Singularities of Solutions of Nonlinear Hyperbolic Systems of Conservation Laws

RONALD J. DIPERNA

Communicated by C. DAFERMOS

1. Introduction

We are concerned with the structure of solutions of systems of conservation laws

$$U_t + F(U)_x = 0, \qquad -\infty < x < \infty, \qquad t > 0$$

which are strictly hyperbolic and genuinely nonlinear in the sense of LAX [7]. Here U takes on values in R^n and F is a nonlinear mapping function from R^n to R^n.

The problem of structure is posed for both generic and arbitrary solutions. For a single genuinely nonlinear equation ($n=1$), it is known (SCHAEFFER [10]) that solutions are generically piecewise smooth. For the general single equation, solutions are also known to be generically piecewise smooth (GUCKENHEIMER [5]). Moreover, the generic stability of shock waves in a single genuinely nonlinear equation has been established by GOLUBITSKY & SCHAEFFER [4]. On the other hand, the problem of generic structure for *systems* is presently open.

In this paper we are concerned with the structure of an arbitrary solution of a system of conservation laws. The paper is divided into three working sections: Section 2, the local decomposition of the solution into elementary waves; Section 3, the classification and propagation of singularities in the solution and the limiting behavior at singularities; Section 4, the propagation of sound waves in domains of continuous flow. We consider solutions which are constructed by the difference scheme of GLIMM [2] and whose initial data have small total variation. Such solutions satisfy the entropy admissibility criterion of LAX [9]. While the analysis is carried out for systems of two equations, it would seem that a straightforward generalization to systems of n equations is likely.

It is well-known that solutions are functions of bounded variation in the sense of CESARI: their first order partial derivatives are locally Borel measures. A solution therefore inherits the regularity and structure common to all such functions (here referred to by the notation BVC). For the purpose of comparison we recall the structure of an arbitrary BVC function, [1], [12]. Let V be a BVC function defined on an open domain in R^m. Each point P of its domain is classified as a regular or irregular point according to the existence or nonexistence of a hyperplane through P with respect to which V has approximate one-sided limits at P. The set I of irregular points is sparse in the sense that it has zero $(m-1)$-dimensional Hausdorff measure. Each regular point proves to be either a point of approximate continuity or a point of approximate jump discontinuity, according to whether as the associated one-sided limits are equal or distinct. The set J of jump points is countably rectifiable with respect to $(m-1)$-dimensional Hausdorff measure in the sense of FEDERER: there exist compact subsets $K_j \subset R^{m-1}$ and univalent Lipschitzian maps ψ_j such that $\psi_j(K_j)$ are disjoint and such that $J = \bigcup_{j=1}^{\infty} \psi_j(K_j) \cup M$, where M is a set with zero $(m-1)$-dimensional Hausdorff measure.

The singular sets of the solution have the following structure (Theorem 3.1). The set I of irregular points is at most countable. The set $I \cup J$ is an at most countable union of Lipschitz continuous curves $\Gamma_n = \{x_n(t), t\}$ such that no two curves intersect in more than two points. Furthermore, the speed of propagation $\dot{x}_n(t)$ of each curve is a function of bounded variation. The curves Γ_n constitute the shock waves of the solution. The set of irregular points consists precisely of the points of collision and formation of shocks, together with the centers of (generalized) compression waves impinging on shocks.

In contrast to an arbitrary BVC function the limiting behavior of the solution at a singular point admits a classical description (Theorem 2.1, Corollary 3.1). At each regular point on a shock wave the solution has one-sided pointwise limits with respect to both the left and right sides of the shock. These limits are distinct and satisfy the Rankine-Hugoniot relations. Moreover, the shock waves Γ_n propagate at classical shock speed in the sense that, with the possible exception of a countable set of points, the derivative $\dot{x}_n(t)$ exists and is given by the classical formula which relates the speed of propagation of a shock to the one-sided limits of the solution along the shock. The exceptional set is contained in the set of irregular points and corresponds to points of wave interaction. At each irregular point on a shock wave the limiting behavior of the solution admits a description in terms of generalized elementary waves, cf. Theorem 2.1.

The above regularity of shock waves is essentially optimal. Even for a single conservation law there exist solutions with the property that the speed of propagation $\dot{x}_n(t)$ of each shock wave Γ_n fails to exist on a dense subset of its domain of definition. In such examples the collision set, and hence the shock set $\cup \Gamma_n$, is everywhere dense in the upper half-plane.

Although a solution may admit an everywhere dense shock set, there is a measure-theoretic sense in which the individual waves Γ_n are isolated and locally dominate the variation of the solution (Corollary 3.1). There exists a neighborhood of each regular point on a shock in which the restriction of the solution to the complement of the shock has arbitrarily small total variation. This property implies the stability of shocks in LAX's sense, namely that nearby characteristics run into the shock when followed in the forward direction of time. The isolated character of shock waves is a consequence of the condition of genuine nonlinearity. The characteristic curves which impinge on a shock do so at a uniform angle depending on the magnitude of the shock. Thus the accumulation of shocks occurs simultaneously with the diminishing of magnitude and the approach of shock speed to characteristic speed. Likewise, there is a measure-theoretic sense in which the set of irregular points consists only of isolated points of wave interaction; cf. Theorem 2.1 and Theorem 3.1.

In contrast to an arbitrary BVC function, the admissible discontinuities of a solution have a natural physical interpretation as either shock waves or points of wave interaction. Moreover, the solution is not merely approximately continuous on the complement of its shock set but is continuous in the pointwise sense (Corollary 2.1). Furthermore, the solution is Lipschitz continuous in any interior component of the set on which it is continuous (Theorem 4.1). The interior regularity is a consequence of the reversibility of the solution in domains of continuous flow and the corresponding geometry of characteristic curves. A quantitative expression of the reversibility of Lipschitz continuous solutions is provided by a theorem of LAX [8] which estimates the Lipschitz constant of uniformly bounded solutions in terms of the equations and the length of the interval of existence of the solution. We note that not much additional regularity is expected since Lipschitz continuous solutions can be constructed whose first order partials derivatives fail to exist on an everywhere dense subset of their domain of definition.

As an immediate corollary of the interior Lipschitz continuity it follows from a theorem of LAX [6] that the discontinuities in the first order partial derivatives of the solution propagate only along characteristic curves. Thus a solution affords a classification of its singularities into shock waves, points of wave interaction and sound waves.

The limiting behavior of the solution at a singular point is derived as a corollary of its local structure. The structure of a solution U in the neighborhood of an arbitrary point (x_0, t_0) is determined by the one-sided limits $U(x_0 \pm 0, t_0)$ of the restriction of U to the line $t = t_0$. If the restriction is discontinuous at x_0 then U admits a local decomposition into generalized elementary waves; cf. Section 2. If the restriction is continuous at x_0 there exist no recognizable waves interacting at (x_0, t_0), in the sense that the solution is not only continuous at (x_0, t_0) as a function of x and t but its restriction to a small neighborhood of (x_0, t_0) has arbitrarily small total variation on almost all space-like and time-like arcs (Corollary 2.2).

The problem of local structure is two-fold: to determine, in a small neighborhood N of a given point (x_0, t_0), the structure of the outgoing solution at (x_0, t_0) (i.e. the restriction of U to $N \cap \{t > t_0\}$) and the structure of the incoming solution at (x_0, t_0) (i.e. the restriction of U to $N \cap \{t < t_0\}$). The solution distinguishes between the positive and negative directions of time through the entropy condition. The structure of the outgoing solution is similar to the structure of the classical solution of the Riemann problem. In general the outgoing solution of a system of two equations admits a decomposition into three wedge-shaped domains of small variation which are separated either by generalized shock waves or generalized (centered) rarefaction waves; cf. Section 2. Moreover, the structure of the outgoing solution is determined by the one-sided limits $U(x_0 \pm 0, t_0)$ of the restriction of U to $t = t_0$. These can be classified into three types according to the relative location of $U(x_0 + 0, t_0)$ and $U(x_0 - 0, t_0)$ in state-space (Corollary 2.1). In general, the incoming solution admits a decomposition into three wedge-shaped domains of small variation which are separated by generalized compression waves; cf. Section 2. In contrast to the outgoing solution, the structure of the incoming solution is not uniquely determined by the limits $U(x_0 \pm 0, t_0)$ since the initial value problem is not properly posed in the backward direction of time.

The main tool in the analysis of the local structure of the solution is the theory of characteristics developed by Glimm & Lax [3]. The use of this theory for both the large-time decay and the local structure of the solution is a reflection of the fact that the equations are invariant under similarity transformations. In Section 2 we recall certain basic results in the theory of characteristics.

With the exception of Theorem 4.1 on interior Lipschitz continuity, the proofs of this paper do not make essential use of the existence of a coordinate system of Riemann invariants, a condition which is special to systems of two equations. It is for this reason that (with the possible exception of Theorem 4.1) a natural generalization of our results to systems of n equations can be expected.

2. Local Structure

Consider a system of two equations

$$\frac{\partial}{\partial t} U + \frac{\partial}{\partial x} F(U) = 0, \qquad U = \begin{Bmatrix} u \\ v \end{Bmatrix}, \qquad F = \begin{Bmatrix} f \\ g \end{Bmatrix} \tag{2.1}$$

which is strictly hyperbolic and genuinely nonlinear in the sense of Lax [7]. For concreteness, the system (2.1) is assumed to satisfy the Glimm-Lax shock interaction condition [3], i.e. the condition that the interaction of two shocks of the same field produces a shock of that field and a rarefaction wave of the opposite field. This condition is known to hold for equations of physical interest, e.g. gas dynamics and shallow water waves.

