SINGULARITIES OF PARALLEL SURFACES

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Abstract

We investigate singularities of all parallel surfaces to a given regular surface. In generic context, the types of singularities of parallel surfaces are cuspidal edge, swallowtail, cuspidal lips, cuspidal beaks, cuspidal butterfly and 3-dimensional $D_4^{\pm}$ singularities. We give criteria for these singularities types in terms of differential geometry (Theorem 3.4 and 3.5).

1 Introduction Classically, a wave front is the locus of points having the same phase of vibration. A wave front is described by Huygens principle: The wave front of a propagating wave of light at any instant conforms to the envelope of spherical wavelets emanating from every point on the wave front at the prior instant (with the understanding that the wavelets have the same speed as the overall wave).

It is well known that a wave front may have singularities at some moment. Singularities of wave fronts are classified in generic context (see [1, p. 336]). The local classification of bifurcations in generic one parameter families of fronts in 3-dimensional spaces are also given in [1, p. 348]. To understand their singularities, it is important to know when the given front is generic and when the given one parameter family is generic.

In the differential geometric context, a wave front can be described as the parallel surface

$$g^t : U \to \boldsymbol{R}^3, \qquad g^t(u,v) := g(u,v) + t\mathbf{n}(u,v),$$

of a regular surface $g : U \to \boldsymbol{R}^3$ at time t. Here U is an open set of $\boldsymbol{R}^2$ and $\mathbf{n}$ denotes the unit normal vector given by $\mathbf{n} = (g_u \times g_v)/\|g_u \times g_v\|$. It is well known that when t is either of the principal radii of curvature at a point of the initial surface g, the parallel surface g^t has a singularity at the corresponding point (see, for example, [13]). So singularities of parallel surfaces should be investigated in terms of differential geometry of the regular map g.

2000 *Mathematics Subject Classification*. Primary 53A05; Secondary 58K05, 58K35.

Key words and phrases. Parallel surface, Versality of distance squared functions.

By Huygens principle, the wave front can be seen as the discriminant set of the distance squared unfolding

$$\Phi^t : U \times \boldsymbol{R}^3 \to \boldsymbol{R}, \qquad (u, v, x, y, z) \mapsto -\frac{1}{2}\left(\|(x, y, z) - g(u, v)\|^2 - {t_0}^2\right),$$

where t_0 is a constant. Porteous [14, 15] investigated the (Thom-Boardman) singularities of the unfolding $(u, v, x, y, z) \mapsto \Phi^t + \frac{1}{2}\|(x, y, z)\|^2$ with $t_0 = 0$. He discovered that the notion of normal vectors, principal radii of curvature, and umbilics correspond to A_1-singularities, A_2-singularities, and D_4-singularities or worse, respectively. Moreover, he discovered the notion of ridge points corresponding to A_3-singularities or worse.

It is now natural to ask a description of the singularity types of g^t in terms of differential geometry, which we answer in this paper. We fix a general regular map g and investigate singularities of g^t for all t. In other words, we investigate changes of singularities due to time evolution of fronts generated by g. To do this we need the notion of sub-parabolic points which is introduced by Bruce and Wilkinson [5] to study singularities of folding maps. The main theorem (Theorem 3.4) states criteria of the singularity types of g^t for all t in terms of differential geometry. For example, we show that, at a first order ridge point, g^t has swallowtail singularity when it is not sub-parabolic where t is the corresponding principal radius of curvature. This is enough to find a normal form when Φ^t is an unfolding of A_1, A_2, and A_3 singularities. This is proved by given a characterization for the unfolding Φ^t to be $\mathcal{K}$-versal in terms of differential geometry.

We now know that Φ^t is not a $\mathcal{K}$-versal unfolding at a sub-parabolic ridge point, a higher order ridge, and an umbilic. At these points, we are interested in the unfolding Φ defined by

$$\Phi : U \times \boldsymbol{R}^4 \to \boldsymbol{R}, \qquad (u, v, x, y, z, t) \mapsto -\frac{1}{2}\left(\|(x, y, z) - g(u, v)\|^2 - t^2\right).$$

Theorem 3.4 also gives a characterization for the unfolding Φ to be $\mathcal{K}$-versal in terms of differential geometry. For example, at a ridge point, we show that Φ is $\mathcal{K}$-versal without any other condition. The parallel surface is the section of discriminant set of this unfolding with the hyperplane defined by $t =$ constant. For A_4-singularities, that is, at a second order ridge point, we also show (Theorem 3.5 (1)) that g^t has cuspidal butterfly when it is not sub-parabolic where t is the corresponding principal radius of curvature. At a sub-parabolic ridge point where Φ^t fails to be $\mathcal{K}$-versal, we show (Theorem 3.5 (2)) the singularities of g^t are cuspidal beaks or cuspidal lips when the corresponding CPC (constant principal curvature) lines are Morse singularities. For D_4-singularities, we also show a similar result (Theorem 3.5 (3)). These results are satisfactory in the context of generic differential geometry.

2 Preliminary from differential geometry

We recall some differential geometric notions and their properties of regular surfaces in Euclidean space, which we need in this paper.

We present the definitions of ridge points, sub-parabolic points and umbilics, and their fundamental properties. We then discuss constant principal curvature (CPC) lines, which are the locus of singular points of the parallel surface. We state a characterization of these notions in terms of the coefficients of a Monge normal form of the surface.

2.1 Fundamental forms Consider a surface g defined by the Monge form:

$$g(u,v)=(u,v,f(u,v)), \qquad f(u,v)=\frac{1}{2}(k_1u^2+k_2v^2)+\sum_{i+j\geq 3}\frac{1}{i!j!}a_{ij}u^iv^j. \tag{2.1}$$

The coefficients of the first fundamental form are given by

$$E=\langle g_u,g_u\rangle=1+{f_u}^2, \qquad F=\langle g_u,g_v\rangle=f_uf_v, \qquad G=\langle g_v,g_v\rangle=1+{f_v}^2.$$

Here subscripts denotes partial derivatives and $\langle\ ,\ \rangle$ denotes the Euclidean inner product of $\boldsymbol{R}^3$. The unit normal vector is given by

$$\mathbf{n}=\frac{1}{\sqrt{1+{f_u}^2+{f_v}^2}}(-f_u,-f_v,1).$$

The coefficients of the second fundamental form are given by

$$L=\langle g_{uu},\mathbf{n}\rangle=\frac{f_{uu}}{\sqrt{1+{f_u}^2+{f_v}^2}},$$
$$M=\langle g_{uv},\mathbf{n}\rangle=\frac{f_{uv}}{\sqrt{1+{f_u}^2+{f_v}^2}},$$
$$N=\langle g_{vv},\mathbf{n}\rangle=\frac{f_{vv}}{\sqrt{1+{f_u}^2+{f_v}^2}}.$$

We consider the matrices of the first fundamental form and the second fundamental form:

$$\mathrm{I}=\begin{pmatrix}E & F\\ F & G\end{pmatrix}, \qquad \mathrm{II}=\begin{pmatrix}L & M\\ M & N\end{pmatrix}.$$

2.2 Principal curvatures We say that κ is a *principal curvature* if there is a non-zero vector (ξ,ζ) such that

$$\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix}=\kappa\begin{pmatrix}E & F\\ F & G\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix} \tag{2.2}$$

or, equivalently,

$$\frac{1}{EG-F^2}\begin{pmatrix}G & -F\\ -F & E\end{pmatrix}\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix}=\kappa\begin{pmatrix}\xi\\ \zeta\end{pmatrix}.$$

This is rewritten as

$$\frac{1}{(1+{f_u}^2+{f_v}^2)^{3/2}}\begin{pmatrix}1+{f_v}^2 & -f_uf_v\\ -f_uf_v & 1+{f_u}^2\end{pmatrix}\begin{pmatrix}f_{uu} & f_{uv}\\ f_{uv} & f_{vv}\end{pmatrix}\begin{pmatrix}\xi\\ \zeta\end{pmatrix}=\kappa\begin{pmatrix}\xi\\ \zeta\end{pmatrix}.$$

The eigenvector (ξ_i, ζ_i) $(i = 1, 2)$ of the equation (2.2) corresponding to the eigenvalue κ_i gives the principal vector $\mathbf{v}_i$. We can choose them so that the tangent vectors $\xi_i g_u + \zeta_i g_v$ are of the unit length.

At a point on the surface where two principal curvatures are distinct, there are two principal vectors and these vectors are mutually orthogonal. These principal vectors are often colored (blue or red) to distinguish between the two vectors. We assume that $\mathbf{v}_1$ is the blue principal vector and $\mathbf{v}_2$ is the red principal vector.

If two principal curvatures are equal at a point on the surface, we call such a point an umbilic. At an umbilic every direction through the umbilic is principal and the umbilic is an isolated singularity of the direction field.

If only one principal curvature is zero, such a point is called a parabolic point. If both principal curvatures are zero, such a point is called a flat umbilic or a planer point.

We can consider the focal surface. Away form umbilics the focal surface consists of two sheets, the blue and red sheets given by $g + \mathbf{n}/\kappa_1$ and $g + \mathbf{n}/\kappa_2$, respectively. The two sheets come together at umbilics. We note that at parabolic points only one of the two sheets exits, and at flat umbilics the common focal point lies at infinity.

The focal surface might have a singular point where the same colored principal curvature has an extreme value along the same colored line of curvature. Such a point on g is called a ridge point and on focal surface a rib. Ridges were first studied in details by Porteous [14].

The locus of points where the principal curvature has extreme value along the other colored line of curvature is also of importance. This locus is called a sub-parabolic line. The sub-parabolic line were studied in details by Bruce and Wilkinson [5] in terms of folding maps. The sub-parabolic line is also the locus of points on the surface whose image is the parabolic line on the same colored sheet of the focal surface. In [12] the sub-parabolic line appear as the locus of points where the other colored line of curvature has the geodesic inflections.

2.3 Ridge points and sub-parabolic points Let $g(p)$ be not an umbilic of a regular surface g, with principal vectors $\mathbf{v}_1$ ('blue'), $\mathbf{v}_2$ ('red') corresponding principal curvature κ_1, κ_2.

We say that the point $g(p)$ is a *ridge point* relative to $\mathbf{v}_i$ ('blue ridge point' for $i = 1$, 'red' for $i = 2$) if $\mathbf{v}_i\kappa_i(p) = 0$, where $\mathbf{v}_i\kappa_i$ is the directional derivative of κ_i in $\mathbf{v}_i$. Moreover, $g(p)$ is a *k-th order ridge point* relative to $\mathbf{v}_i$ if

$$\mathbf{v}_i^{(m)}\kappa_i(p) = 0 \quad (1 \le m \le k) \qquad \text{and} \qquad \mathbf{v}_i^{(k+1)}\kappa_i(p) \ne 0,$$

where $\mathbf{v}_i^{(k)}\kappa_i$ is the k-times directional derivative of κ_i in $\mathbf{v}_i$. The set of ridge points is called a *ridge line* or *ridges*.

Lemma 2.1. *Suppose that a surface g is given in Monge form as in* (2.1)*, and that the origin is not an umbilic.*

(1) *The origin is a first order blue ridge point if and only if*

$$a_{30} = 0 \quad \text{and} \quad 3{a_{21}}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2) \neq 0.$$

(2) *The origin is a second order blue ridge point if and only if*

$$\begin{aligned} &a_{30} = 3{a_{21}}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2) = 0 \quad \text{and} \\ &15{a_{21}}^2 a_{12} + 10a_{21}a_{31}(k_1 - k_2) + a_{50}(k_1 - k_2)^2 \neq 0. \end{aligned}$$

Proof. We remark that the principal curvatures at the origin are k_1, k_2 ($k_1 \neq k_2$) with corresponding principal vectors $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (0, 1)$.

