1)$ if $\lambda \geq 2$; if $\lambda = 1$, its dimension is either 2 if $\cos\omega = 0$ or 1 if not. If $\lambda+1$ does not belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$, we check that in any case the operator of equation (5.5c) is one to one, thus it is onto: the system (5.5) has only polynomial solutions. If $\lambda + 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$, its kernel is one-dimensional, and for $\lambda \geq 2$ we add to the above polynomial space (5.9) a singular function equal to the sum of $\mathbf{grad}(\operatorname{Im}\zeta^{\lambda+1}\log\zeta)$ and of a polynomial: we have found now a solution of type 1. For $\lambda = 1$ finally, $\lambda + 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$ only if $\cos\omega = 0$ and the operator of equation (5.5c) is onto and there is no singularity.

(ii) $\omega = 2\pi$: The arguments are similar. Here $\lambda - 1$ and $\lambda + 1$ never belong to $\Lambda^{\mathrm{Dir}}(\Gamma)$. The dimensions of the polynomial spaces involving boundary conditions are slightly different: $\dim P^{\lambda-1}_{\mathrm{Dir}}(\Gamma) = \lambda - 1$ and the operator of equation (5.5a) has a one-dimensional kernel generated by $\operatorname{Im}\zeta^{\lambda-1}$. Thus it is still onto from $P^{\lambda-1}_{\mathrm{Dir}}(\Gamma) \to Q^{\lambda-3}$.

• The situation for equation (5.5b) is unchanged.

• The dimension of the space $Q^{\lambda-1}\times P^{\lambda-1}_{\rm Dir}(\Gamma)$ is $2\lambda-1$ and the dimension of the space (5.9) is $2(\lambda+2)-2=2\lambda$. The kernel of the operator of equation (5.5c) is generated by $\mathbf{grad}\,\mathrm{Im}\,\zeta^{\lambda-1}$. Thus we have only polynomial solutions.

(iii) $\omega\neq 2\pi$ and $\lambda-1$ belongs to $\Lambda^{\rm Dir}(\Gamma)$: Then the operator of equation (5.5a) has a one-dimensional kernel generated by $\mathrm{Im}\,\zeta^{\lambda-1}$ and it is onto from the space generated by the sum of $P^{\lambda-1}_{\rm Dir}(\Gamma)$ and of $\Phi^{\lambda-1}_{\rm Dir}$ which is the sum of $\mathrm{Im}\,\zeta^{\lambda-1}\log\zeta$ and of a polynomial.

• Corresponding to this new solution q, we find a new solution of equation (5.5b) $\Psi=\mathrm{Re}\,\zeta^{\lambda-1}\log\zeta$.

• Accordingly, we find a new solution $\boldsymbol{V}$ of equation (5.5c) in the form $\boldsymbol{V}=\frac{1}{2\lambda}(\mathrm{Im}\,\zeta^{\lambda}\log\zeta,-\mathrm{Re}\,\zeta^{\lambda}\log\zeta)$, which is a non-polynomial solution of type 3.

The proofs of Lemmas 3.1 and 4.4 are complete.

6 Maxwell corner singularities

In this section we prove Lemma 4.1 for "electric" boundary conditions and its analogue for "magnetic" boundary conditions. Let Γ be a three-dimensional polyhedral cone. We recall that the polar coordinates are denoted by (ρ,ϑ) and $\Gamma=\{(\rho,\vartheta)\mid\rho>0,\ \vartheta\in G\subset\mathbb{S}^2\}$. Let us recall the definition (4.1)

$$\boldsymbol{S}^{\lambda}_{N}(\Gamma)=\big\{\boldsymbol{U}\in X^{\rm loc}_{N}(\overline{\Gamma}^{*})\mid\ \boldsymbol{U}=\rho^{\lambda}\sum_{q=0}^{Q}\log^{q}\!\rho\,\boldsymbol{u}_{q}(\vartheta)\big\}$$

and let us introduce its analogue for magnetic boundary conditions:

$$\boldsymbol{S}^{\lambda}_{T}(\Gamma)=\big\{\boldsymbol{U}\in X^{\rm loc}_{T}(\overline{\Gamma}^{*})\mid\ \boldsymbol{U}=\rho^{\lambda}\sum_{q=0}^{Q}\log^{q}\!\rho\,\boldsymbol{u}_{q}(\vartheta)\big\}.$$

6.a Splitting of the problem

Here we concentrate on the case when λ *is not a positive integer.* Thus, in the electric case, the problem reduces to finding non-zero solutions to

$$\begin{cases}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U}-\mathbf{grad}\,\mathrm{div}\,\boldsymbol{U}=0 & \text{in }\ \Gamma,\\ \boldsymbol{U}\times\boldsymbol{n}=0,\quad \mathrm{div}\,\boldsymbol{U}=0 & \text{on }\ \partial\Gamma,\\ \boldsymbol{U}\in\boldsymbol{S}^{\lambda}_{N}(\Gamma),\quad \mathrm{div}\,\boldsymbol{U}\in S^{\lambda-1}_{\rm Dir}(\Gamma) & \end{cases}\tag{6.1}$$

and concerning the magnetic case:

$$\begin{cases}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{U}-\mathbf{grad}\,\mathrm{div}\,\boldsymbol{U}=0 & \text{in }\ \Gamma,\\ \boldsymbol{U}\cdot\boldsymbol{n}=0,\quad \mathbf{curl}\,\boldsymbol{U}\times\boldsymbol{n}=0 & \text{on }\ \partial\Gamma,\\ \boldsymbol{U}\in\boldsymbol{S}^{\lambda}_{T}(\Gamma),\quad \mathrm{div}\,\boldsymbol{U}\in S^{\lambda-1}_{\rm Neu}(\Gamma) & \end{cases}\tag{6.2}$$

Like in the case of plane sectors, we introduce the auxiliary unknowns

$$\boldsymbol{\Psi} = \mathbf{curl}\,\boldsymbol{U} \quad \text{and} \quad q = \operatorname{div}\boldsymbol{U}.$$

Thus q belongs to $S^{\lambda-1}_{\mathrm{Dir}}(\Gamma)$ or to $S^{\lambda-1}_{\mathrm{Neu}}(\Gamma)$. Concerning $\boldsymbol{\Psi}$ in problem (6.1), we remark that the definition of $\boldsymbol{\Psi}$ implies that $\operatorname{div}\boldsymbol{\Psi} = 0$ and since $\boldsymbol{U}\times\boldsymbol{n} = 0$ on $\partial\Gamma$, then $\boldsymbol{\Psi}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. Moreover, from the equality $\mathbf{curl}\boldsymbol{\Psi} = \mathbf{grad}\operatorname{div}\boldsymbol{U}$ and from the H^1 regularity of $\operatorname{div}\boldsymbol{U}$, we obtain that $\mathbf{curl}\boldsymbol{\Psi}$ belongs to $L^2_{\mathrm{loc}}(\overline{\Gamma}^*)$. Thus the natural space for $\boldsymbol{\Psi}$ is $\boldsymbol{S}^{\lambda}_{T}(\Gamma)$ and it is now clear that problem (6.1) is equivalent to find non-zero solutions to the system

$$-\Delta q = 0 \ \text{ in } \Gamma, \qquad \text{with} \quad q \in S^{\lambda-1}_{\mathrm{Dir}}(\Gamma). \tag{6.3a}$$

$$\mathbf{curl}\boldsymbol{\Psi} = \mathbf{grad}\,q \quad \text{and} \quad \operatorname{div}\boldsymbol{\Psi} = 0 \ \text{ in } \Gamma, \qquad \text{with} \quad \boldsymbol{\Psi} \in \boldsymbol{S}^{\lambda-1}_{T}(\Gamma). \tag{6.3b}$$

$$\mathbf{curl}\,\boldsymbol{U} = \boldsymbol{\Psi} \quad \text{and} \quad \operatorname{div}\boldsymbol{U} = q \ \text{ in } \Gamma, \qquad \text{with} \quad \boldsymbol{U} \in \boldsymbol{S}^{\lambda}_{N}(\Gamma). \tag{6.3c}$$

Now we see that the "electric" and "magnetic" boundary conditions appear simultaneously inside (6.3). Thus we have better to treat both conditions together. The "magnetic" problem (6.2) is equivalent to find non-zero solutions to the system of three problems

$$-\Delta q = 0 \ \text{ in } \Gamma, \qquad \partial_n q = 0 \ \text{ on } \partial\Gamma, \quad \text{with} \quad q \in S^{\lambda-1}_{\mathrm{Neu}}(\Gamma). \tag{6.4a}$$

$$\mathbf{curl}\boldsymbol{\Psi} = \mathbf{grad}\,q \quad \text{and} \quad \operatorname{div}\boldsymbol{\Psi} = 0 \ \text{ in } \Gamma, \qquad \text{with} \quad \boldsymbol{\Psi} \in \boldsymbol{S}^{\lambda-1}_{N}(\Gamma). \tag{6.4b}$$

$$\mathbf{curl}\,\boldsymbol{U} = \boldsymbol{\Psi} \quad \text{and} \quad \operatorname{div}\boldsymbol{U} = q \ \text{ in } \Gamma, \qquad \text{with} \quad \boldsymbol{U} \in \boldsymbol{S}^{\lambda}_{T}(\Gamma). \tag{6.4c}$$

Like for the plane sectors, the solutions of systems (6.3) and (6.4) belong to one of three types:

1. $q = 0$, $\boldsymbol{\Psi} = 0$ and $\boldsymbol{U}$ general non-zero solution of (6.3c), resp. (6.4c).
2. $q = 0$, $\boldsymbol{\Psi}$ general non-zero solution of (6.3b), resp. (6.4b) and $\boldsymbol{U}$ particular solution of (6.3c), resp. (6.4c).
3. q general non-zero solution of (6.3a), resp. (6.4a), $\boldsymbol{\Psi}$ particular solution of (6.3b), resp. (6.4b) and $\boldsymbol{U}$ particular solution of (6.3c), resp. (6.4c).

6.b Explicit solutions of first order problems

The Laplace singularities on polyhedral cones were described in Lemma 2.4. They contain Laplace-Beltrami eigenfunctions and have therefore, in contrast to the two-dimensional case, no analytically known form, in general. But once these Laplace singularities are known, we are able to provide completely explicit formulas for the three types of Maxwell singularities.

This section is devoted to the description of solution formulas for the first order problems (6.3) and (6.4). All these formulas are based on the scalar product or the vector product with the vector $\boldsymbol{x}$, with $\boldsymbol{x}$ denoting the vector of cartesian coordinates (x, y, z), and $\rho = |\boldsymbol{x}|$.