First we shall recall the classical definitions of elementary waves. Let $w_j = w_j(u, v)$, $j = 1, 2$, denote a pair of Riemann invariants for system (2.1). The corresponding characteristic equations for smooth solutions take the form

$$\frac{\partial}{\partial t} w_j + \lambda_j \frac{\partial}{\partial x} w_j = 0, \qquad j = 1, 2,$$

where λ_1 and λ_2 denote the characteristic values of F', $\lambda_1 < \lambda_2$. It is standard to normalize the right eigenvectors r_j of F' so that the condition of genuine nonlinearity is expressed by

$$r_j \cdot \nabla \lambda_j > 0, \qquad j = 1, 2,$$

and then to normalize the invariants so that

$$r_j \cdot \nabla w_j > 0, \qquad j = 1, 2. \tag{2.2}$$

The classical waves can be conveniently described by associating a major and a minor Riemann invariant with each characteristic field. We define the major invariant of the j^{th} field, $M_j(U)$, to be the function w_j and define the minor invariant of the j^{th} field, $m_j(U)$, to be the function w_k, $k \neq j$. The elementary waves are of two types, simple and shock. A j-simple wave is a domain in the $x-t$ plane on which the solution U is smooth and on which the minor invariant $m_j(U)$ is constant. A j-simple wave is classified as a j-rarefaction wave or as a j-compression wave according to whether the j-characteristics within the wave do or do not diverge with increasing time (equivalently, whether the major invariant $M_j(U)$ is an increasing or decreasing function of x for fixed t). A j-shock wave is a curve of discontinuity whose speed of propagation satisfies the j-shock conditions of LAX [7]. Under the normalization (2.2), the major invariant M_j decreases with increasing x across a j-shock. Under the GLIMM-LAX shock interaction condition the minor invariant m_j also decreases with increasing x across a j-shock. Thus, the decreasing variation of both invariants is supported by shocks and compression waves while the increasing variation is supported by rarefaction waves. The change in the minor invariant across a shock is third order in the change in the major invariant independently of any shock interaction condition:

$$[m_j]=O([M_j]^3).$$

The classical solution of the Riemann problem is a self-similar solution $U=U(x/t)$ which consists in general of three constant states such that any two consecutive states are separated by a shock or rarefaction wave. The structure of the solution can be conveniently classified by an ordered pair (W_1, W_2), where $W_j=S$ or $W_j=R$ according to whether the j-wave of the solution is a j-shock or a j-rarefaction wave. If the solution contains only one wave, say a k-wave, then the absence of the j-wave, $j \neq k$, will be indicated by setting $W_j=\phi$.

The structure of the solution of the Riemann problem is determined by the positions in state-space of the initial data. For given data U^+, U^- let $w_j^{\pm}=w_j(U^{\pm})$ and let

$$P^{\pm}=(w_1^{\pm}, w_2^{\pm}).$$

In the plane of Riemann invariants let $S_j=S_j(P^-)$ and $R_j=R_j(P^-)$ denote the shock wave curves and the rarefaction wave curves of the j^{th} field through the point P^-; cf. Figure 1. The structure of the solution U is determined by the location of P^+ either on one of the curves R_j, S_j or in one of the open quadrants Q_j defined by R_j and S_j [11]:

$$(2.3)\qquad \begin{array}{llll} U=(R,R) & \text{if } P^+\in Q_1 & U=(R,\phi) & \text{if } P^+\in R_1 \\ U=(R,S) & \text{if } P^+\in Q_2 & U=(\phi,S) & \text{if } P^+\in S_2 \\ U=(S,S) & \text{if } P^+\in Q_3 & U=(S,\phi) & \text{if } P^+\in S_1 \\ U=(S,R) & \text{if } P^+\in Q_4 & U=(\phi,R) & \text{if } P^+\in R_2. \end{array}$$

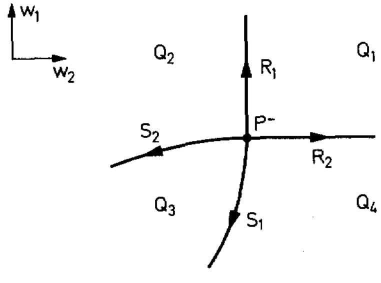

Fig. 1

The solution U contains three constant states U_-, U_m and U_+ such that the middle state U_m is a uniquely defined smooth function of the initial data which will be denoted by

(2.4) $$U_m = U_m(U^+, U^-).$$

The problem of locally decomposing an arbitrary solution into waves requires a generalization of the classical notion of an elementary wave. Let $\sigma_j = \sigma_j(U^+, U^-)$ denote the speed of propagation of a classical j-shock separating U^+ and U^-:

$$\sigma_j = (f^+ - f^-)/(v^+ - v^-) = (g^+ - g^-)/(u^+ - u^-).$$

Definition. A generalized j-characteristic is a Lipschitz continuous curve $X_j(t)$ whose speed of propagation exists, with the possible exception of countably many points, and has the following properties. If $X_j(t)$ is a point of continuity of the restriction $U(\cdot, t)$ then

$$\dot{X}_j(t) = \lambda_j\{U(X_j(t), t)\}.$$

If $X_j(t)$ is a point of discontinuity of the restriction $U(\cdot, t)$ then the limits $U^\pm = U(X_j(t) \pm 0, t)$ satisfy the Rankine-Hugoniot relations and

$$\dot{X}_j(t) = \sigma_j(U^+, U^-).$$

Generalized characteristics can be constructed as the limit of approximate characteristics X_j^h in the Glimm approximate solutions U^h, [3]. The property that generalized characteristics propagate at either shock or characteristic speed follows from the corresponding property for approximate characteristics. The speed of propagation $\dot{X}_j(t)$ of a j-characteristic X_j constructed in this fashion is necessarily a function of bounded variation [3].

Let $X_j(t) \leqq Y_j(t)$ be two j-characteristics which are defined in a neighborhood of a given point (x_0, t_0) and which pass through (x_0, t_0). Let

$$\pi_j = \{(x, t): X_j(t) \leqq x \leqq Y_j(t)\}.$$

A generalized j-simple wave is defined by the limiting behavior in t of the increasing, decreasing and total variations in x of the restriction of the invariants $m_j(U(\cdot, t))$ and $M_j(U(\cdot, t))$ to the interval $[X_j(t), Y_j(t)]$.

Definition. The domain $\pi_j \cap \{t > t_0\}$ is a generalized j-rarefaction wave if $X_j(t) < Y_j(t)$ for $t > t_0$ and if the following properties hold:

(2.5) $$\lim_{t \to t_0} IVM_j[X_j(t), Y_j(t)] > 0$$

(2.6) $$\lim_{t \to t_0} DVM_j[X_j(t), Y_j(t)] = 0$$

(2.7) $$\lim_{t \to t_0} TVm_j[X_j(t), Y_j(t)] = 0.$$

Conditions (2.6) and (2.7) imply that the total strength of all shocks within the wave approaches zero as t approaches t_0. Condition (2.5) is roughly equivalent to the condition that j-characteristics leaving (x_0, t_0) diverge with increasing time. For the equations of isentropic gas dynamics conditions (2.5)–(2.7) imply that the predominant effect of the wave is to rarefy rather than compress the gas.

Definition. The domain $\pi_j \cap \{t<t_0\}$ is a generalized j-compression wave if the following properties hold:

(2.8) $$\lim_{t \to t_0} DVM_j[X_j(t), Y_j(t)] > 0$$

(2.9) $$\lim_{t \to t_0} IVM_j[X_j(t), Y_j(t)] = 0$$

(2.10) $$\lim_{t \to t_0} IVm_j[X_j(t), Y_j(t)] = 0.$$

Condition (2.7) is not imposed on a generalized j-compression wave in order to accommodate j-shocks within the wave. Any combination of classical j-shocks and classical centered j-compression waves interacting at a point qualifies as a generalized j-compression wave. The notion of a j-compression wave which does not contain j-shocks in the limit as t approaches t_0 is introduced in Section 3.

We note that rates cannot be associated with the limits (2.5)–(2.10) which define generalized rarefaction and compression waves, due to the invariance of the equations under similarity transformations.

Let K be an interval contained in R^+.

Definition. A j-shock wave is a j-characteristic $X_j(t)$, $t \in K$, whose strength,

$$\operatorname{str} X_j(t) = |U(X_j(t)+0, t) - U(X_j(t)-0, t)|,$$

has the following property. For every bounded subinterval of K, not containing the end points, there exists a $\delta > 0$ such that $\operatorname{str} X_j(t) \geqq \delta$ if t lies in that subinterval.

We note that the strength of a shock wave is defined except possibly at a countable set of points. The strength of a shock wave at the initial and end points t_1 and t_2 of its domain of definition will be defined by taking limits:

$$\operatorname{str} X_j(t_k) = \lim_{t \to t_k} \operatorname{str} X_j(t).$$

Let Ω be a simply-connected domain with piecewise smooth boundary. Let l_s denote a typical line segment on the line $t=s$ and let l_y denote a typical line segment on the line $x=y$. Let

$$\operatorname{Var}_t(\Omega) = \sup\{TVU(l_t): l_t \subset \Omega\}$$
$$\operatorname{Var}_x(\Omega; G) = \sup\{TVU(l_x): l_x \subset \Omega, x \in G\},$$

where $TVU(l)$ denotes the total variation of U restricted to the line segment l and $G \subset \mathbb{R}$.

Definition. A region Ω is a state of small variation with respect to $(x_0, t_0) \in \bar{\Omega}$ if there exists a set G with zero Lebesgue measure such that

(2.12) $$\lim_{r \to 0} \operatorname{Var}_t(\Omega \cap B_r) + \lim_{r \to 0} \operatorname{Var}_x(\Omega \cap B_r; G) = 0,$$

where B_r is the ball of radius r centered at (x_0, t_0).

In the above situation the limit of $U(x, t)$ as (x, t) approaches (x_0, t_0) within Ω exists and will be denoted by

$$U(x_0, t_0; \Omega).$$

The absence of a restriction on the line segments l_t beyond that required by the inclusion is a consequence of L^1-continuity of the solution in t; cf. (3.1).

