Singularities of Parallel Manipulators: A Geometric Treatment

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***Abstract*—A parallel manipulator is naturally associated with a set of constraint functions defined by its closure constraints. The differential forms arising from these constraint functions completely characterize the geometric properties of the manipulator. In this paper, using the language of differential forms, we provide a thorough geometric study on the various types of singularities of a parallel manipulator, their relations with the kinematic parameters and the configuration spaces of the manipulator, and the role redundant actuation plays in reshaping the singularities and improving the performance of the manipulator. First, we analyze configuration space singularities by constructing a Morse function on some appropriately defined spaces. By varying key parameters of the manipulator, we obtain homotopic classes of the configuration spaces. This allows us to gain insight on configuration space singularities and understand how to choose design parameters for the manipulator. Second, we define parametrization singularities which include actuator and end-effector singularities (or other equivalent definitions) as their special cases. This definition naturally contains the closure constraints in addition to the coordinates of the actuators and the end-effector and can be used to search a complete set of actuator or end-effector singularities including some singularities that may be missed by the usual kinematics methods. We give an intrinsic classification of parametrization singularities and define their topological orders. While a nondegenerate singularity poses no problems in general, a degenerate singularity can sometimes be a source of danger and should be avoided if possible.**

***Index Terms*—Degenerate, differential form, Morse function, singularity, topological order.**

I. Introduction

COMPARED with its serial counterparts, a parallel manipulator (or a closed-chain mechanism or system) has a much more complex structure in terms of its kinematics, dynamics, planning and control. In particular, the configuration space of a parallel manipulator is not even explicitly known, it is implicitly defined by a set of constraint functions introduced by the manipulator's closure constraints. A parallel manipulator also has, in addition to the usual end-effector singularities, different types of singularities such as configuration space singularities and actuator singularities. Understanding the intrinsic nature of the various types of singularities and their relations with the kinematic parameters and the configuration spaces is of ultimate importance in design, planning and control of the system.

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Unlike its serial counterparts, where there have been well established mathematical tools for their analysis, studies on singularities of parallel manipulators were confined to basic issues such as definition, classification and identification of singularities. Furthermore, the mathematical tools used in most studies were directly borrowed from that for serial manipulators and were applicable only to local analysis. The unique structures of parallel mechanisms were not fully explored.

Gosselin and Angeles [1] were perhaps the first to define and study singularities of closed-loop kinematic chains. Based on some derived Jacobian relations, they introduced several notions of singularities which formed a basis of later research. Park and Kim [2] used differential geometric tools to study singularities of parallel mechanisms and provided a finer classification of singularities. In their later works, they proposed the use of redundant actuation as a means of eliminating actuator singularities and improving manipulator performances. A six-axis parallel machine platform was constructed based on this principle [3], [4]. Kock [5], [6] also used redundant actuation to design a two-degree-of-freedom (DOF) planar parallel manipulator for high speed assembly. Similar works could also be found in Nahon and Angeles [7]. It is interesting to note that redundant actuation also appears in multifingered robotic hands [8], [9], and in walking machines [10]. Merlet and others [11]–[17] studied extensively singularities of the Stewart–Gough platform and several of its variants. A good account of recent progress on parallel mechanism research can be found in [18] and references therein. Aside from local analysis, there is some research on global analysis of manipulator singularities. C. Innocenti and Parenti-Castelli [19], [20] studied the problem of planning a path to connect two regular inverse kinematic solutions without intersecting the singular sets; [21] studied the topology of self-motion manifolds of redundant manipulators; Bedrossian [22] performed studies on determination of self motion to take from a singular configuration to a nonsingular configuration; Kieffer [23], based on a Taylor series expansion method, discovered the difference between ordinary singularities, isolated singularities and their bifurcations. Kumar [14], Park [2] and Wen [8] introduced several kinematic manipulability measures for design and control of parallel mechanisms.

The behavior of singularities for parallel manipulator is indeed very complex. An interesting example is offered by that of the Seoul National University (SNU) manipulator, a 3-DOF translational manipulator with the joints of its three subchains arranged in the order of universal–prismatic–universal (UPU). Tsai [24] initiated design of general 3-UPU manipulators and his work was later generalized by Gregorio and Parenti-Castelli [25]. Gregorio *et al.* [26], [27] analyzed singularities of Tsai's manipulator and divided them into rotational and translational singularities. Park and his coworkers [28] identified a configuration space (CS) singularity of the SNU manipulator and observed that the manipulator at its home position exhibits finite motions even with all active prismatic joints locked. Zlatanov *et al.* [29] studied this singularity using screw theory and classified it as a constraint singularity. The same singularity was also identified by Joshi and Tsai [30], Simaan and Shoham [31], and Wolf *et al.* [32] using an augmented Jacobian matrix which took the constraints into account. By interpreting the rows of this matrix as lines, line geometry method, i.e., that in [11], could be used to efficiently find all possible singularities. Even though a good explanation of this singularity of the SNU manipulator has been offered in [28], [29], a concise mathematical formula for detecting these singularities is not available.

The purpose of the paper is twofold. First, in view of the fact that a parallel manipulator is most naturally described by the constraint functions and their differential forms associated with the manipulator's closure constraints, we develop a unified mathematical tool for singularity analysis of parallel mechanisms using the language of differential forms. We give precise and coordinate invariant definitions of configuration space and parametrization singularities with actuator and end-effector singularities as special cases of the latter. We investigate the intrinsic nature of the various singularities of a parallel mechanism, their relations with the kinematic parameters and the configuration space of the manipulator and the role redundant actuation plays in reshaping the singularities and improving the manipulator's performance. We present a detailed classification of parametrization singularities and identify those which are potentially dangerous and should be avoided or eliminated through design.

The paper is organized as follows: In Section II, we introduce notations and review basics of differential forms. We then define and give concise conditions for CS singularity. Based on this, we analyze the "strange" singularity of the SNU manipulator and show that it is a CS singularity. Then, we show that configuration space singularities can be reformulated as the critical points of a Morse function on some appropriately defined spaces. By varying a selected kinematic parameter of the manipulator, different homotopic classes of the configuration space can be obtained. This study allows us to determine permissible ranges of the kinematic parameters in the design phase. In Section III, we define parametrization singularity with end-effector and actuator singularities as two special cases. According to the rank of the constraint functions, we divide parametrization singularity into regular and irregular ones. We explicitly show that one singularity of the SNU manipulator is an irregular actuator singularity. In Section IV, we develop a fine classification of parametrization singularities and show the computation and implication of each class of singularities. In particular, we note that degenerate singularities are sometimes dangerous and should be avoided. Finally, Section V follows with a brief conclusion of the paper.

II. Configuration Space Singularities

A parallel manipulator as shown in Fig. 1 is regarded as a set of open-chains connected in parallel to a common rigid body, known as the end-effector. The joints that connect the links of each open-chain and the chains with the end-effector can be revolute, prismatic, universal or ball-in-socket joints. The configuration space of a revolute joint is S^1, the unit circle, $\mathbb{R}^1$ for a prismatic joint, $S^1 \times S^1$ for a universal joint, and $SO(3)$ for a ball-in-socket joint. Thus, the ambient space E of the manipulator is given by the Cartesian product of the joint spaces of all the joints that make up the manipulator. We denote by $\theta \in \mathbb{R}^n$ the local coordinates of E. The loop (or closure) constraints of the manipulator are denoted by

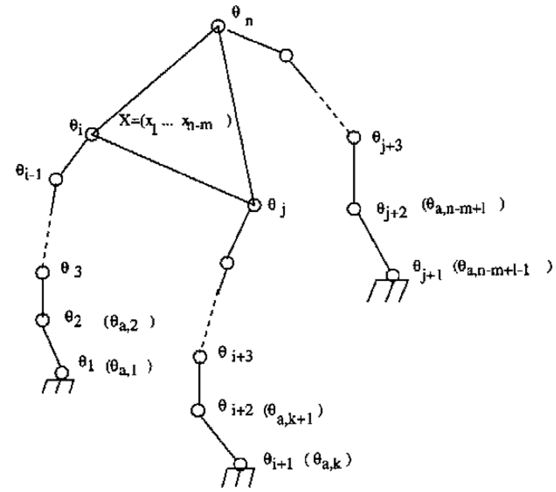

Fig. 1. Coordinate systems for a parallel manipulator.

$$H : E \longrightarrow \mathbb{R}^m,\ \theta \longmapsto H(\theta) = \begin{bmatrix} h_1(\theta) \\ \vdots \\ h_m(\theta) \end{bmatrix} = 0. \qquad (1)$$

Note that the loop constraints are obtained by equating pairwise the end-effector positions from each of the open chains. The preimage $Q := H^{-1}(0)$ is referred to as the configuration space of the manipulator.

Given a function $h : E \to \mathbb{R}$, its differential, denoted dh and given in local coordinates by

$$dh = \sum_{i=1}^{n} \frac{\partial h}{\partial \theta_i} d\theta_i$$

is a one-form, i.e., $dh(\theta) \in T_\theta^* E$. Physically, an element in $T_\theta^* E$ has the meaning of a generalized force, and its pairing with a generalized velocity vector v in $T_\theta E$ gives the virtual power. In particular, for $v = \sum_{i=1}^{n} vi(\partial/\partial\theta_i) \in T_\theta E$

$$\langle dh, v \rangle = \sum_{i=1}^{n} v_i \frac{\partial h}{\partial \theta_i} := v(h)$$

is the directional derivative of h in the direction v.

We refer the readers to [33] for the definition of $dh_1 \wedge dh_2$, a two-form, known as the wedge product of dh_1 with dh_2, and similarly that of $dh_1 \wedge \cdots \wedge dh_m$, an m-form. In local coordinates

$$dh_1 \wedge dh_2 = \sum_{i<j} \left(\frac{\partial h_1}{\partial \theta_i} \frac{\partial h_2}{\partial \theta_j} - \frac{\partial h_2}{\partial \theta_i} \frac{\partial h_1}{\partial \theta_j} \right) d\theta_i \wedge d\theta_j.$$

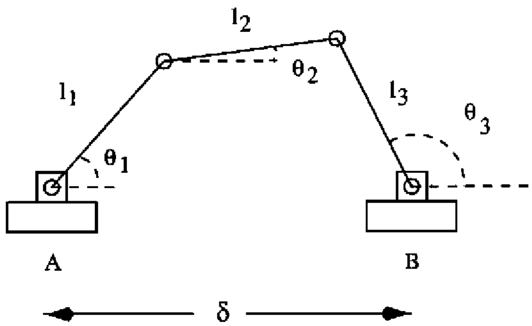

Fig. 2. An example: Four-bar mechanism.

Two functions h_1 and h_2 are said to be linearly independent at $\theta \in E$ if $dh_1(\theta)$ and $dh_2(\theta)$ are linearly independent as two covectors. The latter is translated exactly into the requirement that

$$dh_1 \wedge dh_2|_\theta \neq 0.$$

When $dh_1 \wedge dh_2 = 0$, the intersection of $h_1^{-1}(0)$ with $h_2^{-1}(0)$ is not transversal [34].

Definition 1: A point $\theta \in Q$ is called a CS singularity if

$$dh_1 \wedge \cdots \wedge dh_m|_\theta = 0. \tag{2}$$

The concept of CS singularity was initiated by Park and Kim [2] which provides a geometric interpretation of the uncertainty configurations [35] and the constraint singularities [29]. A basic result of differentiable manifolds [33] shows that if $\forall\theta \in Q$, $dh_1 \wedge \cdots \wedge dh_m|_\theta \neq 0$, then Q is a submanifold of dimension $n - m$ of E.