We begin with three series of formulas. First we give product laws: $\boldsymbol{a}$ and $\boldsymbol{b}$ denoting vector fields and γ being a scalar function on $\mathbb{R}^3$, we have

$$\begin{aligned}
\mathbf{grad}(\boldsymbol{a}\cdot\boldsymbol{b}) &= (\boldsymbol{a}\cdot\mathbf{grad})\,\boldsymbol{b} + (\boldsymbol{b}\cdot\mathbf{grad})\,\boldsymbol{a} + \boldsymbol{a}\times\mathbf{curl}\,\boldsymbol{b} + \boldsymbol{b}\times\mathbf{curl}\,\boldsymbol{a}, && \text{(6.5a)}\\
\mathbf{curl}(\boldsymbol{a}\times\boldsymbol{b}) &= (\boldsymbol{b}\cdot\mathbf{grad})\,\boldsymbol{a} - (\boldsymbol{a}\cdot\mathbf{grad})\,\boldsymbol{b} + \boldsymbol{a}\,\operatorname{div}\boldsymbol{b} - \boldsymbol{b}\,\operatorname{div}\boldsymbol{a}, && \text{(6.5b)}\\
\operatorname{div}(\boldsymbol{a}\times\boldsymbol{b}) &= \boldsymbol{b}\cdot\mathbf{curl}\,\boldsymbol{a} - \boldsymbol{a}\cdot\mathbf{curl}\,\boldsymbol{b}, && \text{(6.5c)}\\
\mathbf{curl}(\gamma\boldsymbol{a}) &= \gamma\,\mathbf{curl}\,\boldsymbol{a} + \mathbf{grad}\,\gamma\times\boldsymbol{a}, && \text{(6.5d)}\\
\operatorname{div}(\gamma\boldsymbol{a}) &= \gamma\operatorname{div}\boldsymbol{a} + \mathbf{grad}\,\gamma\cdot\boldsymbol{a}. && \text{(6.5e)}
\end{aligned}$$

Now, using the above formulas for the field $\boldsymbol{x}$ which satisfies

$$\operatorname{div}\boldsymbol{x} = 3, \quad \mathbf{curl}\,\boldsymbol{x} = 0, \quad \boldsymbol{x}\cdot\mathbf{grad} = \rho\partial_\rho \quad \text{and} \quad \mathbf{grad}\,\boldsymbol{x} = \mathrm{I},$$

we obtain for any field $\boldsymbol{a}$ and scalar q

$$\begin{aligned}
\mathbf{grad}(\boldsymbol{a}\cdot\boldsymbol{x}) &= (\rho\partial_\rho + 1)\boldsymbol{a} + \boldsymbol{x}\times\mathbf{curl}\,\boldsymbol{a}, && \text{(6.6a)}\\
\mathbf{curl}(\boldsymbol{a}\times\boldsymbol{x}) &= (\rho\partial_\rho + 2)\boldsymbol{a} - \boldsymbol{x}\,\operatorname{div}\boldsymbol{a}, && \text{(6.6b)}\\
\operatorname{div}(\boldsymbol{a}\times\boldsymbol{x}) &= \boldsymbol{x}\cdot\mathbf{curl}\,\boldsymbol{a}, && \text{(6.6c)}\\
\mathbf{curl}(q\boldsymbol{x}) &= \mathbf{grad}\,q\times\boldsymbol{x}, && \text{(6.6d)}\\
\operatorname{div}(q\boldsymbol{x}) &= (\rho\partial_\rho + 3)q. && \text{(6.6e)}
\end{aligned}$$

Finally, with $\gamma = \rho^2$ and $\boldsymbol{a} = \mathbf{grad}\,q$, (6.5d) and (6.5e) yield

$$\begin{aligned}
\mathbf{curl}(\rho^2\,\mathbf{grad}\,q) &= -2\,\mathbf{grad}\,q\times\boldsymbol{x}, && \text{(6.6f)}\\
\operatorname{div}(\rho^2\,\mathbf{grad}\,q) &= 2\rho\partial_\rho q + \rho^2\Delta q. && \text{(6.6g)}
\end{aligned}$$

We need the following spaces of pseudo-homogeneous functions

$$S_0^\lambda(\Gamma) = \big\{\Phi\in L^2_{\mathrm{loc}}(\overline{\Gamma}^*) \mid \ \Phi = r^\lambda \sum \log^q r\ \phi_q(\theta)\big\} \quad \text{and} \quad S_1^\lambda(\Gamma) = S_0^\lambda(\Gamma)\cap H^1_{\mathrm{loc}}(\overline{\Gamma}^*).$$

The above formulas allow us to solve first order problems in the subspaces of homogeneous elements of our pseudo-homogeneous spaces S_1^λ, $\boldsymbol{S}_N^\lambda$ and $\boldsymbol{S}_T^\lambda$:

$$\mathring{S}_1^\lambda(\Gamma) = \big\{\Phi\in S_1^\lambda(\Gamma) \mid \ \Phi = \rho^\lambda\phi(\vartheta)\big\},$$

$$\mathring{\boldsymbol{S}}_N^\lambda(\Gamma) = \big\{\boldsymbol{U}\in\boldsymbol{S}_N^\lambda(\Gamma) \mid \ \boldsymbol{U} = \rho^\lambda\boldsymbol{v}(\vartheta)\big\} \quad \text{and} \quad \mathring{\boldsymbol{S}}_T^\lambda(\Gamma) = \big\{\boldsymbol{U}\in\boldsymbol{S}_T^\lambda(\Gamma) \mid \ \boldsymbol{U} = \rho^\lambda\boldsymbol{v}(\vartheta)\big\}.$$

As an easy consequence of formula (6.6a), we can solve the equation $\mathbf{grad}\,\Phi = \boldsymbol{U}$ with Dirichlet or Neumann boundary conditions:

Lemma 6.1 *Let* $\boldsymbol{U}$ *belong to* $\boldsymbol{S}_N^\lambda(\Gamma)$ *or* $\boldsymbol{S}_T^\lambda(\Gamma)$.
(i) *Then* $\boldsymbol{U}\cdot\boldsymbol{x}$ *belongs to* $S_1^{\lambda+1}(\Gamma)$.
(ii) *We assume that* $\lambda\neq -1$ *and that* $\mathbf{curl}\,\boldsymbol{U}=0$. *If moreover* $\boldsymbol{U}$ *is homogeneous, i.e.* $\boldsymbol{U}\in \mathring{\boldsymbol{S}}_N^\lambda(\Gamma)$, *resp.* $\mathring{\boldsymbol{S}}_T^\lambda(\Gamma)$, *then* Φ *defined as*

$$\Phi=\frac{\boldsymbol{U}\cdot\boldsymbol{x}}{\lambda+1}\quad\in\quad \mathring{S}_1^{\lambda+1}(\Gamma), \tag{6.7}$$

solves the equation $\mathbf{grad}\,\Phi=\boldsymbol{U}$, *with zero Dirichlet, resp. Neumann boundary conditions on* $\partial\Gamma$.

PROOF. *(i)* A first consequence of formula (6.6a) is that $\mathbf{grad}(\boldsymbol{U}\cdot\boldsymbol{x})$ belongs to $S_0^\lambda(\Gamma)^3$, thus $\boldsymbol{U}\cdot\boldsymbol{x}$ has the correct regularity outside the corner of Γ.

(ii) As an obvious consequence of the fact that if $\boldsymbol{U}$ belongs to $\mathring{\boldsymbol{S}}^\lambda(\Gamma)$, then $\rho\partial_\rho\boldsymbol{U}=\lambda\boldsymbol{U}$, we obtain that $\mathbf{grad}\,\Phi=\boldsymbol{U}$. Moreover, if $\boldsymbol{U}\times\boldsymbol{n}=0$ on $\partial\Gamma$, then $\boldsymbol{U}\cdot\boldsymbol{x}=0$ on $\partial\Gamma$ as a simple consequence of the fact that $\boldsymbol{x}$ is a tangential field. As for the Neumann boundary condition in the case when $\boldsymbol{U}\cdot\boldsymbol{n}=0$, it is only a consequence of the formula $\partial_n\Phi=\boldsymbol{n}\cdot\boldsymbol{U}$. ∎

Similarly formulas (6.6b) and (6.6c) yield a solution of the equation $\mathbf{curl}\,\boldsymbol{U}=\boldsymbol{\Psi}$:

Lemma 6.2 *Let* $\boldsymbol{\Psi}$ *belong to* $\boldsymbol{S}_T^{\lambda-1}(\Gamma)$, *resp.* $\boldsymbol{S}_N^{\lambda-1}(\Gamma)$.
(i) *Then* $\boldsymbol{\Psi}\times\boldsymbol{x}$ *belongs to* $\boldsymbol{S}_N^\lambda(\Gamma)$, *resp.* $\boldsymbol{S}_T^\lambda(\Gamma)$.
(ii) *We assume that* $\lambda\neq-1$ *and that* $\operatorname{div}\boldsymbol{\Psi}=0$. *If moreover* $\boldsymbol{\Psi}$ *is homogeneous, i.e.* $\boldsymbol{\Psi}\in\mathring{\boldsymbol{S}}_T^{\lambda-1}(\Gamma)$, *resp.* $\mathring{\boldsymbol{S}}_N^{\lambda-1}(\Gamma)$, *then* $\boldsymbol{U}$ *defined as*

$$\boldsymbol{U}=\frac{\boldsymbol{\Psi}\times\boldsymbol{x}}{\lambda+1}\quad\in\quad\mathring{\boldsymbol{S}}_N^\lambda(\Gamma)\quad\textit{resp.}\quad\mathring{\boldsymbol{S}}_T^\lambda(\Gamma), \tag{6.8}$$

solves the equation $\mathbf{curl}\,\boldsymbol{U}=\boldsymbol{\Psi}$. *Moreover* $\operatorname{div}\boldsymbol{U}=\frac{1}{\lambda+1}\,\boldsymbol{x}\cdot\mathbf{curl}\boldsymbol{\Psi}$.