The local decomposition of the solution into elementary waves in a neighborhood of a given point (x_0, t_0) requires separate consideration of the incoming and outgoing solution at (x_0, t_0), i.e. the restriction of the solution to $N \cap \{t<t_0\}$ and $N \cap \{t>t_0\}$ respectively. In Theorem 2.1 the structure of the incoming and outgoing solutions is described in terms of j-characteristics $X_j \leqq Y_j, j=1,2$, passing through (x_0, t_0) and in terms of the following domains which are complementary to the waves π_j (cf. Figure 2):

$$\Omega_- = \{(x,t): x<X_1(t) \text{ if } t\geqq t_0,\ x<X_2(t) \text{ if } t<t_0\}$$

$$\Omega_+ = \{(x,t): x>Y_2(t) \text{ if } t\geqq t_0,\ x<Y_1(t) \text{ if } t<t_0\}$$

$$\Omega_m = \{(x,t): Y_1(t)<x<X_2(t)\}$$

$$\Omega_n = \{(x,t): Y_2(t)<x<X_1(t)\}.$$

Theorem 2.1. *Through each point (x_0, t_0), $t_0>0$, there pass generalized j-characteristics $X_j(t) \leqq Y_j(t)$, $j=1,2$, which are defined on an interval containing t_0 in its interior and which satisfy the following properties:*

1) *The domains $\Omega_\pm$, Ω_m, Ω_n are states of small variation with respect to (x_0, t_0).*

2) *Either $\pi_j \cap \{t<t_0\}$ is a generalized j-compression wave or*

$$\lim_{t\to t_0} TVU[X_j(t), Y_j(t)] = 0, \qquad t<t_0.$$

3) *Either $\pi_j \cap \{t>t_0\}$ is a generalized j-rarefaction wave or $\pi_j \cap \{t>t_0\}$ equals $X_j(t)$ and $X_j(t)$ is a generalized j-shock such that* str $X_j(t_0+0)>0$ *or*

$$\lim_{t\to t_0} TVU[X_j(t), Y_j(t)] = 0, \qquad t>t_0.$$

A typical example is illustrated in Figure 2.

It is a straightforward corollary of Theorem 2.1 that the structure of the outgoing solution at (x_0, t_0) is determined by the limits

$$U^\pm \stackrel{\text{def}}{=} U(x_0 \pm 0, t_0)$$

in the same way as the solution of the Riemann problem with initial data $U^\pm$. A further expression of the property of locally finite propagation speed is the

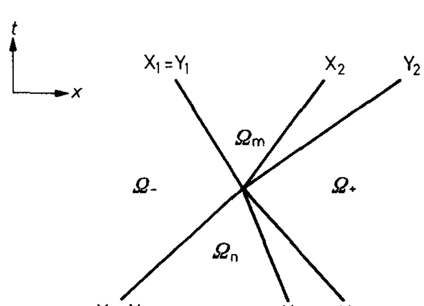

Fig. 2

fact that

$$U^{\pm}=U(x_0, t_0; \Omega_{\pm}).$$

Corollary 2.1. *Let* $w_j^{\pm}=w_j(U^{\pm})$ *and* $P^{\pm}=(w_1^{\pm}, w_2^{\pm})$. *The structure of the outgoing solution at* (x_0, t_0) *satisfies the classification given in* (2.3) *for the Riemann problem. The limiting value of* U *at* (x_0, t_0) *with respect to the middle state is determined as in the Riemann problem:* $U(x_0, t_0; \Omega_m)=U_m(U^+, U^-)$.

Corollary 2.2. *If the restriction of* U *to* $t=t_0$ *is continuous at* x_0, *then any neighborhood of* (x_0, t_0) *is a region of small variation with respect to* (x_0, t_0). *In particular,* U *is continuous at* (x_0, t_0) *as a function of* x *and* t.

Remarks. In the case where P^+ lies on one of the shock or rarefaction wave curves through P^- the outgoing solution consists of only one wave. For example, if P^- lies on R_1 then the union of Ω_m, π_2 and Ω_+ is a state of small variation with respect to (x_0, t_0) and the outgoing solution is classified as (R_1, ϕ).

It follows from the structure of the incoming solution at (x_0, t_0) that the increasing variation of both Riemann invariants approaches zero as t approaches t_0 from below. Therefore,

$$w_j(U(x_0+0, t_0))-w_j(U(x_0-0, t_0))\leqq 0, \qquad j=1, 2.$$

These inequalities imply in particular that the outgoing solution cannot assume the forms (R_1, ϕ), (ϕ, R_2) or (R_1, R_2). Centered rarefaction waves exist at times $t>0$ only as a result of their birth from the interaction of shocks and/or compression waves. A classification of the structure of the incoming solution similar to that of the outgoing solution is not possible since the incoming waves are not uniquely determined by the limits U^+ and U^-.

Preliminaries. The standard measure of the magnitude ε_j of a classical j-wave is the jump in the major invariant from right to left across the wave:

$$\varepsilon_j=M_j^+-M_j^-.$$

Under the normalization (2.2) shock waves have negative magnitude and rarefaction waves positive. In the GLIMM approximate solutions U_h the balance between the total amount of j-shock $(-)$ and j-rarefaction wave $(+)$ entering (E) and leaving (L) a given region Λ is expressed by the approximate conservation laws for waves [3]:

$$L_j^{\pm}(\Lambda)=E_j^{\pm}(\Lambda)\mp C_j(\Lambda)+O(\tau)\,Q(\Lambda). \tag{2.13}$$

Here Q and C_j denote the total amount of wave interaction and the total amount of j-wave cancellation within Λ, and τ denotes the oscillation of U_h within Λ. The domain Λ is an arbitrary union of diamond-shaped regions formed by connecting the random mesh points of the difference scheme with line segments; cf. [3]. Similar conservation laws hold in the limit for the solution U.

The conservation laws (2.13) also hold for domains bounded by pairs of approximate characteristics [3]. In fact, the trajectory of an approximate j-characteristic is determined by this very condition and by the requirement that j-rarefaction waves of U_h do not cross approximate j-characteristics. A j-rarefaction wave in U_h cannot enter or leave a region through that part of its boundary which consists of approximate j-characteristics. The prescription [3] of the trajectory of approximate j-characteristics in the forward time direction applies with only a simple modification to the backward time direction and yields in the limit generalized j-characteristics with the same properties.

Our analysis of the local structure of the solution employs estimates of Glimm & Lax [3] which connect the rate of spreading of a pair of j-characteristics to the amount of j-shock and j-rarefaction wave contained between them. In order to minimize the influence of characteristics of the opposite field, the distance between j-characteristics is measured at times displaced in the direction in which k-waves propagate, $k \neq j$. Consider a sequence of approximate j-characteristics $X_j^h \leq Y_j^h$. Let σ_t^h denote the family of space-like polygonal arcs which lie between X_j^h and Y_j^h and which consist of space-like edges of diamonds in U_h. Let t_h^* denote the terminal time of the arc σ_t^h which originates at time t. Figure 3 illustrates the case $j=1$.

Let $\omega_j^\pm(\sigma_t^h)$ denote the total amount of j-shock and j-rarefaction wave in U_h which crosses σ_t^h between but not on X_j^h and Y_j^h. If X_j^h and Y_j^h coalesce, the quantities $\omega_j^\pm$ are defined to be zero. After passing to a subsequence, the characteristics X_j^h and Y_j^h converge uniformly to generalized j-characteristics X_j and Y_j, and the arcs σ_t^h converge to space-like line segments σ_t which propagate at a speed equal to the fixed ratio of mesh lengths $\Delta x/\Delta t$. In addition, the limiting amounts of j-shock and j-rarefaction waves crossing σ_t, namely

$$\omega_j^\pm(\sigma_t) = \lim \omega_j^\pm(\sigma_t^h),$$

exist and are functions of bounded variation in t [3]. The rate at which X_j and Y_j approach is influenced predominantly by the balance of j-shocks and j-rarefaction waves:

$$\dot{Y}_j(t^*) - \dot{X}_j(t) = \mu_j \Delta_j(t) + O(1)(\omega_k^+ + |\omega_k^-|), \tag{2.14}$$

where

$$\begin{aligned} \Delta_j(t) &= \omega_j^+ + \omega_j^- - (1/2 + O(\tau))(\operatorname{str} X_j + \operatorname{str} Y_j) + O(\tau)(\omega_j^+ + |\omega_j^-|) \\ \mu_j &= \frac{\partial}{\partial w_j}\lambda_j(0,0) > 0. \end{aligned} \tag{2.15}$$

Here $t^* = t^*(t)$ denotes the time at which the line segment σ_t intersects Y_j. The factor of 1/2 enters equation (2.15) as a consequence of the fact that the speed of

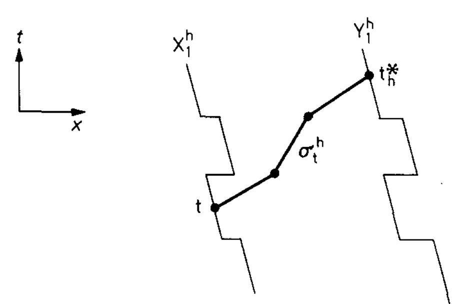

Fig. 3

propagation of a weak shock is the average of the characteristic speeds on either side up to second order terms:

$$\dot{X}_j(t)=\lambda_j\{U(X_j(t)\pm 0, t)\}\pm\tfrac{1}{2}\mu_j(1+O(\tau))\operatorname{str} X_j(t). \tag{2.16}$$

The action of k-waves on the spreading of j-characteristics is minimized by measuring distances at times displaced in the direction of k-waves [3]. If $D^*(t)=Y_j(t^*)-X_j(t)$, then

$$D^*(T)\leqq \text{const.}\left\{D^*(t)+\mu_j\int_t^T|\Delta_j(t)|\,dt\right\} \tag{2.17}$$

where both of the constants are of the form $1+O(TVU_0)$. Note that $\Delta_j(t)$ depends only on j-waves and that

$$D^*(t)=(1+O(\tau))|Y_j(t)-X_j(t)|.$$

In [3] it is shown using estimate (2.17) that either j-characteristics diverge with increasing time or the total amount of j-rarefaction wave between them decays modulo contributions from wave interactions:

$$\text{const.}\,\omega_j^+\{L(T)\}\leqq\frac{|L(T)|}{T-t}+O(\tau)\,Q\{\pi_j\cap(t\leqq T)\},\qquad T>t. \tag{2.18}$$

Here $L(T)$ denotes the interval between X_j and Y_j at time T. An analogous argument shows that either j-characteristics diverge with decreasing time or the amount of j-shock and j-compression wave between them decays modulo contributions from wave interactions and cancellations:

$$\text{const.}\,|\omega_j^-\{L(t)\}|\leqq\frac{|L(t)|}{T-t}+O(\tau)\,Q\{\pi_j\cap(t\leqq T)\}+\text{const.}\,C\{\pi_j\cap(t\leqq T)\}. \tag{2.19}$$

The presence of a cancellation term in (2.19) is a consequence of the following fact. As time decreases the strength of a j-shock may increase due to cancellations with j-rarefaction waves. In the forward time direction the cancellation of waves favors estimate (2.18). We recall that the measures C and Q are obtained as the w^*-limit of measures C_h and Q_h which are defined by associating with the center of each diamond in U_h a point mass equal to the amount of cancellation and interaction, respectively, which occurs in that diamond.