The principal curvatures are the eigenvalues of $\mathrm{I}^{-1}\mathrm{II}$. So the principal curvature κ_1 is expressed as

$$\begin{aligned} \kappa_1(u, v) = k_1 + a_{30}u + a_{21}v + \frac{1}{2(k_1 - k_2)}\{[2{a_{21}}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2)]u^2 \\ + 2[2a_{21}a_{12} + a_{31}(k_1 - k_2)]uv + [2{a_{12}}^2 + (a_{22} - k_1{k_2}^2)(k_1 - k_2)]v^2\} + O(u, v)^3, \end{aligned} \tag{2.3}$$

and we have

$$\frac{\partial^3 \kappa_1}{\partial u^3}(0, 0) = \frac{6{a_{21}}^2(-a_{30} + a_{12}) + 6a_{21}a_{31}(k_1 - k_2) + (a_{50} - 18a_{30}{k_1}^2)(k_1 - k_2)^2}{6(k_1 - k_2)^2}.$$

Let (ξ_1, ζ_1) be the eigenvector of $\mathrm{I}^{-1}\mathrm{II}$ with the eigenvalue κ_1. It thus follows form (2.2) that there is a real number $\mu \neq 0$ such that

$$(\xi_1, \zeta_1) = \mu(N - \kappa_1 G, -M + \kappa_1 F).$$

Selection of the vector (ξ_1, ζ_1) in order for the tangent vector $\xi_1 g_u + \zeta_1 g_v$ to be of the unit length shows that the principal vector $\mathbf{v}_1$ is expressed as

$$\mathbf{v}_1(u, v) = \left(1 + O(u, v)^2\right)\frac{\partial}{\partial u} + \left(\frac{1}{k_1 - k_2}(a_{21}u + a_{12}v) + O(u, v)^2\right)\frac{\partial}{\partial v}, \tag{2.4}$$

and that

$$\frac{\partial^2 \zeta_1}{\partial u^2}(0, 0) = \frac{2a_{21}(a_{12} - a_{30}) + a_{31}(k_1 - k_2)}{2(k_1 - k_2)^2}.$$

Therefore, we have

$$\begin{aligned} \mathbf{v}_1\kappa_1(0, 0) &= \frac{\partial \kappa_1}{\partial u}(0, 0) = a_{30}, \\ {\mathbf{v}_1}^2\kappa_1(0, 0) &= \frac{\partial^2 \kappa_1}{\partial u^2}(0, 0) + \frac{\partial \kappa_1}{\partial v}(0, 0)\frac{\partial \zeta_1}{\partial u}(0, 0) = \frac{3{a_{21}}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2)}{k_1 - k_2}. \end{aligned}$$

Moreover, when $\mathbf{v}_1\kappa_1(0,0) = \mathbf{v}_1{}^2\kappa_1(0,0) = 0$, we obtain

$$\begin{aligned}\mathbf{v}_1{}^3\kappa_1(\mathbf{0}) &= \frac{\partial^3\kappa_1}{\partial u^3}(\mathbf{0}) + 3\frac{\partial^2\kappa_1}{\partial u\partial v}(\mathbf{0})\frac{\partial\zeta_1}{\partial u}(\mathbf{0}) + \frac{\partial\kappa_1}{\partial v}(\mathbf{0})\left(\frac{\partial\zeta_1}{\partial u}(\mathbf{0})\frac{\partial\zeta_1}{\partial v}(\mathbf{0}) + \frac{\partial^2\zeta_1}{\partial u^2}(\mathbf{0})\right)\\ &= \frac{15a_{21}{}^2a_{12} + 10a_{21}a_{31}(k_1-k_2) + a_{50}(k_1-k_2)^2}{(k_1-k_2)^2}.\end{aligned}$$

□

Lemma 2.2. *Suppose that a surface g is given in Monge form as in* (2.1)*, and that the origin is a blue ridge point. Then the blue ridge line through the origin fails to be smooth at the origin if and only if*

$$3a_{21}{}^2 + (a_{40} - 3k_1{}^3)(k_1 - k_2) = 3a_{21}a_{12} + a_{31}(k_1-k_2) = 0.$$

Proof. It follows form (2.3) and (2.4) that the equation of the blue ridge line through the origin is expressed as

$$[3a_{21}{}^2 + (a_{40} - 3k_1{}^3)(k_1-k_2)]u + [3a_{21}a_{12} + a_{31}(k_1-k_2)]v + \cdots = 0. \tag{2.5}$$

This equation implies the assertion. □

We turn to sub-parabolic points. A point $g(p)$ which is not an umbilic is a *sub-parabolic point* relative to $\mathbf{v}_i$ ('blue sub-parabolic point' for $i = 1$, 'red' for $i = 2$) if $\mathbf{v}_i\kappa_j(p) = 0$ $(i \neq j)$. The set of sub-parabolic points is called a *sub-parabolic line*.

Lemma 2.3. *Suppose that a surface g is given in Monge form as in* (2.1)*, and that the origin is not an umbilic. Then the origin is a red sub-parabolic point if and only if $a_{21} = 0$.*

Proof. Since the principal vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, it follows from (2.4) that the principal vector $\mathbf{v}_2$ is expressed the following form:

$$\mathbf{v}_2(u,v) = \left(\frac{1}{k_2-k_1}(a_{21}u + a_{12}v) + O(u,v)^2\right)\frac{\partial}{\partial u} + \left(1 + O(u,v)^2\right)\frac{\partial}{\partial v}. \tag{2.6}$$

From (2.3) and (2.6), the directional derivative $\mathbf{v}_2\kappa_1$ is given by

$$\begin{aligned}\mathbf{v}_2\kappa_1(u,v) = a_{21} &+ \frac{a_{21}(2a_{12}-a_{30}) + a_{31}(k_1-k_2)}{k_1-k_2}u\\ &+ \frac{a_{12}(2a_{12}-a_{30}) + (a_{22} - k_1k_2{}^2)(k_1-k_2)}{k_1-k_2}v + O(u,v)^2.\end{aligned} \tag{2.7}$$

This equation implies the assertion. □

We can deduce from (2.7) that the equation of the red sub-parabolic line through the origin has the form

$$a_{31}(k_1-k_2)u + [a_{12}(2a_{12}-a_{30}) + (a_{22} - k_1k_2{}^2)(k_1-k_2)]v + \cdots = 0. \tag{2.8}$$

2.4 Umbilics *Umbilics* of a regular surface are points where the two principal curvatures coincide. At these points the principal direction field is singular and the lines of curvature fail to cross at right angle. The classification of generic umbilics is due to Darboux [6]. He gave three configurations of the lines of curvature. The three configurations were given the names lemon, star, and monstar by Berry and Hannay [2]. Their classification was provided by Gutierrez and Sotomayor [7].

Suppose that the origin is an umbilic of a surface g, and that g is given in Monge form

$$g(u,v) = (u,v,f(u,v)), \quad f(u,v) = \frac{k}{2}(u^2+v^2) + \sum_{i+j\geq 3} \frac{1}{i!\,j!} a_{ij} u^i v^j, \tag{2.9}$$

where k is the common value for the principal curvatures at the origin.

At an umbilic the cubic part f_3 of f in (2.9) determines its type. An umbilic of the surface g is said to be *elliptic* or *hyperbolic* if f_3 has three real roots or one real root, respectively. Moreover, An umbilic is said to be *right-angled* if the root directions of the quadratic form which is the determinant of the Hessian matrix of f_3 are mutually orthogonal with respect to the standard scalar product on $\boldsymbol{R}^2$. Such an umbilic necessarily is a hyperbolic umbilic.

We shall present the conditions for types of umbilics in terms of the coefficients of the Monge form. We set

$$\Gamma := \begin{vmatrix} a_{30} & 2a_{21} & a_{12} & 0 \\ 0 & a_{30} & 2a_{21} & a_{12} \\ a_{21} & 2a_{12} & a_{03} & 0 \\ 0 & a_{21} & 2a_{12} & a_{03} \end{vmatrix}, \quad \text{and} \quad \Gamma' := \begin{vmatrix} 1 & 0 & 1 \\ a_{30} & a_{21} & a_{12} \\ a_{21} & a_{12} & a_{03} \end{vmatrix}.$$

The discriminant of f_3 is given by $-\Gamma$. Hence, the origin is an elliptic umbilic or hyperbolic umbilic if and only if $\Gamma < 0$ or $\Gamma > 0$, respectively. Moreover, the determinant of the Hessian matrix of f_3 is given by

$$-36[({a_{21}}^2 - a_{30}a_{12})u^2 + (a_{21}a_{12} - a_{30}a_{03})uv + ({a_{12}}^2 - a_{21}a_{03})v^2].$$

It follows that the origin is a right-angled umbilic if and only if $\Gamma' = 0$.

It is shown in [15] that there is one ridge line passing through a hyperbolic umbilic and three ridge lines passing through an elliptic umbilic. It is also shown in [15] that ridge lines change their color as they pass through a generic umbilic.

It is known that when there is one direction for lines of curvature at an umbilic, there is one sub-parabolic line through the umbilic in the same direction, while, when there are three directions for lines of curvature at an umbilic, there are three sub-parabolic lines through the umbilic in the same three directions [5, 12].

2.5 Constant principal curvature lines We set

$$\Sigma_c := \{(u,v) \in U\ ;\ \kappa_i(u,v) = c \text{ for some } i\}.$$

We call Σ_c the *constant principal curvature (CPC) line with the value of c.* There are two CPC lines $\Sigma_{\kappa_1(p)}$ (colored by blue) and $\Sigma_{\kappa_2(p)}$ (colored by red) locally through a non-umbilical point $g(p)$. We recall that a point $p \in U$ is a singular point of the parallel surface g^t at distance t if and only if $t = 1/\kappa_i(p)$ for some i. This means that the set of singular points of g^t is the CPC line $\Sigma_{\kappa_i(p)}$.

Firstly, we investigate the CPC lines away form umbilics. Suppose that a surface g is given in Monge form as in (2.1). From (2.3), $\kappa_1(u,v) = k_1$ is expressed by the equation

$$\begin{aligned}0 = a_{30}u + a_{21}v + \frac{1}{2(k_1-k_2)}\{&[2a_{21}{}^2 + (a_{40} - 3k_1{}^3)(k_1-k_2)]u^2 \\ &+ 2[2a_{21}a_{12} + a_{31}(k_1-k_2)]uv + [2a_{12}{}^2 + (a_{22} - k_1k_2{}^2)(k_1-k_2)]v^2\} + \cdots .\end{aligned} \tag{2.10}$$

Hence, the CPC line Σ_{k_1} is locally given by the equation (2.10) at the origin. The equation (2.10) shows that the CPC line Σ_{k_1} is singular at the origin if and only if $a_{30} = a_{21} = 0$, that is, the origin is a blue ridge point and a red sub-parabolic point (Lemma 2.1 and 2.3).

Lemma 2.4. *Suppose that the origin is a blue ridge point which is not a red sub-parabolic point.*

(1) *The CPC line Σ_{k_1} is transverse to the blue ridge line at the origin if and only if the order of the ridge is one.*

(2) *the CPC line Σ_{k_1} is tangential to the blue ridge line at the origin if and only if the order of the ridge is more than one.*

Proof. It follows from (2.5) and (2.10) that the CPC line Σ_{k_1} is transverse to the blue ridge line at the origin if and only if

$$3a_{21}{}^2 + (a_{40} - 3k_1{}^3)(k_1 - k_2) \neq 0.$$

On the other hand, both lines are tangential at the origin if and only if

$$3a_{21}{}^2 + (a_{40} - 3k_1{}^3)(k_1 - k_2) = 0.$$

Hence, the statement of the lemma follows from Lemma 2.1 □

Lemma 2.5. *Suppose that the origin is a blue ridge point and red sub-parabolic point. Then the CPC line Σ_{k_1} is locally either an isolated point or two intersecting smooth curves at the origin, if the blue ridge line crosses the red sub-parabolic line at the origin.*

Proof. First we remark that

$$\frac{\partial \kappa_1}{\partial u}(0,0) = a_{30} = 0 \quad \text{and} \quad \frac{\partial \kappa_1}{\partial v}(0,0) = a_{21} = 0.$$

The equations of the blue ridge line (2.5) and the red sub-parabolic line (2.8) reduce

$$(a_{40} - 3{k_1}^3)(k_1 - k_2)u + a_{31}(k_1 - k_2)v + \cdots = 0$$

and

$$a_{31}(k_1 - k_2)u + [2{a_{12}}^2 + (a_{22} - k_1{k_2}^2)(k_1 - k_2)]v + \cdots = 0,$$

respectively. From these equations, the blue ridge line crosses the red sub-parabolic line at the origin if and only if

$$(a_{40} - 3{k_1}^3)(k_1 - k_2)[2{a_{12}}^2 + (a_{22} - k_1{k_2}^2)(k_1 - k_2)] - {a_{31}}^2(k_1 - k_2)^2 \neq 0.$$

In addition, from (2.3), the determinant of the Hessian matrix of κ_1 at $(0,0)$ is given by

$$(a_{40} - 3{k_1}^3)(k_1 - k_2)[2{a_{12}}^2 + (a_{22} - k_1{k_2}^2)(k_1 - k_2)] - {a_{31}}^2(k_1 - k_2)^2.$$

By the Morse lemma (see, for example, [3]), we complete the proof. □

Secondly, we investigate the CPC line near an umbilic.