PROOF. The regularity of $\boldsymbol{U}$ is a direct consequence of formulas (6.6b) and (6.6c). The boundary condition $\boldsymbol{n}\times(\boldsymbol{\Psi}\times\boldsymbol{x})=0$ is satisfied if $\boldsymbol{n}\cdot\boldsymbol{\Psi}=0$ on $\partial\Gamma$ due to the equality $\boldsymbol{n}\times(\boldsymbol{\Psi}\times\boldsymbol{x})=\boldsymbol{\Psi}(\boldsymbol{n}\cdot\boldsymbol{x})-\boldsymbol{x}(\boldsymbol{n}\cdot\boldsymbol{\Psi})$. And the boundary condition $\boldsymbol{n}\cdot(\boldsymbol{\Psi}\times\boldsymbol{x})=0$ is satisfied if $\boldsymbol{n}\times\boldsymbol{\Psi}=0$ on $\partial\Gamma$ due to the equality $\boldsymbol{n}\cdot(\boldsymbol{\Psi}\times\boldsymbol{x})=\boldsymbol{x}\cdot(\boldsymbol{n}\times\boldsymbol{\Psi})$. Part *(i)* is proved and part *(ii)* is now obvious. ∎

The third step is the solution of the equations $\mathbf{curl}\,\boldsymbol{U}=0$, $\operatorname{div}\boldsymbol{U}=q$, which is done with the help of formulas (6.6d)-(6.6g):

Lemma 6.3 *Let* q *belong to* $S_1^{\lambda-1}(\Gamma)$, *such that* $\Delta q\in S_0^{\lambda-3}(\Gamma)$ *and satisfying Dirichlet, resp. Neumann boundary conditions on* $\partial\Gamma$.
(i) *Then* $2q\boldsymbol{x}+\rho^2\,\mathbf{grad}\,q$ *belongs to* $\boldsymbol{S}_N^\lambda(\Gamma)$, *resp.* $\boldsymbol{S}_T^\lambda(\Gamma)$.
(ii) *We assume that* $\lambda\neq-\frac12$ *and that* $\Delta q=0$. *If moreover* q *is homogeneous, i.e.*

$q \in \overset{\circ}{S}{}_1^{\lambda-1}(\Gamma)$, *then* $\boldsymbol{U}$ *defined as*

$$\boldsymbol{U} = \frac{2q\boldsymbol{x} + \rho^2 \,\mathbf{grad}\, q}{4\lambda + 2} \quad \in \quad \overset{\circ}{\boldsymbol{S}}{}_N^{\lambda}(\Gamma) \quad \textit{resp.} \quad \overset{\circ}{\boldsymbol{S}}{}_T^{\lambda}(\Gamma), \tag{6.9}$$

solves the equations $\mathbf{curl}\, \boldsymbol{U} = 0$ *and* $\operatorname{div} \boldsymbol{U} = q$.

6.c The three types of Maxwell singularities generated by the Laplacian

In the case of plane sectors, we have seen that only two types of Maxwell singularities do exist and that they are generated by the Laplace operator: type 1, corresponding to the exponents $\frac{k\pi}{\omega} - 1$ and the singular functions of the form $\mathbf{grad}\, \Phi$ with Φ Dirichlet singularity for the Laplace operator, and type 3, corresponding to the exponents $\frac{k\pi}{\omega} + 1$. Now for three-dimensional cones, relying on the solution formulas (6.7)-(6.9) we are going to exhibit the three types which are generated by the Laplacian (Dirichlet or Neumann). In the next subsection, we will describe the remaining singularities which are generated by the *topology* of Γ.

In the following lemmas, we show the link between the sets of Maxwell singularity exponents $\Lambda_N(\Gamma)$ and $\Lambda_T(\Gamma)$ and those of the Laplacian, see $\Lambda_{\mathrm{Dir}}(\Gamma)$ and $\Lambda_{\mathrm{Neu}}(\Gamma)$ in Lemmas 2.1 and 2.2. We also prove that the singularities of type 1, 2 and 3 can be expressed with the help of the corresponding spaces of Laplace singular functions $Z^{\lambda}_{\mathrm{Dir}}(\Gamma)$ and $Z^{\lambda}_{\mathrm{Neu}}(\Gamma)$, *except in particular geometrical situations when* $\lambda = -1$.

Lemma 6.4 *We assume that* $\lambda \neq -1$. *Then* (i) *is equivalent to* (ii)*:*
(i) $\boldsymbol{U} \in \boldsymbol{S}_N^{\lambda}(\Gamma)$ *is a solution of* (6.3) *of type 1,*
(ii) $\lambda + 1$ *belongs to* $\Lambda_{\mathrm{Dir}}(\Gamma)$ *and* $\boldsymbol{U} = \mathbf{grad}\, \Phi$ *where* Φ *belongs to* $Z^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$.
Similarly, (iii) *is equivalent to* (iv)*:*
(iii) $\boldsymbol{U} \in \boldsymbol{S}_T^{\lambda}(\Gamma)$ *is a solution of* (6.4) *of type 1,*
(iv) $\lambda + 1$ *belongs to* $\Lambda_{\mathrm{Neu}}(\Gamma)$ *and* $\boldsymbol{U} = \mathbf{grad}\, \Phi$ *where* Φ *belongs to* $Z^{\lambda+1}_{\mathrm{Neu}}(\Gamma)$.

PROOF. **1.** In a first step, we investigate the non-zero homogeneous solutions of (6.3) of type 1, i.e. solutions of

$$\mathbf{curl}\, \boldsymbol{U} = 0 \quad \text{and} \quad \operatorname{div} \boldsymbol{U} = 0 \ \text{ in } \Gamma \qquad \text{with} \quad \boldsymbol{U} \in \overset{\circ}{\boldsymbol{S}}{}_N^{\lambda}(\Gamma).$$

Using Lemma 6.1, we immediately obtain that $\boldsymbol{U} = \mathbf{grad}\, \Phi$ with $\Phi = \frac{1}{\lambda+1} \boldsymbol{U} \cdot \boldsymbol{x}$. Thus

$$\Phi \in S_1^{\lambda+1} \quad \text{and} \quad \Phi = 0 \quad \text{on} \quad \partial\Gamma.$$

Moreover the condition $\operatorname{div} \boldsymbol{U} = 0$ yields that $\Delta\Phi = 0$. In other words, Φ is a Dirichlet singularity for Δ, thus $\lambda+1$ belongs to $\Lambda_{\mathrm{Dir}}(\Gamma)$ and $\Phi \in Z^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$. The converse statement is straightforward: for any $\Phi \in Z^{\lambda+1}_{\mathrm{Dir}}(\Gamma)$, $\boldsymbol{U}$ defined as $\mathbf{grad}\, \Phi$ is a singularity of type 1.

Concerning the magnetic boundary condition, the same arguments lead to $\mathbf{grad}\,\Phi = \boldsymbol{U}$, where Φ is still defined by (6.7) and satisfies

$$\Phi \in S_1^{\lambda+1} \quad \text{and} \quad \partial_n \Phi = 0 \quad \text{on} \quad \partial\Gamma,$$

and $\Delta\Phi = 0$. Thus $\lambda + 1$ belongs to $\Lambda_{\mathrm{Neu}}(\Gamma)$.

2. In a second step, we prove that there is no logarithmic term in any solution of type 1. It suffices to study a solution of type 1 with one logarithmic term, i.e. of the form

$$\boldsymbol{U} = \boldsymbol{U}^0 + \boldsymbol{U}^1 \log\rho, \quad \text{with} \quad \boldsymbol{U}^0,\ \boldsymbol{U}^1 \in \mathring{\boldsymbol{S}}_N^{\lambda}(\Gamma).$$

Since $\mathbf{curl}\,\boldsymbol{U}$ is the sum of $\mathbf{curl}\,\boldsymbol{U}^1 \log\rho$ and of a field with each component in $\mathring{S}^{\lambda-1}(\Gamma)$, we deduce that $\mathbf{curl}\,\boldsymbol{U}^1 = 0$. Thus we obtain that $\boldsymbol{U}^1$ is itself a solution of type 1 of the same problem. Then instead of (6.7), we set

$$\Phi = \frac{\boldsymbol{U}\cdot\boldsymbol{x}}{\lambda+1} - \frac{\boldsymbol{U}^1\cdot\boldsymbol{x}}{(\lambda+1)^2}, \tag{6.10}$$

and we deduce from the previous remark that $\mathbf{grad}\,\Phi = \boldsymbol{U}$, and Φ satisfies the Dirichlet (or Neumann) conditions on $\partial\Gamma$ and $\Delta\Phi = 0$. Therefore Φ belongs to $Z^{\lambda+1}(\Gamma)$, but since we do not consider polynomial right hand sides here, there is no logarithmic term in Φ, hence $\boldsymbol{U}^1 = 0$. ∎

If $\boldsymbol{U}$ is a singularity of type 2, then $\boldsymbol{\Psi}$ is a singularity of type 1 with a permutation of the roles of electric and magnetic boundary conditions. Moreover, when $\boldsymbol{\Psi}$ is known, Lemma 6.2 provides a formula for $\boldsymbol{U}$ (note that here $\mathrm{div}\boldsymbol{\Psi} = 0$, thus formula (6.8) yields a divergence free $\boldsymbol{U}$). Thus we obtain:

Lemma 6.5 *We assume that* $\lambda \notin \{-1, 0\}$. *Then* (i) *is equivalent to* (ii)*:*
(i) $\boldsymbol{U} \in \boldsymbol{S}_N^{\lambda}(\Gamma)$ *is a solution of* (6.3) *of type 2,*
(ii) λ *belongs to* $\Lambda_{\mathrm{Neu}}(\Gamma)$ *and* $\mathbf{curl}\,\boldsymbol{U} = \mathbf{grad}\,\Phi$ *where* Φ *belongs to* $Z_{\mathrm{Neu}}^{\lambda}(\Gamma)$.
Similarly (iii) *is equivalent to* (iv)*:*
(iii) $\boldsymbol{U} \in \boldsymbol{S}_T^{\lambda}(\Gamma)$ *is a solution of* (6.4) *of type 2,*
(iv) λ *belongs to* $\Lambda_{\mathrm{Dir}}(\Gamma)$ *and* $\mathbf{curl}\,\boldsymbol{U} = \mathbf{grad}\,\Phi$ *where* Φ *belongs to* $Z_{\mathrm{Dir}}^{\lambda}(\Gamma)$.
In each case, representatives of type 2 are given by $\boldsymbol{U} = \frac{1}{\lambda+1}\,\mathbf{grad}\,\Phi \times \boldsymbol{x}$.

Finally we have directly from equations (6.3a) and (6.4a) the necessary conditions for the existence of a non-zero q and we combine lemmas 6.2 and 6.3 to obtain formulas for $\boldsymbol{\Psi}$ and $\boldsymbol{U}$:

Lemma 6.6 *We assume that* $\lambda \notin \{-\frac{1}{2}, 0\}$. *Then* (i) *is equivalent to* (ii)*:*
(i) $\boldsymbol{U} \in \boldsymbol{S}_N^{\lambda}(\Gamma)$ *is a solution of* (6.3) *of type 3,*
(ii) $\lambda - 1$ *belongs to* $\Lambda_{\mathrm{Dir}}(\Gamma)$ *and* $\mathrm{div}\,\boldsymbol{U} = q$ *where* q *belongs to* $Z_{\mathrm{Dir}}^{\lambda-1}(\Gamma)$.
Similarly (iii) *is equivalent to* (iv)*:*
(iii) $\boldsymbol{U} \in \boldsymbol{S}_T^{\lambda}(\Gamma)$ *is a solution of* (6.4) *of type 3,*

(iv) λ *belongs to* $\Lambda_{\mathrm{Neu}}(\Gamma)$ *and* $\operatorname{div}\boldsymbol{U} = q$ *where* q *belongs to* $Z_{\mathrm{Neu}}^{\lambda-1}(\Gamma)$.
In each case, representatives of type 3 are given by $\boldsymbol{\Psi} = \frac{1}{\lambda}\,\mathbf{grad}\, q \times \boldsymbol{x}$ *and by* $\boldsymbol{U} = \frac{1}{\lambda(2\lambda+1)}\big((2\lambda-1)q\boldsymbol{x} - \rho^2\,\mathbf{grad}\, q\big)$.