Analysis of Local Structure. Fix a point (x_0, t_0) with $t_0>0$. One approach to the localization problem is to consider the measures $\frac{\partial}{\partial x}w_j^h$ generated by the restriction of the Riemann invariants w_j^h to the line $t=t_0$. However, it is known only that the approximate solutions converge pointwise a.e. on $t=t_0$ (after passing to a subsequence). Thus if x_0 is, for example, a point of discontinuity of the restriction $U(\cdot, t_0)$, one is guaranteed only that the difference between the associated increasing and decreasing variation measures $\frac{\partial}{\partial x}w_j^{h\pm}$ evaluated on a small interval containing x_0 is close to the limiting jump $w_j(x_0+0, t_0)-w_j(x_0-0, t_0)$ for small h. Hence, on face value the conservation laws for waves do not imply that waves of a particular family will dominate near (x_0, t_0) due to the presence of the interaction term. Nevertheless, the invariance of the equations under similarity transformations together with the large-time behavior of the solution suggest that the cancellation process will dominate the interaction process in the limit despite the existence of a point mass of Q at (x_0, t_0).

The main idea of the proof of Theorem 2.1 is to construct the generalized j-characteristics X_j and Y_j as the limits of approximate characteristics X_j^h and Y_j^h whose x-intercepts on the line $t=t_0$ approach x_0 at a rate which is slow enough to allow the cancellation process to dominate the interaction process.

Lemma 2.1. *There exist generalized j-characteristics* $X_j(t) \leqq Y_j(t)$, $j=1,2$, *passing through* (x_0, t_0) *which are defined on some interval containing* t_0 *and which have the property that the domains* $\Omega_\pm$, Ω_m *and* Ω_n *are states of small variation with respect to* (x_0, t_0).

Proof. Let $\omega_{j,h}^\pm$ denote the measures defined by setting $\omega_{j,h}^\pm(I)$ equal to the absolute value of the total amount of j-shock $(-)$ and j-rarefaction wave $(+)$ in U_h which crosses the x-interval I at time t_0. After passing to a subsequence, let $\omega_j^\pm$ denote the w^*-limit of $\omega_{j,h}^\pm$. Let

$$D_r = \{(x, t): 0 < |x - x_0| + \delta\,|t - t_0| < r\},$$

where δ equals the ratio of mesh lengths $\Delta x/\Delta t$. The boundary of D_r consists of space-like arcs. Let

$$I_r = D_r \cap \{t = t_0\}.$$

Choose a sequence D_n of domains with radii r_n such that $r_{n+1} < r_n$ and such that

$$\omega_j^\pm(I_n) + Q(D_n) + C(D_n) \leqq 1/n.$$

Since any finite positive measure admits at most countably many disjoint sets of non-zero weight, we may assume without loss of generality that

$$\omega_j^\pm(\partial I_n - \{x_0\}) + Q(\partial D_n) + C(\partial D_n) = 0$$

for all n. For each fixed n and every $m < n$, define annular domains $D_{n,m}$ by

$$D_{n,m} = \{(x, t): r_m < |x - x_0| + \delta\,|t - t_0| < r_n\}$$

(see Figure 4). It follows from w^*-convergence that for each fixed n the total amount of wave crossing $D_{n,m}$ at time $t = t_0$ and the total amount of wave cancellation and interaction within $D_{n,m}$ is small if h is sufficiently small: there exist $h(n, m)$ such that

$$\omega_{j,h}^\pm(D_{n,m} \cap \{t = t_0\}) + Q_h(D_{n,m}) + C_h(D_{n,m}) \leqq 2/n \tag{2.20}$$

if $h \leqq h(n, m)$.

Consider the subsequence $U_m = U_{h_m}$ where

$$h_m = \min\{h(n, m): n + 1 \leqq m\}. \tag{2.21}$$

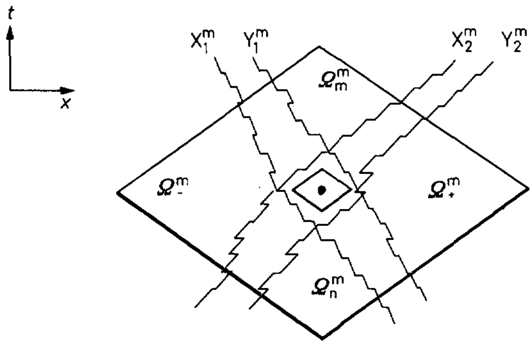

Fig. 4

Let X_j^m and Y_j^m denote the approximate j-characteristics in U_m which pass through the points (x_m, t_0) and (y_m, t_0) defined by

$$x_m = x_0 - [r_m]\,\Delta x, \qquad y_m = x_0 + [r_m]\,\Delta x,$$

where the square bracket denotes the smallest integer greater than or equal to r_m. The characteristics X_j^m and Y_j^m do not cross the inner boundary of $D_{n,m}$ since $\partial D_{n,m}$ consists of space-like arcs and both points (x_m, t_0) and (y_m, t_0) lie in $D_{n,m}$; cf. Figure 4. Let $\Omega_\pm^m$, Ω_m^m and Ω_n^m denote the open regions determined by X_j^m and Y_j^m as in Figure 4.

Fix $\varepsilon > 0$ and choose $n < 1/\varepsilon$. It follows from (2.20), (2.21) and the approximate conservation laws for waves that the total variation of U_l is $O(\varepsilon)$ in all of the regions $\Omega_\pm^m$, Ω_m^m, Ω_n^m provided $l \geqq m$. The lemma follows by passing to a subsequence of approximate solutions for which X_j^h and Y_j^h converge.

The limiting behavior of the outgoing solution at (x_0, t_0) as t approaches t_0 is determined by the rate at which X_j and Y_j approach. If X_j and Y_j are separated by a distance of the order $t - t_0$ then they bound a generalized j-rarefaction wave. If not, there are two possibilities: either X_j and Y_j coincide on some interval $[t_0, t_1]$ as a generalized j-shock wave or the total x-variation of U between X_j and Y_j approaches zero as t approaches t_0. Let

$$L_j = \lim_{t \to t_0} \{Y_j(t) - X_j(t)\}/(t - t_0), \qquad t > t_0.$$

Lemma 2.2. *If* $L_j > 0$ *then* $\pi_j \cap \{t > t_0\}$ *is a generalized j-rarefaction wave. If* $L_j = 0$ *then either*

$$\lim_{t \to t_0} TVU[X_j(t), Y_j(t)] = 0, \qquad t > t_0$$

or $X_j(t) = Y_j(t)$ *on some interval* $[t_0, t_1]$ *and* $\operatorname{str} X_j(t_0 + 0) > 0$.

Proof. Suppose X_j and Y_j do not coincide on any interval of the form $[t_0, t_1]$. We claim that

$$\operatorname{str} X_j(t_0 + 0) = \operatorname{str} Y_j(t_0 + 0) = 0. \tag{2.22}$$

Assume on the contrary that str $X_j(t_0+0)=\gamma>0$. The condition of genuine nonlinearity guarantees that all j-characteristics near X_j are directed toward X_j at a uniform angle depending only on γ, [3]. There exist positive constants $\sigma_1(\gamma)$ and $\sigma_2(\gamma)$ with the following property. If t is near t_0 then, with the possible exception of countably many points $t>t_0$,

$$\lambda_j\{U(x,t)\}+\sigma_1<\dot{X}_j(t) \tag{2.23}$$

provided in the case $j=1$ that

$$X_1(t)\leqq x<\min\{X_1(t)+\sigma_2, X_2(t)\} \tag{2.24}$$

and in the case $j=2$ that

$$X_2(t)\leqq x<X_2(t)+\sigma_2.$$

The inequality (2.23) is established during the proof of Lemma 3.4 of [3]; cf. pp. 55–63. While the hypotheses of Lemma 3.4 require that certain measures do not admit point masses at (x_0,t_0) the proof of (2.23) holds virtually without modification. The restriction in (2.24) that x be less than X_2 is imposed simply to bound x away from the wave π_2.

With the possible exception of countably many points, the shock stability conditions are satisfied along Y_j, [3]:

$$\lambda_j\{U(Y_j(t)+0,t)\}\leqq \dot{Y}_j(t)\leqq \lambda_j\{U(Y_j(t)-0,t)\}. \tag{2.25}$$

It follows from (2.23) and (2.25) that X_j and Y_j cannot intersect at t_0 if they are separated at times arbitrarily close to t_0. This completes the proof of (2.22).

It is an immediate corollary of the proof of Lemma 2.1 that

$$\lim_{t\to t_0}\omega_k^{\pm}[X_j(t), Y_j(t)]=0 \tag{2.26}$$

if $k\neq j$. Therefore, only j-waves need be considered. Suppose $L_j>0$. Then X_j and Y_j intersect only at t_0 and it is only necessary to consider waves which are contained strictly between X_j and Y_j by (2.22). It will be shown that

$$\lim_{t\to t_0}\omega_j^-(X_j(t), Y_j(t))=0, \qquad t>t_0 \tag{2.27}$$

$$\lim_{t\to t_0}\omega_j^+(X_j(t), Y_j(t))>0, \qquad t>t_0. \tag{2.28}$$

Fix $\varepsilon>0$ and let $T>t>t_0$. Applying (2.19) to X_j and Y_j yields

$$\begin{aligned}\text{const.}\,|\omega_j^-(X_j(t), Y_j(t))|\leqq&\{Y_j(t)-X_j(t)\}/(T-t)+\text{const.}\,C(\pi_j\cap\{t\leqq s\leqq T\})\\&+O(\tau)\,Q(\pi_j\cap\{t\leqq s\leqq T\}).\end{aligned} \tag{2.29}$$

Choose T sufficiently close to t_0 so that the second and third terms on the right of (2.29) are both less than $\varepsilon/3$ for $t>t_0$, and then choose t so that the first term is less than $\varepsilon/3$. This proves (2.27).