Theorem 2.6. (1) *The CPC line Σ_k is locally an isolated point at the elliptic umbilic, where k is the common value for the principal curvatures at the umbilic.*

(2) *The CPC line Σ_k is locally two intersecting smooth curves at a hyperbolic umbilic. The locally two curves change their color as they pass through the hyperbolic umbilic.*

Proof. We suppose that the origin is an umbilic of a surface g, and that the surface g is given in Monge form as in (2.9). The principal curvatures are the roots of the quadric equation

$$(EG - F^2)\kappa^2 - (EN - 2FM + GL)\kappa + (LN - M^2) = 0.$$

Replacing κ by k which is the common value for the principal curvatures at the origin, we can express the equation in the form

$$(a_{30}a_{12} - {a_{21}}^2)u^2 + (a_{30}a_{03} - a_{21}a_{12})uv + (a_{21}a_{03} - {a_{12}}^2)v^2 + \cdots = 0. \tag{2.11}$$

The locus of this equation is the CPC line Σ_k. We denote the quadric part of (2.11) by $\alpha u^2 + 2\beta uv + \gamma v^2$. Then we have $\beta^2 - \alpha\gamma = \Gamma/4$, where Γ is as in Section 2.4. Hence, the CPC line Σ_k at an umbilic is locally either an isolated point if $\Gamma < 0$ (i.e., the origin is an elliptic umbilic) or two smooth intersecting curves if $\Gamma > 0$ (i.e., the origin is a hyperbolic umbilic).

We investigate the case of hyperbolic umbilics in detail. For a hyperbolic umbilic, we may assume that g is locally given in the form

$$g(u,v) = (u, v, f(u,v)), \quad f(u,v) = \frac{k}{2}(u^2+v^2) + \frac{P}{6}u(u^2+2Quv+Rv^2) + \cdots \tag{2.12}$$

for some P, Q, and R with $P \neq 0$ and $Q^2 - R < 0$. Then the principal curvatures κ_1 (maximum curvature), κ_2 (minimum curvature) are expressed as

$$\begin{aligned} \kappa_1(u,v) &= k + \frac{1}{6}\Big(P[(R+3)u + 2Qu] \\ &\quad + |P|\sqrt{[16Q^2+(R-3)^2]u^2 + 12Q(R+1)uv + 4(Q^2+R^2)v^2}\Big) + \cdots, \\ \kappa_2(u,v) &= k + \frac{1}{6}\Big(P[(R+3)u + 2Qu] \\ &\quad - |P|\sqrt{[16Q^2+(R-3)^2]u^2 + 12Q(R+1)uv + 4(Q^2+R^2)v^2}\Big) + \cdots. \end{aligned} \tag{2.13}$$

Therefore, the locally two smooth curves change their color as they through the hyperbolic umbilic. □

REMARK 2.7. (1) A simple calculation gives $\Gamma' = \alpha + \gamma$, where Γ' is as in Section 2.4. It follows that the tangents to the locally two smooth curves of the CPC line through the right-angled umbilic are mutually orthogonal. We note that the right-angled umbilic necessarily is a hyperbolic umbilic.

(2) Equation (2.11) shows that the CPC line Σ_k is approximated by a conic near the origin when the origin is not a parabolic umbilic.

Finally, We investigate bifurcations of the CPC lines at an umbilic. We start with the case of an elliptic umbilic. There are three ridge lines through the elliptic umbilic. The bifurcation of the CPC lines at the elliptic umbilic is shown in Figure 1 (i), (ii) (cf. [4], Figure 2). We now turn to the case of a hyperbolic umbilic. We may assume that the surface given in the from (2.12). There is one ridge line through the hyperbolic umbilic. Calculations show that the ridge line is tangent to $2Qu + Rv = 0$ at the origin (cf. [15], corollary (iii) of Theorem 11.10) and that the locally two smooth curves of the CPC line Σ_k are tangent to $[QR \pm \sqrt{R^2(-Q^2+R)}]u + R^2v = 0$. Thus it follows that the bifurcation of the CPC lines at the hyperbolic umbilic is given in Figure 1 (iii)–(v) (cf. [4], Figure 2), in the generic context.

As shown in Figure 1, there are three intersection points of the CPC line and the same colored ridge line near an elliptic umbilic, and there is one such intersection point near a hyperbolic umbilic, in the generic context.

3 Singularities of parallel surfaces

In this section we present our main theorem.

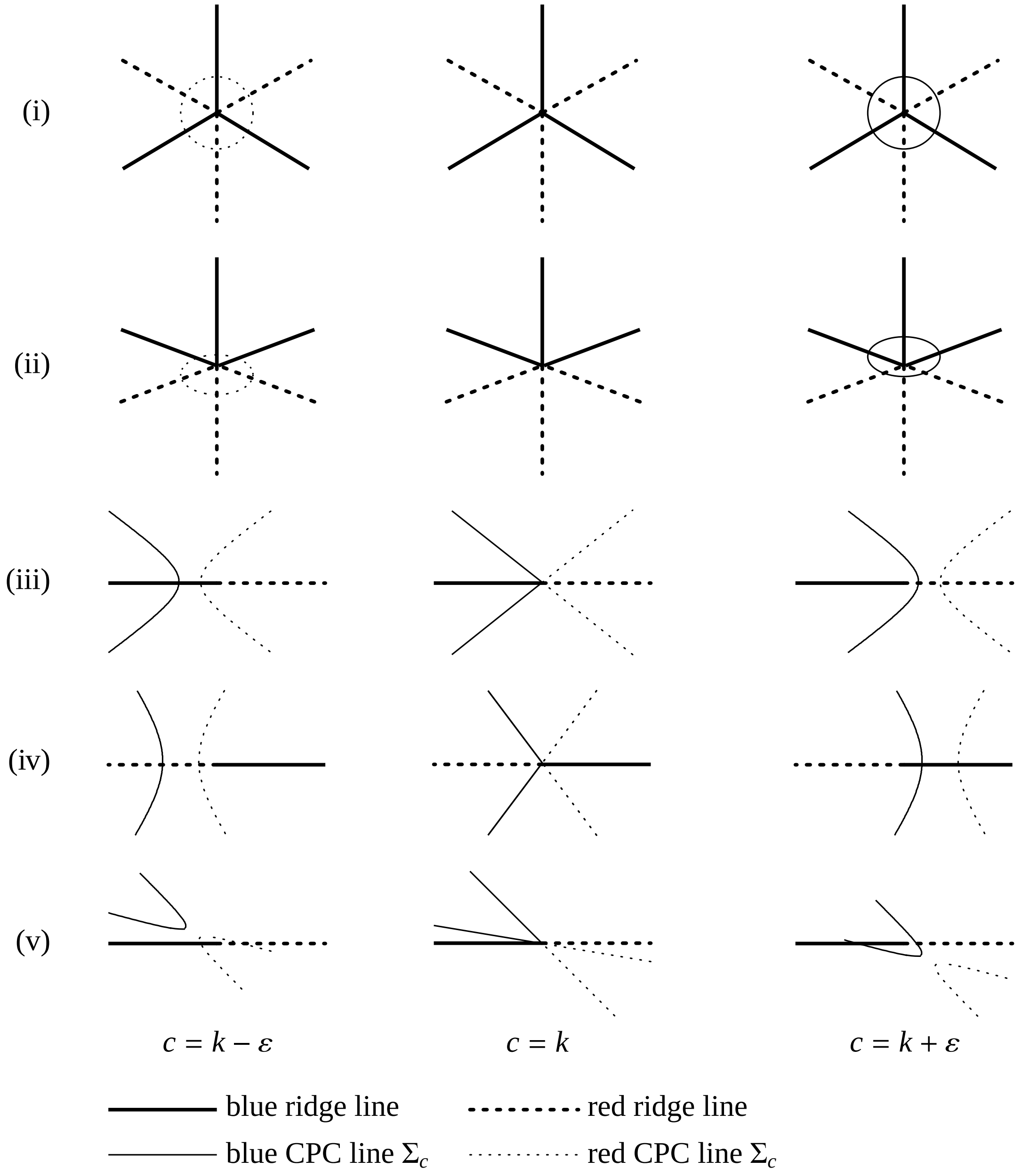

Figure 1: Bifurcations of the CPC lines near an elliptic umbilic (i) and (ii), and a hyperbolic umbilic (iii)–(v) , where ε is a small positive number.

3.1 Augmented distance squared functions Let $f : (\boldsymbol{R}^n, \mathbf{0}) \to (\boldsymbol{R}, \mathbf{0})$ be a smooth function germ. We say that a smooth function germ $F : (\boldsymbol{R}^n \times \boldsymbol{R}^r, \mathbf{0}) \to (\boldsymbol{R}, 0)$ is an *unfolding* of f if $F(\mathbf{u}, \mathbf{0}) = f(\mathbf{u})$. We define the *discriminant set* of F by

$$\mathcal{D}(F) = \left\{ \mathbf{x} \in \boldsymbol{R}^r \ ; \ F(\mathbf{u}, \mathbf{x}) = \frac{\partial F}{\partial u_1}(\mathbf{u}, \mathbf{x}) = \cdots \frac{\partial F}{\partial u_n}(\mathbf{u}, \mathbf{x}) = 0 \text{ for some } \mathbf{u} \in U \right\},$$

where $(\mathbf{u}, \mathbf{x}) = (u_1, \ldots, u_n, x_1, \ldots, x_r) \in (\boldsymbol{R}^n \times \boldsymbol{R}^r, \mathbf{0})$. We say that the F is a $\mathcal{K}$*-versal unfolding* if any unfolding $G : (\boldsymbol{R}^n \times \boldsymbol{R}^s, \mathbf{0}) \to (\boldsymbol{R}, 0)$ of f is representable in the form

$$G(\mathbf{u}, \mathbf{y}) = h(\mathbf{u}, \mathbf{y}) \cdot F(\Psi(\mathbf{u}, \mathbf{y}), \psi(\mathbf{y})),$$

where $\Psi : (\boldsymbol{R}^n \times \boldsymbol{R}^s, \mathbf{0}) \to (\boldsymbol{R}^n, \mathbf{0})$ is a smooth map germ with $\Psi(\mathbf{u}, \mathbf{0}) = \mathbf{u}$, $\psi : (\boldsymbol{R}^s, \mathbf{0}) \to (\boldsymbol{R}^r, \mathbf{0})$ is a smooth map germ with $\psi(\mathbf{0}) = 0$ and $h : (\boldsymbol{R}^n \times \boldsymbol{R}^s, \mathbf{0}) \to \boldsymbol{R}$ is a smooth function germ with $h(\mathbf{0}, \mathbf{0}) \neq 0$ (cf. [1, §8]). This condition is equivalent to that

$$\mathcal{E}_n = \left\langle \frac{\partial f}{\partial u_1}, \cdots, \frac{\partial f}{\partial u_n}, f \right\rangle_{\mathcal{E}_n} + \left\langle \left.\frac{\partial F}{\partial x_1}\right|_{\boldsymbol{R}^n \times \{\mathbf{0}\}}, \cdots, \left.\frac{\partial F}{\partial x_r}\right|_{\boldsymbol{R}^n \times \{\mathbf{0}\}} \right\rangle_{\boldsymbol{R}} + \mathcal{M}_n^{k+1}$$

when $f(\mathbf{u})$ is k-determined (see [17, §3] and [11, p.75]). Here, $\mathcal{E}_n$ is the set of smooth function germs $(\boldsymbol{R}^n, \mathbf{0}) \to \boldsymbol{R}$, which is the local ring with the unique maximal ideal $\mathcal{M}_n = \{f \in \mathcal{E}_n \ ; \ f(\mathbf{0}) = 0\}$. We say that two function germs f and $g : (\boldsymbol{R}^n, \mathbf{0}) \to (\boldsymbol{R}, \mathbf{0})$ are $\mathcal{K}$*-equivalent* if there exist a diffeomorphism germ $\psi : (\boldsymbol{R}^n, \mathbf{0}) \to (\boldsymbol{R}^n, \mathbf{0})$ and a smooth function germ $h : (\boldsymbol{R}^n, \mathbf{0}) \to \boldsymbol{R}$ with $h(\mathbf{0}) \neq 0$ such that $g(\mathbf{u}) = h(\mathbf{u}) \cdot f \circ \psi(\mathbf{u})$. If $F, G : (\boldsymbol{R}^n \times \boldsymbol{R}^r, \mathbf{0}) \to (\boldsymbol{R}, \mathbf{0})$ are $\mathcal{K}$-versal unfoldings of $\mathcal{K}$-equivalent function germs f, g, respectively. Then, there exist a diffeomorphism germ $\tilde{\Psi} : (\boldsymbol{R}^n \times \boldsymbol{R}^r, \mathbf{0}) \to (\boldsymbol{R}^n \times \boldsymbol{R}^r, \mathbf{0})$, $(\mathbf{u}, \mathbf{x}) \mapsto (\Psi(\mathbf{u}, \mathbf{x}), \psi(\mathbf{x}))$ and a smooth function germ $h : (\boldsymbol{R}^n \times \boldsymbol{R}^r, \mathbf{0}) \to \boldsymbol{R}$ with $h(\mathbf{0}, \mathbf{0}) \neq 0$ such that

$$G(\mathbf{u}, \mathbf{x}) = h(\mathbf{u}, \mathbf{x}) \cdot F(\Psi(\mathbf{u}, \mathbf{x}), \psi(\mathbf{x})).$$

(cf. [1, §8]). Moreover, calculation shows that $\mathcal{D}(F) = \psi(\mathcal{D}(G))$.