Example 1: A Four-Bar Mechanism: Consider a four-bar mechanism with link lengths l_1, l_2 and l_3 as shown in Fig. 2. Let δ be the separation between the two fixed bases, and θ_1, θ_2 and θ_3 the angle coordinates (another angle variable is eliminated from the angle constraint). The loop constraints are given by

$$h_1 = l_1 \sin\theta_1 + l_2 \sin\theta_2 - l_3 \sin\theta_3 = 0 \tag{3}$$

$$h_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_3 \cos\theta_3 - \delta = 0. \tag{4}$$

Without loss of generality, we assume that $l_1 - l_2 > l_3$ and $l_2 > l_3$. Computing dh_i, $i = 1, 2$ and $dh_1 \wedge dh_2$ gives

$$\begin{aligned} dh_1 \wedge dh_2 = &\, l_1 l_2 (s_1 c_2 - c_1 s_2) d\theta_1 \wedge d\theta_2 \\ &+ l_1 l_3 (c_1 s_3 - s_1 c_3) d\theta_1 \wedge d\theta_3 \\ &+ l_2 l_3 (c_2 s_3 - c_3 s_2) d\theta_2 \wedge d\theta_3 \end{aligned}$$

where $s_i = \sin\theta_i$ and $c_i = \cos\theta_i$ for $i = 1, 2, 3$. Thus, CS singularities are obtained by solving

$$\sin(\theta_i - \theta_j) = 0, \quad i < j$$

and the solutions are (modular a free parameter) $\theta_i = 0$ or π, $i = 1, 2, 3$. In view of the constraint h_2, the singularity points as a function of the separation distance δ are given in Table I.

To visualize the configuration space Q of the mechanism and see how Q changes as a function of δ, we draw two circles O_1 and O_2 with their centers at the origin, and radius $(l_1 - l_2)$ and $(l_1 + l_2)$, respectively, and another circle O_3 with center at $(\delta, 0)$ and radius l_3, as shown in Fig. 3. Obviously, the annulus formed

TABLE I
CONFIGURATION SPACE SINGULARITY POINTS VERSUS PARAMETER δ

Singularity Points	Parameter Relation	Parameter Value	Description of Q	Morse Index
		$\delta \in (\delta_4, \infty)$	Empty set	
$p_1 = (0, 0, \pi)$	$l_1 + l_2 + l_3 = \delta_4$	$\delta = \delta_4$	A single point ●	$Mi = 2$
		$\delta \in (\delta_3, \delta_4)$	Unit circle ○	
$p_2 = (0, 0, 0)$	$l_1 + l_2 - l_3 = \delta_3$	$\delta = \delta_3$	"Figure 8" ∞	$Mi = 1$
		$\delta \in (\delta_2, \delta_3)$	Two separate circles ○ ○	
$p_3 = (0, \pi, \pi)$	$l_1 - l_2 + l_3 = \delta_2$	$\delta = \delta_2$	"Figure 8" ∞	$Mi = 1$
		$\delta \in (\delta_1, \delta_2)$	Unit circle ○	
$p_4 = (0, \pi, 0)$	$l_1 - l_2 - l_3 = \delta_1$	$\delta = \delta_1$	A point ●	$Mi = 0$
		$\delta \in (0, \delta_1)$	Empty set	

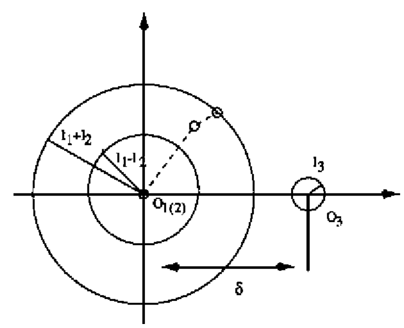

Fig. 3. Relative position between the annulus and the circle O_3.

by O_1 and O_2 is the workspace of the tip point of link 2 obtained by rotating the first two links, and O_3 is the trajectory of the same point obtained by rotating about θ_3. The loop constraints are satisfied if and only if O_3 intersects the annulus. Define $\delta_1 = l_1 - l_2 - l_3$, $\delta_2 = l_1 - l_2 + l_3$, $\delta_3 = l_1 + l_2 - l_3$, and $\delta_4 = l_1 + l_2 + l_3$ (see Table I). When δ is large and far away to the right, Q is empty. As δ approaches δ_4, Q becomes a single point, a circle for $\delta \in (\delta_3, \delta_4)$, a "figure 8" at δ_3, two separate circles for $\delta \in (\delta_2, \delta_3)$, back to "figure 8" at δ_2, a circle for $\delta \in (\delta_1, \delta_2)$, a single point at δ_1 and back to the empty set for $\delta < \delta_1$.

A. CS Singularities of the 3-UPU Manipulator

In this subsection, we will study the CS singularities of a well studied spatial mechanism: the 3-UPU manipulator as shown in Figs. 4 and 5. As implied by its name, the manipulator consists of three serial chains with their joints arranged in the order of UPU, where only the three prismatic joints are actuated. By simply applying the Gruebler's mobility formula, we see that the mechanism has three DOFs.

3-UPU manipulators with their simple kinematic structure have attracted many researchers. Tsai [24] and Gregorio and Parenti-Castelli [25] provided conditions for such a manipulator

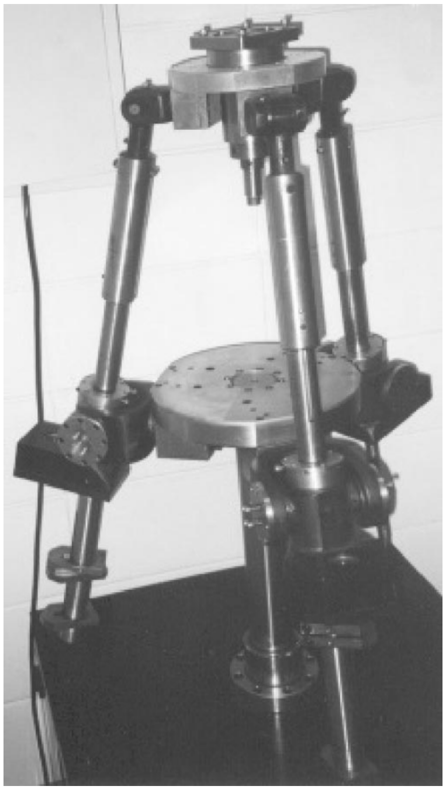

Fig. 4. The SNU 3-UPU manipulator.

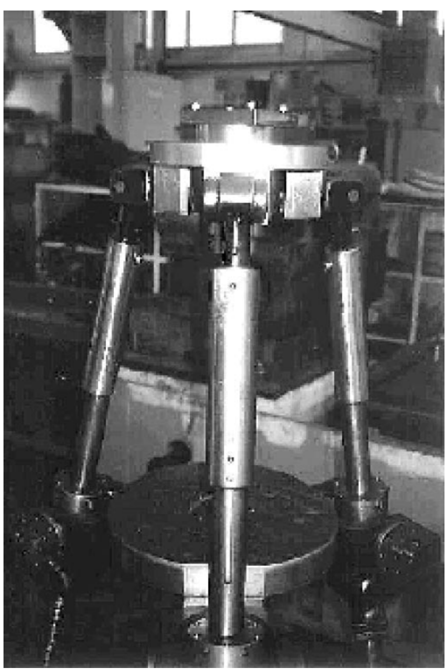

Fig. 5. The SNU 3-UPU manipulator.

to be purely translational. Han *et al.* [28] performed singularity analysis of such a manipulator in detail. They found that for the axis arrangement as the SNU manipulator (see Figs. 4 and 6), the mechanism will exhibit a "strange" singularity at the home position, where the end-effector is free to rotate even when all the prismatic joints are locked. They pointed out that "strangeness" of the singularity lies in the fact that it can not be detected

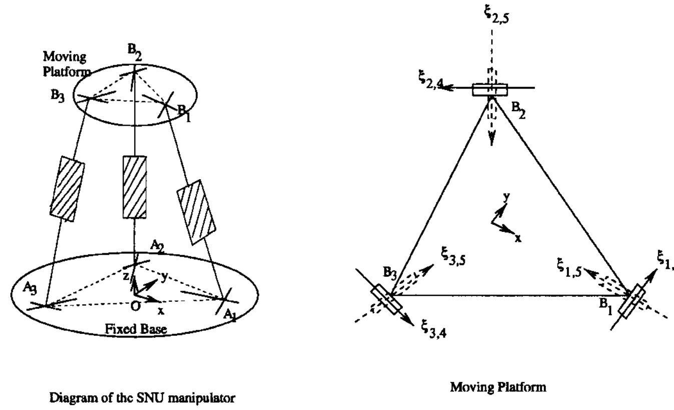

Fig. 6. The SNU manipulator with the {first, second} and {fourth, fifth} axles of the three serial chains lying in two parallel planes.

from the kinematics relation, $Zv = \dot{\theta}_a$, where $v \in \mathbb{R}^6$ is the generalized velocity of the end-effector, $\theta_a = [\theta_{a,1}, \theta_{a,2}, \theta_{a,3}]^T$ the joint angles of the three actuators, and Z the Jacobian matrix whose rows represent the screws of the three prismatic joints. Z is full rank at the home position and thus fails to predict the observed behavior, (see [29] for more detail). Han *et al.* [28] identified such a singularity as a CS singularity by checking the rank (and the condition number) of the Jacobian matrix of the loop closure equations. Zlatanov, Bonev and Gosselin [29] obtained the same results by exploring the rank of the constraint screws. This singularity was also identified by Joshi and Tsai [30] using an augmented Jacobian matrix which took the constraints into account. By interpreting the rows of this matrix as lines, line geometry method, i.e., that in [11], could be used to efficiently find all possible singularities. In this section and the next we show that this singularity is not only a CS singularity, but also an irregular actuator singularity. We also show the equivalence between the HKKP method [28] and the ZBG method [29].

Following the notations of Murray *et al.* [36], let $\Theta_i = [\theta_{i,1}, \cdots, \theta_{i,5}]^T$, $i = 1, \cdots, 3$ be the joint angles of the three serial chains. Denote by $SE(3)$ the configuration of the end-effector relative to the world frame and $se(3)$ its Lie algebra. An element of $SE(3)$ can be represented in terms of homogeneous coordinates as

$$\begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}, \; R \in SO(3), \; P \in \mathbb{R}^3$$

and that of $se(3)$ by

$$\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix}, \; w, v \in \mathbb{R}^3$$

where for $w = [w_1, w_2, w_3]^T$, we have

$$\hat{w} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}.$$

Let $\vee$ be the identification map

$$\vee : se(3) \rightarrow \mathbb{R}^6, \; \hat{\xi} \rightarrow [v^T, w^T]^T.$$

The forward kinematics of the ith chain is described by the product of exponential formula

$$g_i(\Theta_i) = e^{\hat{\xi}_{i,1}\theta_{i,1}} \cdots e^{\hat{\xi}_{i,5}\theta_{i,5}} g_i(0),\ i = 1, \cdots, 3$$

where $\hat{\xi}_{i,j} \in se(3)$ is the screw representing the jth joint in the ith chain.