Remark 6.7 For the sake of comparison, let us consider the case when Γ is a dihedron of opening ω. Then $\Lambda_{\mathrm{Dir}}(\Gamma) = \Lambda_{\mathrm{Neu}}(\Gamma) = \{\frac{k\pi}{\omega} + \ell,\ \ k,\ell \in \mathbb{Z},\ \ k \neq 0\}$, *cf* [12, Ch.18.C]. In this case, for finding explicit expressions for the singular functions, one can choose between the formulas given in Lemma 4.4 and those of Lemma 6.6. They do, however, not give the same results because of the non-trivial influence of homogeneous polynomials in the tangential variable z along the edge. ■

6.d The Maxwell singularities generated by the topology

It essentially remains to investigate the solutions of type 1 for $\lambda = -1$, i.e. the elements $\boldsymbol{U}$ in $\boldsymbol{S}_N^{-1}(\Gamma)$, resp. $\boldsymbol{S}_T^{-1}(\Gamma)$ with zero curl and divergence. The existence of such solutions depends on the topology of the spherical domain G which generates the cone Γ. We are going to prove that we have singularity spaces in $\lambda = -1$, $\boldsymbol{Z}_N^{-1}(\Gamma)$ and $\boldsymbol{Z}_T^{-1}(\Gamma)$, if and only if G is not simply connected, and that their dimensions are equal to the dimension of the homology space of G.

Lemma 6.8 *Let us assume that* G *is simply connected. If* $\boldsymbol{U}$ *belongs to* $\boldsymbol{S}_N^{-1}(\Gamma)$*, resp.* $\boldsymbol{S}_T^{-1}(\Gamma)$ *and satisfies* $\mathbf{curl}\,\boldsymbol{U} = 0$ *and* $\operatorname{div}\boldsymbol{U} = 0$*, then* $\boldsymbol{U} = 0$.

PROOF. Since Γ is simply connected, we derive from the condition $\mathbf{curl}\,\boldsymbol{U} = 0$ that $\boldsymbol{U}$ is the gradient of some function Φ. Then we use a formula of integration of $\boldsymbol{U}$ along paths like (5.7): we fix $\boldsymbol{x}_0 \in \Gamma$ and write in polar coordinates $(\rho,\vartheta) = (|\boldsymbol{x}|, \boldsymbol{x}/|\boldsymbol{x}|)$ and $(\rho_0,\vartheta_0) = (|\boldsymbol{x}_0|, \boldsymbol{x}_0/|\boldsymbol{x}_0|)$:

$$\Phi(\rho,\vartheta) - \Phi(\rho_0,\vartheta_0) = \int_{\rho_0}^{\rho} \frac{\boldsymbol{x}_0}{\rho_0}\cdot \boldsymbol{U}(\rho',\vartheta_0)\,d\rho' \ +\ \rho\int_{\gamma(\vartheta_0,\vartheta)} \boldsymbol{U}(\rho,\vartheta')\cdot d\vartheta', \tag{6.11}$$

where the second integral is a path integral along a curve $\gamma(\vartheta_0,\vartheta)$ from ϑ_0 to ϑ in G. From (6.11), we find that Φ belongs to $S_1^0(\Gamma)$. The condition $\operatorname{div}\boldsymbol{U} = 0$ yields that $\Delta\Phi = 0$ and the boundary conditions on $\boldsymbol{U}$ give either the Dirichlet conditions on Φ, or the Neumann condition. In the first case, we find that $\Phi = 0$ since the eigenvalues μ_j^{Dir} are all > 0, and in the second case, we find that Φ is a constant, thus in any case $\boldsymbol{U} = 0$. ■

If the spherical domain G is not simply connected, its boundary ∂G is not connected. Let $\partial_j G$, $j = 1,\ldots,J+1$ be its connected components ($J \geq 1$). We assume that G itself is connected (if not, the cones corresponding to each of its connected components can be considered separately). Then there exist J regular and non-intersecting cuts σ_j, $j = 1,\ldots,J$ such that $G^0 := G \setminus \cup_{j=1}^{J}\sigma_j$ is simply connected.

The singularities of degree -1 that we are investigating are closely linked with the kernels (6.12) and (6.13) of the tangential curl and divergence, $\mathrm{curl}_\top$ and $\mathrm{div}_\top$. These operators are tangential to the sphere $\mathbb{S}^2$ and can be defined with the help of the usual $\mathbf{curl}$ and div on three-dimensional fields: we first introduce $L^2_\top(G)$ as the subspace of $L^2(G)^3$ spanned by the fields $\boldsymbol{v}$ tangential to the sphere, i.e. satisfying $\boldsymbol{v}\cdot\boldsymbol{x}=0$. If $\boldsymbol{v}$ belongs to $L^2_\top(G)$, we can introduce $\widehat{\boldsymbol{v}}$ as any homogeneous extension of $\boldsymbol{v}$ to the cone Γ: we fix any μ and $\widehat{\boldsymbol{v}}(\rho,\vartheta)$ is defined as $\rho^\mu\boldsymbol{v}(\vartheta)$. Then

- $\mathrm{div}_\top\boldsymbol{v}$ is the restriction on $\rho=1$ of $\mathrm{div}\,\widehat{\boldsymbol{v}}$,
- $\mathrm{curl}_\top\boldsymbol{v}$ is the restriction on $\rho=1$ of $\boldsymbol{x}\cdot\mathbf{curl}\,\widehat{\boldsymbol{v}}$.

Then the two kernels are defined in a classical way:

$$K_N(G)=\big\{\boldsymbol{v}\in L^2_\top(G)\mid \mathrm{curl}_\top\boldsymbol{v}=0,\ \mathrm{div}_\top\boldsymbol{v}=0\ \text{ in } G\,,\ \boldsymbol{v}\times\boldsymbol{n}=0\ \text{ on } \partial G\big\} \quad (6.12)$$

and

$$K_T(G)=\big\{\boldsymbol{v}\in L^2_\top(G)\mid \mathrm{curl}_\top\boldsymbol{v}=0,\ \mathrm{div}_\top\boldsymbol{v}=0\ \text{ in } G\,,\ \boldsymbol{v}\cdot\boldsymbol{n}=0\ \text{ on } \partial G\big\}. \quad (6.13)$$

Their description involves the tangential gradient $\mathbf{grad}_\top$, and also (alternatively) the tangential vectorial curl $\mathbf{curl}_\top$, which are defined for any scalar function ϕ in $L^2(G)$ with the help of any homogeneous extension $\widehat{\phi}$ of ϕ to Γ as follows:

- $\mathbf{grad}_\top\phi$ is the restriction on $\rho=1$ of $\mathbf{grad}\,\widehat{\phi}-(\mathbf{grad}\,\widehat{\phi}\cdot\boldsymbol{x})\,\boldsymbol{x}$,
- $\mathbf{curl}_\top\phi$ is the restriction on $\rho=1$ of $\mathbf{curl}(\widehat{\phi}\,\boldsymbol{x})$.

There holds the following description of the spaces $K_N(G)$ and $K_T(G)$ (see [7] for a classical presentation and [2] for the case of less regular domains). In the definitions (6.14) and (6.15) below, the c_j denote arbitrary constant functions, n_j is a unitary normal to σ_j in $\mathbb{S}^2$ and $[\,\cdot\,]_{\sigma_j}$ the jump across σ_j along n_j:

Lemma 6.9
(i) *The space* $K_N(G)$ *is generated by the tangential gradients* $\mathbf{grad}_\top\phi$ *where* $\phi\in P_{\mathrm{Dir}}(G)$,

$$P_{\mathrm{Dir}}(G)=\big\{\phi\in H^1(G)\mid \begin{array}{l}\Delta_G\phi=0\ \textit{ in } G,\\ \phi=c_j\ \textit{ on } \partial_jG,\ 1\le j\le J+1\big\}.\end{array} \quad (6.14)$$

The dimension of $P_{\mathrm{Dir}}(G)$ *is* $J+1$ *and the dimension of* $K_N(G)$ *is* J.

(ii) *The space* $K_T(G)$ *is generated by the* L^2 *extensions* $\widetilde{\mathbf{grad}}_\top\phi$ *to* G *of the tangential gradients* $\mathbf{grad}_\top\phi$ *on* G^0 *where* $\phi\in P_{\mathrm{Neu}}(G)$,

$$P_{\mathrm{Neu}}(G)=\big\{\phi\in H^1(G^0)\mid \begin{array}{l}\Delta_G\phi=0\ \textit{ in } G,\ \partial_n\phi=0\ \textit{ on } \partial G,\\ {[\phi]}_{\sigma_j}=c_j,\ [\partial_{n_j}\phi]_{\sigma_j}=0,\ 1\le j\le J\big\}.\end{array} \quad (6.15)$$

The dimension of $P_{\mathrm{Neu}}(G)$ *is* J *and the dimension of* $K_T(G)$ *is* J, *too.*

Relying on the definitions of the "tangential" operators $\mathbf{curl}_\top$, $\mathbf{grad}_\top$, $\operatorname{curl}_\top$, $\operatorname{div}_\top$ and on the relations (6.6b)-(6.6d), it is easy to show that for any $\phi \in L^2(G)$ and any $\boldsymbol{v} \in L^2_\top(G)$

$$\mathbf{curl}_\top\phi = (\mathbf{grad}_\top\phi) \times \boldsymbol{x}, \quad \operatorname{div}_\top\boldsymbol{v} = -\operatorname{curl}_\top(\boldsymbol{v} \times \boldsymbol{x}), \quad \operatorname{curl}_\top\boldsymbol{v} = \operatorname{div}_\top(\boldsymbol{v} \times \boldsymbol{x}).$$

Then we can prove the following

Corollary 6.10 *The space $K_N(G)$ is generated by the extended tangential curls $\widetilde{\mathbf{curl}}_\top\phi$ where ϕ spans the space $P_{\mathrm{Neu}}(G)$ and the space $K_T(G)$ is generated by $\mathbf{curl}_\top\phi$ where ϕ spans the space $P_{\mathrm{Dir}}(G)$. Moreover, we have the relations*

$$K_T = \boldsymbol{x} \times K_N \quad \textit{and} \quad K_N = \boldsymbol{x} \times K_T.$$

Thus, the kernels $K_N(G)$ and $K_T(G)$ are gradients (or curls) of harmonic functions belonging to the spaces $P_{\mathrm{Dir}}(G)$ and $P_{\mathrm{Neu}}(G)$. We extend the elements of these spaces to homogeneous functions of degree 0 on the cone Γ as $\Phi(\rho,\vartheta) = \phi(\vartheta)$ and thus define the spaces $P_{\mathrm{Dir}}(\Gamma)$ and $P_{\mathrm{Neu}}(\Gamma)$. We note that any $\Phi \in P_{\mathrm{Dir}}(\Gamma)$ has its traces constant on each connected component of $\partial\Gamma$ and similarly that the jumps of any $\Phi \in P_{\mathrm{Neu}}(\Gamma)$ across the cuts Σ_j of Γ corresponding to σ_j are constant too.