In order to investigate singularities of parallel surfaces, we consider the functions

$$\Phi^t : U \times \boldsymbol{R}^3 \to \boldsymbol{R}, \qquad \text{defined by} \qquad (u, v, x, y, z) \mapsto -\frac{1}{2}\left(\|(x, y, z) - g(u, v)\|^2 - {t_0}^2 \right),$$

where $t_0 \in \boldsymbol{R} \setminus \{0\}$, and

$$\Phi : U \times \boldsymbol{R}^4 \to \boldsymbol{R}, \qquad \text{defined by} \qquad (u, v, x, y, z, t) \mapsto -\frac{1}{2}\left(\|(x, y, z) - g(u, v)\|^2 - t^2 \right).$$

We call them *augmented distance squared functions*.

Calculating the discriminant set of Φ^t, we have

$$\mathcal{D}(\Phi^t) = \{(x, y, z) \in \boldsymbol{R}^3 \ ; \ (x, y, z) = g(u, v) + t_0 \mathbf{n}(u, v) \text{ for some } (u, v) \in \boldsymbol{R}^2\},$$

which is the parallel surface of g at a distance t_0. Besides, the discriminant set of Φ is given by

$$\mathcal{D}(\Phi) = \{(x, y, z, t) \in \boldsymbol{R}^3 \ ; \ (x, y, z) = g(u, v) + t\mathbf{n}(u, v) \text{ for some } (u, v) \in \boldsymbol{R}^2\}.$$

Its intersection with the hyperplane $t = t_0$ is the parallel surface of g at distance t_0.

We take a point $q_0 = (x_0, y_0, z_0)$ or $q_0 = (x_0, y_0, z_0, t_0)$ where

$$(x_0, y_0, z_0) = g(u_0, v_0) + t_0\mathbf{n}(u_0, v_0), \qquad t_0 = \frac{1}{\kappa_i(u_0, v_0)}$$

possibly with $\kappa_1(u_0, v_0) = \kappa_2(u_0, v_0)$, and set

$$\varphi(u, v) = \Phi^t(u, v, q_0) \qquad \text{or} \qquad \varphi(u, v) = \Phi(u, v, q_0).$$

Then the augmented distance functions Φ and Φ^t are the unfoldings of φ.

If φ is $\mathcal{K}$-equivalent to A_2 (resp. A_3) and Φ^t is a $\mathcal{K}$-versal unfolding of φ, then the discriminant set of Φ^t is locally diffeomorphic to the discriminant set of the versal unfolding $G : (U \times \boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}, 0)$,

$$G(u, v, x, y, z) = u^3 \pm v^2 + x + yu \quad (\text{resp. } G(u, v, x, y, z) = u^4 \pm v^2 + x + yu + zu^2)$$

of $g(u, v) = u^3 \pm v^2$ (resp. $g(u, v) = u^4 \pm v^2$). The singularity of the discriminant set of G is the cuspidal edge (resp. swallowtail).

Here, the *cuspidal edge* is a set locally diffeomorphic to the image of a map germ CE : $(\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$, $(u, v) \mapsto (u, v^2, v^3)$ and the *swallowtail* is a a set locally diffeomorphic to the image of a map germ $SW : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$, $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$. The pictures of the cuspidal edge and the swallowtail are shown in Figure 2.

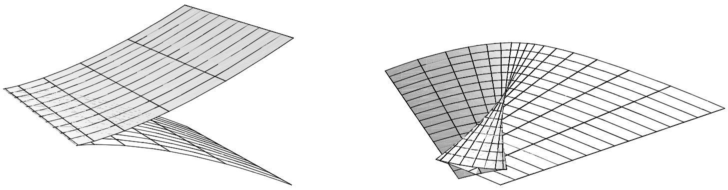

Figure 2: From left to right: Cuspidal edge, Swallowtail.

If φ is $\mathcal{K}$-equivalent to A_4 (resp. $D_4^{\pm}$) and $\Phi : (U \times \boldsymbol{R}^4, (u_0, v_0, q_0)) \to (\boldsymbol{R}, 0)$ is a $\mathcal{K}$-versal unfolding of φ, then the discriminant set of Φ is locally diffeomorphic to the discriminant set of the versal unfolding $G : (U \times \boldsymbol{R}^4, \mathbf{0}) \to (\boldsymbol{R}, 0)$,

$$G(u, v, x, y, z, t) = u^4 \pm v^2 + x + yu + zu^2 + tu^3 \quad (\text{resp. } G(u, v, x, y, z, t) = u^2 v \pm v^3 + x + yu + zv + tv^2)$$

of $g(u, v) = u^4 \pm v^2$ (resp. $g(u, v) = u^2 v \pm v^3$). The singularity of the discriminant set of G is a butterfly (resp. $D_4^{\pm}$ singularities).

Here, the *butterfly* is a set locally diffeomorphic to the image of a map germ $BF : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^4, \mathbf{0})$, $(u, v, w) \mapsto (u, 5v^4 + 2uv + 3v^2 w, 4v^5 + uv^2 + 2v^3 w, w)$ and the 4-*dimensional* $D_4^{\pm}$ *singularity* is a set locally diffeomorphic to the image of a map germ $FD^{\pm} : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^4, \mathbf{0})$, $(u, v, w) \mapsto (uv, u^2 + 2vw \pm 3v^2, 2u^2 v + v^2 w \pm 2v^3, w)$.

3.2 Criteria for singularities of fronts in $\boldsymbol{R}^3$ It is well known that the parallel surface g^t is a front. Fronts were first studied in details by Arnol'd and Zakalyukin. They showed that the generic singularities of fronts in $\boldsymbol{R}^3$ are cuspidal edges and swallowtails. Moreover, they showed that the singularities of the bifurcations in generic one parameter families of fronts in $\boldsymbol{R}^3$ are cuspidal lips, cuspidal beaks, cuspidal butterflies and 3-dimensional $D_4^{\pm}$ singularities (cf. [1]).

Here, the *cuspidal lips* is a set locally diffeomorphic to the image of a map germ $CLP : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$, $(u, v) \mapsto (3u^4 + 2u^2 v^2, u^3 + uv^2, v)$, the *cuspidal beaks* is a set locally diffeomorphic to the image of a map germ $CBK : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$, $(u, v) \mapsto (3u^4 - 2u^2 v^2, u^3 - uv^2, v)$, the *cuspidal butterfly* is a set of the image of a map germ $CBF : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$, $(u, v) \mapsto (4u^5 + u^2 v, 5u^4 + 2uv, v)$ and the 3-*dimensional* D_4^{+} *singularity* (resp. D_4^{-} *singularity*) is a set of the image of a map germ $TD^{+} : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$, $(u, v) \mapsto (uv, u^2 + 3v^2, u^2 v + v^3)$ (resp. $TD^{-} : (u, v) \mapsto (uv, u^2 - 3v^2, u^2 v - v^3)$). Their pictures are shown in Figure 3.

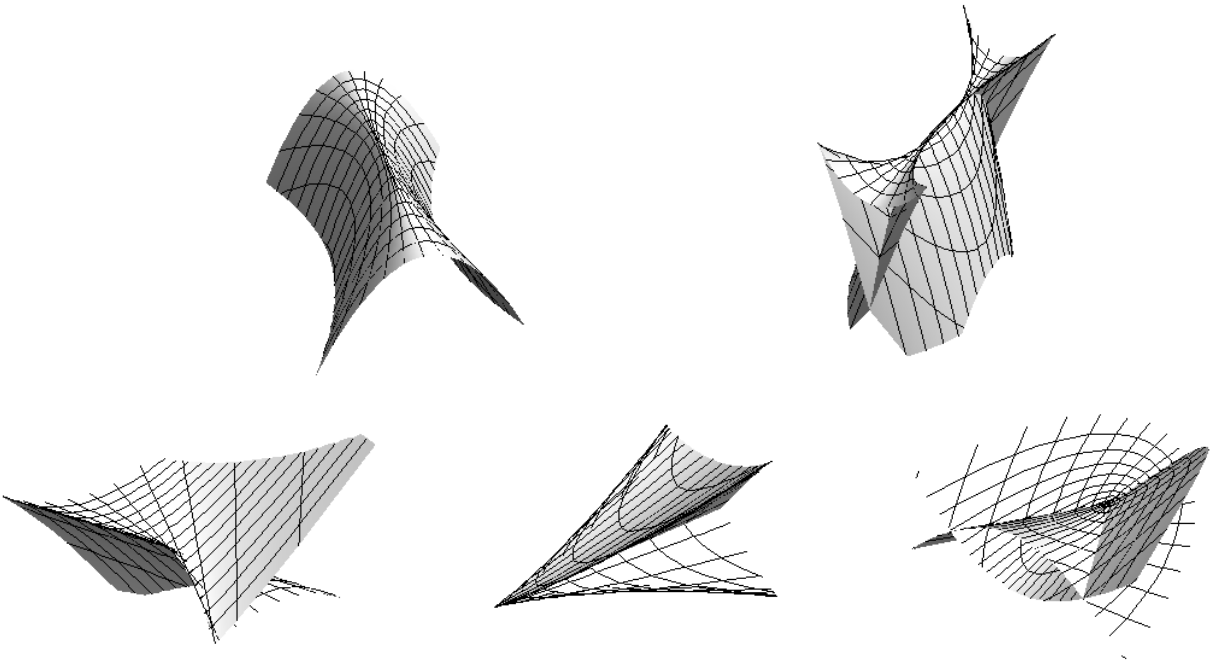

Figure 3: From top left to bottom right: Cuspidal lips, Cuspidal beaks, Cuspidal butterfly, 3-dimensional D_4^{+} singularity, 3-dimensional D_4^{-} singularity.

Recently, criteria for these singularities are shown in [8], [9], [10], [16]. To present these criteria, we prepare basic notions of fronts in $\boldsymbol{R}^3$. A smooth map $f : U \to \boldsymbol{R}^3$ is called a *front* if there exists a unit vector field ν of $\boldsymbol{R}^3$ along f such that $L_f = (f, \nu) : U \to T_1\boldsymbol{R}^3$ is a Legendrian immersion, where $T_1\boldsymbol{R}^3$ is the unit tangent bundle of $\boldsymbol{R}^3$ (cf. [1], see also [10]). For a front f, we define a function $\lambda : U \to \boldsymbol{R}$ by $\lambda(u, v) = \det(f_u, f_v, \nu)$. The function λ is called a *discriminant function* of f. The set of singular points $S(f)$ of f is the zero set of λ. A singular point $p \in U$ of f is said to be *non-degenerate* if $d\lambda(p) \neq 0$. Let p be a non-degenerate singular point of a front f. Then $S(f)$ is parameterized by a regular curve $\gamma(t) : (-\varepsilon, \varepsilon) \to U$ near p. Moreover, there exists a a unique direction $\eta(t) \in T_{\gamma(t)}U$ up to scalar multiplications such that $df(\eta(t)) = 0$. We call $\eta(t)$ the *null direction*. Under these notations, we present the criterion for the cuspidal butterfly.