Note that $(g_i^{-1}dg_i)^\vee = \sum_{j=1}^{5} Ad^{-1}_{e^{\hat{\xi}_{i,j}\theta_{i,j}} \cdots e^{\hat{\xi}_{1,5}\theta_{i,5}} g_i(0)} \xi_{i,j} d\theta_{i,j} \in se^*(3)$ is exactly the Maurer–Cartan form [37], [38] with its action on a permissible velocity $\sum_{j=1}^{5} \dot{\theta}_{i,j}(\partial/\partial\theta_{i,j}) \in se(3)$. The end-effector velocity is given by

$$v = \sum_{j=1}^{5} Ad^{-1}_{e^{\hat{\xi}_{i,j}\theta_{i,j}} \cdots e^{\hat{\xi}_{i,5}\theta_{i,5}} g_i(0)} \xi_{i,j} \dot{\theta}_{i,j} := J_i \dot{\Theta}_i$$

with J_i the Jacobian of the ith chain. Express the manipulator's loop closure constraints as

$$g_1(\Theta_1) = g_2(\Theta_2) = g_3(\Theta_3)$$

and equate the Maurer–Cartan form of the three chains, we have

$$\begin{aligned} \omega_\Theta &= \begin{bmatrix} \omega_{\Theta,1} \\ \vdots \\ \omega_{\Theta,12} \end{bmatrix} \\ &:= \begin{bmatrix} (g_1^{-1}dg_1)^\vee - (g_2^{-1}dg_2)^\vee \\ (g_1^{-1}dg_1)^\vee - (g_3^{-1}dg_3)^\vee \end{bmatrix} \\ &= \begin{bmatrix} J_1 d\Theta_1 - J_2 d\Theta_2 \\ J_1 d\Theta_1 - J_3 d\Theta_3 \end{bmatrix} = 0. \end{aligned}$$

We can prove that at the home position of the SNU manipulator

$$\omega_{\Theta,1} \wedge \cdots \wedge \omega_{\Theta,12} = 0. \quad (5)$$

While for Tsai's manipulator (see Fig. 7), the left-hand side of (5) is not zero. Figs. 8 and 9 represent the inverse condition number of the SNU manipulator and that of Tsai's manipulator in a neighborhood of the home position. This shows that the home position of the SNU manipulator is a CS singularity. Appendix A gives J_i, $i = 1, \cdots, 3$ for the SNU manipulator and Tsai's manipulator at their home positions. The derivation is similar to that of HKKP [28].

To establish the equivalence between the HKKP method [28] and the ZBG method [29], we first consider CS singularities in the space $se(3) \times se(3) \times se(3)$ with the map $J = \mathrm{diag}(J_1, J_2, J_3) : TE = T\mathbb{R}^{15} \to se(3) \times se(3) \times se(3)$

$$\dot{\Theta} = \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{\Theta}_3 \end{bmatrix} \to \dot{X} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = J\dot{\Theta}$$

where $X = [X_1^T, X_2^T, X_3^T]^T \in \mathbb{R}^{18}$ with $X_i \in \mathbb{R}^6$ being the Cartesian coordinates of the end-effector of the ith subchain. Since $J \in \mathbb{R}^{18\times 15}$, there exists a set of Pfaffin constraints such that

$$A\dot{X} := \begin{bmatrix} a_1^T & 0 & 0 \\ 0 & a_2^T & 0 \\ 0 & 0 & a_3^T \end{bmatrix} J\dot{\Theta} = 0$$

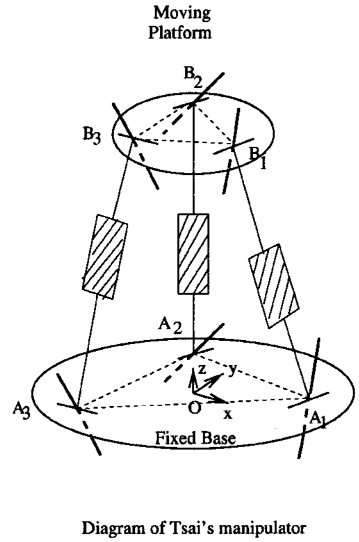

Fig. 7. Tsai's manipulator with the planes formed by the {first, second} axle of the three serial chains being different, and so do the planes formed by the {fourth, fifth} axle.

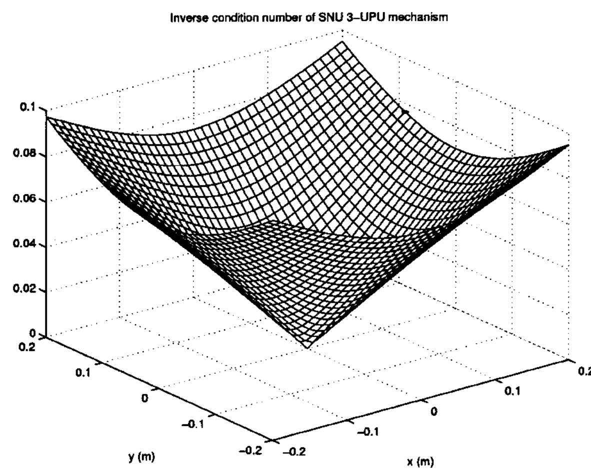

Fig. 8. The inverse condition number of the SNU manipulator.

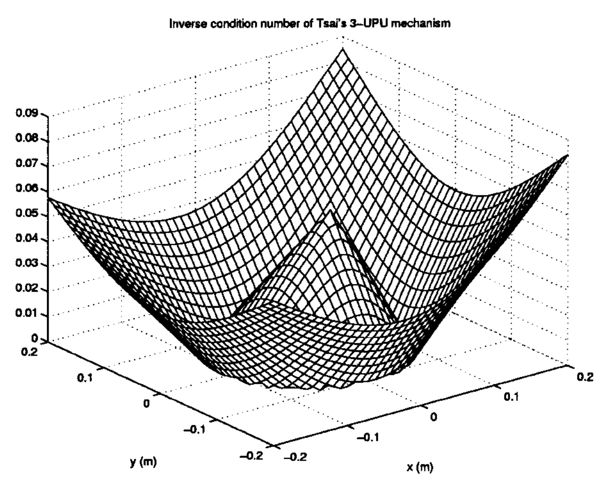

Fig. 9. The inverse condition number of Tsai's manipulator.

where $a_i \in \mathbb{R}^6$, $i = 1, \cdots, 3$ is the constraint screw of the ith chain satisfying $a_i^T J_i = 0$. Here, for simplicity, we assume that J_i, $i = 1, \cdots, 3$ are of full rank. Second, the loop constraints impose another set of Pfaffin constraints on $se(3) \times se(3) \times se(3)$ such that

$$\begin{bmatrix} I & -I & 0 \\ I & 0 & -I \end{bmatrix} \dot{X} = 0.$$

Collectively, we have the following complete set of Pfaffin constraints:

$$\omega_X = \begin{bmatrix} \omega_{X,1} \\ \vdots \\ \omega_{X,15} \end{bmatrix} = \begin{bmatrix} a_1^T dX_1 \\ a_2^T dX_2 \\ a_3^T dX_3 \\ dX_1 - dX_2 \\ dX_1 - dX_3 \end{bmatrix} = 0.$$

It is shown in Appendix B that

$$\omega_{X,1} \wedge \cdots \wedge \omega_{X,15} = 0 \Leftrightarrow a_1^T dX_1 \wedge a_2^T dX_1 \wedge a_3^T dX_1 = 0. \quad (6)$$

Another proof in Appendix C also shows that

$$\omega_{\Theta,1} \wedge \cdots \wedge \omega_{\Theta,12} = 0 \Leftrightarrow \omega_{X,1} \wedge \cdots \wedge \omega_{X,15} = 0. \quad (7)$$

In other words, a CS singularity of HKKP [28] is equivalent to the linear dependence of the $a_i's$, $i = 1, \cdots, 3$, coinciding with the definition of ZBG [29]. For the SNU manipulator at its home position, we have

$$a_i = [0,\ 0,\ 0,\ 0,\ 0,\ 1]^T,\ i = 1, \cdots, 3$$

and thus, (6) is zero. On the other hand, for Tsai's manipulator, we have

$$\begin{aligned} a_1 &= [0,\ 0,\ 0,\ 0.1155,\ 0,\ -0.9933]^T \\ a_2 &= [0,\ 0,\ 0,\ -0.0578,\ 0.1,\ -0.9933]^T \\ a_3 &= [0,\ 0,\ 0,\ 0.0578,\ 0.1,\ 0.9933]^T \end{aligned}$$

and the left-hand side of (6) is not zero, and the home position of Tsai's manipulator is not singular.

B. CS Singularity: Another View

Observe from Example 1 that away from CS singularities or when $\delta \in (\delta_3, \delta_4)$ or $\delta \in (\delta_2, \delta_3)$, $Q = H^{-1}(0)$ is indeed a manifold of dimension one. However, the topological structure of these two one-dimensional manifolds are quite different, with one being connected and the other not. Also, note that the topology of the configuration space changes precisely at the CS singularity points. From this change, we can obtain an initial design guide for selection of the parameter δ. For instance, δ should be chosen so that, first of all, CS singularities are avoided, and secondly, the resulting configuration space is either a unit circle ($\delta \in (\delta_3, \delta_4)$) or two separate circles ($\delta \in (\delta_2, \delta_3)$). Other design criteria can be later introduced to fine-tune the parameter within a selected range. This topic will be addressed in another research on optimal design of parallel manipulators using LMI technique and semi-definite programming [39].

To develop a general understanding of CS singularities and their relations with some key kinematic parameters of a parallel manipulator, we apply Morse theory [34] where CS singularities are viewed as the critical points of an appropriately defined function with its value being the design parameter. By varying the design parameter from one critical value to another, we obtain the homotopy classes of the configuration space and useful information on suitable ranges of the design parameter. We use again the four-bar mechanism to illustrate the application of Morse theory for study of CS singularities and kinematic parameter design.

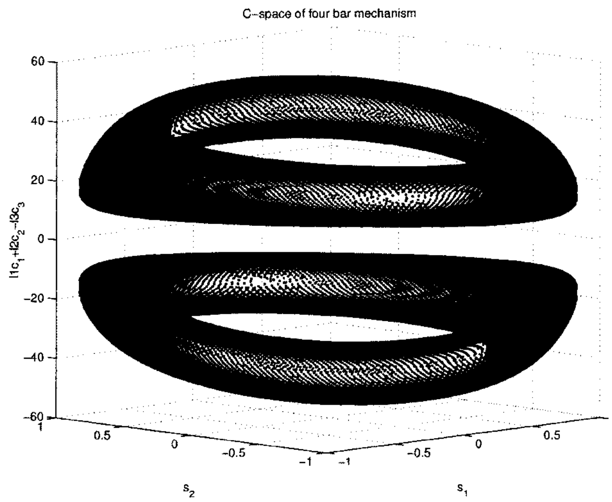

Fig. 10. Torus in $(s_1, s_2, l_1 c_1 + l_2 c_2 - l_3 c_3)$.

Example 2: Four-Bar Mechanism Revisited: Assume that dh_1 does not vanish (this is in fact true for all value of δ), and $\tilde{Q} = h_1^{-1}(0)$ is a manifold of dimension 2. Topologically, $\tilde{Q}$ is a torus as shown in Figs. 10 and 11.