For any $\Phi \in P_{\mathrm{Dir}}(\Gamma)$, the gradient $\mathbf{grad}\,\Phi$ is a homogeneous function of degree -1 whose radial component is 0: we have

$$\mathbf{grad}\,\Phi(\rho,\vartheta) = \frac{1}{\rho}\ \mathbf{grad}_\top\phi(\vartheta).$$

The field $\boldsymbol{U} = \mathbf{grad}\,\Phi$ belongs to $\boldsymbol{S}^{-1}_N(\Gamma)$ and satisfies $\mathbf{curl}\,\boldsymbol{U} = 0$ like all gradients, and $\operatorname{div}\boldsymbol{U} = \Delta\Phi = \rho^{-2}\Delta_G\phi = 0$ by construction. Similarly for $\Phi \in P_{\mathrm{Neu}}(\Gamma)$, the extended gradient $\boldsymbol{U} = \widetilde{\mathbf{grad}}\,\Phi$ belongs to $\boldsymbol{S}^{-1}_T(\Gamma)$ and satisfies $\mathbf{curl}\,\boldsymbol{U} = 0$ and $\operatorname{div}\boldsymbol{U} = 0$.

Lemma 6.11 *Let us assume that G is not simply connected. Then $\boldsymbol{Z}^{-1}_N(\Gamma)$ is the space of the fields of the form $\boldsymbol{U} = \mathbf{grad}\,\Phi$, where $\Phi \in P_{\mathrm{Dir}}(\Gamma)$. Correspondingly, $\boldsymbol{Z}^{-1}_T(\Gamma)$ is the space of the fields of the form $\boldsymbol{U} = \widetilde{\mathbf{grad}}\,\Phi$, where $\Phi \in P_{\mathrm{Neu}}(\Gamma)$.*

PROOF. We have just proved that any field of the form $\boldsymbol{U} = \mathbf{grad}\,\Phi$, where Φ is a non-zero element of $P_{\mathrm{Dir}}(\Gamma)$ is a non-trivial element of $\boldsymbol{Z}^{-1}_N(\Gamma)$. Conversely let $\boldsymbol{U}$ belong to $\boldsymbol{Z}^{-1}_N(\Gamma)$. Then, for a non-zero field $\boldsymbol{U}_P$, $P \geq 0$, we have

$$\boldsymbol{U} = \rho^{-1}\big(\boldsymbol{U}_0 + \cdots + \log^P\rho\ \boldsymbol{U}_P\big).$$

Since $\mathbf{curl}\,\boldsymbol{U} = 0$, $\boldsymbol{U}$ is a gradient $\mathbf{grad}\,\Phi$ in the simply connected domain Γ^0 generated by the spherical domain G^0. Using formula (6.11) with paths $\gamma(\vartheta_0,\vartheta)$ contained in G^0, we obtain that Φ belongs to $S^0_1(\Gamma^0)$ and can be expanded into

$$\Phi = \Phi_0 + \cdots + \log^Q\rho\ \Phi_Q, \quad \text{with} \quad P \leq Q \leq P+1.$$

Since $\boldsymbol{U}\times\boldsymbol{n}$ is zero on $\partial\Gamma$, the traces of Φ on $\partial\Gamma$ are constant on each of its connected components $\partial_j\Gamma$. Thus the traces of Φ_0 are constant on each $\partial_j G$ and the traces of Φ_q for $q\geq 1$ are zero. Since $\mathbf{curl}\,\boldsymbol{U}$ and $\operatorname{div}\boldsymbol{U}$ are zero in Γ, the jumps of $\boldsymbol{U}$ across the cuts Σ_j generated by σ_j are zero. Thus $[\Phi]_{\Sigma_j}$ is constant and $[\partial_{n_j}\Phi]_{\Sigma_j}$ is zero. With the conditions on the traces on $\partial_j\Gamma$, this yields that $[\Phi]_{\Sigma_j}$ is zero. Moreover $\operatorname{div}\boldsymbol{U}=0$ in Γ gives $\Delta_G\Phi_Q=0$.

Therefore Φ_Q belongs to $P_{\mathrm{Dir}}(G)$, and if $Q\geq 1$ we have moreover that the traces of Φ_Q on ∂G are all zero, thus $\Phi_Q=0$. Whence $Q=0$ and Φ_0 belongs to $P_{\mathrm{Dir}}(G)$, so $\boldsymbol{U}$ has the desired form.

The proof for $\boldsymbol{Z}_T^{-1}(\Gamma)$ is similar. ■

6.e Corner singularities: a synthesis

We summarize all the results in the following table, where we omit the reference to the cone Γ in the notation of spaces:

Type	λ	$>$	Generator	$\boldsymbol{U}$	$\boldsymbol{\Psi}=\mathbf{curl}\,\boldsymbol{U}$	$q=\operatorname{div}\boldsymbol{U}$
1_Δ	$\lambda+1\in\Lambda^{\mathrm{Dir}}$	-1	$\Phi_{\mathrm{Dir}}\in Z^{\lambda+1}_{\mathrm{Dir}}$	$\mathbf{grad}\,\Phi_{\mathrm{Dir}}$	0	0
2_Δ	$\lambda\in\Lambda^{\mathrm{Neu}}$	0	$\Phi_{\mathrm{Neu}}\in Z^{\lambda}_{\mathrm{Neu}}$	$\dfrac{\mathbf{grad}\,\Phi_{\mathrm{Neu}}\times\boldsymbol{x}}{\lambda+1}$	$\mathbf{grad}\,\Phi_{\mathrm{Neu}}$	0
3_Δ	$\lambda-1\in\Lambda^{\mathrm{Dir}}$	1	$q\in Z^{\lambda-1}_{\mathrm{Dir}}$	$\dfrac{(2\lambda-1)q\boldsymbol{x}-\rho^2\,\mathbf{grad}\,q}{\lambda(2\lambda+1)}$	$\dfrac{\mathbf{grad}\,q\times\boldsymbol{x}}{\lambda}$	q
1_{Top}	-1		$\Phi_{\mathrm{Dir}}\in P_{\mathrm{Dir}}$	$\mathbf{grad}\,\Phi_{\mathrm{Dir}}$	0	0
2_{Top}	0		$\Phi_{\mathrm{Neu}}\in P_{\mathrm{Neu}}$	$\widetilde{\mathbf{grad}}\Phi_{\mathrm{Neu}}\times\boldsymbol{x}$	$\widetilde{\mathbf{grad}}\Phi_{\mathrm{Neu}}$	0

Alternative formulation

2_{Top}	0		$\Phi_{\mathrm{Dir}}\in P_{\mathrm{Dir}}$	$\rho\,\mathbf{grad}\,\Phi_{\mathrm{Dir}}$	$\dfrac{\boldsymbol{x}}{\rho}\times\mathbf{grad}\,\Phi_{\mathrm{Dir}}$	0

Table 1

The alternative formulation is obtained with the help of Corollary 6.10. The adaptation of this table to magnetic boundary conditions is left to the reader.

6.f Corner singularities for non-zero frequency ω

Going back to the primitive Maxwell equations (0.5), we see that for a regular current density $\boldsymbol{J}$, the divergences of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are regular too, thus only the singularities of types 1 and 2 can occur and they exchange each other between the electric and magnetic fields (here λ denotes the degree of homogeneity of the generator and is either the degree of $\boldsymbol{E}$ or $\boldsymbol{H}$):

Type	Generator	λ	$\boldsymbol{E}$	$\boldsymbol{H}$
Δ (electric)	$\Phi_{\rm Dir} \in Z^{\lambda}_{\rm Dir}$	$\lambda \in \Lambda^{\rm Dir}$	$\mathbf{grad}\,\Phi_{\rm Dir}$	$-i\omega\,\dfrac{\mathbf{grad}\,\Phi_{\rm Dir} \times \boldsymbol{x}}{\lambda+1}$
Δ (magnetic)	$\Phi_{\rm Neu} \in Z^{\lambda}_{\rm Neu}$	$\lambda \in \Lambda^{\rm Neu}$	$i\omega\,\dfrac{\mathbf{grad}\,\Phi_{\rm Neu} \times \boldsymbol{x}}{\lambda+1}$	$\mathbf{grad}\,\Phi_{\rm Neu}$
Top (electric)	$\Phi_{\rm Dir} \in P_{\rm Dir}$	0	$\mathbf{grad}\,\Phi_{\rm Dir}$	$-i\omega\;\mathbf{grad}\,\Phi_{\rm Dir} \times \boldsymbol{x}$
Top (magnetic)	$\Phi_{\rm Neu} \in P_{\rm Neu}$	0	$i\omega\,\widetilde{\mathbf{grad}}\Phi_{\rm Neu} \times \boldsymbol{x}$	$\widetilde{\mathbf{grad}}\Phi_{\rm Neu}$

Table 2

This table gives the *principal parts* of the singularities (as can be seen from (0.9) or (0.10) the operators are not homogeneous and according to the general theory [17, 12] the singularities themselves have an asymptotic expansion).

6.g Pseudo-Maxwell corner singularities

There is, in general, no simple relation between the singular functions of the pseudo-Maxwell problems (see in §4.f the singularity spaces $\widetilde{\boldsymbol{Z}}^{\lambda}_{N}(\Gamma)$) and those of the Dirichlet or Neumann problems for the Laplace operator. In particular, our previous explicit constructions do not work here, and the classification into types 1, 2, 3 does not make sense. Let us explain two reasons for this.

First, the solutions of the first-order systems (6.3b) and (6.3c) do not belong to H^1 near a non-convex edge. Thus, independently of the corner exponent λ, the X_N-singular functions of all 3 types will not belong to H^1.

Second, the Laplace-Dirichlet problem (6.3a) for q is now posed with only L^2 regularity required. This problem (with $S_0^\lambda(\Gamma)$ defined in §6.b)

$$-\Delta q = 0 \text{ in } \Gamma \quad \text{and} \quad q = 0 \text{ on } \partial\Gamma \qquad \text{with } q \in S_0^{\lambda-1}(\Gamma) \tag{6.16}$$

does now not select a discrete set of exponents λ.