Theorem 3.1 ([8]). *Let $f : U \to \boldsymbol{R}^3$ be a front and p a non-degenerate singular point of f. Then the germ of the front f at p is $\mathcal{A}$-equivalent to the map germ CBF if and only if $\eta\lambda(p) = \eta^2\lambda(p) = 0$ and $\eta^3\lambda(p) \neq 0$.*

Here, two map germs $f_1, f_2 : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}^3, \boldsymbol{0})$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\psi_1 : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}^2, \boldsymbol{0})$ and $\psi_2 : (\boldsymbol{R}^3, \boldsymbol{0}) \to (\boldsymbol{R}^3, \boldsymbol{0})$ such that $\psi_2 \circ f_1 = f_2 \circ \psi_1$, and $\eta\lambda$ denotes the directional derivative of λ in the direction of η.

We now turn to degenerate singularities. Let p be a degenerate singular point of the front f. If $\mathrm{rank}(df_p) = 1$, then there exists the non-zero vector field η near p such that if $q \in S(f)$ then $df_q(\eta(q)) = 0$. Criteria for degenerate singularities are as follows:

Theorem 3.2 ([9]). *Let $f : U \to \boldsymbol{R}^3$ be a front and p a degenerate singular point of f.*

1. *The germ of the front f at p is $\mathcal{A}$-equivalent to the map germ CLP if and only if* $\mathrm{rank}(df_p) = 1$ *and the* $\det(\mathrm{Hess}\,\lambda(p)) > 0$, *where* $\det(\mathrm{Hess}\,\lambda(p))$ *denotes the determinant of the Hessian matrix of λ at p.*
2. *The germ of the front f at p is $\mathcal{A}$-equivalent to the map germ CBK if and only if* $\mathrm{rank}(df_p) = 1$, $\det(\mathrm{Hess}\,\lambda(p)) < 0$ *and* $\eta^2\lambda(p) \neq 0$.

Theorem 3.3 ([16]). *Let $f : U \to \boldsymbol{R}^3$ be a front and pa degenerate singular point of f. Then the germ of the front f at p is $\mathcal{A}$-equivalent to the map germ TD^+ (resp. TD^-) if and only if* $\mathrm{rank}(df)_p = 0$ *and* $\det(\mathrm{Hess}\,\lambda(p)) < 0$ (*resp.* $\det(\mathrm{Hess}\,\lambda(p)) > 0$).

3.3 Singularities of parallels surfaces Now we are ready to state our main theorem.

Theorem 3.4. *Let $g : U \to \boldsymbol{R}^3$ be a regular surface and g^t the parallel surface of g at distance t, where U is an open subset of $\boldsymbol{R}^2$. Assume that Φ, Φ^t, and φ is defined as in Section* 3.1.

1. *If $g(u_0, v_0)$ is neither a ridge point relative to the principal vector $\mathbf{v}_i$ nor an umbilic, and $\kappa_i(u_0, v_0) \neq 0$, then φ has an A_2 singularity at (u_0, v_0). In this case, Φ^t is a $\mathcal{K}$-versal unfolding of φ. Moreover, the parallel surface g^t at distance $t = 1/\kappa_i(u_0, v_0)$ is locally diffeomorphic to the cuspidal edge at $g^t(u_0, v_0)$.*

(2) *If $g(u_0, v_0)$ is a first order ridge point relative to the principal vector $\mathbf{v}_i$, and $\kappa_i(u_0, v_0) \neq 0$, then φ has an A_3 singularity at (u_0, v_0). In this case, Φ^t is a $\mathcal{K}$-versal unfolding of φ if and only if $g(u_0, v_0)$ is not a sub-parabolic point relative to the other principal vector $\mathbf{v}_j$. Moreover, the parallel surface g^t at distance $t = 1/\kappa_i(u_0, v_0)$ is locally diffeomorphic to the swallowtail at $g^t(u_0, v_0)$.*

(3) *If $g(u_0, v_0)$ is a second order ridge point relative to the principal vector $\mathbf{v}_i$, and $\kappa_i(u_0, v_0) \neq 0$, then φ has an A_4 singularity at (u_0, v_0). In this case, Φ is a $\mathcal{K}$-versal unfolding of φ if and only if (u_0, v_0) is a regular point of the ridge line relative to the same principal vector $\mathbf{v}_i$. Moreover, the parallel surface g^t at distance $t = 1/\kappa_i(u_0, v_0)$ is the section of the discriminant set $\mathcal{D}(\Phi)$, which is locally diffeomorphic to the butterfly, with the hyperplane $t = 1/\kappa_i(u_0, v_0)$.*

(4) *If $g(u_0, v_0)$ is a hyperbolic umbilic and not a flat umbilic, then φ has a D_4^+ singularity at (u_0, v_0). In this case, Φ is a $\mathcal{K}$-versal unfolding of φ if and only if $g(u_0, v_0)$ is not a right-angled umbilic. Moreover, the parallel surface g^t at distance $t = 1/\kappa_i(u_0, v_0)$ is the section of the discriminant set $\mathcal{D}(\Phi)$, which is locally diffeomorphic to the 4-dimensional D_4^+ singularity, with the hyperplane $t = 1/\kappa_i(u_0, v_0)$.*

(5) *If $g(u_0, v_0)$ is an elliptic umbilic and not a flat umbilic, then φ has a D_4^- singularity at (u_0, v_0). In this case, Φ is a $\mathcal{K}$-versal unfolding of φ. Moreover, the parallel surface g^t at distance $t = 1/\kappa_i(u_0, v_0)$ is the section of the discriminant set $\mathcal{D}(\Phi)$, which is locally diffeomorphic to the 4-dimensional D_4^- singularity, with the hyperplane $t = 1/\kappa_i(u_0, v_0)$.*

A proof of this theorem is given in Section 5.

Again, we remark that the parallel surfaces g^t of a regular surface g are the front. Since the unit normal vector of the parallel surface g^t coincides with the unit normal vector $\mathbf{n}$ of the initial surface g, the discriminant function of g^t is given by

$$\lambda(u, v) = \det(g^t_u(u, v), g^t_v(u, v), \mathbf{n}(u, v)).$$

Moreover, the Jacobian matrix J_{g^t} of g^t is given by

$$J_{g^t} = J_g - tJ_g\mathrm{I}^{-1}\mathrm{II} = J_g(I_2 - t\mathrm{I}^{-1}\mathrm{II}), \tag{3.1}$$

where J_g is the Jacobian matrix of g and I_2 is the 2×2 identity matrix. Applying criteria for singularities of fronts (Theorem 3.1–3.3) to the parallel surface g^t, we obtain Theorem 3.5 as corollaries of these criteria.

Theorem 3.5. *Let $g : U \to \boldsymbol{R}^3$ be a regular surface and g^t the parallel surface of g at distance t, where U is an open subset of $\boldsymbol{R}^2$.*

(1) *Suppose that $g(p)$ is a second order ridge point relative to the principal vector $\mathbf{v}_i$ which is not a sub-parabolic point relative to the other principal direction $\mathbf{v}_j$, and that $\kappa_i(p) \neq 0$. Then the parallel surface g^t at distance $t = 1/\kappa_i(p)$ is locally diffeomorphic to the cuspidal butterfly at $g^t(p)$.*

(2) *Suppose that* $g(p)$ *is a ridge point relative to the principal direction* $\mathbf{v}_i$ *and a sub-parabolic point relative to the other principal direction* $\mathbf{v}_j$*, and that* $\kappa_i(p) \neq 0$*. Then the parallel surface* g^t *at distance* $t = 1/\kappa_1(p)$ *is locally diffeomorphic to the cuspidal lips* (*resp. cuspidal beaks*) *at* $g^t(p)$ *if* $\det(\mathrm{Hess}_{(\mathbf{v}_1,\mathbf{v}_2)}\kappa_i(p)) > 0$ (*resp.* $\det(\mathrm{Hess}_{(\mathbf{v}_1,\mathbf{v}_2)}\kappa_i(p)) < 0$ *and the order of ridge is one*)*, where* $\mathrm{Hess}_{(\mathbf{v}_1,\mathbf{v}_2)}\kappa_i$ *is the Hessian matrix of* κ_i *with respect to* $\mathbf{v}_1$ *and* $\mathbf{v}_2$.

(3) *Suppose that* $g(p)$ *is an umbilic which is not a flat umbilic. Then the parallel surface* g^t *at distance* $t = 1/\kappa_1(0,0) = 1/\kappa_2(0,0)$ *is locally diffeomorphic to a 3-dimensional* D_4^+ *singularity* (*resp.* D_4^- *singularity*) *at* $g^t(p)$ *if* $g(p)$ *is a hyperbolic umbilic* (*resp. elliptic umbilic*).

Proof. (1) We may assume that $p = (0,0)$ and that the initial regular surface g given in Monge form as in (2.1). We remark that $k_1 \neq k_2$. Now we prove in the case $t = 1/\kappa_1(0,0) = 1/k_1$. From Lemma 2.1 and 2.3, we have

$$\begin{gathered} a_{30} = 3{a_{21}}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2) = 0, \\ 15a_{21}^2a_{12} + 10a_{21}a_{31}(k_1 - k_2) + a_{50}(k_1 - k_2)^2 \neq 0, \quad \text{and} \quad a_{21} \neq 0. \end{gathered} \tag{3.2}$$

Suppose that $t = 1/k_1$. Then we have $\lambda(0,0) = 0$. Moreover, from (3.2), we have

$$\lambda_u(0,0) = \frac{a_{30}(k_2 - k_1)}{{k_1}^2} = 0 \quad \text{and} \quad \lambda_v(0,0) = \frac{a_{21}(k_2 - k_1)}{{k_1}^2} \neq 0. \tag{3.3}$$

It turns out that $(0,0)$ is a non-degenerate singular point of g^t. Therefore, the set of singular points of g^t is a locally smooth curve near $(0,0)$, which is the CPC line Σ_{k_1}, and there exists a null direction η with $dg^t(\eta) = 0$ along this smooth curve. It follows form (3.1) that the null direction η has the same direction as the principal vector $\mathbf{v}_1$. From (3.2), we have

$$\begin{aligned} \mathbf{v}_1\lambda(0,0) &= -\frac{a_{30}(k_1 - k_2)^2}{{k_1}^2} = 0, \\ {\mathbf{v}_1}^2\lambda(0,0) &= -\frac{(k_1 - k_2)^2[a_{30}(a_{30} - 3a_{12}) + 3{a_{21}}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2)]}{{k_1}^2} = 0, \quad \text{and} \\ {\mathbf{v}_1}^3\lambda(0,0) &= -\frac{(k_1 - k_2)^2[15a_{21}^2a_{12} + 10a_{21}a_{31}(k_1 - k_2) + a_{50}(k_1 - k_2)^2]}{{k_1}^2} \neq 0. \end{aligned}$$

Therefore, we obtain that $\eta\lambda(0,0) = \eta^2\lambda(0,0) = 0$, $\eta^3\lambda(0,0) \neq 0$.

If the two map germs are $\mathcal{A}$-equivalent, their images are locally diffeomorphic. Hence, by Theorem 3.1 the parallel surface g^t at distance $t = 1/k_1$ is locally diffeomorphic to the cuspidal butterfly.

(2) We may assume that $p = (0,0)$ and that the initial regular surface g given in Monge form as in (2.1). We remark that $k_1 \neq k_2$. Now we prove in the case $t = 1/\kappa_1(0,0) = 1/k_1$. From Lemma 2.1 and 2.3, we have

$$a_{30} = a_{21} = 0. \tag{3.4}$$

Suppose that $t = 1/k_1$. Then we have $\lambda(0,0) = 0$ and

$$J_{g^t}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & (k_1 - k_2)/k_1 \\ 0 & 0 \end{pmatrix}.$$

Moreover, from (3.4), we have $\lambda_u(0,0) = \lambda_v(0,0) = 0$. It follows that $(0,0)$ is a degenerate singular point of g^t with $\mathrm{rank}(dg^t_p) = 1$. Using (3.4), we obtain that

$$\det(\mathrm{Hess}\lambda(0,0)) = \frac{(k_1 - k_2)^2}{{k_1}^4} \begin{vmatrix} a_{40} - 3{k_1}^3 & a_{31} \\ a_{31} & \dfrac{2{a_{12}}^2 + (a_{22} - k_1{k_2}^2)}{k_1 - k_2} \end{vmatrix}$$

and

$$\det(\mathrm{Hess}_{(\mathbf{v}_1,\mathbf{v}_2)}\kappa_1(0,0)) = \begin{vmatrix} a_{40} - 3{k_1}^3 & a_{31} \\ a_{31} & \dfrac{2{a_{12}}^2 + (a_{22} - k_1{k_2}^2)}{k_1 - k_2} \end{vmatrix}. \tag{3.5}$$

Therefore, the sign of $\det(\mathrm{Hess}\lambda(0,0))$ is the same as the sign of $\det(\mathrm{Hess}_{(\mathbf{v}_1,\mathbf{v}_2)}\kappa_1(0,0))$. Besides, since $\mathrm{rank}(dg^t_p) = 1$, there exists a non-zero vector η with $dg^t_p(\eta) = 0$. From (3.1), the non-zero vector η has the same direction as the principal vector $\mathbf{v}_1$. Hence, $\eta^2\lambda(0,0) \neq 0$ if and only if ${\mathbf{v}_1}^2\lambda(0,0) \neq 0$. From (3.4), we have

$${\mathbf{v}_1}^2\lambda(0,0) = -\frac{(a_{40} - 3{k_1}^3)(k_1 - k_2)^3}{{k_1}^2}.$$

Therefore, this shows that $(0,0)$ is a first order blue ridge point if and only if $\eta^2\lambda(0,0) \neq 0$ (cf. Lemma 2.1). Applying Theorem 3.2 to the argument indicated above, we obtain (2).