Rearrange h_2 so that

$$h_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 - l_3 \cos\theta_3 = \delta$$

and let $\tilde{h}_2$ be the restriction of h_2 to $\tilde{Q}$, i.e.,

$$\tilde{h}_2 : \tilde{Q} \longrightarrow \mathbb{R} : \theta \longmapsto h_2(\theta) = \delta.$$

Note that $\tilde{h}_2$ gives the height of the torus, as shown in Fig. 10 and 11. Furthermore, $\tilde{h}_2^{-1}(\delta) = h_1^{-1}(0) \cap h_2^{-1}(\delta) = Q$. A point $p \in \tilde{Q}$ is called a critical point of $\tilde{h}_2$ if

$$v(\tilde{h}_2) = \langle d\tilde{h}_2, v \rangle = \langle dh_2, v \rangle = 0 \quad \forall v \in T_p\tilde{Q}.$$

The value of $\tilde{h}_2$ at a critical point is called a critical value. Since

$$T_p\tilde{Q} = \{z \in T_pE | \langle z, dh_1 \rangle = 0\}$$

we see that $p \in \tilde{Q}$ is a critical point if and only if there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$dh_2 = \lambda dh_1.$$

The later is true if and only if the CS singularity condition $dh_1 \wedge dh_2 = 0$ is satisfied. Thus, CS singularity points are translated exactly into critical points of $\tilde{h}_2$. Let

$$\tilde{Q}^a = \tilde{h}_2^{-1}(-\infty, a] = \{p \in \tilde{Q} : \tilde{h}_2(p) \le a\}.$$

We state the following useful result from [34].

Proposition 1: Let $a < b$ and suppose that the set $\tilde{h}_2^{-1}[a, b]$, consisting of all $p \in \tilde{Q}$ with $a \le \tilde{h}_2(p) \le b$, is compact, and contains no critical points of $\tilde{h}_2$. Then, $\tilde{Q}^a$ is diffeomorphic to

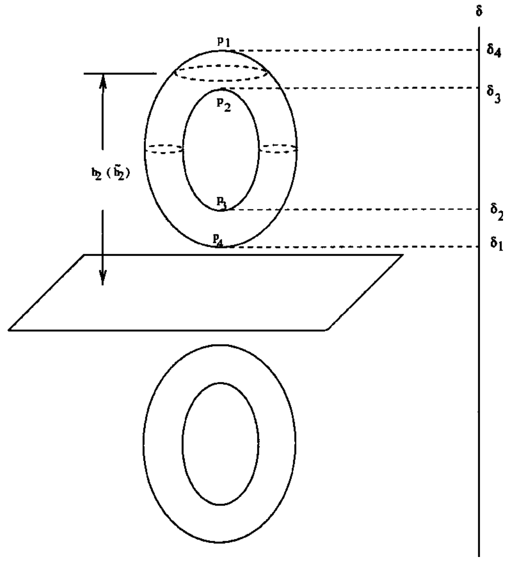

Fig. 11. Diagram of the torus-topology of the four-bar mechanism.

$\tilde{Q}^b$. Furthermore, $\tilde{Q}^a$ is a deformation retract of $\tilde{Q}^b$, so that the inclusion map $\tilde{Q}^a \rightarrow \tilde{Q}^b$ is a homotopy equivalence.

In other words, the parameter δ should be chosen to lie in a range $[a, b]$ so that $\tilde{h}_2^{-1}[a, b]$ contains no critical points of $\tilde{h}_2$.

Given a critical point $p \in \tilde{Q}$, it is important to know whether it is an isolated critical point or not. This can be answered by a result of Morse Lemma [34] which states that, if the Hessian $D^2\tilde{h}_2$ of $\tilde{h}_2$ at p is nondegenerate, then p is an isolated critical point. To compute the Hessian, $D^2\tilde{h}_2$, we assume that $\tilde{Q}$ is parametrized by (θ_1, θ_2), and let

$$\theta_3 = \theta_3(\theta_1, \theta_2).$$

From (3) we obtain

$$\frac{\partial\theta_3}{\partial\theta_i} = \frac{-\left(\frac{\partial h_1}{\partial\theta_i}\right)}{\left(\frac{\partial h_1}{\partial\theta_3}\right)} = \frac{l_i c_i}{l_3 c_3}, \; i = 1, 2. \tag{8}$$

Using (8) and applying the chain rule to

$$\tilde{h}_2(\theta_1, \theta_2) = h_2\left(\theta_1, \theta_2, \theta_3(\theta_1, \theta_2)\right)$$

yields

$$\begin{aligned} D^2\tilde{h}_2 &= \begin{bmatrix} \frac{\partial^2\tilde{h}_2}{\partial\theta_1\partial\theta_2} & \frac{\partial^2\tilde{h}_2}{\partial\theta_1\partial\theta_2} \\ \frac{\partial^2\tilde{h}_2}{\partial\theta_1\partial\theta_2} & \frac{\partial^2\tilde{h}_2}{\partial\theta_2^2} \end{bmatrix} \\ &= \begin{bmatrix} -l_1c_1 - \frac{l_1 s_1 s_2}{c_3} + \frac{l_1^2 c_1^2}{l_3 c_3^3} & \frac{l_1 l_2 c_1 c_2}{l_3 c_3^3} \\ \frac{l_1 l_2 c_1 c_2}{l_3 c_3^3} & -l_2c_2 - \frac{l_2 s_2 s_3}{c_3} + \frac{l_2^2 c_2^2}{l_3 c_3^3} \end{bmatrix}. \end{aligned} \tag{9}$$

Apparently, the Hessians at all four critical points are nondegenerate, and they are thus isolated critical points. The Morse index of $\tilde{h}_2$ is defined as the number of negative eigenvalues of $D^2\tilde{h}_2$. This index is independent of the local coordinates for $\tilde{Q}$. The Morse index of $D^2\tilde{h}_2$ at each of the four critical points is given in the last column of Table I. Note that in Fig. 10, a good visualization of $\tilde{Q}$ is obtained as follows. Let Φ be a differentiable map

$$\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \rightarrow \begin{bmatrix} s_1 \\ s_2 \\ l_1c_1 + l_2c_2 - l_3c_3 \end{bmatrix}$$

that maps $\tilde{Q}$ to a new manifold $\hat{Q}$. $\hat{Q}$ has the same topology as $\tilde{Q}$ since the critical points of $\tilde{h}_2$ on $\tilde{Q}$ are mapped to those on $\hat{Q}$, and the Morse index of

$$\begin{aligned} \tilde{D}^2\tilde{h}_2 &:= \begin{bmatrix} \frac{\partial^2\tilde{h}_2}{\partial s_1^2} & \frac{\partial^2\tilde{h}_2}{\partial s_1\partial s_2} \\ \frac{\partial^2\tilde{h}_2}{\partial s_1\partial s_2} & \frac{\partial^2\tilde{h}_2}{\partial s_2^2} \end{bmatrix} \\ &= \mathrm{diag}\left(c_1^{-1}, c_2^{-1}\right) D^2\tilde{h}_2 \mathrm{diag}\left(c_1^{-1}, c_2^{-1}\right) \end{aligned} \tag{10}$$

is equal to that of $D^2\tilde{h}_2$ because $c_i = \pm 1 \neq 0 (i = 1, 2)$ at the critical points. $\hat{Q}$ is depicted in Fig. 10 instead of $\tilde{Q}$ with $l_1 = 30, l_2 = 15$, and $l_3 = 10$.

To conclude the four-bar example, we note that other kinematic parameters can also be studied by manipulating the constraint functions. For instance, to consider l_3 we simply let

$$\begin{aligned} h_1 &= \frac{l_1s_1 + l_2s_2}{l_1c_1 + l_2c_2 - \delta} - \tan\theta_3 = 0 \\ h_2 &= (l_1s_1 + l_2s_2)^2 + (l_1c_1 + l_2c_2 - \delta)^2 = l_3^2. \end{aligned}$$

The general theory for understanding the relation between a selected kinematic parameter and the topology of the configuration space proceeds in a similar manner as that of the four-bar example. We assume, for simplicity, that the first $(m - 1)$ functions are linearly independent for all $p \in \tilde{Q}$, where $\tilde{Q} = \tilde{H}^{-1}(0)$ and

$$\tilde{H} : E \longrightarrow \mathbb{R}^{m-1}, \theta \longmapsto \begin{bmatrix} h_1(\theta) \\ \vdots \\ h_{m-1}(\theta) \end{bmatrix}.$$

Then, $\tilde{Q}$ is a manifold of dimension $(n - m + 1)$. Let $h_m(\theta) = \delta$, and define

$$\tilde{h}_m : \tilde{Q} \longrightarrow \mathbb{R}, \; \theta \longmapsto h_m(\theta) = \delta$$

to be the restriction of h_m to $\tilde{Q}$. Since the tangent space $T_p\tilde{Q}$ has the form

$$T_p\tilde{Q} = \{v \in T_pE | \langle v, dh_i \rangle = 0, \quad i = 1, \cdots m - 1\}$$

it is not difficult to see that a point $p \in \tilde{Q}$ is a critical point of $\tilde{h}_m$ if and only if it is a CS singularity. The study of CS singularities is translated into study of the critical points of the Morse function $\tilde{h}_m$. By varying the parameter δ over $\mathbb{R}$, we obtain homotopy classes of the configuration space. To determine whether a critical point is isolated or not, the Hessian of $\tilde{h}_m$ can be computed as in the previous examples.

III. Parametrization Singularity

Consider the configuration space $Q = H^{-1}(0) \subset E$ of a parallel manipulator. We wish to specify Q or parametrize it with a suitable set of parameters. In general, if Q does not contain a

CS singularity, then it is a manifold of dimension $n - m$, and a minimal number of $n - m$ parameters is needed to locally parametrize Q. For parallel manipulators, there are two natural choices of parametrization variables: the angle coordinates of the active joints and the coordinates of the end-effector.

Just like the polar coordinates of the unit sphere that becomes singular at its boundary of definition, any set of parametrization variables for Q could encounter singularities. In this section, we study singularities of a parallel manipulator when the angle coordinates of the active joints and the coordinates of the end-effector are used to parametrize Q. Here singularities could arise either because the parametrization variables reach their limits (or not properly chosen) or\and Q becomes singular simultaneously. Traditionally, actuator singularities [2] (or singularity of type 2 [1], redundant output [40], forward singularity and uncontrollable singularity) and end-effector singularities [2] (or singularity of type 1 [1], redundant input [40] and inverse singularity) have been separately defined. But, we see here that these singularities are just two special cases of parametrization singularities.

We will use again the language of differential forms to give precise conditions of parametrization singularities. We will study these singularities when Q is a manifold (i.e., away from CS singularities) and when Q is singular. We call the former case regular P-singularities and the latter case irregular P-singularities. Physical significance of these singularities will also be discussed. In the section that follows, a finer classification of the regular P-singularities will be presented.

A. Regular Parametrization Singularity

Assuming that $Q = H^{-1}(0) \subset E$ does not contain a CS singularity and is thus a manifold of dimension $(n - m)$, we say that the manipulator is *nominally* actuated if the number of actuated joints is equal to $(n - m)$, and redundantly actuated if it is strictly larger than $(n - m)$. For a nominally actuated system, the set of active joint angles, denoted $\theta_a \in \mathbb{R}^{n-m}$, serves as a natural candidate for the coordinates of Q. Normally, every coordinate system has its limitations as parametrization singularities are inevitable at certain points in Q. It is important in kinematic analysis and control to know the singularities of each coordinate system. To define precisely parametrization singularities and give exact conditions for occurance of these singularities, we consider the case of a unit sphere parametrized by the horizontal axes.