Inequality (2.28) is established by applying (2.17) to X_j and Y_j: this gives

$$D_j^*(t)\leqq\text{const.}\,\mu_j\int_{t_0}^{t}|\Delta_j(t)|\,dt$$

where

$$\Delta_j(t)=\omega_j^+ +\omega_j^- +(1/2+O(\tau))(\operatorname{str} X_j+\operatorname{str} Y_j)+O(\tau)\,\omega_j^+ +O(\tau)|\omega_j^-|.$$

Since $D_j^*(t)=(1+O(\tau))\,|X_j(t)-Y_j(t)|$ and since Δ_j is a function of bounded variation it follows that

$$L_j \leqq \text{const.}\,\mu_j \lim_{t\to t_0} |\Delta_j(t)|, \qquad t>t_0.$$

The limit of the second, third and fifth terms of Δ_j equals zero by (2.22) and (2.27). Hence

$$L_j \leqq \text{const.}\,\mu_j \lim_{t\to t_0} \omega_j^+(t).$$

This proves (2.28) since $L_j>0$.

Consider the case $L_j=0$. Suppose X_j and Y_j coincide on some interval of the form $[t_0, t_1]$. Then either $\operatorname{str} X_j(t_0+0)>0$ or $\operatorname{str} X_j(t_0-0)=0$. In both cases the proof of the lemma is complete. Suppose X_j and Y_j do not coincide on any such interval. In view of (2.22), (2.26) and (2.27) it is only necessary to prove that

$$\lim_{t\to t_0} \omega_j^+(X_j(t), Y_j(t))=0, \qquad t>t_0.$$

Fix $\varepsilon>0$ and let $T>t>t_0$. Applying (2.18) to X_j and Y_j yields

(2.30) $$\text{const.}\,\omega_j^+(X_j(T), Y_j(T)) \leqq \{Y_j(T)-X_j(T)\}/(T-t)+O(\tau)\,Q(\pi_j \cap \{t\leqq s\leqq T\}).$$

Choose T sufficiently close to t_0 so that the second term on the right of (2.30) is less than $\varepsilon/2$ for $t>t_0$ and so that

$$\{Y_j(T)-X_j(T)\}/(T-t_0)<\varepsilon/3.$$

Then choose $t>t_0$ so that the first term on the right of (2.30) is less than $\varepsilon/2$. This completes the proof of the lemma.

In order to complete the proof of Theorem 2.1 it remains only to establish the second of the three assertions. The second statement follows immediately from the definition of generalized compression wave, equation (2.26) and the third order property of the minor invariants, i.e. $[m_j]=O([M_j]^3)$.

3. Classification and Structure of Singularities

It is well-known that solutions U constructed by the difference scheme of Glimm are functions of bounded variation in the sense of Cesari. This property is guaranteed by the estimates:

$$TVU(\cdot, t)\leqq \text{const.}\, TVU_0$$

and

(3.1) $$\int_{-\infty}^{\infty} |U(x,t_1)-U(x,t_2)|\,dx \leqq \text{const.}\,|t_1-t_2|.$$

Here the constant depends only on the equations. We shall briefly recall certain basic definitions and results in BVC theory [1], [12].

Let $u=u(y)$ be a real-valued measurable function defined on R^m. The function u is said to have an approximate limit $lu(y)$ at the point y if the complement of the inverse image of every open set $\mathcal{O}$ containing $lu(y)$ has Lebesgue density zero at y:

$$\lim_{r\to 0} |u^{-1}(\mathcal{O})^c \cap B(y,r)|/|B(y,r)| = 0,$$

where B denotes a ball of radius r centered at y and $|\cdot|$ denotes m-dimensional Lebesgue measure. If $u(y)=lu(y)$ then u is said to be approximately continuous at y.

The singularities of an arbitrary BVC function can be classified using the notion of an approximate limit with respect to a half-space. The function u is said to have an approximate limit $l_\nu u(y)$ at y with respect to the half-space

$$H_\nu(y) = \{z \mid (z-y, \nu) > 0\}$$

if its restriction to H_ν has an approximate limit at y:

$$\lim_{r\to 0} |u^{-1}(O)^c \cap H_\nu(y) \cap B(y,r)|/|B(y,r)| = 0$$

where O is any open set containing $l_\nu u(y)$. Consider, for example, a piecewise smooth function u defined on $\mathbb{R}^2$. At each point y which lies on exactly one curve of discontinuity, there exist approximate limits $l_\nu u(y) \neq l_\nu u(y)$ with respect to both of the complementary half-planes determined by the normal ν to the curve at y. At a point of non-tangential intersection of two or more curves of discontinuity the function u does not have approximate limits with respect to any pair of complementary half-planes.

Let u be a BVC function defined on an open domain Ω in R^n. The points of Ω are classified as regular or irregular according to the existence or nonexistence of approximate limits with respect to some pair of complementary half-spaces:

$$R \equiv \{y : \exists\, l_{\pm\nu} u(y) \text{ for some } \nu\}$$
$$I \equiv R^c.$$

The set I of irregular points has vanishing $(m-1)$-dimensional Hausdorff measure. The set R of regular points is naturally partitioned into subsets

$$A = \{y \in R : l_\nu u(y) = l_{-\nu} u(y)\}$$
$$J = \{y \in R : l_\nu u(y) \neq l_{-\nu} u(y)\}.$$

After a modification on a set of zero m-dimensional Lebesgue measure u has the following properties. At each point of A, u is approximately continuous. In particular, u has identical approximate limits with respect to each and every half-space $H_\sigma(y)$. At each point of the jump set J, u has approximate limits $l_{\pm\sigma} u(y)$ if and only if $\sigma = \pm\nu$.

The following definitions are preliminary to the classification of singularities in a solution. Let Γ be a countable collection of shock waves Γ_j^n, $j=1,2$, with the following property. Any two distinct waves Γ_j^n and Γ_j^m of the j^{th} field intersect at most at their initial and end points. Let $c_{12}(\Gamma)$ denote the points of interaction of shock waves of opposite fields at which there exist at least two shocks with non-zero strength:

$$c_{12}(\Gamma) = \bigcup \{P = \Gamma_1^n \cap \Gamma_2^m : \operatorname{str} \Gamma_1^n(P) > 0, \operatorname{str} \Gamma_2^m(P) > 0\}.$$

Let $c_j(\Gamma)$ denote the points of interaction of shocks of the j^{th} field at which there exist at least two shocks with non-zero strength:

$$c_j(\Gamma) = \bigcup_{n \neq m} \{P = \Gamma_j^n \cap \Gamma_j^m : \operatorname{str} \Gamma_j^n(P) > 0, \operatorname{str} \Gamma_j^m(P) > 0\}.$$

Let π_j be a generalized j-compression wave centered at (x_0, t_0). The wave π_j will be called *pure* if it satisfies either

$$(3.2) \qquad \lim_{t \to t_0} TV m_j(X_j(t), Y_j(t)] = 0 \quad \text{or} \quad \lim_{t \to t_0} TV m_j[X_j(t), Y_j(t)) = 0.$$

The conditions (3.2) imply that the maximum strength of all j-shocks within π_j approaches zero as t approaches t_0, with the possible exception of one of the edges X_j or Y_j. The freedom of allowing a shock to bound a pure compression wave is a technical convenience. Let $c_0(\Gamma)$ denote the set of all centers of compression waves which lie on some shock wave of Γ. Let $c(\Gamma)$ denote the *collision set* of Γ:

$$C(\Gamma) = \bigcup_{j=0}^{2} c_j(\Gamma) \cup c_{12}(\Gamma).$$

Let $f(\Gamma)$ denote the set of all initial points P of shock waves Γ_j^n such that $\operatorname{str} \Gamma_j^n(P) > 0$, i.e. the *formation set* of Γ.

Theorem 3.1. *Let U be a solution which is constructed by the difference scheme of* GLIMM *and which has initial data with small total variation. Let J and I denote respectively the set of jump points and the set of irregular points of the solution U considered as BVC function defined on $\{(x,t): t>0\}$. There exists a countable collection Γ of shock waves Γ_j^n with the following properties:*

1) *Any two distinct waves Γ_j^n and Γ_j^m of the same field intersect at most at their initial and end points.*

2) $J = \bigcup_{n=1}^{\infty} \Gamma_j^n - I.$

3) $I = c(\Gamma) \cup f(\Gamma).$

4) *U is continuous in the classical pointwise sense at each point of $(J \cup I)^c$.*

Remarks. 1. The sets $c(\Gamma)$ and $f(\Gamma)$ need not be disjoint. A shock wave of one field may form at a point of interaction of two or more shocks of the opposite field. Similarly a shock may form at the center of a pure compression wave of the same or opposite field.

2. The center of every pure compression wave in U is an irregular point. Such centers necessarily lie on some shock wave in Γ.

3. It is necessary to restrict the formation set of Γ to include only those initial points at which the corresponding shock has non-zero strength. Unless a classical compression wave is centered, the initial point of the shock to which it gives rise will be a point of continuity of the solution. In this situation the shock initially has zero strength.