(3) We may assume that $p = (0,0)$ and that the initial regular surface g given in Monge form as in (2.9). We remark that $\kappa_1(0,0) = \kappa_2(0,0) = k$. Suppose that $t = 1/k$. Then we have $\lambda(0,0) = 0$,

$$\lambda_u(0,0) = t(kt-1)(a_{30} + a_{21}) = 0, \quad \lambda_v(0,0) = t(kt-1)(a_{12} + a_{03}) = 0, \quad \text{and}$$

$$J_{g^t}(0,0) = \begin{pmatrix} 1 - kt & 0 \\ 0 & 1 - kt \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $(0,0)$ is a degenerate singular point of g^t with $\mathrm{rank}(dg^t_p) = 0$. Moreover, we have

$$\det(\mathrm{Hess}\lambda(0,0)) = -\frac{1}{k^4}({a_{30}}^2{a_{03}}^2 - 6a_{30}a_{21}a_{12}a_{03} + 4a_{30}{a_{12}}^3 + 4{a_{21}}^3a_{03} - 3{a_{21}}^2{a_{12}}^2) = -\frac{1}{k^4}\Gamma,$$

where Γ is as in Section 2.4. It follows that $\det(\mathrm{Hess}\lambda(0,0)) < 0$ (resp. $\det(\mathrm{Hess}\lambda(0,0)) > 0$) if and only if $g(0,0)$ is a hyperbolic umbilic (resp. elliptic umbilic). Therefore, using Theorem 3.3, we obtain (3). □

Remark 3.6. Suppose that $g(p)$ is a ridge point relative to the principal direction $\mathbf{v}_i$ and a sub-parabolic point relative to the other principal direction $\mathbf{v}_j$. It follow from (2.5), (2.8) and (3.5) that $\det(\mathrm{Hess}_{(\mathbf{v}_1,\mathbf{v}_2)}\kappa_i(p)) = 0$ if and only if the ridge line relative to $\mathbf{v}_i$ and the sub-parabolic line relative to $\mathbf{v}_j$ are tangent at p.

These theorems imply that the configuration of CPC lines, ridge lines, and sub-parabolic lines determines types of singularities of parallel surfaces. For example, it follows from Theorem 3.4 (1) and Lemma 2.4 (1) that if the CPC line $\Sigma_{\kappa_i(u_0,v_0)}$ $(\kappa_i(u_0, v_0) \neq 0)$ does not meet the ridge line relative $\mathbf{v}_i$ at (u_0, v_0) then the parallel surface g^t at distance $t = 1/\kappa_i(u_0, v_0)$ is the cuspidal edge at $g^t(u_0, v_0)$. Moreover, it follows from Theorem 3.4 (2) and Lemma 2.4 (1) that if CPC line $\Sigma_{\kappa_i(u_0,v_0)}$ $(\kappa_i(u_0, v_0) \neq 0)$ crosses the ridge line relative to the principal vector $\mathbf{v}_i$ and does not cross the sub-parabolic line relative to the other principal vector $\mathbf{v}_j$ at (u_0, v_0) then the parallel surface g^t at distance $t = 1/\kappa_i(u_0, v_0)$ is the swallowtail at $g^t(u_0, v_0)$. Therefore, Figure 1 (i) and (ii) show that there are three swallowtails near $g^t(u_0, v_0)$ on the parallel surface g^t at distance $t = 1/(\kappa_i(u_0, v_0) \pm \varepsilon)$ if $g(u_0, v_0)$ is an elliptic umbilic which is not flat. Similarly, Figure 1 (iii)–(v) show that there is one swallowtail near $g^t(u_0, v_0)$ on the parallel surface g^t at distance $t = 1/(\kappa_i(u_0, v_0) \pm \varepsilon)$ if $g(u_0, v_0)$ is a hyperbolic umbilic which is either flat nor right-angled. These bifurcations of parallel surfaces near umbilics are depicted in Figure 4. These are also shown in [1, p. 384].

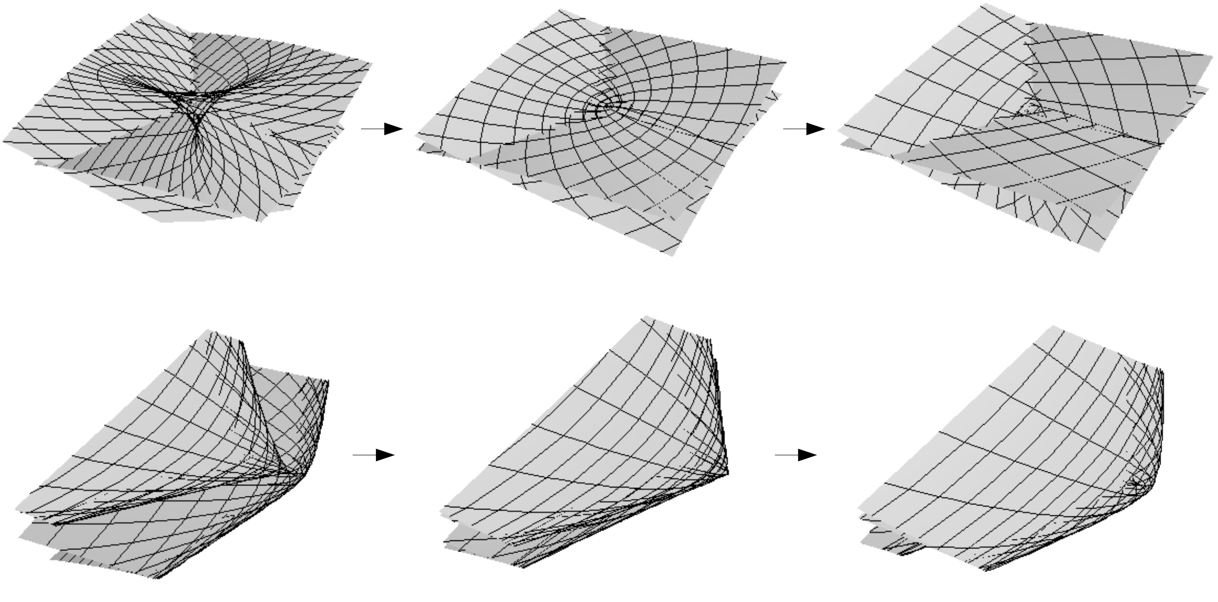

Figure 4: From top to bottom: Elliptic umbilic, Hyperbolic umbilic.

4 Criteria for A_1, A_2, A_3, A_4 and $D_4^{\pm}$ singularities

Before we present proof of Theorem 3.4, we shall provide a convenient criteria for $A_{\leq 4}$ and D_4 singularities in this section.

We consider the function $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$ whose Taylor expansion at $(0, 0)$ is

$$f(u, v) = \sum_{i,j} \frac{1}{i!\,j!} c_{ij} u^i v^j.$$

4.1 Criteria for A_k-singularities ($k \leq 4$) We assume that f is singular at $(0, 0)$ (i.e., $c_{10} = c_{01} = 0$). It is well known that the function f has an A_1-singularity at $(0, 0)$ if and only if

$$\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix}$$

is of full rank. Now we set

$$c_n(u, v) := \sum_{i+j=n} \frac{1}{i!\,j!} c_{ij} u^i v^j.$$

It is easy to see that the following conditions are equivalent.

(1) The matrix $\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix}$ is of rank 1.

(2) There exists a non-zero vector (λ, μ) such that $\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(3) There exist a non-zero vector (λ, μ) and non-zero real number s such that

$$\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix} = s \begin{pmatrix} \mu^2 & -\lambda\mu \\ -\lambda\mu & \lambda^2 \end{pmatrix}. \tag{4.1}$$

The rank of the Hesse's matrix of f is 1 if and only if one of these conditions holds. Under this assumption, we have the followings.

Theorem 4.1. (1) *The function f is A_2-singularity at $(0, 0)$ if and only if $c_3(\lambda, \mu) \neq 0$.*

(2) *The function f is A_3-singularity at $(0, 0)$ if and only if $c_3(\lambda, \mu) = 0$,*

$$\hat{c}_4(\lambda, \mu) := c_4(\lambda, \mu) + \frac{1}{8s} \begin{vmatrix} \mu^2 & -\lambda\mu & \lambda^2 \\ c_{30} & c_{21} & c_{12} \\ c_{21} & c_{12} & c_{03} \end{vmatrix} \neq 0.$$

(3) *The function f is A_4-singularity at $(0, 0)$ if and only if $c_3(\lambda, \mu) = \hat{c}_4(\lambda, \mu) = 0$ and one of the following conditions holds.*

(*a*) $\lambda \neq 0$, $c_5(\lambda, \mu) - \dfrac{1}{s\lambda^2} c_{4v}(\lambda, \mu) c_{3v}(\lambda, \mu) + \dfrac{1}{2s^2\lambda^4} c_{3v}(\lambda, \mu)^2 c_{3vv}(\lambda, \mu)$,

(*b*) $\mu \neq 0$, $c_5(\lambda, \mu) - \dfrac{1}{s\mu^2} c_{4u}(\lambda, \mu) c_{3u}(\lambda, \mu) + \dfrac{1}{2s^2\mu^4} c_{3u}(\lambda, \mu)^2 c_{3uu}(\lambda, \mu)$.

Here, (λ, μ) is a non-zero vector and s is a non-zero real number that satisfy (4.1).

Proof. (1) If $\lambda \neq 0$, the coefficient of u^2, v^2, and u^3 in $f(u, v + (\mu/\lambda)u)$ are 0, $s\lambda^2/2$, and $c_3(\lambda, \mu)/\lambda^3$ respectively. Hence, we obtain the result. The case that $\mu \neq 0$ is similar.

(2) We assume that $c_3(\lambda, \mu) = 0$. Suppose that $\lambda \neq 0$. Setting $c = c_{3v}(\lambda, \mu)/(s\lambda^4)$, we obtain that the coefficients of v^2, u^2v, and u^4 in $f(u, v + (\mu/\lambda)u - cu^2)$ are $s\lambda^2/2$, 0, and

$$\frac{1}{\lambda^4}\left(c_4(\lambda, \mu) - \frac{1}{2s\lambda^2}c_{3v}(\lambda, \mu)^2\right), \tag{4.2}$$

respectively. Since

$$\lambda^2 \begin{vmatrix} \lambda^2 & -\lambda\mu & \mu^2 \\ c_{30} & c_{21} & c_{12} \\ c_{21} & c_{12} & c_{03} \end{vmatrix} + 4c_{3v}(\lambda, \mu)^2 = 6c_{3vv}(\lambda, \mu)c_3(\lambda, \mu),$$

$\hat{c}_4(\lambda, \mu) \neq 0$ implies that (4.2) is not zero. The case that $\mu \neq 0$ is similar.

(3) We keep the notation above and assume $c_3(\lambda, \mu) = \hat{c}_4(\lambda, \mu) = 0$. We shall consider case (a). (Case (b) is similar and we omit the detail.) If $\lambda \neq 0$, the coefficients of v^2, u^2v, u^4, and u^5 in $f(u, v + (\mu/\lambda)u - cu^2)$ are $s\lambda^2/2$, 0, 0, and

$$\frac{1}{\lambda^5}\left(c_5(\lambda, \mu) - \frac{1}{s\lambda^2}c_{4v}(\lambda, \mu)c_{3v}(\lambda, \mu) + \frac{1}{2s^2\lambda^4}c_{3v}(\lambda, \mu)^2 c_{3vv}(\lambda, \mu)\right),$$

respectively. The case that $\mu \neq$ is similar.