Proposition 6.12 *Let Γ be a non-convex polyhedral cone. Then the Dirichlet problem* (6.16) *has non-trivial solutions for any $\lambda \in \mathbb{C}$.*

PROOF. This is a Laplace-Beltrami eigenvalue problem on $G = \Gamma \cap \mathbb{S}^2$. We look for eigenfunctions in $L^2(G)$. By duality (and the "very weak" definition of the Dirichlet problem (6.16), see (1.7)), we see that such eigenfunctions span the orthogonal complement of the image of $\mathring{H}^1(G) \cap H^2(G)$ under the adjoint operator. Now we know that in the presence of non-convex corners of G, one never has H^2 regularity for this Laplace-Beltrami Dirichlet eigenvalue problem. Thus, in addition to the $\mathring{H}^1(G)$ eigenfunctions that may exist if $\lambda - 1 \in \Lambda^{\mathrm{Dir}}(\Gamma)$, we find as many $L^2(G)$ eigenfunctions and therefore solutions to (6.16) as there are non-convex edges meeting at $\boldsymbol{c}$. ■

7 Variational formulations of Maxwell's equations

In this section, we discuss some commonly used variational formulations of the time-harmonic Maxwell equations. We give a proof of Theorem 0.1. Since the proof works in a more general setting, we consider general inhomogeneous materials here. We also prove a generalization of the regularity Theorem 1.2 for the divergence. The domain Ω is a 3D corner domain as defined in the preliminaries (§0).

7.a Time harmonic Maxwell's equations

The following assumptions correspond to the modelling of general linear, anisotropic inhomogeneous materials that can have a nonvanishing conductivity.

Let ε and μ two complex 3×3 matrices with L^∞ elements on Ω such that their symmetric part is positive in the sense that there exist $\rho_0 > 0$ such that for all $x \in \Omega$ and for all $\xi \in \mathbb{C}^3$:

$$\operatorname{Re}(\varepsilon(x)\xi \cdot \bar{\xi}) \geq \rho_0 |\xi|^2 \quad \text{and} \quad \operatorname{Re}(\mu(x)\xi \cdot \bar{\xi}) \geq \rho_0 |\xi|^2.$$

The classical time harmonic Maxwell equations describing electromagnetic radiation of frequency ω in a body occupying Ω, with permeability μ and permittivity ε are

$$\mathbf{curl}\,\boldsymbol{E} - i\omega\,\mu\boldsymbol{H} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{H} + i\omega\,\varepsilon\boldsymbol{E} = \boldsymbol{J} \quad \text{in} \quad \Omega. \tag{7.1a}$$

Here $\boldsymbol{E}$ is the electric part and $\boldsymbol{H}$ the magnetic part of the electromagnetic field. The right hand side $\boldsymbol{J}$ is the current density, where a current obeying Ohm's law can be subtracted giving a nonzero imaginary part of ε. As boundary conditions on $\partial\Omega$ we consider only those of the perfect conductor ($\boldsymbol{n}$ denotes the unit outer normal on $\partial\Omega$):

$$\boldsymbol{E}\times\boldsymbol{n}=0 \quad\text{and}\quad \mu\boldsymbol{H}\cdot\boldsymbol{n}=0 \quad\text{on}\quad \partial\Omega. \tag{7.1b}$$

If the body is formed by several different homogeneous media, ε and μ are piecewise constant and there are internal transmission conditions at the interfaces contained in the functional formulation. Equations (7.1a) hide equations on the divergence of the fields, as soon as ω is not 0 : taking the divergence of (7.1a) leads to

$$\operatorname{div}(\varepsilon\boldsymbol{E})=\frac{1}{i\omega}\operatorname{div}\boldsymbol{J} \quad\text{and}\quad \operatorname{div}(\mu\boldsymbol{H})=0. \tag{7.1c}$$

In pure radiation problems, the charge density is zero, hence $\operatorname{div}\boldsymbol{J}=0$.

Let us assume that $\boldsymbol{J}\in H(\operatorname{div};\Omega)$, i.e. $\boldsymbol{J}$ belongs to $L^2(\Omega)^3$ and its divergence $\operatorname{div}\boldsymbol{J}$ belongs to $L^2(\Omega)$. The equations (7.1a) and (7.1c) yield immediately that if $\boldsymbol{E}$ and $\boldsymbol{H}$ are in $L^2(\Omega)^3$, then they belong respectively to the following spaces

$$\boldsymbol{E}\in H(\mathbf{curl};\Omega)\cap H(\operatorname{div};\varepsilon;\Omega) \qquad\text{and}\qquad \boldsymbol{H}\in H(\mathbf{curl};\Omega)\cap H(\operatorname{div};\mu;\Omega),$$

where $H(\operatorname{div};\rho;\Omega)$ is the space $\{\boldsymbol{u}\in L^2(\Omega)^3 \mid \operatorname{div}(\rho\boldsymbol{u})\in L^2(\Omega)\}$. Taking into account the boundary conditions (7.1b), we obtain that

$$\boldsymbol{E}\in X_N \qquad\text{and}\qquad \boldsymbol{H}\in X_T,$$

where we define now

$$X_N=\{\boldsymbol{u}\in H(\mathbf{curl};\Omega)\cap H(\operatorname{div};\varepsilon;\Omega) \mid \quad \boldsymbol{u}\times\boldsymbol{n}=0 \quad\text{on}\quad \partial\Omega\}$$

and

$$X_T=\{\boldsymbol{u}\in H(\mathbf{curl};\Omega)\cap H(\operatorname{div};\mu;\Omega) \mid \quad (\mu\boldsymbol{u})\cdot\boldsymbol{n}=0 \quad\text{on}\quad \partial\Omega\}.$$

These are our variational spaces.

7.b Variational formulation for the electric field

We construct first a commonly used coercive variational formulation containing a "regularization" or "penalization" parameter s (see [16]). Choose a test field $\boldsymbol{E}'\in X_N$. As a consequence of the assumptions, μ is invertible. Let us integrate the first equation of (7.1a) versus $(\mu^T)^{-1}\boldsymbol{E}'$, and the second versus $i\omega\boldsymbol{E}'$. Since for $\boldsymbol{E}'\in X_N$ and $\boldsymbol{H}\in H(\mathbf{curl};\Omega)$, there holds:

$$\int_\Omega \boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{E}'\,dx=\int_\Omega \mathbf{curl}\,\boldsymbol{H}\cdot\boldsymbol{E}'\,dx,$$

we obtain

$$\boldsymbol{E}\in X_N,\quad \forall\boldsymbol{E}'\in X_N,\quad \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{E}'-\omega^2\,\varepsilon\boldsymbol{E}\cdot\boldsymbol{E}'=i\omega\int_\Omega \boldsymbol{J}\cdot\boldsymbol{E}'. \tag{7.2}$$

Taking into account the equation (7.1c) on the divergence of $\boldsymbol{E}$, we introduce a parameter $s > 0$ and the new right hand sides

$$\boldsymbol{f}[\boldsymbol{J}, s](\boldsymbol{v}) = i\omega \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} + \frac{s}{i\omega} \int_\Omega \operatorname{div} \boldsymbol{J} \; \operatorname{div} \bar{\varepsilon} \boldsymbol{v} \qquad \text{and} \qquad g[\boldsymbol{J}] = \frac{1}{i\omega} \operatorname{div} \boldsymbol{J}. \tag{7.3}$$

Then we define the following variational problem ($\bar{\varepsilon}$ is the complex conjugate of ε)

$$\boldsymbol{u} \in X_N, \; \forall \boldsymbol{v} \in X_N, \; \int_\Omega \mu^{-1} \operatorname{\mathbf{curl}} \boldsymbol{u} \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} + s \operatorname{div} \varepsilon \boldsymbol{u} \; \operatorname{div} \bar{\varepsilon} \boldsymbol{v} - \omega^2 \varepsilon \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{f}(\boldsymbol{v}), \tag{7.4}$$

and its saddle-point version, which involves a Lagrange multiplier p (a pseudo-pressure):

$$(\boldsymbol{u}, p) \in X_N \times L^2(\Omega), \quad \forall (\boldsymbol{v}, q) \in X_N \times L^2(\Omega),$$

$$\left\{ \begin{aligned} \int_\Omega \mu^{-1} \operatorname{\mathbf{curl}} \boldsymbol{u} \cdot \operatorname{\mathbf{curl}} \boldsymbol{v} + s \operatorname{div} \varepsilon \boldsymbol{u} \; \operatorname{div} \bar{\varepsilon} \boldsymbol{v} + p \; \operatorname{div} \bar{\varepsilon} \boldsymbol{v} - \omega^2 \varepsilon \boldsymbol{u} \cdot \boldsymbol{v} \; &= \; \boldsymbol{f}(\boldsymbol{v}), \\ \int_\Omega \operatorname{div} \varepsilon \boldsymbol{u} \, q \; &= \; \int_\Omega g \, q. \end{aligned} \right. \tag{7.5}$$

The following statement describes the equivalence between problems (7.1), (7.4) and (7.5) if $\boldsymbol{f}$ and g are defined in (7.3). The essential argument relies on the properties of the operator

$$\begin{array}{rccc} \Delta_\varepsilon^{\mathrm{Dir}} : & \mathring{H}^1(\Omega) & \longrightarrow & H^{-1}(\Omega) \\ & \varphi & \longmapsto & \operatorname{div} \varepsilon \operatorname{\mathbf{grad}} \varphi. \end{array} \tag{7.6}$$

The assumption about ε implies that the sesquilinear form associated with $-\Delta_\varepsilon^{\mathrm{Dir}}$ is coercive on $\mathring{H}^1(\Omega)$:

$$\operatorname{Re} \int_\Omega \varepsilon \operatorname{\mathbf{grad}} \varphi \cdot \overline{\operatorname{\mathbf{grad}} \varphi} \geq \rho_0 |\varphi|^2_{H^1(\Omega)} .$$

Thus the operator $-\Delta_\varepsilon^{\mathrm{Dir}}$ is invertible from its Dirichlet domain

$$D(\Delta_\varepsilon^{\mathrm{Dir}}) = \{ \varphi \in \mathring{H}^1(\Omega) \mid \quad \Delta_\varepsilon^{\mathrm{Dir}} \varphi \in L^2(\Omega) \}$$

onto $L^2(\Omega)$ and has a discrete spectrum.

Theorem 7.1 *We assume* $\omega \neq 0$. *Let* $\boldsymbol{J} \in H(\operatorname{div}; \Omega)$ *; for a fixed* $s > 0$, $\boldsymbol{f} = \boldsymbol{f}[\boldsymbol{J}, s]$ *and* $g = g[\boldsymbol{J}]$ *as defined in* (7.3).

(i) *If* $(\boldsymbol{E}, \boldsymbol{H})$ *solves* (7.1)*, then* $\boldsymbol{u} = \boldsymbol{E}$ *solves* (7.4) *and* $(\boldsymbol{u}, p) = (\boldsymbol{E}, 0)$ *solves* (7.5).