The shock set $\bigcup \Gamma_j^n$ of the solution may be everywhere dense in $\{(x,t): t>0\}$. For example, in a scalar conservation law with convex flux, monotonically decreasing data with a dense set of discontinuities will give rise to an everywhere dense shock set. However, there is a sense in which the shock waves Γ_j^n are isolated. We shall say that a shock wave $X(t)$ is *isolated* with respect to an interior point $(X(t_0), t_0)$ if there exists a neighborhood of $(X(t_0), t_0)$ in which the total x-variation of U is arbitrarily small on the complement of Γ, i.e. if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$TVU[X(t)-\delta, X(t)) + TVU(X(t), X(t)+\delta] < \varepsilon \quad \text{if } |t-t_0| < \delta.$$

At such a point the limiting behavior of the solution can be described classically. Let X be a shock wave of U defined on $[t_1, t_2]$. Let

$$X^+ = \{(x, t): x > X(t), t_1 < t < t_2\}$$
$$X^- = \{(x, t): x < X(t), t_1 < t < t_2\}.$$

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let X be a j-shock wave of Γ. Let $(x_0, t_0) \in X - I$ be an interior point of X. Then*

1) *X is isolated with respect to (x_0, t_0).*

2) *The speed of propagation $\dot{X}(t_0)$ exists and equals $\sigma_j\{U(x_0+0, t_0), U(x_0-0, t_0)\}$.*

3) *The domains $X^{\pm}$ are states of small variation with respect to (x_0, t_0).*

4) *The limits $U\{(x_0, t_0); X^{\pm}\}$ exist, equal $U(x_0 \pm 0, t_0)$, and satisfy the Rankine-Hugoniot relations.*

5) *There exists a neighborhood of (x_0, t_0) in which all generalized j-characteristics run into X when followed in the forward direction of time.*

It follows from the proof of Theorem 3.1 that the shock waves Γ_j^n exist as the limit of approximate j-characteristics in the approximating solutions U_h. Thus the speed of propagation of each shock wave Γ_j^n is a function of bounded variation [3]. In particular, each shock is continuously differentiable with the possible exception of a countable set of points. This regularity is optimal even for a scalar conservation law. In the example cited above of a solution with an everywhere dense shock set, the speed of propagation of each shock wave fails to exist at a (countable) dense set of points.

Proof of Theorem 3.1. The structure of J will be considered first. For reference we recall the following fact which was established as part of the proof of Lemma 2.2.

Assertion 1. Let X and Y be generalized j-characteristics at least one of which exists as the limit of approximate characteristics. Suppose $X(t_0) = Y(t_0)$. Then either X and Y coincide on an interval of the form $[t_0, t_1]$ or

$$\operatorname{str} X(t_0+0) = \operatorname{str} Y(t_0+0) = 0.$$

Fix a point (x_0, t_0), $t_0 > 0$. A shock wave X will be said to be incoming at (x_0, t_0) if the following properties hold: the domain of definition of X contains an interval of the form $[t_1, t_0]$, $X(t_0) = x_0$ and $\operatorname{str} X(t_0 - 0) > 0$.

Assertion 2. Let X_j and Y_j be the generalized j-characteristics constructed in Theorem 2.1. Then one of the following properties holds:

1. There exists no j-shock which is incoming at (x_0, t_0). Either π_j is a pure j-compression wave or

$$\lim_{t \to t_0} TVU[X_j(t), Y_j(t)] = 0.$$

2. There exists exactly one j-shock wave X which is incoming at (x_0, t_0). In this case X satisfies one of the following:

(a) $X_j(t) < X(t) < Y_j(t)$ for t near t_0. Either at least one of the sets

$$\{(x, t): X_j(t) \leqq x \leqq X(t)\}, \qquad \{(x, t): X(t) \leqq x \leqq Y_j(t)\}$$

is a pure j-compression wave or

(3.3) $$\lim_{t \to t_0} TVU[X_j(t), X(t)) + \lim_{t \to t_0} TVU(X(t), Y_j(t)] = 0, \qquad t < t_0.$$

(b) $X_j(t) = X(t) < Y_j(t)$ for t near t_0. Either $\{(x, t): X(t) \leqq x \leqq Y_j(t)\}$ is a pure j-compression wave or

(3.4) $$\lim_{t \to t_0} TVU(X(t), Y_j(t)] = 0, \qquad t < t_0.$$

(c) $X_j(t) < X(t) = Y_j(t)$ for t near t_0. Either $\{(x, t): X_j(t) \leqq x \leqq X(t)\}$ is a pure j-compression wave or

(3.5) $$\lim_{t \to t_0} TVU[X_j(t), X(t)) = 0, \qquad t < t_0.$$

3. There exist at least two shock waves which are incoming at (x_0, t_0).

The proof of Assertion 2 is straightforward and details are left to the reader. We note that if X is a shock wave in U then X is necessarily the limit of approximate characteristics in U_h. More precisely, if X is a j-shock wave such that $\operatorname{str} X(t_0 + 0) > 0$ then there exists an interval $[t_0, t_1]$ on which X coincides with the generalized j-characteristic X_j through $(X(t_0), t_0)$ which is constructed in Theorem 2.1. Assertion 1 implies that X and X_j coincide on the intersection of the domain of definitiin of X with the set $\{t \geqq t_0\}$.

Assertion 3. Suppose $(x_0, t_0) \in J$. Then there exists precisely one shock wave $X(t)$ with the following properties: $X(t)$ is defined on an interval containing t_0 in its interior, $X(t_0) = x_0$ and $X(t)$ is isolated with respect to (x_0, t_0).

Proof. Theorem 2.1 together with the fact that (x_0, t_0) is regular implies that $U(x_0 + 0, t_0)$ lies on one of the shock wave curves through $U(x_0 - 0, t_0)$, say the shock wave curve of the first kind. Consider the 1-characteristics X_1 and Y_1 through (x_0, t_0) which are constructed in Theorem 2.1. Since (x_0, t_0) is a jump point, $X_1(t) = Y_1(t)$ on some interval $[t_0, t_1]$ and $\operatorname{str} X_1(t_0 + 0) > 0$. Since (x_0, t_0) is regular,

$$\lim_{t \to t_0} TVU[X_2(t), Y_2(t)] = 0, \qquad t < t_0,$$

where X_2 and Y_2 are the 2-characteristics constructed in Theorem 2.1. Thus, only 1-waves need be analyzed.

Consider the alternatives of Assertion 2. We claim that there exists exactly one 1-shock $\tilde{X}(t)$ which is incoming at (x_0, t_0) and moreover that either (3.3), (3.4) or (3.5) holds. If, on the contrary, the first alternative held then (x_0, t_0) would not be regular. If the second alternative held and there existed a pure 1-compression wave centered at (x_0, t_0), then $U(x_0+0, t_0)$ would lie in the fourth open quadrant with respect to $U(x_0-0, t_0)$ as defined by the shock and rarefaction wave curves through $U(x_0-0, t_0)$. In this situation the outgoing solution at (x_0, t_0) would contain a 2-rarefaction wave, contradicting the fact that (x_0, t_0) is regular. If the third alternative held, then $U(x_0+0, t_0)$ would again lie in the fourth open quadrant with respect to $U(x_0-0, t_0)$.

The desired shock wave $X(t)$ is obtained by choosing σ_1 and σ_2 close to t_0 and setting $X(t)=\tilde{X}(t)$ for t in $[\sigma_1, t_0]$ and $X(t)=X_1(t)$ for t in $[t_0, \sigma_2]$. This completes the proof of the assertion.

Next we shall construct a countable family of shock waves X_n such that $J=UX_n-I$. Choose a countable dense set $G=\{t_n\}$ such that $I\cap\{(x,t): t\in G\}$ is empty. The existence of G is guaranteed by the fact that I has zero 1-dimensional Hausdorff measure. Let $\{(x_n, t_n)\}$ be an enumeration of the points of discontinuity of the functions $U(\cdot, t_0)$, $t_0\in G$. A shock wave X_n is constructed through each point (x_n, t_n) as follows. First, consider the shock wave X through (x_n, t_n) which is constructed in Assertion 3. Let $[t_1, t_2]$ denote the domain of definition of X. Define $X_n(t)=X(t)$ for t in $[t_1, t_2]$. The domain of definition of X_n is extended in the following way. Consider a characteristic Y of the same field as X which passes through (x_n, t_n) and which is defined for $t\geqq t_n$. As a consequence of Assertion 1, X and Y coincide on the interval $[t_n, t_2]$. Normalize $\operatorname{str} Y(t)$ to be left-continuous and let $[t_n, \sigma_n]$ denote the component of $\{t\geqq t_n: \operatorname{str} Y(t\pm 0)>0\}$ which is open in the relative topology and which contains t_n. Define $X_n(t)=Y(t)$ for t in $[t_n, \sigma_n]$. The curve X_n is a shock wave since for every proper subinterval of its domain of definition there exists a $\delta>0$ such that $\operatorname{str} X_n(t)\geqq\delta$ on that subinterval.

We note that the shock wave X_n is *maximal* in the forward direction of time in the following sense. If $\operatorname{str} X_n(\sigma_n-0)>0$ then there does not exist an outgoing shock wave of the same field at the point $(X_n(\sigma_n), \sigma_n)$ which has non-zero strength at σ_n. This property is a consequence of Assertion 1 and the definition of the domain of X_n.

In order to prove that $J=UX_n-I$ we first observe that U is approximately continuous at each point (x_0, t_0) in $(I\cup J)^c$. At such a point it follows from the structure of the outgoing solution that $U(\cdot, t_0)$ is continuous at x_0. Therefore by Corollary 2.2, U is continuous at (x_0, t_0) as a function of x and t. This establishes the fourth statement of the theorem and the inclusion $X_n-I\subset J$.

Consider the inclusion $J\subset X_n-I$. Let $(x_0, t_0)\in J$ and consider the shock wave X through (x_0, t_0) constructed in Assertion 3. Choose a jump point (x_n, t_n) on X with $t_n<t_0$. The shock waves X_n and X coincide for $t\geqq t_n$ by Assertion 1. Thus $(x_0, t_0)\in X_n$ and we conclude that $J=UX_n-I$.

Assertion 4. The set of irregular points is at most countable.