□

4.2 Criterion for $D_4^\pm$-singularity We assume that $c_{10} = c_{01} = c_{20} = c_{11} = c_{02} = 0$. Then f is at least D_4-singularity at $(0, 0)$. We have the following.

Theorem 4.2. *The function f is D_4^+-singularity (resp. D_4^--singularity) at $(0, 0)$ if and only if*

$$\begin{vmatrix} c_{30} & 2c_{21} & c_{12} & 0 \\ 0 & c_{30} & 2c_{21} & c_{12} \\ c_{21} & 2c_{12} & c_{03} & 0 \\ 0 & c_{21} & 2c_{12} & c_{03} \end{vmatrix} \tag{4.3}$$

takes positive values (resp. negative values).

Proof. The function f is D_4^+-singularity or D_4^--singularity at $(0, 0)$ if the cubic part c_3 of f has one real root or three real roots, respectively. The discriminant Δ of c_3 is given by

$$\Delta = -\frac{1}{48}({a_{30}}^2{a_{03}}^2 - 6a_{03}a_{21}a_{12}a_{30} + 4a_{30}{a_{12}}^3 + 4{a_{21}}^3a_{03} - 3{a_{21}}^2{a_{12}}^2).$$

Expanding (4.3), we have

$$\begin{vmatrix} c_{30} & 2c_{21} & c_{12} & 0 \\ 0 & c_{30} & 2c_{21} & c_{12} \\ c_{21} & 2c_{12} & c_{03} & 0 \\ 0 & c_{21} & 2c_{12} & c_{03} \end{vmatrix} = -48\Delta,$$

and we complete the proof. □

5 Singularities of φ and $\mathcal{K}$-versality Let g be given in Monge from as (2.1). If we write down Φ as

$$\Phi = c_{00} + xu + yv + \frac{1}{2}(\hat{k}_1 u^2 + \hat{k}_2 v^2) + \sum_{i+j\geq 3} \frac{1}{i!j!} c_{ij} u^i v^j,$$

then we obtain that

$$\begin{aligned} &c_{00} = \frac{t^2 - x^2 - y^2 - z^2}{2}, \quad \hat{k}_i = k_i z - 1 \ \ (i = 1, 2), \quad c_{ij} = a_{ij} z \ \ (i + j = 3), \\ &c_{40} = a_{40} z - 3{k_1}^2, \quad c_{31} = a_{31} z, \quad c_{22} = a_{22} z - k_1 k_2, \quad c_{13} = a_{13} z, \\ &c_{04} = a_{04} z - 3{k_2}^2, \quad c_{50} = a_{50} z - 10 k_1 a_{30}, \quad c_{05} = a_{05} z - 10 k_2 a_{03}. \end{aligned}$$

We recall that we take a point $q_0 = (x_0, y_0, z_0)$ or $q_0 = (x_0, y_0, z_0, t_0)$, where

$$(x_0, y_0, z_0) = g(u_0, v_0) + t_0 \mathbf{n}(u_0, v_0) \quad \text{and} \quad t_0 = \frac{1}{\kappa_i(u_0, v_0)},$$

and that we set $\varphi(u, v) = \Phi(u, v, q_0)$ or $\varphi(u, v) = \Phi^t(u, v, q_0)$. Now we assume that $(u_0, v_0) = (0, 0)$. So we have

$$(x_0, y_0, z_0) = \left(0, 0, \frac{1}{k_i}\right), \quad t_0 = \frac{1}{k_i}.$$

We note that Φ (resp. Φ^t) is a $\mathcal{K}$-versal unfolding of φ if and only if

$$\begin{aligned} &\mathcal{E}_2 = \langle \varphi, \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times q_0}, \Phi_y|_{\boldsymbol{R}^2\times q_0}, \Phi_z|_{\boldsymbol{R}^2\times q_0}, \Phi_t|_{\boldsymbol{R}^2\times q_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^{k+1} \\ &(\text{resp. } \mathcal{E}_2 = \langle \varphi, \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi^t_x|_{\boldsymbol{R}^2\times q_0}, \Phi^t_y|_{\boldsymbol{R}^2\times q_0}, \Phi^t_z|_{\boldsymbol{R}^2\times q_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^{k+1}) \end{aligned}$$

when φ is k-determined.

5.1 A_2-singularity

Proof of Theorem 3.4 (1). Using Theorem 4.1 (1), we have φ is $\mathcal{K}$-equivalent to A_2 at $(0, 0)$ if and only if one of the following conditions holds:

(a) $\hat{k}_1 = 0, \hat{k}_2 \neq 0, c_{30} \neq 0$;

(b) $\hat{k}_1 \neq 0, \hat{k}_2 = 0, c_{03} \neq 0$.

We consider Case (a). (Case (b) is similar and we omit the detail.) Conditions (a) are equivalent to

$$z_0 = 1/k_1, \quad k_1 \neq k_2, \quad a_{30} \neq 0$$

in the original coefficients of the Monge form. From Lemma 2.1, this implies the first assertion. We next remark that A_2-singularity is 3-determined. To show $\mathcal{K}$-versality of Φ^t, it is enough to verify that

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle {\Phi^t}_x|_{\boldsymbol{R}^2\times q_0}, {\Phi^t}_y|_{\boldsymbol{R}^2\times q_0}, {\Phi^t}_z|_{\boldsymbol{R}^2\times q_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^4. \tag{5.1}$$

Then the coefficients of $u^i v^j$ of functions appearing in (5.1) are given by the following table:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3
Φ^t_x	0	$\boxed{1}$	0	0	0	0	0	0	0	0
Φ^t_y	0	0	$\boxed{1}$	0	0	0	0	0	0	0
Φ^t_z	$\boxed{-z_0}$	0	0	$\frac{1}{2}k_1$	0	$\frac{1}{2}k_2$	$\frac{1}{6}a_{30}$	$\frac{1}{2}a_{21}$	$\frac{1}{2}a_{12}$	$\frac{1}{6}a_{03}$
φ_u	0	0	0	$\boxed{\frac{1}{2}c_{30}}$	c_{21}	$\frac{1}{2}c_{12}$	$\frac{1}{6}c_{40}$	$\frac{1}{2}c_{31}$	$\frac{1}{2}c_{22}$	$\frac{1}{6}c_{13}$
φ_v	0	0	$\hat{k}_2$	$\frac{1}{2}c_{21}$	c_{12}	$\frac{1}{2}c_{03}$	$\frac{1}{6}c_{31}$	$\frac{1}{2}c_{22}$	$\frac{1}{2}c_{13}$	$\frac{1}{6}c_{04}$
φ	0	0	0	0	0	$\frac{1}{2}\hat{k}_2$	$\frac{1}{6}c_{30}$	$\frac{1}{2}c_{21}$	$\frac{1}{2}c_{12}$	$\frac{1}{6}c_{03}$
$u\varphi_u$	0	0	0	0	0	0	$\boxed{\frac{1}{2}c_{30}}$	c_{21}	$\frac{1}{2}c_{12}$	0
$v\varphi_u$	0	0	0	0	0	0	0	$\frac{1}{2}c_{30}$	c_{21}	$\frac{1}{2}c_{12}$
$u\varphi_v$	0	0	0	0	$\boxed{\hat{k}_2}$	0	$\frac{1}{2}c_{21}$	c_{12}	$\frac{1}{2}c_{03}$	0
$v\varphi_v$	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	$\frac{1}{2}c_{21}$	c_{12}	$\frac{1}{2}c_{03}$
$u^2\varphi_v$	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0
$uv\varphi_v$	0	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0
$v^2\varphi_v$	0	0	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$

By Gauss's elimination method using boxed elements as pivots, we show the matrix presented by this table is full rank, and we obtain (5.1) □

Remark 5.1. The function Φ^t is an $\mathcal{R}^+$-versal unfolding of φ if and only if

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi^t_x|_{\boldsymbol{R}^2 \times q_0}, \Phi^t_y|_{\boldsymbol{R}^2 \times q_0}, \Phi^t_z|_{\boldsymbol{R}^2 \times q_0} \rangle_{\boldsymbol{R}} + \langle 1 \rangle_{\boldsymbol{R}} + \langle u, v \rangle^4$$

when φ is 3-determined. By using the table appearing in the proof of Theorem 3.4 (1), it follows that Φ^t is also an $\mathcal{R}^+$-versal unfolding of φ when φ is A_2-singularity.

5.2 A_3-singularity

Proof of Theorem 3.4 (2). Using Theorem 4.1 (2), we have φ is $\mathcal{K}$-equivalent to A_3 at $(0,0)$ if and only if one of the following conditions holds:

(a) $\hat{k}_1 = 0, \hat{k}_2 \neq 0, c_{30} = 0, \hat{k}_2 c_{40} - 3{c_{21}}^2 \neq 0$;

(b) $\hat{k}_1 \neq 0, \hat{k}_2 = 0, c_{03} = 0, \hat{k}_1 c_{04} - 3{c_{12}}^2 \neq 0$.

We consider Case (a). (Case (b) is similar and we omit the detail.) Conditions (a) are equivalent to

$$z_0 = 1/k_1, \quad k_1 \neq k_2, \quad a_{30} = 0, \quad 3{a_{21}}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2) \neq 0$$

in the original coefficients of the Monge form. Form Lemma 2.1, these conditions imply the first assertion. We next remark that A_3-singularity is 4-determined. To show $\mathcal{K}$-versality of Φ^t, it is enough to verify that

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi^t_x|_{\boldsymbol{R}^2 \times q_0}, \Phi^t_y|_{\boldsymbol{R}^2 \times q_0}, \Phi^t_z|_{\boldsymbol{R}^2 \times q_0}, \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5. \tag{5.2}$$

Setting $c = c_{21}/(2\hat{k}_2)$ and replacing v by $v - cu^2$, we see that the coefficients of $u^i v^j$ of functions appearing in (5.2) are given by the following table:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3	u^4
Φ^t_x	0	$\boxed{1}$	0	0	0	0	0	0	0	0	0
Φ^t_y	0	0	1	$\boxed{-c}$	0	0	0	0	0	0	0
Φ^t_z	$\boxed{-z_0}$	0	0	$\frac{1}{2}k_1$	0	$\frac{1}{2}k_2$	0	$*$	$*$	$*$	$*$
φ_u	0	0	0	0	0	$\frac{1}{2}c_{12}$	$\boxed{\frac{1}{6}\hat{c}_{40}}$	$\frac{1}{2}\hat{c}_{31}$	$\frac{1}{2}\hat{c}_{22}$	$\frac{1}{6}c_{13}$	$\frac{1}{24}\hat{c}_{50}$
φ_v	0	0	$\boxed{\hat{k}_2}$	0	c_{12}	$\frac{1}{2}c_{03}$	$\frac{1}{6}\hat{c}_{31}$	$\frac{1}{2}\hat{c}_{22}$	$\frac{1}{2}c_{13}$	0	$\frac{1}{24}\hat{c}_{41}$
φ	0	0	0	0	0	$\frac{1}{2}\hat{k}_2$	0	0	$\frac{1}{2}c_{12}$	$\frac{1}{6}c_{03}$	$\frac{1}{24}\hat{c}_{40}$
$u\varphi_u$	0	0	0	0	0	0	0	0	$\frac{1}{2}c_{12}$	0	$\boxed{\frac{1}{6}\hat{c}_{40}}$
$v\varphi_u$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}c_{12}$	0
$u\varphi_v$	0	0	0	0	$\boxed{\hat{k}_2}$	0	0	c_{12}	$\frac{1}{2}c_{03}$	0	$\frac{1}{6}\hat{c}_{31}$
$v\varphi_v$	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0	c_{12}	$\frac{1}{2}c_{03}$	0
$u^2\varphi_v$	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0	0
$uv\varphi_v$	0	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0
$v^2\varphi_v$	0	0	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0

	u^iv^j $(i+j \leq 3)$	u^4	u^3v	u^2v^2	uv^3	v^4
$u^3\varphi_u$	0	0	$\boxed{\hat{k}_2}$	0	0	0
$u^2v\varphi_u$	0	0	0	$\boxed{\hat{k}_2}$	0	0
$uv^2\varphi_u$	0	0	0	0	$\boxed{\hat{k}_2}$	0
$v^3\varphi_u$	0	0	0	0	0	$\boxed{\hat{k}_2}$