(ii) *If* $\boldsymbol{u}$ *solves* (7.4) *and* ω^2/s *is not an eigenvalue of the Dirichlet operator* $-\Delta_\varepsilon^{\mathrm{Dir}}$ *on* Ω*, then* $(\boldsymbol{E}, \boldsymbol{H}) = (\boldsymbol{u}, (i\omega\mu)^{-1} \operatorname{\mathbf{curl}} \boldsymbol{u})$ *solves* (7.1).

(iii) *If* $(\boldsymbol{u}, p)$ *solves* (7.5)*, then* $p = 0$ *and* $(\boldsymbol{E}, \boldsymbol{H}) = (\boldsymbol{u}, (i\omega\mu)^{-1} \operatorname{\mathbf{curl}} \boldsymbol{u})$ *solves* (7.1).

PROOF. *(i)* was proved while stating problems (7.4) and (7.5).

(ii) In (7.4) let us take as test functions all fields $\boldsymbol{v} = \mathbf{grad}\,\overline{\varphi}$ with $\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}})$, which ensures that $\mathbf{grad}\,\varphi \in X_N$. Let us denote by $\langle a, b\rangle_\Omega := \int_\Omega a\,\overline{b}$ the hermitian scalar product on $L^2(\Omega)$. Using the expression (7.3) of the right hand side, and the identities, valid for $\varphi \in \mathring{H}^1(\Omega)$

$$\big\langle \varepsilon\boldsymbol{u}\,, \mathbf{grad}\,\varphi\big\rangle_\Omega = -\big\langle\, \operatorname{div}\varepsilon\boldsymbol{u}\,, \varphi\big\rangle_\Omega \quad\text{and}\quad \big\langle \boldsymbol{J}\,, \mathbf{grad}\,\varphi\big\rangle_\Omega = -\big\langle\, \operatorname{div}\boldsymbol{J}\,, \varphi\big\rangle_\Omega\,,$$

we easily arrive at

$$\big\langle\, \operatorname{div}\varepsilon\boldsymbol{u} - g\,,\, s\Delta_\varepsilon^{\mathrm{Dir}}\varphi + \omega^2\varphi\big\rangle_\Omega = 0$$

for all $\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}})$. Thus if ω^2/s is not an eigenvalue of $-\Delta_\varepsilon^{\mathrm{Dir}}$, we find that $\operatorname{div}\varepsilon\boldsymbol{u} = g$. We deduce that $\boldsymbol{u}$ solves problem (7.2). Thus:

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E} - \omega^2\,\varepsilon\boldsymbol{E} = i\omega\boldsymbol{J}.$$

Setting $\boldsymbol{H} = (i\omega\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{u}$, we arrive at (7.1).

(iii) We obtain similarly that

$$\forall\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle p\,, \Delta_\varepsilon^{\mathrm{Dir}}\varphi\big\rangle_\Omega = 0,$$

whence $p = 0$, since $\Delta_\varepsilon^{\mathrm{Dir}}$ is invertible. Thus $\boldsymbol{u}$ solves (7.2), and $(\boldsymbol{u}, (i\omega\mu)^{-1}\,\mathbf{curl}\,\boldsymbol{u})$ solves (7.1). ∎

Both formulations (7.4) and (7.5) are strongly elliptic in the following sense. The norm $\|\cdot\|_{X_N}$ of X_N is given by

$$\|\boldsymbol{u}\|_{X_N}^2 = \int_\Omega |\,\mathbf{curl}\,\boldsymbol{u}|^2 + |\operatorname{div}\varepsilon\boldsymbol{u}|^2 + |\boldsymbol{u}|^2.$$

The principal part of the sesquilinear form associated with problem (7.4)

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\overline{\mathbf{curl}\,\boldsymbol{v}} + s\operatorname{div}\varepsilon\boldsymbol{u}\;\overline{\operatorname{div}\varepsilon\boldsymbol{v}}$$

is coercive on X_N. Moreover, concerning the saddle-point formulation (7.5), we introduce

$$b(p, \boldsymbol{v}) = \int_\Omega p\,\overline{\operatorname{div}\varepsilon\boldsymbol{v}}$$

and, as an easy consequence of the invertibility of $\Delta_\varepsilon^{\mathrm{Dir}}$, we have the Babuška–Brezzi inf-sup condition, for a constant $\beta > 0$:

$$\forall p \in L^2(\Omega), \quad \sup_{\boldsymbol{v}\in X_N} \frac{b(p, \boldsymbol{v})}{\|\boldsymbol{v}\|_{X_N}} \geq \beta\,\|p\|_{L^2(\Omega)}\,.$$

Thus both variational formulations (7.4) and (7.5) are suitable for theoretical and numerical solution methods of the Maxwell boundary value problem. For almost all $s > 0$, in particular for sufficiently large s, one has equivalence with the original problem.

7.c Variational formulation for the magnetic field

We describe the situation for the magnetic field $\boldsymbol{H}$ with less details, since there are numerous symmetries and Theorem 7.1 yields already information for $\boldsymbol{H}$.

We have already seen that a suitable variational space for $\boldsymbol{H}$ is X_T. The first variational formulation is obtained by integrating the second equation of (7.1a) versus $(\varepsilon^T)^{-1}\boldsymbol{H}'$, and the first one versus $i\omega\boldsymbol{H}'$, for any $\boldsymbol{H}' \in X_T$:

$$\int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H}\cdot\mathbf{curl}\,\boldsymbol{H}' - \omega^2\,\mu\boldsymbol{H}\cdot\boldsymbol{H}' = \int_\Omega \varepsilon^{-1}\boldsymbol{J}\cdot\mathbf{curl}\,\boldsymbol{H}' =: \boldsymbol{h}(\boldsymbol{H}'). \tag{7.7}$$

Taking account of the equation (7.1c) $\operatorname{div}\mu\boldsymbol{H} = 0$, we obtain the following variational problem

$$\boldsymbol{u}\in X_T,\ \forall\boldsymbol{v}\in X_T,\ \int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + s\operatorname{div}\mu\boldsymbol{u}\,\operatorname{div}\bar\mu\boldsymbol{v} - \omega^2\,\mu\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{h}(\boldsymbol{v}), \tag{7.8}$$

and its saddle-point version:

$$(\boldsymbol{u},p)\in X_T\times L^2(\Omega)/\mathbb{C},\quad \forall(\boldsymbol{v},q)\in X_T\times L^2(\Omega)/\mathbb{C},$$
$$\left\{\begin{array}{rcl} \displaystyle\int_\Omega \varepsilon^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} + s\operatorname{div}\mu\boldsymbol{u}\,\operatorname{div}\bar\mu\boldsymbol{v} + p\,\operatorname{div}\bar\mu\boldsymbol{v} - \omega^2\,\mu\boldsymbol{u}\cdot\boldsymbol{v} &=& \boldsymbol{h}(\boldsymbol{v}),\\ \displaystyle\int_\Omega \operatorname{div}\mu\boldsymbol{u}\,q &=& 0. \end{array}\right. \tag{7.9}$$

The Laplace-like operator which plays a similar role as $\Delta_\varepsilon^{\mathrm{Dir}}$ is the Neumann operator $\Delta_\mu^{\mathrm{Neu}}$ defined from its domain

$$D(\Delta_\mu^{\mathrm{Neu}}) = \{\varphi\in H^1(\Omega)\mid \quad \operatorname{div}\mu\,\mathbf{grad}\,\varphi\in L^2(\Omega)\quad\text{and}\quad \partial_n\varphi = 0 \text{ on } \partial\Omega\}$$

by $\Delta_\mu^{\mathrm{Neu}}\varphi = \operatorname{div}\mu\,\mathbf{grad}\,\varphi$. The operator $-\Delta_\mu^{\mathrm{Neu}}$ is invertible from $D(\Delta_\mu^{\mathrm{Neu}})/\mathbb{C}$ onto $L^2_0(\Omega)$ (the subspace orthogonal to constants) and has a discrete spectrum.

Theorem 7.2 *We assume* $\omega\neq 0$. *Let* $\boldsymbol{J}\in H(\operatorname{div};\Omega)$; $\boldsymbol{h}$ *is defined from* $\boldsymbol{J}$ *in* (7.7). *For a fixed* $s>0$:

(i) *If* $(\boldsymbol{E},\boldsymbol{H})$ *solves* (7.1)*, then* $\boldsymbol{u}=\boldsymbol{H}$ *solves* (7.8) *and* $(\boldsymbol{u},p)=(\boldsymbol{H},c)$ *solves* (7.9) *for any* $c\in\mathbb{C}$.

(ii) *If* $\boldsymbol{u}$ *solves* (7.8) *and* ω^2/s *is not an eigenvalue of* $-\Delta_\mu^{\mathrm{Neu}}$ *on* Ω*, then* $(\boldsymbol{E},\boldsymbol{H}) = (\frac{i}{\omega}\,\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{u}-\boldsymbol{J}),\boldsymbol{u})$ *solves* (7.1).

(iii) *If* $(\boldsymbol{u},p)$ *solves* (7.9)*, then* p *is a constant and* $(\frac{i}{\omega}\,\varepsilon^{-1}(\mathbf{curl}\,\boldsymbol{u}-\boldsymbol{J}),\boldsymbol{u})$ *solves* (7.1).

7.d Symmetric roles of divergences and pressures. Regularity

Under a weak assumption of regularity on the right hand sides, we obtain the H^1 regularity for the divergence $\operatorname{div}\varepsilon\boldsymbol{u}$ in (7.4) and the pressure p in (7.5). This comes from the fact that $\operatorname{div}\varepsilon\boldsymbol{u}$ and p are solutions of independent boundary value problems.

Let $\varepsilon^* = \bar{\varepsilon}^T$ be the hermitian adjoint of ε.

Theorem 7.3 *Let $s > 0$, $\boldsymbol{f} \in L^2(\Omega)^3$ and $g \in L^2(\Omega)$. We assume that ω^2/s is not an eigenvalue of the Dirichlet operator $-\Delta_\varepsilon^{\mathrm{Dir}}$ on Ω.*

(i) *If $\boldsymbol{u}$ solves* (7.4)*, then $\operatorname{div}\varepsilon\boldsymbol{u}$ is the solution q of the Dirichlet problem*

$$\begin{cases} q \in \mathring{H}^1(\Omega), \\ s\Delta_{\varepsilon^*}^{\mathrm{Dir}} q + \omega^2 q = -\operatorname{div}\boldsymbol{f}. \end{cases} \tag{7.10}$$

(ii) *If $(\boldsymbol{u}, p)$ solves* (7.5)*, then $p + sg$ is the solution q of the Dirichlet problem*

$$\begin{cases} q \in \mathring{H}^1(\Omega), \\ \Delta_{\varepsilon^*}^{\mathrm{Dir}} q = -\operatorname{div}\boldsymbol{f} - \omega^2 g. \end{cases} \tag{7.11}$$

Thus, if moreover g belongs to $H^1(\Omega)$, the pseudo-pressure p belongs to $H^1(\Omega)$ too. As a straightforward consequence of Theorem 7.3, we rediscover the compatibility condition between $\boldsymbol{f}$ and g (compare (7.3)), which ensures that $\operatorname{div}\varepsilon\boldsymbol{u} = g$ and $p = 0$ respectively:

Corollary 7.4 *Under the assumptions of Theorem* 7.3*, the three following conditions are equivalent:*

(i) *g is solution of the Dirichlet problem* (7.10)*;*

(ii) *If $\boldsymbol{u}$ solves* (7.4)*, then $\operatorname{div}\varepsilon\boldsymbol{u} = g$;*

(iii) *If $(\boldsymbol{u}, p)$ solves* (7.5)*, then $p = 0$.*

In particular, if we search for divergence-ε free solutions of (7.4), the necessary and sufficient condition on $\boldsymbol{f}$ is that $\boldsymbol{f}$ is divergence free.