Example 3: Parametrization of S^2: A unit sphere in $\mathbb{R}^3$ is specified by

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R} : (x_1, x_2, x_3) \longmapsto x_1^2 + x_2^2 + x_3^2 - 1 = 0.$$

Let $Q = H^{-1}(0)$. A local coordinate system of Q at $p \in Q$ is a local diffeomorphism

$$\psi : Q \longrightarrow \mathbb{R}^2 : x \longmapsto \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$$

where $\psi_i(\cdot)$, $i = 1, 2$ are known as the coordinate functions of p. In general, ψ_i, $i = 1, 2$ could be any two of x_j, $j = 1, \cdots, 3$. Here, we simply choose $\psi_i(x) = x_i$, $i = 1, 2$ without loss of generality.

Definition 2: A point $p \in Q$ is called a **parametrization singularity** (or P-singularity) if $T_p\psi$ drops rank, i.e., there exists $0 \neq v \in T_pQ$ such that

$$T_p\psi \cdot v = 0. \tag{11}$$

In the current setting, the above condition is equivalent to

$$\langle d\psi_i, v\rangle = \langle dx_i, v\rangle = 0, \quad i = 1, 2. \tag{12}$$

On the other hand, we also have from the definition of T_pQ that

$$\langle dH, v\rangle = 0. \tag{13}$$

Equations (12) and (13) show that $d\psi_1$, $d\psi_2$, and dH are linearly dependent, and hence

$$dH \wedge d\psi_1 \wedge d\psi_2 = dH \wedge dx_1 \wedge dx_2 = 0. \tag{14}$$

Note that, by expanding (14), yields

$$\begin{aligned}\left(\frac{\partial H}{\partial x_1}dx_1 + \frac{\partial H}{\partial x_2}dx_2 + \frac{\partial H}{\partial x_3}dx_3\right) \wedge dx_1 \wedge dx_2 &= \frac{\partial H}{\partial x_3}dx_1 \\ &\quad \wedge dx_2 \wedge dx_3 \\ &= 0.\end{aligned}$$

The above condition is equivalent to that in Gosselin [1] and Park [2], which states that $p \in Q$ is a P-singularity if and only if $(\partial H/\partial x_p)$ of the following expression drops rank:

$$\frac{\partial H}{\partial x_a}dx_a + \frac{\partial H}{\partial x_p}dx_p = 0 \tag{15}$$

where $x_a = (x_1, x_2)$ and $x_p = x_3$. From the Implicit Function Theorem we know that, away from P-singularities, the passive joints can be expressed uniquely in terms of the active joints which can in turn be measured using sensors integrated with the actuators.

(11), (14) and (15) give three equivalent conditions for P-singularities, from which we conclude that the P-singularities of the (x_1, x_2)-coordinates coincide with the equator $(x_3 = 0)$ of the unit sphere.

Generalizing from the unit sphere example, we have the following.

Proposition 2: Let $\psi_i : E \to \mathbb{R}$, $i = 1, \cdots n - m$, be a set of local coordinate functions on Q. A point $p \in Q$ is a P-singularity of ψ if and only if the n-form in (16) vanishes at p

$$dh_1 \wedge \cdots \wedge dh_m \wedge d\psi_1 \wedge \cdots \wedge d\psi_{n-m}|_p = 0. \tag{16}$$

In particular, if we let $\psi(\theta) = \theta_a \in \mathbb{R}^{n-m}$, the joint angles of the active joints, then the condition for **actuator singularities** (or A-singularity) is given by

$$dh_1 \wedge \cdots \wedge dh_m \wedge d\theta_{a,1} \wedge \cdots \wedge d\theta_{a,n-m}|_p = 0 \tag{17}$$

and if we let $\psi(\theta) = x \in \mathbb{R}^{n-m}$, the local coordinates of the end-effector, then the condition for **end-effector singularities** (or E-singularity) is given by

$$dh_1 \wedge \cdots \wedge dh_m \wedge dx_1 \wedge \cdots \wedge dx_{n-m}|_p = 0. \tag{18}$$

Because of inevitable P-singularities, a single set of $(n - m)$ active joints can not cover an entire manifold Q of dimension

$(n-m)$. To provide a complete coordinate system, more active joints need to be introduced so that a new combination of the active joints can serve as a valid coordinate system when another set is singular. Consider, for instance, the case of a unit sphere. If we "actuate" joint x_3 as well, then the unit sphere is completely covered by the three coordinate systems (x_i, x_j), $1 \leq i < j \leq 3$. This is because the three equators defined by $x_i = 0$, $i = 1, \cdots, 3$, i.e. the P-singularities of these three coordinate charts, have no common intersection point. In other words, for a point $p \in Q$ to be an A-singularity of (x_1, x_2, x_3), we need to have

$$dH \wedge dx_i \wedge dx_j = 0, \quad 1 \leq i < j \leq 3.$$

The only solution of the above equation is $p = (0, 0, 0)$, which is not in Q.

In general, when there are a total $(n - m + l)$ active joints, a point $p \in Q$ is an A-singularity if no combination of $(n - m)$ active joints exists that makes p a regular point.

Corollary 1: A point $p \in Q$ is an A-singularity of $(n-m+l)$ active joints $(\theta_{a,1}, \cdots \theta_{a,n-m+l})$ (or l redundant actuators) if and only if

$$dh_1 \wedge \cdots \wedge dh_m \wedge d\theta_{a,i_1} \wedge \cdots d\theta_{a,i_{m-n}}|_p = 0 \qquad (19)$$

where $1 \leq i_1 < \cdots < i_{n-m} \leq n - m + l$. In other words, all C^{n-m}_{n-m+l} n-forms vanish simultaneously at $p \in Q$.

Remark 1: All results contained in this paper are coordinates invariant, an added advantage of using differential forms!

B. Irregular P-Singularity

Obviously, at a CS singularity, the condition (16) and (17) for P- or A-singularity is automatically satisfied. In this case, the set of 1-forms $\{dh_1, \cdots, dh_m, d\psi_1, \cdots, d\psi_{n-m}\}$ or $\{dh_1, \cdots, dh_m, d\theta_{a,1}, \cdots, d\theta_{a,n-m}\}$ may drop rank by one or more. We call these singularities irregular P- or A-singularities. At an irregular A-singularity, Q is not a manifold, the mechanism may instantaneously gain one or more DOF, and the originally nominally actuated system becomes underactuated. The home position of the SNU manipulator is an irregular A-singularity as it is a CS singularity. This explains why the end-effector can experience finite motion even with all active joints locked.

For a redundantly actuated manipulator, a definition of irregular A-singularity is given by

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_n}|_p = 0 \qquad (20)$$

where $\omega_{ij} \in \{dh_1, \cdots, dh_m, d\theta_{a,1}, \cdots, d\theta_{a,n-m+l}\}$, $j = 1, \cdots, n$. In other words, all C^n_{n+l} n-forms vanish simultaneously at $p \in Q$. From (17), we see that irregular actuator singularities can not be eliminated by actuating a new set of joints while keeping the number of actuators fixed. However, from (20), it is possible to avoid them through redundant actuation.

Remark 2: To completely identify actuator and end-effector singularities, it is necessary to take into account the constraints in addition to the kinematics as was done in (16), (17) and (18).

To conclude this section, we highlight following two advantages of our approach to singularity analysis: 1) The use of exterior algebra (also called Grassmann algebra [41]) is a good generalization of other tools such as matrix analysis and screw theory, as shown in Section 2.1; 2) naturally taking into account the constraint functions in addition to the selected actuator and end-effector coordinates. The former allows us to analyze singularities of a wide class of parallel manipulators and the latter allows us to completely search for all actuator and end-effector singularities, including the "strange" A-singularity of the SNU manipulator.

IV. Geometric Structure of Regular P-Singularities

When a parallel manipulator is at a regular P-singularity, the set of passive joints becomes undetermined. Depending on the nature of the singularity, the manipulator may still be able to exhibit finite motions even when all actuators are locked. This can sometimes be a source of danger as the mechanism may collapse, harming the mechanism itself or humans in the workspace of the mechanism. It is thus important to be able to classify all singularities and identify these which are potentially dangerous.

A. Degrees of Deficiency and Stratified Structure

Consider a parallel manipulator with actuated joints $(\theta_{a,1}, \cdots, \theta_{a,n-m+l})$. Define the set of regular P-singularity points

$$Q_s = \Big\{ p \in Q \,\big|\, dh_1 \wedge \cdots \wedge dh_m \wedge d\theta_{a,i_1} \wedge \cdots \wedge d\theta_{a,i_{n-m}}\big|_p = 0, \\ dh_1 \wedge \cdots \wedge dh_m|_p \neq 0, 1 \leq i_1 < \cdots < i_{n-m} \leq n-m+l \Big\}$$

and the set of annihilation vectors at $p \in Q_s$

$$T_pV = \{v \in T_pQ | \langle v, dh_i \rangle = \langle v, d\theta_{a,j} \rangle = 0, \\ i = 1, \cdots m, j = 1, \cdots n - m + l\}.$$

The dimension d of T_pV measures the deficiency of these one-forms and is called the degree of deficiency (DoD) of the P-singularity. Since the $dh_i's$, $i = 1, \cdots m$, are linearly independent, $d \leq n - m$. On the other hand, d is also equal to the co-rank of the Jacobian matrix $J_p = (\partial H / \partial \theta_p) \in \mathbb{R}^{m \times (m-l)}$, with $\theta_p = (\theta_{p,1}, \cdots \theta_{p,m-l})$ being the angles of the passive joints, we have $d \leq m - l$. Thus, an upper bound on d is given by

$$d \leq d_0 := \min(n - m, m - l).$$

Let

$$Q_{sk} = \{p \in Q_s | \dim(T_pV) = k\}$$

be the set of singularities of DoD $k(1 \leq k \leq d_0)$. A result of [42] shows that Q_{sk} is generically a manifold of codimension $(l + k)k$. We also have $Q_s = \cup_{k=1}^{d_0} Q_{sk}$.

Define

$$\Delta sk = \bigcup_{p \in Q_{sk}} T_pV$$

and

$$\Delta_s = \cup_{k=1}^{d_0} \Delta_{sk}.$$

Δ_{sk} is a distribution of annihilation spaces of dimension k on Q_{sk}, and Δ_s is a distribution of annihilation spaces on Q_s, with not necessarily constant ranks. Note that the dimensions of Q_{sk}

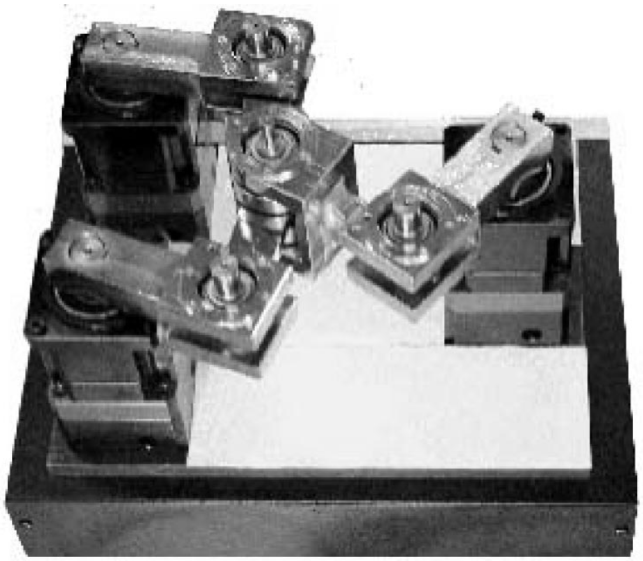

Fig. 12. 2-DOF parallel manipulator.

and Δ_{sk} are not necessarily the same. With these notions, we are able to give a classification of P-singularity points [42].