,

where

$$\hat{c}_{40} = (\hat{k}_2 c_{40} - 3{c_{21}}^2)/\hat{k}_2, \quad \hat{c}_{31} = (\hat{k}_2 c_{31} - 3c_{21}c_{12})/\hat{k}_2, \quad \hat{c}_{22} = (\hat{k}_2 c_{22} - c_{21}c_{03})/\hat{k}_2,$$
$$\hat{c}_{50} = (\hat{k}_2^2 c_{50} - 10\hat{k}_2 c_{21}c_{31} + 15{c_{21}}^2 c_{12})/\hat{k}_2^2, \quad \hat{c}_{41} = (\hat{k}_2^2 c_{41} - 6\hat{k}_2 c_{21}c_{22} + 3{c_{21}}^2 c_{03})/\hat{k}_2^2,$$

and so on. The coefficients mentioned by "$*$" are not important. The condition $c = 0$ is equivalent to $a_{21} = 0$, that is, the origin is a sub-parabolic point relative to $\mathbf{v}_2$ (see Lemma 2.3). Therefore, by using Gauss's elimination method using boxed elements as pivots, it follows that the matrix presented by this table is full rank, that is, (5.2) holds if and only if the origin is not a sub-parabolic point relative to $\mathbf{v}_2$. □

Remark 5.2. The function Φ^t is an $\mathcal{R}^+$-versal unfolding of φ if and only if

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} + \langle \Phi^t_x|_{\boldsymbol{R}^2\times q_0}, \Phi^t_y|_{\boldsymbol{R}^2\times q_0}, \Phi^t_z|_{\boldsymbol{R}^2\times q_0} \rangle_{\boldsymbol{R}} + \langle 1 \rangle_{\boldsymbol{R}} + \langle u, v \rangle^5$$

when φ is 4-determined. By using the table appearing in the proof of Theorem 3.4 (2), it follows that Φ^t is an $\mathcal{R}^+$-versal unfolding of φ when φ is A_3-singularity.

5.3 A_4-singularity

Proof of Theorem 3.4 (3). From Theorem 4.1 (3), φ is $\mathcal{K}$-equivalent to A_4 at $(0,0)$ if and only if one of the following conditions holds:

(a) $\hat{k}_1 = 0$, $\hat{k}_2 \neq 0$, $c_{30} = 0$, $\hat{k}_2 c_{40} - 3{c_{21}}^2 = 0$, $\hat{k}_2^2 c_{50} - 10\hat{k}_2 c_{21} c_{31} + 15{c_{21}}^2 c_{12} \neq 0$;

(b) $\hat{k}_1 \neq 0$, $\hat{k}_2 = 0$, $c_{03} = 0$, $\hat{k}_1 c_{04} - 3{c_{12}}^2 = 0$, $\hat{k}_1^2 c_{05} - 10\hat{k}_1 c_{12} c_{13} + 15c_{21}{c_{12}}^2 \neq 0$.

We work on Case (a). (Case (b) is similar and we omit the detail.) Conditions (a) are equivalent to

$$\begin{aligned} &z_0 = 1/k_1, \quad k_1 \neq k_2, \quad a_{30} = 0, \quad 3a_{21}^2 + (a_{40} - 3{k_1}^3)(k_1 - k_2) = 0, \\ &15{a_{21}}^2 a_{12} + 10a_{21}a_{31}(k_1 - k_2)^2 + a_{50}(k_1 - k_2)^2 \neq 0, \end{aligned}$$

in the original coefficients of the Monge form. By Lemma 2.1, we obtain the first assertion. We next remark that A_4-singularity is 5-determined. To show $\mathcal{K}$-versality of Φ, it is enough to verify that

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times q_0}, \Phi_y|_{\boldsymbol{R}^2\times q_0}, \Phi_z|_{\boldsymbol{R}^2\times q_0}, \Phi_t|_{\boldsymbol{R}^2\times q_0} \rangle_{\boldsymbol{R}} + \langle u, v \rangle^6 \tag{5.3}$$

Setting $c = c_{21}/(2\hat{k}_2)$ and replacing v by $v - cu^2$, we see that the coefficients of $u^i v^j$ of functions appearing in (5.3) are given by the following table:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3	u^4	u^5
Φ_x	0	$\boxed{1}$	0	0	0	0	0	0	0	0	0	0
Φ_y	0	0	$\boxed{1}$	$-c$	0	0	0	0	0	0	0	0
Φ_z	$-z_0$	0	0	$\boxed{\frac{1}{2}k_1}$	0	$\frac{1}{2}k_2$	0	$*$	$*$	$*$	$*$	$*$
Φ_t	$\boxed{t_0}$	0	0	0	0	0	0	0	0	0	0	0
φ_u	0	0	0	0	0	$\frac{1}{2}c_{12}$	0	$\frac{1}{2}\hat{c}_{31}$	$\frac{1}{2}\hat{c}_{22}$	$\frac{1}{6}c_{13}$	$\boxed{\frac{1}{24}\hat{c}_{50}}$	$*$
φ_v	0	0	$\hat{k}_2$	0	c_{12}	$\frac{1}{2}c_{03}$	$\boxed{\frac{1}{6}\hat{c}_{31}}$	$\frac{1}{2}\hat{c}_{22}$	$\frac{1}{2}c_{13}$	0	$\frac{1}{24}\hat{c}_{41}$	$*$
φ	0	0	0	0	0	$\frac{1}{2}\hat{k}_2$	0	0	$\frac{1}{2}c_{12}$	$\frac{1}{6}c_{03}$	0	$\frac{1}{120}\hat{c}_{50}$
$u\varphi_u$	0	0	0	0	0	0	0	0	$\frac{1}{2}c_{12}$	0	0	$\boxed{\frac{1}{24}\hat{c}_{50}}$
$v\varphi_u$	0	0	0	0	0	0	0	0	0	$\frac{1}{2}c_{12}$	0	0
$u\varphi_v$	0	0	0	0	$\boxed{\hat{k}_2}$	0	0	c_{12}	$\frac{1}{2}c_{03}$	0	$\frac{1}{6}\hat{c}_{31}$	$\frac{1}{24}\hat{c}_{41}$
$v\varphi_v$	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0	c_{12}	$\frac{1}{2}c_{03}$	0	0
$u^2\varphi_v$	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0	0	$\frac{1}{6}\hat{c}_{31}$
$uv\varphi_v$	0	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0	0
$v^2\varphi_v$	0	0	0	0	0	0	0	0	0	$\boxed{\hat{k}_2}$	0	0

	u^iv^j $(i+j\leq 3)$	u^4	u^3v	u^2v^2	uv^3	v^4	u^5
$u^3\varphi_v$	0	0	$\boxed{\hat{k}_2}$	0	0	0	0
$u^2v\varphi_v$	0	0	0	$\boxed{\hat{k}_2}$	0	0	0
$uv^2\varphi_v$	0	0	0	0	$\boxed{\hat{k}_2}$	0	0
$v^3\varphi_v$	0	0	0	0	0	$\boxed{\hat{k}_2}$	0

	u^iv^j $(i+j\leq 4)$	u^5	u^4v	u^3v^2	u^2v^3	uv^4	v^5
$u^4\varphi_v$	0	0	$\boxed{\hat{k}_2}$	0	0	0	0
$u^3v\varphi_v$	0	0	0	$\boxed{\hat{k}_2}$	0	0	0
$u^2v^2\varphi_v$	0	0	0	0	$\boxed{\hat{k}_2}$	0	0
$uv^3\varphi_v$	0	0	0	0	0	$\boxed{\hat{k}_2}$	0
$v^4\varphi_v$	0	0	0	0	0	0	$\boxed{\hat{k}_2}$

We claim that Φ is $\mathcal{K}$-versal when $\hat{c}_{31} \neq 0$. The condition $\hat{c}_{31} \neq 0$ is equivalent to

$$3a_{12}a_{21} + a_{31}(k_1 - k_2) \neq 0$$

in the original coefficients of the Monge form. From Lemma 2.2, Φ is $\mathcal{K}$-versal unfolding of φ if and only if $(0,0)$ is a regular point of the ridge line relative to $\mathbf{v}_1$. □

5.4 $D_4^{\pm}$-singularity

Proof of Theorem 3.4 (4). From Theorem 4.2, φ is $\mathcal{K}$-equivalent to D_4^+ at $(0,0)$ if

$$\hat{k}_1 = \hat{k}_2 = 0, \text{ and } \begin{vmatrix} c_{30} & 2c_{21} & c_{12} & 0 \\ 0 & c_{30} & 2c_{21} & c_{12} \\ c_{21} & 2c_{12} & c_{03} & 0 \\ 0 & c_{21} & 2c_{12} & c_{03} \end{vmatrix} > 0$$

These conditions are equivalent to

$$k_1 = k_2 = \frac{1}{z_0}, \text{ and } \begin{vmatrix} a_{30} & 2a_{21} & a_{12} & 0 \\ 0 & a_{30} & 2a_{21} & a_{12} \\ a_{21} & 2a_{12} & a_{03} & 0 \\ 0 & a_{21} & 2a_{12} & a_{03} \end{vmatrix} > 0$$

in the original coefficients of the Monge form. Therefore, φ is $\mathcal{K}$-equivalent to D_4^+ at $(0,0)$ if the origin is an elliptic (see Section 2.4). Since $D_4^{\pm}$-singularity is 3-determined, Φ is $\mathcal{K}$-versal unfolding of φ if and only if

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2} + \langle \Phi_x|_{\boldsymbol{R}^2\times q_0}, \Phi_y|_{\boldsymbol{R}^2\times q_0}, \Phi_z|_{\boldsymbol{R}^2\times q_0}, \Phi_t|_{\boldsymbol{R}^2\times q_0} \rangle_{\boldsymbol{R}} + \langle u, v\rangle^4 \tag{5.4}$$

The coefficients of u^iv^j of functions appearing in (5.4) are given by the following tables:

	1	u	v	u^2	uv	v^2	u^3	u^2v	uv^2	v^3
Φ_x	0	1	0	0	0	0	0	0	0	0
Φ_y	0	0	1	0	0	0	0	0	0	0
Φ_z	$-z_0$	0	0	$\frac{1}{2}k_1$	0	$\frac{1}{2}k_2$	$\frac{1}{6}a_{30}$	$\frac{1}{2}a_{21}$	$\frac{1}{2}a_{12}$	$\frac{1}{6}a_{03}$
Φ_t	t_0	0	0	0	0	0	0	0	0	0
Φ_u	0	0	0	$\frac{1}{2}c_{30}$	c_{21}	$\frac{1}{2}c_{12}$	$\frac{1}{6}c_{40}$	$\frac{1}{2}c_{31}$	$\frac{1}{2}c_{22}$	$\frac{1}{6}c_{13}$
Φ_v	0	0	0	$\frac{1}{2}c_{21}$	c_{12}	$\frac{1}{2}c_{03}$	$\frac{1}{6}c_{31}$	$\frac{1}{2}c_{22}$	$\frac{1}{2}c_{13}$	$\frac{1}{6}c_{04}$
$u\Phi_u$	0	0	0	0	0	0	$\frac{1}{2}c_{30}$	c_{21}	$\frac{1}{2}c_{12}$	0
$v\Phi_u$	0	0	0	0	0	0	0	$\frac{1}{2}c_{30}$	c_{21}	$\frac{1}{2}c_{12}$
$u\Phi_v$	0	0	0	0	0	0	$\frac{1}{2}c_{21}$	c_{12}	$\frac{1}{2}c_{03}$	0
$v\Phi_v$	0	0	0	0	0	0	0	$\frac{1}{2}c_{21}$	c_{12}	$\frac{1}{2}c_{03}$

Thus we obtain that Φ is $\mathcal{K}$-versal if and only if

$$\begin{vmatrix} 1 & 0 & 1 \\ c_{30} & c_{21} & c_{12} \\ c_{21} & c_{12} & c_{03} \end{vmatrix} \neq 0.$$

This condition is equivalent to

$$\begin{vmatrix} 1 & 0 & 1 \\ a_{30} & a_{21} & a_{12} \\ a_{21} & a_{12} & a_{03} \end{vmatrix} \neq 0$$

in the original coefficients of the Monge form. Hence, we complete the proof. □

Since the proof of Theorem 3.4 (5) is completely parallel to that of Theorem 3.4 (4), we omit the detail. We remark that an elliptic umbilic is not right-angled.

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