PROOF OF THEOREM 7.3. *(i)* If $\boldsymbol{u}$ solves (7.4), then taking as test functions $\boldsymbol{v} = \mathbf{grad}\,\overline{\varphi}$ with $\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}})$ we obtain

$$\forall\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle \operatorname{div}\varepsilon\boldsymbol{u}\,,\, s\Delta_\varepsilon^{\mathrm{Dir}}\varphi + \omega^2\varphi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\varphi \big\rangle_\Omega. \tag{7.12}$$

But the solution of (7.10) satisfies

$$\forall\psi \in \mathring{H}^1(\Omega), \qquad -\big\langle s\varepsilon^*\,\mathbf{grad}\,q\,,\, \mathbf{grad}\,\psi \big\rangle_\Omega + \big\langle \omega^2 q\,,\psi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\psi \big\rangle_\Omega,$$

whence

$$\forall\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle q\,,\, s\Delta_\varepsilon^{\mathrm{Dir}}\varphi + \omega^2\varphi \big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\varphi \big\rangle_\Omega.$$

Thus $\operatorname{div}\varepsilon\boldsymbol{u} - q$ is orthogonal to the range of $s\Delta_\varepsilon^{\mathrm{Dir}} + \omega^2$, which is the whole $L^2(\Omega)$.
(ii) If $(\boldsymbol{u}, p)$ solves (7.5), we obtain similarly that

$$\forall\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle sg + p\,,\, \Delta_\varepsilon^{\mathrm{Dir}}\varphi\big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\varphi\big\rangle_\Omega - \big\langle \omega^2 g\,,\varphi\big\rangle_\Omega\,,$$

and that the solution of (7.11) satisfies

$$\forall\varphi \in D(\Delta_\varepsilon^{\mathrm{Dir}}), \qquad \big\langle q\,, \Delta_\varepsilon^{\mathrm{Dir}}\varphi\big\rangle_\Omega = \big\langle \boldsymbol{f}\,,\, \mathbf{grad}\,\varphi\big\rangle_\Omega - \big\langle \omega^2 g\,,\varphi\big\rangle_\Omega\,.$$

Whence $sg + p - q$ is orthogonal to the range of $\Delta_\varepsilon^{\mathrm{Dir}}$, which is the whole $L^2(\Omega)$. ■

We have similar statements concerning the "magnetic" problems (7.8) and (7.9): if $\boldsymbol{u}$ solves (7.8), then $\operatorname{div}\mu\boldsymbol{u}$ is the solution in $H^1(\Omega)$ (with zero mean value if $\omega = 0$) of a Neumann problem. If $(\boldsymbol{u}, p)$ solves (7.9), then $p + sg$ is similarly solution of a Neumann problem.

Remark 7.5 There is another equivalent variational formulation of Maxwell equation. It satisfies an inf-sup condition and is used in finite-element approximations, see NEDELEC [24], GIRAULT - RAVIART [14] and also [2].
Let $\mathring{H}(\mathbf{curl};\Omega)$ be the space of the fields $\boldsymbol{u} \in H(\mathbf{curl};\Omega)$ such that $\boldsymbol{u}\times\boldsymbol{n} = 0$ on $\partial\Omega$. With this definition, we can introduce the alternative saddle-point version, which is a replacement for (7.5) when $s = 0$:

$$(\boldsymbol{u}, p) \in \mathring{H}(\mathbf{curl};\Omega)\times\mathring{H}^1(\Omega), \quad \forall(\boldsymbol{v}, q) \in \mathring{H}(\mathbf{curl};\Omega)\times\mathring{H}^1(\Omega),$$
$$\left\{\begin{array}{rcl} \displaystyle\int_\Omega \mu^{-1}\,\mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} - \mathbf{grad}\,p\cdot\bar\varepsilon\boldsymbol{v} - \omega^2\,\varepsilon\boldsymbol{u}\cdot\boldsymbol{v} &=& \boldsymbol{f}_0(\boldsymbol{v}), \\ \displaystyle -\int_\Omega \varepsilon\boldsymbol{u}\;\mathbf{grad}\,q &=& \displaystyle\int_\Omega g\,q. \end{array}\right. \tag{7.13}$$

For any $s > 0$ and $\boldsymbol{v} \in X_N$, we define, *cf* (7.3)

$$\boldsymbol{f}_s(\boldsymbol{v}) = \boldsymbol{f}_0(\boldsymbol{v}) + s\int_\Omega g\;\operatorname{div}\bar\varepsilon\boldsymbol{v}.$$

Then, if $\boldsymbol{f}_0 \in L^2(\Omega)^3$ and $g \in \mathring{H}^1(\Omega)$, we can prove, as a consequence of Theorem 7.3 and of the density of $\mathcal{D}(\Omega)^3$ in $\mathring{H}(\mathbf{curl};\Omega)$ that $(\boldsymbol{u}, p)$ solves (7.13) if and only if $(\boldsymbol{u}, p)$ solves (7.5) with $\boldsymbol{f} = \boldsymbol{f}_s$. ■

References

[1] M. S. AGRANOVICH, M. I. VISHIK. Elliptic problems with a parameter and parabolic problems of general type. *Russian Math. Surveys* **19** (1964) 53–157.

[2] C. AMROUCHE, C. BERNARDI, M. DAUGE, V. GIRAULT. Vector potentials in three-dimensional nonsmooth domains. *Math. Meth. Appl. Sci.* **21** (1998) 823–864.

[3] F. ASSOUS, P. CIARLET, E. SONNENDRÜCKER. Résolution des équations de Maxwell dans un domaine avec un coin rentrant. *C. R. Acad. Sc. Paris, Série I* **323** (1996) 203–208.

[4] M. BIRMAN, M. SOLOMYAK. L^2-theory of the Maxwell operator in arbitrary domains. *Russ. Math. Surv.* **42 (6)** (1987) 75–96.

[5] M. BIRMAN, M. SOLOMYAK. On the main singularities of the electric component of the electro-magnetic field in regions with screens. *St. Petersbg. Math. J.* **5 (1)** (1993) 125–139.

[6] A. BONNET-BENDHIA, C. HAZARD, S. LOHRENGEL. A singular field method for the solution of Maxwell's equations in polyhedral domains. Rapport de Recherche 299, ENSTA 1997. To appear in SIAM J. Appl. Math.

[7] M. CESSENAT. Exemples en électromagnétisme. In R. DAUTRAY, J.-L. LIONS, editors, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, chapter IX, part. A. Masson, Paris 1988.

[8] M. COSTABEL. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Math. Methods Appl. Sci.* **12 (4)** (1990) 365–368.

[9] M. COSTABEL. A coercive bilinear form for Maxwell's equations. *J. Math. Anal. Appl.* **157 (2)** (1991) 527–541.

[10] M. COSTABEL, M. DAUGE. Singularités des équations de Maxwell dans un polyèdre. *C. R. Acad. Sc. Paris, Série I* **324** (1997) 1005–1010.

[11] M. COSTABEL, M. DAUGE, S. NICAISE. Singularities of Maxwell interface problems. Preprint 98-24, Université de Rennes 1 (1998). To appear in *Modél. Math. Anal. Numér.* `http://www.maths.univ-rennes1.fr/~dauge/` .

[12] M. DAUGE. *Elliptic Boundary Value Problems in Corner Domains – Smoothness and Asymptotics of Solutions*. Lecture Notes in Mathematics, Vol. 1341. Springer-Verlag, Berlin 1988.

[13] N. FILONOV. Système de Maxwell dans des domaines singuliers. Thesis, Université de Bordeaux 1, 1996.

[14] V. GIRAULT, P. RAVIART. *Finite Element Methods for the Navier–Stokes Equations, Theory and Algorithms*. Springer series in Computational Mathematics, 5. Springer-Verlag, Berlin 1986.

[15] P. GRISVARD. *Boundary Value Problems in Non-Smooth Domains*. Pitman, London 1985.

[16] C. HAZARD, M. LENOIR. On the solution of time-harmonic scattering problems for Maxwell's equations. *SIAM J. Math. Anal.* **27 (6)** (1996) 1597–1630.

[17] V. A. KONDRAT'EV. Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.* **16** (1967) 227–313.

[18] V. A. KOZLOV, V. G. MAZ'YA, J. ROSSMANN. Conic singularities of solutions problems in hydrodynamics of a viscous fluid with a free surface. *Matematica Scandinavia,* To appear (1998).

[19] M. KŘIČEK, P. NEITTAANMÄKI. On the validity of Friedrichs' inequalities. *Math. Scand.* **54** (1984) 17–26.

[20] S. LOHRENGEL. Etude théorique et résolution numérique des équations de Maxwell dans un domaine non régulier. Thèse, Université Pierre et Marie Curie 1998.

[21] V. G. MAZ'YA, B. A. PLAMENEVSKII. L^p estimates of solutions of elliptic boundary value problems in a domain with edges. *Trans. Moscow Math. Soc.* **1** (1980) 49–97.

[22] V. G. MAZ'YA, B. A. PLAMENEVSKII. Coefficients in the asymptotics of the solutions of an elliptic boundary value problem in a cone. *Amer. Math. Soc. Trans. (2)* **123** (1984) 57–88.

[23] M. MOUSSAOUI. Espaces $H(\mathrm{div}, \mathrm{rot}, \Omega)$ dans un polygone plan. *C. R. Acad. Sc. Paris, Série I* **322** (1996) 225–229.

[24] J. NEDELEC. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* **35** (1980) 315–341.

[25] J. SARANEN. Über das Verhalten der Lösungen der Maxwellschen Randwertaufgabe in Gebieten mit Kegelspitzen. *Math. Methods Appl. Sci.* **2** (1980) 235–250.

[26] J. SARANEN. On an inequality of Friedrichs. *Math. Scand.* **51** (1982) 310–322.