Definition 3: A point $p \in Q_{sk}$ is called a first-order singularity point if and only if there does not exist a vector $v \in \Delta_{sk}$ that is also tangent to Q_{sk}. Otherwise, p is called a second-order singularity.

Second-order singularity has a special property that it may still be close to the singular curve under a perturbation along an annihilation vector (e.g. the vector v that is tangent to Q_{sk}).

Example 4: P-Singularities of a 2-DOF Parallel Manipulator: Consider a 2-DOF parallel manipulator shown in Figs. 12 and 13, where the ambient space is parametrized by (θ_1, θ_2, θ_3, ϕ_1, ϕ_2, ϕ_3), and the actuated joints by (θ_1, θ_2). By equating pairwise the end-effector position coordinates from each of the three arms, we obtain the following constraint functions:

$$\begin{aligned}
h_1(p) &= x_{a_1} + l_2\cos\theta_1 + l_1\cos\phi_1 - x_{a_2} - l_2\cos\theta_2 - l_1\cos\phi_2\\
h_2(p) &= y_{a_1} + l_2\sin\theta_1 + l_1\sin\phi_1 - y_{a_2} - l_2\sin\theta_2 - l_1\sin\phi_2\\
h_3(p) &= x_{a_1} + l_2\cos\theta_1 + l_1\cos\phi_1 - x_{a_3} - l_2\cos\theta_3 - l_1\cos\phi_3\\
h_4(p) &= y_{a_1} + l_2\sin\theta_1 + l_1\sin\phi_1 - y_{a_3} - l_2\sin\theta_3 - l_1\sin\phi_3.
\end{aligned}$$

Actuator singularities are obtained by solving the equation

$$dh_1 \wedge dh_2 \wedge dh_3 \wedge dh_4 \wedge d\theta_1 \wedge d\theta_2 = 0$$

or

$$\sin(\phi_1 - \phi_2)\sin(\theta_3 - \phi_3) = 0.$$

There are two solutions given by, respectively

$$p_1 : \phi_1 = \phi_2 + \pi \quad \text{or} \quad p_2 : \theta_3 = \phi_3.$$

Consider first singularity point p_1, where the singularity manifold Q_{s1} is obtained by solution of the constraint function $H = 0$ and $\phi_1 - \phi_2 - \pi = 0$. The result is a one-dimensional curve, with a basis for its tangent space $T_{p1}Q_{s1}$ given by

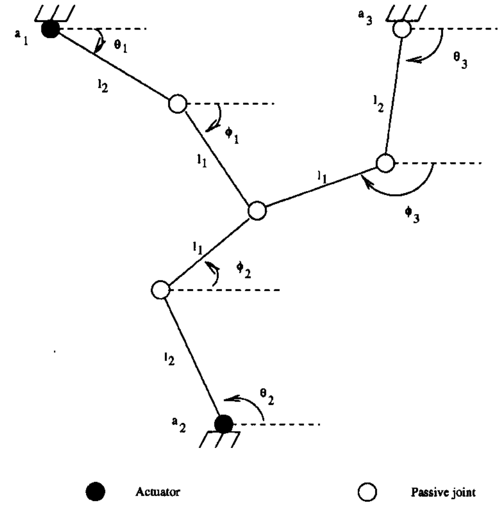

Fig. 13. Dimension of the 2-DOF parallel manipulator.

$$Y = \gamma_1\frac{\partial}{\partial\theta_1} + \gamma_1\frac{\partial}{\partial\theta_2} + \gamma_2\frac{\partial}{\partial\theta_3} + \gamma_3\frac{\partial}{\partial\phi_1} + \gamma_3\frac{\partial}{\partial\phi_2} + \frac{\partial}{\partial\phi_3}$$

where

$$\gamma_1 = \frac{4l_1\cos\left(\frac{1}{2}(\phi_1+\phi_2-2\theta_2)\right)\sin\left(\frac{\phi_1-\phi_2}{2}\right)\sin(\phi_3-\theta_3)}{-2l_2(\sin(\phi_2-\theta_2)\sin(\theta_1-\theta_3)-\sin(\phi_1-\theta_1)\sin(\theta_2-\theta_3))}$$

$$\gamma_2 = \frac{l_1(\sin(\phi_3-\theta_1)\sin(\phi_2-\theta_2)-\sin(\phi_1-\theta_1)\sin(\phi_3-\theta_2))}{l_2(\sin(\phi_2-\theta_2)\sin(\theta_1-\theta_3)-\sin(\phi_1-\theta_1)\sin(\theta_2-\theta_3))}$$

$$\gamma_3 = -\frac{2\sin(\theta_1-\theta_2)\sin(\phi_3-\theta_3)}{\alpha}$$

$$\begin{aligned}
\alpha = {} & \cos(\phi_2+\theta_1-\theta_2-\theta_3) - \cos(\phi_1-\theta_1+\theta_2-\theta_3)\\
& + \cos(\phi_1-\theta_1-\theta_2+\theta_3) - \cos(\phi_2-\theta_1-\theta_2+\theta_3).
\end{aligned}$$

A basis for Δ_{s1} can also be calculated from the definition, and is given by

$$\begin{aligned}
V = {} & \csc(\phi_1-\theta_3)\sin(\phi_3-\theta_3)\frac{\partial}{\partial\phi_1} - \csc(\phi_1-\theta_3)\sin(\phi_3-\theta_3)\\
& \times\frac{\partial}{\partial\phi_2} - \frac{l_1\csc(\phi_1-\theta_3)\sin(\phi_1-\phi_3)}{l_2}\frac{\partial}{\partial\theta_3} + \frac{\partial}{\partial\phi_3}.
\end{aligned}$$

V is tangent to Q_{s1} if

$$\theta_3 = \phi_3.$$

In other words, most points of the singularity curve are first-order singularities, see, e.g., Fig. 14(a). Only one point given by the above condition is a second-order singularity. Fig. 14(b) shows the configuration of the second-order singularity.

Consider next singularity point p_2. The singularity curve Q_{s2} is determined by the constraint $H = 0$ and $\theta_3 - \phi_3 = 0$. The basis for its tangent space $T_{p2}Q_{s2}$ is given by

$$\hat{Y} = \hat{\gamma}_1\frac{\partial}{\partial\theta_1} + \hat{\gamma}_2\frac{\partial}{\partial\theta_2} + \frac{\partial}{\partial\theta_3} + \hat{\gamma}_3\frac{\partial}{\partial\phi_1} + \hat{\gamma}_4\frac{\partial}{\partial\phi_2} + \frac{\partial}{\partial\phi_3}$$

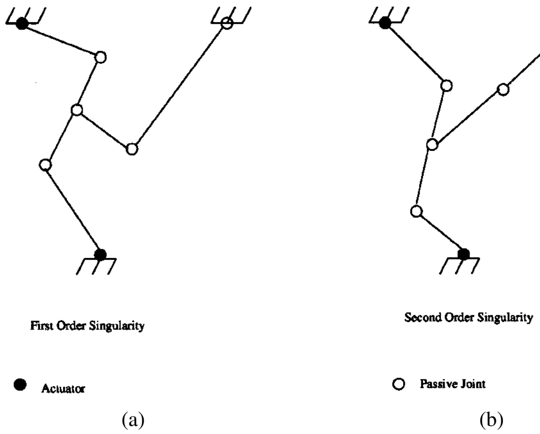

Fig. 14. (a) First-order singularity. (b) Second-order singularity.

where

$$\hat{\gamma}_1 = \frac{(l_1+l_2)\csc(\phi_1-\theta_1)\sin(\phi_1-\theta_3)}{l_2}$$
$$\hat{\gamma}_2 = \frac{(l_1+l_2)\csc(\phi_2-\theta_2)\sin(\phi_2-\theta_3)}{l_2}$$
$$\hat{\gamma}_3 = -\frac{(l_1+l_2)\csc(\phi_1-\theta_1)\sin(\theta_1-\theta_3)}{l_2}$$
$$\hat{\gamma}_4 = -\frac{(l_1+l_2)\csc(\phi_2-\theta_2)\sin(\theta_2-\theta_3)}{l_2}.$$

A basis of Δ_{s2} is calculated as

$$\hat{V} = \frac{l_1}{l_2}\frac{\partial}{\partial\theta_3} + \frac{\partial}{\partial\phi_3}.$$

Thus, no solution exists for $\hat{V}$ to be tangent to Q_{s2}, and all these points are first-order singularity points.

B. Degenerate and Nondegenerate Singularities

A second-order singularity can be further classified into a nondegenerate singularity if it is isolated and degenerate singularity if it is continuous. More precisely, we know that generically Q_{sk} is a $(l+k)k$-codimensional submanifold of Q_s, and $\Delta_{sk} = \text{Span}(Y_1,\cdots,Y_k)$ is a k-dimensional subdistribution of annihilation vectors on Q_{sk}.

Definition 4: A second-order singularity $p \in Q_{sk}$ is degenerate if and only if there is a constant rank k_1 (with $k_1 < k$) sub-distribution $\tilde{\Delta}_{sk1} \subset \Delta_{sk}$ such that

1) $\tilde{\Delta}_{sk_1}(p) \subset T_pQ_{sk}$
2) $\tilde{\Delta}_{sk1} = \text{Span}(Y_1,\cdots Y_{k1})$ can be integrated to form a degenerate singular manifold Q_d.

Note that the second condition requires $\tilde{\Delta}_{sk1}$ to satisfy the Frobenious involutive condition. Degenerate second-order singularities may be a continuous curve or a surface of higher dimensions. A degenerate A-singularity is sometimes dangerous because the mechanism may collapse along the integral manifold Q_d even if all active joints are locked. Similarly, a mechanism with a degenerate E-singularity will exhibit finite internal motions without affecting the end-effector. It is for these reasons that degenerate singularities should be eliminated in the design, or to be avoided to the least.

A hierarchic diagram of our classification of singularities is given in Fig. 15.

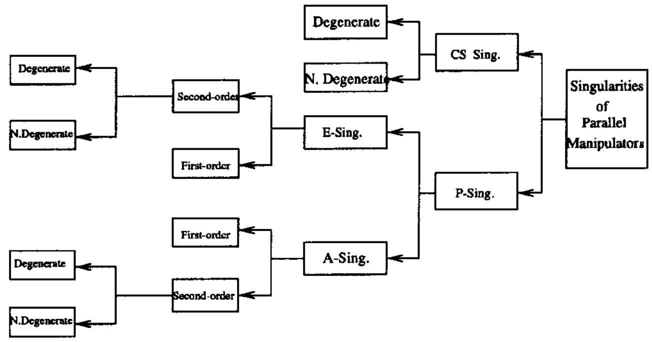

Fig. 15. A hierarchic diagram of singularities, A-sing.: actuator singularity. E-sing.: end-effector singularity. P-sing.: parametrization singularity. N. Degenerate: Nondegenerate.