Proof. Let $(x_0, t_0)\in I$. We claim that either there exists some pure compression wave centered at (x_0, t_0) or there exist at least two distinct shock waves which are incoming at (x_0, t_0). These alternatives are not mutually exclusive. There are two situations to consider. First, suppose there exists no incoming shock wave at (x_0, t_0). Then Assertion 2 implies that there exists a pure compression wave centered at (x_0, t_0). Otherwise x_0 would be a point of continuity of $U(\cdot, t_0)$.

Second, suppose there exists precisely one shock of (say) the j^{th} field which is incoming at (x_0, t_0). If there exists a pure compression wave of the k^{th} field, $k \neq j$, centered at (x_0, t_0) then no further argument is necessary. If not, there necessarily exists a pure compression wave of the j^{th} field centered at (x_0, t_0). The contrary assumption would imply that (x_0, t_0) is a jump point. This establishes the claim.

Clearly, in both cases the interaction measure Q has a point mass at (x_0, t_0). Since Q is a Borel measure, Q can admit only countably many point masses. This completes the proof of the assertion.

It remains to redefine the domains of definition of the shock waves X_n so that the first and third statements of the theorem hold. Let $\{(z_n, s_n)\}$ be an enumeration of the set of irregular points. Let $Z_{n,j}$, $j=1,2$, denote the outgoing shock waves through (z_n, s_n) which are constructed in Theorem 2.1. We recall that $Z_{n,j}$ are defined for $t \geqq s_n$ and $\operatorname{str} Z_{n,j}(s_n+0)>0$. There exists at least one such wave at (z_n, s_n) by Theorem 2.1. Extend $Z_{n,j}$ to be maximal in the forward direction of time by the method used to extend the shock waves X_n through the jump points (x_n, t_n). If possible extend $Z_{n,j}$ backward for some interval $[\sigma_n, s_n]$ by choosing one of the incoming j-shocks at (z_n, s_n). Choose σ_n sufficiently close to s_n so that $\operatorname{str} Z_{n,j}(\sigma_n+0)>0$.

Consider the countable collections of waves $\{X_n\}$ and $\{Z_{n,j}\}$ and the corresponding points $\{(x_n, t_n)\}$ and $\{(z_n, s_n)\}$. By relabeling, let $\{X_n\}$ and $\{(x_n, t_n)\}$ denote an enumeration of $\{X_n\} \cup \{Z_{n,j}\}$ and $\{(x_n, t_n)\} \cup \{(z_n, t_n)\}$, respectively. The new collection $\{X_n\}$ will be used to construct the shock waves Γ^n required for the theorem. For simplicity the subscript j, indicating the field of the shock, will be suppressed.

First, certain countable families $Y_n = \{Y_n^k : k \text{ an integer}\}$ of shock waves will be defined which have the following properties:

P_1. For a fixed n, the shock waves Y_n^k are all defined on intervals of the form $[a_n^k, b]$ or $[a_n^k, \infty)$ and satisfy

$$Y_n^k \supset Y_n^j \quad \text{and} \quad a_n^k < a_n^j \qquad \text{if } k \geqq j.$$

P_2. Y_n^k and Y_m^j intersect in at most one point if $n \neq m$.

P_3. $\operatorname{str} Y_n^k(a_n^k+0)>0$.

P_4. For every integer p,

$$F_p \stackrel{\text{def}}{=} \bigcup_{n,k \leqq p} Y_n^k = \bigcup_{n \leqq p} X_n.$$

P_5. If the end point of Y_n^k intersects the intial point of Y_m^j at, say, time t and if Y_n^k and Y_m^j are shock waves of the same field, then $\operatorname{str} Y_n^k(t-0)$ and/or $\operatorname{str} Y_m^j(t+0)$ equals zero.

P_6. If X is an arbitrary shock wave in U then one of the following holds: a) X intersects F_p in at most a finite number of points. b) There exists a time $t=\sigma$ satisfying the following three properties: $X(\sigma)$ is an interior point of X, X is included in F_p for $t \geqq \sigma$, and X intersects F_p in at most a finite number of points for $t<\sigma$. c) X is included in F_p.

The above indices n and k form subsequences of integers. The shock waves Γ^n are defined by

$$\Gamma^n = \lim_{k\to\infty} Y_n^k.$$

The limit exists since the total variation of the speed of propagation of Y_n^k is uniformly bounded in k for fixed n.

The construction of Y_n^k is inductive. Let $Y_1^1 = X_1$. Assume that by the p^{th} step the waves Y_n^j have been defined and that they satisfy properties P_1 to P_6. The step $p+1$ is as follows. Consider the shock wave X_{p+1}. Let r and s denote, respectively, the maximum of the indices n and k which occur in the collection of all waves Y_n^k defined by the p^{th} step.

Case 1. Suppose X_{p+1} intersects F_p in at most a finite number of points. Set $Y_{r+1}^1 = X_{p+1}$ and proceed to step $p+2$.

Case 2. Suppose X_{p+1} is included in F_p for $t \geqq \sigma$ and intersects F_p in at most a finite number of points for $t<\sigma$. Suppose further that $P = \{X_{p+1}(\sigma), \sigma\}$ is an interior point of X_{p+1}. If P is an interior point of some shock wave in F_p, then define Y_{r+1}^1 to be the restriction of X_{p+1} to $t \leqq \sigma$ and proceed to step $p+2$. If not, then P is necessarily the initial point of some shock wave Y_n^s which has been constructed by the p^{th} step and which is of the same field as X_{p+1}. In this subcase define Y_n^{s+1} to be the union of Y_n^s with the restriction of X_{p+1} to $t \leqq \sigma$ and proceed to step $p+2$.

Case 3. Suppose X_{p+1} is included in F_p. Proceed to step $p+2$.

It follows from the construction above that the shock waves Γ^n and Γ^m, $n \neq m$, intersect in at most two points. It is straightforward to show that $I \subset c(\Gamma) \cup f(\Gamma)$. Consider the opposite inclusion. Clearly $c(\Gamma) \subset I$. Let $(x_0, t_0) \in f(\Gamma)$ and suppose (x_0, t_0) is not an irregular point. Since (x_0, t_0) is the initial point of a shock wave Γ^n satisfying $\operatorname{str} \Gamma_n(t_0+0) > 0$, it follows that (x_0, t_0) is a jump point. Therefore, (x_0, t_0) is an interior point of some shock wave X by Assertion 3. Since Γ^n is defined as the limit of Y_n^k, there exists an index k such that Y_n^k and X coincide on a certain interval of time. Let p_k denote that particular step of the induction at which Y_n^k is constructed. Choose a regular point (x_m, t_m) on X such that $t_m < t_0$ and such that the corresponding shock wave X_m is considered at a step $p_m > p_k$. Let C denote the collection of all shock waves X_l which are considered between steps p_k and p_m and which pass through regular points (x_l, t_l) on X. Note that X_m and Y_n^k coincide on some interval of time since X_m and X coincide for $t \geqq t_m$. Let p denote the first step between p_k and p_m at which a shock wave $\tilde{X}_l$ in C coincides with some shock wave $Y_n^j, j \geqq k$, on some interval of time.

Let τ denote the t-component of the initial point of Y_n^j. We claim that $\tilde{X}_l$ intersects F_{p-1} in at most a finite number of points within $\{t \leqq \tau\}$. Suppose not. Then there exists a time $\sigma < \tau$ such that $\tilde{X}_l$ is included in F_{p-1} for $t \geqq \sigma$. Hence, by property P_6, there exists a shock wave in F_{p-1} having an end point with non-zero strength which coincides with the initial point of Y_n^j. This contradicts property P_5. Thus, at step p a shock wave Y_n^{j+1} is constructed as the union of Y_n^j with the restriction of X_l to $t \leqq \tau$. This fact implies that (x_0, t_0) is an interior point of Γ^n. We conclude from this contradiction that (x_0, t_0) is irregular. This completes the proof of the theorem.

4. Interior Regularity

For a single genuinely nonlinear equation ($n=1$), the interior Lipschitz continuity of the solution is a consequence of two properties: the invariance of the solution along characteristics and the fact that characteristics are straight lines. The main step in the corresponding result for systems is the proof that the Riemann invariants w_j are constant along generalized j-characteristics. The fact that the spreading of characteristics of one field is not significantly impaired by the action of characteristics of the opposite fields follows from an estimate of GLIMM & LAX [3]; cf. (4.4) and (4.5).

Theorem 4.1. *The solution U is Lipschitz continuous on any open component of the set on which it is continuous.*

The proof is partitioned into lemmas. Let B denote an open ball on which U is continuous. Let X_j and Y_j be two j-characteristics within B such that $X_j(t) < Y_j(t)$. Let $\omega_j^{\pm}(t) = \omega_j^{\pm}(\sigma_t)$ denote the total amount of j-shock $(-)$ and j-rarefaction wave $(+)$ which crosses the space-like line segment σ_t joining X_j and Y_j; cf. Section 2.

Lemma 4.1. *If $t_2 > t_1$ then $|\omega_j^{\pm}(t_2)| \leqq |\omega_j^{\pm}(t_1)|$.*

Proof. Let $K \subset B$ denote a closed ball containing X_j and Y_j. Let $M(h, K)$ denote the maximum of the strength of all elementary waves of U_h which intersect K and put

$$M(K) = \overline{\lim_{h \to 0}}\, M(h, K).$$

Neglecting cancellation terms in the conservation law for waves yields

$$|\omega_j^{\pm}(t_2)| \leqq |\omega_j^{\pm}(t_1)| + \text{const.}\, M(K)\, Q(K);$$

cf. estimate (3.53) of [3].

We shall show that $M(K) = 0$. Assume on the contrary that there exists a subsequence U_k of solutions which contain elementary waves α_k such that α_k intersect K and such that

$$|\text{magnitude}\, \alpha_k| \geqq 2\delta > 0.$$

By passing to a further subsequence we may assume that the waves α_k converge to some point (x_0, t_0) in K. We recall that the length of α_k is $O(k)$. Since (x_0, t_0) is a point of continuity of U, the conservation laws for waves imply the existence of a neighborhood N of (x_0, t_0) with the following property: The total x-variation of U_h restricted to any horizontal line segment within N is less than $\delta + \text{const.}\, Q\{(x_0, t_0)\}$ if h is sufficiently small. Therefore, for small k

$$|\text{magnitude}\, \alpha_k| \leqq \delta + \text{const.}\, Q\{(x_0, t_0)\}.$$

We assert that Q does not have a point mass in B. Fix $\varepsilon > 0$ and let

$$L(t) = \{(x, s) : x_0 - \beta \leqq x \leqq x_0 + \beta,\ s = t\}.$$

It follows from Theorem 2.1 that there exists a $\beta > 0$ such that

$$TV_x U_h\{L(t)\} \leqq \varepsilon \quad \text{if } t_0 - \beta < t < t_0.$$

The Q_h-measure of the domain of determinancy $D(L)$ of any line segment L on $t=\text{const.}$ depends quadratically on the x-variation of U_h restricted to L [3]: thus

$$Q_h\{D(L)\} \leqq \text{const.}\, (TV_x U_h\{L\})^2.$$

Hence $Q\{(x_0, t_0)\}=0$ and the proof of the lemma is complete.

Let X_j and Y_j be two j-characteristics in B passing through (x_0, t_0) and (y_0, t_0), respectively, where $x_0 < y_0$.

Lemma 4.2. *The characteristics X_j and Y_j do not intersect within B for $t<t_0$.*

Proof. Consider the case $j=1$. Assume on the contrary that $X_1(t) < Y_1(t)$ for $\tau < t \leqq t_0$ but that $X_1(\tau) = Y_1(\tau)$. It follows from the proof of Theorem 2.1 that $\omega_1^-(\tau+0)=0$. By use of the relation

$$w_1(Y_1(t^*), t^*) - w_1(X_1(t), t) = \omega_1^+(t) + \omega_1^-(t) + O\{\omega_1^-(t)^3\}$$

and the continuity of U it follows that $\omega_1^+(\tau+0)=0$. Here $t^*=t^*(t)$ denotes the time at which the space-like segment σ_t originating on X_1 intersects Y_1. Hence by Lemma 4.1,

$$\omega_1^{\pm}(t)=0 \qquad \text{for } \tau < t \leqq t_0.$$

Therefore $\Delta_1(t)=0$ for $\tau<t\leqq t_0$ and estimate (2.17) implies that

$$0 < D_1^*(t_0) \leqq \text{const.}\, D_1^*(t) \qquad \text{for } \tau < t \leqq t_0.$$

This contradicts the assumption that $X_1(\tau)=Y_1(\tau)$ and completes the proof of the lemma.

Fix a point (x_0, t_0) in B and let $X_j(t) = X_j(t; y)$ be a j-characteristic which passes through (y, t_0) and which exists as the limit of a sequence of approximate j-characteristics $X_j^h(t)$. Since B is open there exists a time $t_1 < t_0$ such that every characteristic is contained in B for $t_1 < t \leqq t_0$ provided y is sufficiently close to x_0.

Lemma 4.3. *The invariant w_j is constant along every characteristic $X_j(t; y)$ for $t_1 < t \leqq t_0$.*

Proof. By Lemma 4.2 the characteristics $X_j(t; x)$ and $Y_j(t; y)$ do not intersect if $x \neq y$ provided $t_1 < t \leqq t_0$. Hence, there exist at most countably many characteristics $X_j(t; y)$ which have either non-zero Q-measure or non-zero C-measure. Let

$$X_j = \{(x, t) : x = X_j(t), t_1 < t \leqq t_0\}.$$

We shall first show that w_j is constant along $X_j(t)$ provided that

$$Q\{X_j\} = C\{X_j\} = 0. \tag{4.1}$$

Consider a characteristic X_j satisfying (4.1) and let D_h denote the union of all diamonds in U_h which intersect $X_j^h(t)$. The total variation of w_j^h along X_j^h is bounded in terms of the following quantities [3]: the total amount $E_j(X_j^h)$ of j-shock entering X_j^h, the total amount $T_k(D_h)$ of k-wave, $k \neq j$, which is contained in D_h, the maximum $M(D_h)$ of the strength of all elementary waves of U_h in D_h, and the total amount of cancellation and interaction in D_h. Thus

$$TV w_j^h(X_j^h) \leqq E_j(X_j^h) + M(D_h)^2\, T_k(D_h) + C_h(D_h) + O(\tau)\, Q_h(D_h). \tag{4.2}$$

Condition (4.1) implies that the third and fourth terms on the right hand side of (4.2) approach zero as h approaches zero. The proof of Lemma 4.1 shows that $M(D_h)$ approaches zero. Hence the second term of (4.2) approaches zero since $T_k(D_h)$ is uniformly bounded in h. The first term on the right of (4.2) is analyzed using the following estimate [3]:

$$\text{(4.3)} \qquad \operatorname{str} X_j^h(t_0-0) = \operatorname{str} X_j^h(t_1+0) - O\{C_h(D_h)\} + |E_j(X_j^h)| + O(\tau)\, Q_h(D_h).$$

Since U is continuous, the strength of X_j^h tends to zero and equation (4.3) implies that $E_j(X_j^h)$ approaches zero. We conclude that w_j is constant along X_j if X_j has zero Q-measure and zero C-measure.

Since the collection of j-characteristics $X_j(t; y)$ satisfying (4.1) is dense, it follows that w_j is constant along every characteristic curve $X_j(t; y)$. This completes the proof of the lemma.

Proof of Theorem 4.1. Fix a point (x_0, t_0) within B. Choose $t_1 < t_0 < t_2$ such that the j-characteristic $X_j(t; y)$ defined by

$$X_j(t_2; y) = y$$

lies in B for $t_1 \leqq t \leqq t_2$ provided y is close to the particular value y_0 which satisfies $X_j(t_0; y_0) = x_0$.

Consider a small increment $\Delta x > 0$ and suppose

$$\Delta w \equiv w_j(x_0 + \Delta x, t_0) - w_j(x_0, t) \geqq 0.$$

Let $\phi(x) = w_j(x, t_1)$ and let $z(x)$ denote the x-coordinate of the point of intersection of the line $t = t_1$ with the characteristic X_j which passes through (x, t_0). Since w_j is constant along X_j, it follows that

$$\Delta w \equiv \phi(z(x_0 + \Delta x)) - \phi(z(x_0)) \stackrel{\text{def}}{=} \Delta\phi.$$

Letting $\Delta z = z(x_0 + \Delta x) - z(x_0)$, we have

$$\frac{\Delta w}{\Delta x} = \left(\frac{\Delta\phi}{\Delta z}\right)\left(\frac{\Delta z}{\Delta x}\right).$$

By virtue of the continuity of U and the invariance of w_j along j-characteristics, the distance $D_j^*(t)$ satisfies

$$\text{(4.4)} \qquad \text{const.}\, \{D_j^*(\tau_1) + [w_j]\, (\tau_2 - \tau_1)\} \leqq D_j^*(\tau_2)$$

$$\text{(4.5)} \qquad D_j^*(\tau_2) \leqq \text{const.}\, \{D_j^*(\tau_1) + [w_j]\, (\tau_2 - \tau_1)\},$$

where $\tau_2 > \tau_1$ and where $[w_j]$ denotes the jump in w_j between the j-characteristics which define D_j^*. The estimates (4.4) and (4.5) are established in the proof of Theorem 1.1, [3]. Since the distance $D_j^*(t)$ satisfies

$$D_j^*(t) = (1 + O(\tau))\, D_j(t),$$

where $D_j(t)$ denotes the horizontal distance, it follows from (4.4) that

$$\Delta z \leqq \text{const.}\, \{\Delta x - \Delta w(t_0 - t_1)\}.$$

Thus,

$$\frac{\Delta w}{\Delta x} \leqq \frac{\Delta \phi}{\Delta z}\left\{\text{const.} - \frac{\Delta w}{\Delta x}(t_0 - t_1)\right\}$$

and

$$\frac{\Delta w}{\Delta x} \leqq \left\{\text{const.}\,\frac{\Delta \phi}{\Delta z}\right\} \Big/ \left\{1 + (t_0 - t_1)\frac{\Delta \phi}{\Delta z}\right\} \leqq \text{const.}/(t_0 - t_1).$$

If $\Delta w < 0$ then a similar estimate follows by considering $w(x_1\, t_2)$ and estimate (4.5). The case where $\Delta x < 0$ is analogous. The Lipschitz continuity of U in the t-drection follows from the form of the equations (2.1).

Note. This research was supported in part by NSF Grant GP-34322 X.

References

1. Federer, H., Geometric Measure Theory. New York: Springer 1969.
2. Glimm, J., Solutions in the large for nonlinear hyperbolic systems of conservation laws. Comm. Pure Appl. Math. **18**, 697–715 (1965).
3. Glimm, J., & P. D. Lax, Decay of solutions of nonlinear hyperbolic conservation laws. Mem. Amer. Math. Soc. **101** (1970).
4. Golubitsky, M., & D. G. Schaeffer, Stability of shock waves for a single conservation law. To appear.
5. Guckenheimer, J., Solving a single conservation law. To appear.
6. Lax, P. D., Nonlinear hyperbolic equations. Comm. Pure Appl. Math. **6**, 231–258 (1953).
7. Lax, P. D., Hyperbolic systems of conservation laws II. Comm. Pure Appl. Math. **10**, 537–566 (1957).
8. Lax, P. D., Development of singularities of solutions of nonlinear hyperbolic partial differential equations. J. Math. Phys. **5**, 611–613 (1964).
9. Lax, P. D., Shock Waves and Entropy, "Contributions to Nonlinear Functional Analysis", ed. E. A. Zarantonello, pp. 603–634. New York: Academic Press 1971.
10. Schaeffer, D. G., A regularity theorem for conservation laws. Advances in Math. **11**, 368–386 (1973)
11. Smoller, J. A., On the solution of the Riemann problem with general step data for an extended class of hyperbolic systems. Michigan Math. J. **16**, 201–210 (1969).
12. Volpert, A. I., The spaces *BV* and quasilinear equations. Math. USSR Sb. **2**, 257–267 (1967).

Department of Mathematics
University of Michigan
Ann Arbor

(Received July 28, 1975)