It should be noted that Definition 4 is too abstract to be useful in practical situations. Alternative conditions in terms of local coordinates can be developed for verifying degeneracy of a singularity point. Park [2] first explored the Hessian of some Morse functions and developed sufficient conditions for an A-singularity to be nondegenerate. Here, we follow the idea initiated by Park and study the coefficients of the Taylor series expansions of the most obvious Morse functions on $Q: h_i, i = 1,\cdots,m$.

Let $\theta_s \in Q_{sk}$ be an A-singularity, $Y = \{Y_1,\cdots,Y_k\}$ be a basis of Δ_{sk}. Then, a variation vector $v_s \in \Delta_{sk}$ can be written as

$$v_s = \begin{bmatrix}\dot{\theta}_a\\ \dot{\theta}_p\end{bmatrix} = \begin{bmatrix}0\\ \dot{\theta}_p\end{bmatrix} = Y\cdot\alpha, \quad \alpha\in\mathbb{R}^k.$$

$\theta_s \in Q_{sk}$ is a degenerate A-singularity if and only if

$$h_i(\theta_s + \epsilon v_s) - h_i(\theta_s) = 0, \quad i = 1,\cdots,m \tag{21}$$

for small ϵ. Taking the Taylor series expansion of (21) yields

$$h_i(\theta_s+\epsilon v_s) - h_i(\theta_s) = \epsilon\sum_{j=1}^{n}\frac{\partial h_i}{\partial\theta_j}|_{\theta_s}v_{s,j} + \frac{\epsilon^2}{2!}\sum_{j,l=1}^{n}\frac{\partial^2 h_i}{\partial\theta_j\partial\theta_l}|_{\theta_s} \times \sum_{t_1,t_2=1}^{k} Y_{j,t_1}Y_{l,t_2}\alpha_{t_1}\alpha_{t_2} + \cdots = 0. \tag{22}$$

Note that the first term on the right-hand side of (22) drops out since v_s is an annihilation vector. Also, for arbitrarily small ϵ, the remaining coefficients in (22) are required to be zero for $i = 1,\cdots,m$.

$$\left[\frac{\partial^2 h_i}{\partial\theta^2}\right]_{\theta_s}(Y\alpha,Y\alpha) = \alpha^T Y^T\left[\frac{\partial^2 h_i}{\partial\theta^2}\right]_{\theta_s}Y\alpha = 0 \tag{23}$$
$$\left[\frac{\partial^3 h_i}{\partial\theta^3}\right]_{\theta_s}(Y\alpha,Y\alpha,Y\alpha) = 0 \tag{24}$$
$$\vdots \tag{25}$$

where $[((\partial^3 h_i)/(\partial\theta^3)]_{\theta_s}$ is a tensor of order 3 and $1/3![((\partial^3 h_i)/(\partial\theta^3))]_{\theta_s}(Y\alpha,Y\alpha,Y\alpha)$ gives the coefficients of ϵ^3. Thus, degeneracy of the singularity θ_s is equivalent to the existence of a nonzero solution to (23), (24) and (25). In other words, if there exists an i such that the quadratic form $Y^T[((\partial^2 h_i)/(\partial\theta^2))]_{\theta s}Y \in \mathbb{R}^{k\times k}$ is positive definite or negative definite, then the considered singularity is nondegenerate. This kind of singularity is also called elliptic as the quadratic

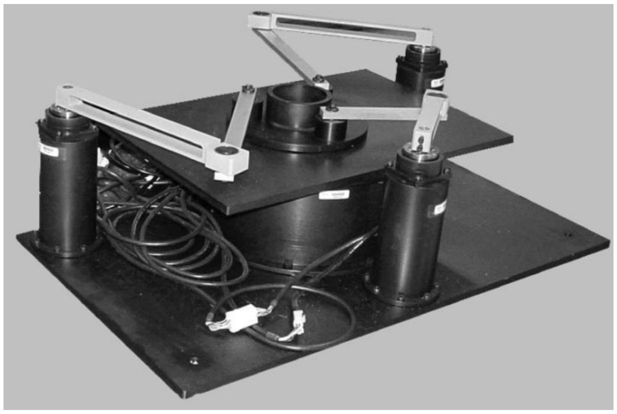

Fig. 16. 3-DOF parallel manipulator.

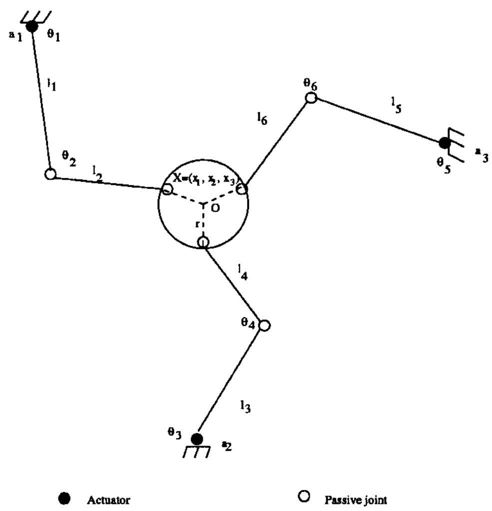

Fig. 17. Dimension of the 3-DOF parallel manipulator.

form has an elliptic Dupin indicatrix [43]. The quadratic form associated with a degenerate singularity should be hyperbolic, parabolic, or planar.

Example 5: Degenerate Singularities of a 3-DOF Parallel Manipulator: Consider a 3-DOF parallel manipulator shown in Fig. 16 and 17, where the ambient space is parametrized by $\theta = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6]^T$. The configuration of the end-effector is described by the Cartesian coordinates $X = [x_1, x_2, x_3]^T$ with (x_1, x_2) the displacement of the center O, and x_3 the orientation angle. $\theta_a = [\theta_1, \theta_3, \theta_5]^T$ are chosen to be the active joints. The loop closure constraints are given by

$$\begin{aligned} h_1 &= (x_{a_1} + l_1 c_1 + l_2 c_{1+2} - x_{a_2} - l_3 c_3 - l_4 c_{3+4})^2 \\ &\quad + (y_{a_1} + l_1 s_1 + l_2 s_{1+2} - y_{a_2} - l_3 s_3 - l_4 s_{3+4})^2 - 3r^2 = 0 \\ h_2 &= (x_{a_1} + l_1 c_1 + l_2 c_{1+2} - x_{a_3} - l_5 c_5 - l_6 c_{5+6})^2 \\ &\quad + (y_{a_1} + l_1 s_1 + l_2 s_{1+2} - y_{a_3} - l_5 s_5 - l_6 s_{5+6})^2 - 3r^2 = 0 \\ h_3 &= (x_{a_2} + l_3 c_3 + l_4 c_{3+4} - x_{a_3} - l_5 c_5 - l_6 c_{5+6})^2 \\ &\quad + (y_{a_2} + l_3 s_3 + l_4 s_{3+4} - y_{a_3} - l_5 s_5 - l_6 s_{5+6})^2 - 3r^2 = 0 \end{aligned}$$

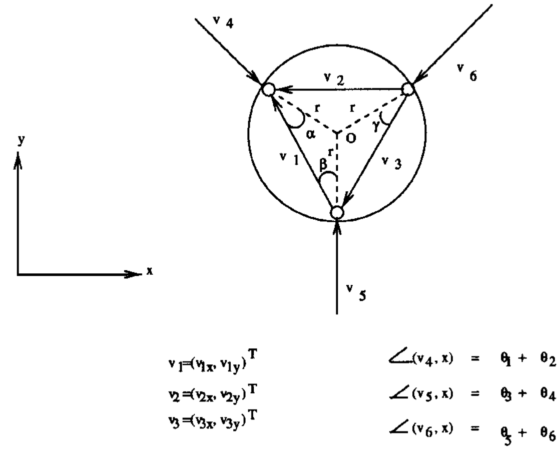

Fig. 18. Vectors of 3-DOF parallel manipulator.

where $s_{i+j} = \sin(\theta_i + \theta_j)$, and $c_{i+j} = \cos(\theta_i + \theta_j)$. A-singularities are obtained from

$$dh_1 \wedge dh_2 \wedge dh_3 \wedge d\theta_1 \wedge d\theta_3 \wedge d\theta_5 = 0.$$

The solution is a surface determined by the following equations:

$$\begin{aligned} h_4 &= v_{1x} v_{3x} v_{2y} s_{3+4} s_{5+6-1-2} + v_{1x} v_{2y} v_{3y} c_{5+6} s_{1+2-3-4} \\ &\quad + v_{1x} v_{2x} v_{3y} s_{1+2} s_{3+4-5-6} + v_{3x} v_{1y} v_{2y} c_{1+2} s_{3+4-5-6} \\ &\quad + v_{1y} v_{2x} v_{3x} s_{5+6} s_{1+2-3-4} + v_{2x} v_{1y} v_{3y} c_{3+4} s_{5+6-1-2} \\ &= 0 \end{aligned} \tag{26}$$

$$H = [h_1, h_2, h_3]^T = 0 \tag{27}$$

where

$$\begin{aligned} v_{1x} &= x_{a_1} + l_1 c_1 + l_2 c_{1+2} - x_{a_2} - l_3 c_3 - l_4 c_{3+4} \\ v_{1y} &= y_{a_1} + l_1 s_1 + l_2 s_{1+2} - y_{a_2} - l_3 s_3 - l_4 s_{3+4} \\ v_{2x} &= x_{a_1} + l_1 c_1 + l_2 c_{1+2} - x_{a_3} - l_5 c_5 - l_6 c_{5+6} \\ v_{2y} &= y_{a_1} + l_1 s_1 + l_2 s_{1+2} - y_{a_3} - l_5 s_5 - l_6 s_{5+6} \\ v_{3x} &= x_{a_2} + l_3 c_3 + l_4 c_{3+4} - x_{a_3} - l_5 c_5 - l_6 c_{5+6} \\ v_{3y} &= y_{a_2} + l_3 s_3 + l_4 s_{3+4} - y_{a_3} - l_5 s_5 - l_6 s_{5+6}. \end{aligned}$$

Since $v_1 = (v_{1x}, v_{1y})^T$, $v_2 = (v_{2x}, v_{2y})^T$, $v_3 = (v_{3x}, v_{3y})^T$, v_4, v_5, v_6 are vectors as shown in Fig. 18, an obvious solution of (26) is given by

$$\angle(-v_4, v_1) = \alpha, \quad \angle(v_1, v_5) = \beta \quad \angle(v_3, v_6) = \gamma$$

or equivalently

$$\begin{aligned} v_{1x} &= r(c_{3+4} - c_{1+2}), \quad v_{2x} = r(c_{5+6} - c_{1+2}) \\ v_{3x} &= r(c_{5+6} - c_{3+4}), \quad v_{1y} = r(s_{3+4} - s_{1+2}) \\ v_{2y} &= r(s_{5+6} - s_{1+2}) \quad v_{3y} = r(s_{5+6} - s_{3+4}) \end{aligned}$$

where $\angle(v_i, v_j)$ represents the angle from v_j to v_i. The singular configuration of the manipulator is depicted in Fig. 19 where the

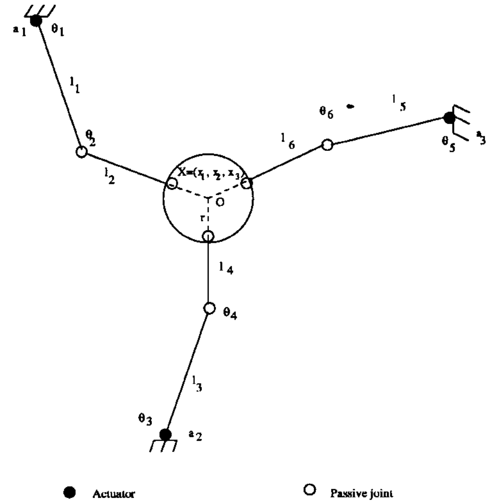

Fig. 19. Nondegenerate P-singularity.

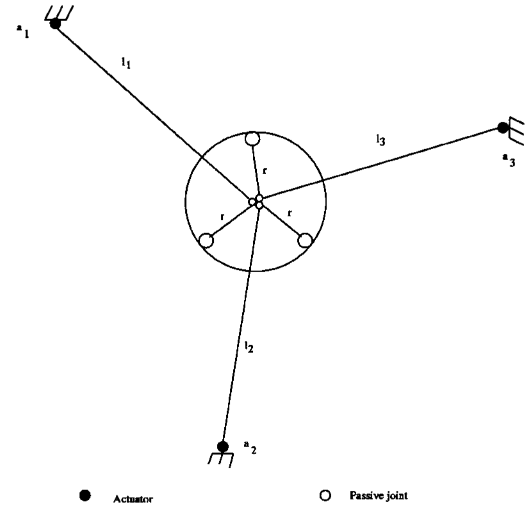

Fig. 20. Degenerate P-singularity.

lines of the link l_2, l_4, l_6 pass through the center O of the manipulator simultaneously. The annihilation velocity at this configuration is computed as

$$V_s = \frac{1}{l_2}\frac{\partial}{\partial\theta_2} + \frac{1}{l_4}\frac{\partial}{\partial\theta_4} + \frac{1}{l_6}\frac{\partial}{\partial\theta_6}$$

or in terms of end-effector velocity as $\dot{X} = (\partial/\partial x_3)$, that is, the end-effector is allowed to perform an infinitesimal rotation about the center O.

To determine the degeneracy of this singularity, the Jacobian $(\partial h_1/\partial\theta_p)$ and the Hessian $((\partial^2 h_1)/(\partial\theta_p^2))$ (similar for others) are needed as shown in the equation at the bottom of the page. It is not difficult to verify that $((\partial h_1)/(\partial\theta_p))V_s = 0$ and $((\partial^2 h_1)/(\partial\theta_p^2))$ is positive semi-definite. Furthermore, if $\dot{\theta}_p \neq 0$, there exists an i such that $\dot{\theta}_p^T((\partial^2 h_i)/\partial\theta_p^2))\dot{\theta}_p \neq 0$. This shows that the singularity is nondegenerate. On the other hand, one degenerate singularity can be obtained by simply letting $l_2 = l_4 = l_6 = -r$. The Hessian matrix of h_1 (similar for others) is calculated as

$$\frac{\partial^2 h_1}{\partial\theta_p^2} = \begin{bmatrix} 2r^2 c_{3+4-1-2} & -2r^2 c_{3+4-1-2} & 0 \\ -2r^2 c_{3+4-1-2} & 2r^2 c_{3+4-1-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The annihilation vector turns out to be a constant vector $V_s = (\partial/\partial\theta_2) + (\partial/\partial\theta_4) + (\partial/\partial\theta_6)$. This singularity is shown in Fig. 20.

V. Conclusion

This paper presented a geometric framework for analyzing the singularities of a parallel manipulator. Using the differential forms associated with the constraint functions, we derived simple conditions for configuration space singularities. Topological structure of these singularities and their relations with the kinematic parameters of the system were investigated systematically using Morse function theory. We gave an intrinsic definition of parametrization singularities, and showed that actuator and end-effector singularities are just two of its special cases. We applied the proposed approach to the analysis of the observed "strange" singularity of the SNU manipulator and showed that it is in fact a CS singularity and an irregular A-singularity. We derived conditions for classifying parametrization singularities into first-order and second-order singularities. The latter can be further classified into nondegenerate and degenerate singularities. The danger associated with a degenerate A- or E-singularity was described.

It should be noted that the mathematical tool presented in this paper is a powerful one even for general analysis of parallel manipulators. In a forth coming paper, we will extend this tool to geometric control of parallel manipulator based systems.

Appendix A

Jacobian Matrices of the SNU Manipulator and Tsai's Manipulator at the Home Configuration

The Jacobian matrices of the three chains of the SNU manipulator at its home position are given by the first equation at the

$$\frac{\partial h_1}{\partial\theta_p} = \begin{bmatrix} 2l_2 r s_{3+4-1-2} & -2l_4 r s_{3+4-1-2} & 0 \end{bmatrix}$$

$$\frac{\partial^2 h_1}{\partial\theta_p^2} = \begin{bmatrix} 2l_2 r(1 - c_{3+4-1-2}) + 2l_2^2 & -2l_2 l_4 c_{3+4-1-2} & 0 \\ -2l_2 l_4 c_{3+4-1-2} & 2l_4 r(1 - c_{3+4-1-2}) + 2l_4^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

bottom of the page, and that of Tsai's manipulator by the second equation at the bottom of the page. The initial configurations of the end-effector for these two manipulators are the same and given by

$$g_i(0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 496.6555 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad i = 1, \cdots, 3.$$

APPENDIX B

PROOF OF (6)

First, we note that the Pfaffin constraints ω_X can be manipulated into

$$\omega_X = \begin{bmatrix} a_1^T dX_1 \\ a_2^T dX_1 \\ a_3^T dX_1 \\ dX_1 - dX_2 \\ dX_1 - dX_3 \end{bmatrix} = 0.$$

$$J_1(0) = \begin{bmatrix} 0 & 0 & -0.1155 & -496.6555 & 0 \\ 0 & 0 & 0 & 0 & 496.6555 \\ 0 & 259.8076 & 0.9933 & 202.0726 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_2(0) = \begin{bmatrix} 0 & 0 & 0.0577 & 248.3277 & -430.1163 \\ 0 & 0 & -0.1000 & -430.1163 & -248.3277 \\ 0 & 259.8076 & 0.9933 & 202.0726 & 0 \\ -0.5000 & -0.8660 & 0 & -0.8660 & -0.5000 \\ 0.8660 & -0.5000 & 0 & -0.5000 & 0.8660 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_3(0) = \begin{bmatrix} 0 & 0 & 0.0577 & 248.3277 & -430.1163 \\ 0 & 0 & 0.1000 & 430.1163 & -248.3277 \\ 0 & 259.8076 & 0.9933 & 202.0726 & 0 \\ -0.5000 & 0.8660 & 0 & 0.8660 & -0.5000 \\ -0.8660 & -0.5000 & 0 & -0.5000 & -0.8660 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_1(0) = \begin{bmatrix} 0 & 0 & -0.1155 & 0 & -496.6555 \\ 0 & -30 & 0 & 470 & 0 \\ 259.8076 & 0 & 0.9933 & 0 & 202.0726 \\ 0 & 0.9933 & 0 & 0.9933 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0.1155 & 0 & 0.1155 & 0 \end{bmatrix}$$

$$J_2(0) = \begin{bmatrix} 0 & 25.9808 & 0.0577 & -407.0319 & 248.3277 \\ 0 & 15 & -0.1000 & -235 & -430.1163 \\ 259.8076 & 0 & 0.9933 & 0 & 202.0726 \\ -0.8660 & -0.4967 & 0 & -0.4967 & -0.8660 \\ -0.5000 & 0.8602 & 0 & 0.8602 & -0.5000 \\ 0 & 0.1155 & 0 & 0.1155 & 0 \end{bmatrix}$$

$$J_3(0) = \begin{bmatrix} 0 & -25.9808 & 0.0577 & 407.0319 & 248.3277 \\ 0 & 15 & 0.1000 & -235 & 430.1163 \\ 259.8076 & 0 & 0.9933 & 0 & 202.0726 \\ 0.8660 & -0.4967 & 0 & -0.4967 & 0.8660 \\ -0.5000 & -0.8602 & 0 & -0.8602 & -0.5000 \\ 0 & 0.1155 & 0 & 0.1155 & 0 \end{bmatrix}$$

If

$$a_1^T dX_1 \wedge a_2^T dX_1 \wedge a_3^T dX_1 = 0,$$

then

$$\omega_{X,1} \wedge \cdots \wedge \omega_{X,15} = a_1^T dX_1 \wedge a_2^T dX_1 \wedge a_3^T dX_1 \wedge \omega_{X,4} \wedge \cdots \wedge \omega_{X,15} = 0.$$

To prove the converse, we expand $\omega_{X,1} \wedge \cdots \wedge \omega_{X,15}$ as the sum of linearly independent forms. One form is

$$a_1^T dX_1 \wedge a_2^T dX_1 \wedge a_3^T dX_1 \wedge dX_{2,1} \wedge \cdots \wedge dX_{3,6}.$$

If $\omega_{X,1} \wedge \cdots \wedge \omega_{X,15} = 0$, we have necessarily that

$$a_1^T dX_1 \wedge a_2^T dX_1 \wedge a_3^T dX_1 = 0.$$

Appendix C

Proof of (7)

Consider the map

$$g = g_1 \times g_2 \times g_3 : \mathbb{R}^{15} \to SE(3) \times SE(3) \times SE(3) : \Theta \to (g_1(\Theta_1), g_2(\Theta_2), g_3(\Theta_3)).$$

J is precisely the tangent map of $g (J = g_*)$. It is not difficult to see that

$$\omega_{\Theta,j} = g^* \omega_{X,j+3} \quad j = 1, \cdots, 12$$
$$0 = g^* \omega_{X,k}, \quad k = 1, \cdots, 3.$$

If $\dot{\Theta}$ is an annihilating vector of $\omega_{\Theta,j}, j = 1, \cdots, 12$, then

$$\omega_{\Theta,j}(\dot{\Theta}) = 0 = g^* \omega_{X,j+3}(\dot{\Theta}) = \omega_{X,j+3}(g_* \dot{\Theta}), \quad j = 1, \cdots, 12$$
$$g^* \omega_{X,k}(\dot{\Theta}) = 0 = \omega_{X,k}(g_* \dot{\Theta}), \quad k = 1, \cdots, 3$$

which shows that $g_* \dot{\Theta}$ is an annihilating vector of $\omega_{X,j} j = 1, \cdots, 15$. Conversely, if $\dot{X}$ annihilates $\omega_{X,j}$, $j = 1, \cdots, 15$, then from

$$\omega_{X,k}(\dot{X}) = 0, \quad k = 1, \cdots, 3$$

$\dot{X} = g_* \dot{\Theta}$ has a unique solution for $\dot{\Theta}$. We derive from the remaining equations that

$$\omega_{X,j+3}(\dot{X}) = \omega_{X,j+3}(g_* \dot{\Theta}) = g^* \omega_{X,j+3}(\dot{\Theta}) = \omega_{\Theta,j}(\dot{\Theta}) = 0$$

where $j = 1, \cdots, 12$. This clearly shows that $\dot{\Theta}$ annihilates $\omega_{\Theta,j}, j = 1, \cdots, 12$. Further, we can easily prove that the dimensions of the annihilation spaces for these two set of forms are equal and then (7) derives.

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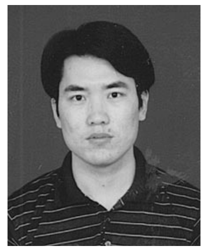

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