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SINGULARITIES OF TANGENT LIGHTCONE MAP OF A TIMELIKE SURFACE IN MINKOWSKI 4-SPACE

LINGLING KONG AND DONGHE PEI

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Abstract. In the paper, we will define tangent lightcone map, tangent lightcone curvature and tangent lightcone height function. Then we study the geometry of the timelike surfaces in Minkowski 4-space through their contact with spacelike hyperplane and give the classification of singularities of tangent lightcone map based on the Legendrian singularity theory of Arnol'd.

1. Introduction. In [9, 10, 11], the authors and others studied the submanifolds in Minkowski space. In this paper, we study the surface in Minkowski 4-space from a different view point. As it was expected, the situation presents certain peculiarities when it is compared with the Euclidean case and the case of the papers [9, 10, 11]. For instance, in our case it is always possible to choose two lightlike tangent directions along the surface as a frame of its tangent bundle. By using this, we define a Lorentzian invariant $\mathcal{K}_t(1, \pm 1)$ and call it the tangent lightcone curvature of the timelike surface. As a preparation for the further study on the relationships between the de Sitter Gauss curvature and tangent lightcone curvature, we study singularities of tangent lightcone map of a timelike surface in Minkowski 4-space and reveal the relationships between such singularities and geometric invariants of these surfaces under the action of Lorentzian group. For this purpose, we need to develop local differential geometry of timelike surfaces in Minkowski 4-space similarly as it was done for surfaces in Euclidean 4-space [12].

We shall assume throughout the whole paper that all maps and manifolds are C^∞ unless the contrary is explicitly stated.

Let $\boldsymbol{R}^4 = \{(x_1, x_2, x_3, x_4)\,;\, x_1, x_2, x_3, x_4 \in \boldsymbol{R}\}$ be a 4-dimensional vector space. For any vectors $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4)$ in $\boldsymbol{R}^4$, the *pseudo-scalar product* of $\mathbf{x}$ and $\mathbf{y}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 + \sum_{i=2}^4 x_i y_i$. We call $(\boldsymbol{R}^4, \langle\ ,\ \rangle)$ a *Minkowski* 4-space and write $\boldsymbol{R}^4_1$ instead of $(\boldsymbol{R}^4, \langle\ ,\ \rangle)$.

We say that a vector $\mathbf{x}$ in $\boldsymbol{R}^4_1 \setminus \{\mathbf{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle$ is positive, zero or negative, respectively. The norm of a vector $\mathbf{x} \in \boldsymbol{R}^4_1$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$. For any $\mathbf{x}, \mathbf{y} \in \boldsymbol{R}^4_1$, we say $\mathbf{x}$ *pseudo-perpendicular* to $\mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

We fix an orientation and a timelike orientation of $\boldsymbol{R}^4_1$ (i.e., a 4-volume form dV, and future timelike vector field, have been chosen). Let $X : U \to \boldsymbol{R}^4_1$ be an embedding, where U is an open subset of $\boldsymbol{R}^2$. We denote $M = X(U)$ and identify M with U by the embedding X.

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We say M a *timelike surface* if the tangent plane T_pM of M is a timelike plane (i.e., Lorentz plane) (cf. [20]) for any point $p \in M$. In this case, the normal space N_pM is a spacelike plane. Let $\{\mathbf{e}_1(x, y), \mathbf{e}_2(x, y)\}$ be a pseudo-orthonormal frame of T_pM and $\{\mathbf{e}_3(x, y), \mathbf{e}_4(x, y)\}$ be a orthonormal frame of N_pM, where $p = X(x, y)$, $\mathbf{e}_1$ is a timelike vector and $\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are spacelike vectors.

We shall now establish the fundamental formula for a timelike surface in $\boldsymbol{R}^4_1$ by methods similar to those in [10].

We can write $dX = \sum_{i=1}^4 \omega_i \cdot \mathbf{e}_i$ and $d\mathbf{e}_i = \sum_{j=1}^4 \omega_{ij} \cdot \mathbf{e}_j$ by the 1-forms ω_i and ω_{ij} given by $\omega_i = \delta(\mathbf{e}_i)\langle dX, \mathbf{e}_i\rangle$ and $\omega_{ij} = \delta(\mathbf{e}_j)\langle d\mathbf{e}_i, \mathbf{e}_j\rangle$, where

$$\delta(\mathbf{e}_i) = \langle \mathbf{e}_i, \mathbf{e}_i\rangle = \begin{cases} 1, & i = 2, 3, 4; \\ -1, & i = 1. \end{cases}$$

We have Codazzi type equations:

$$\begin{cases} d\omega_i = \sum_{j=1}^4 \delta(\mathbf{e}_i)\delta(\mathbf{e}_j)\omega_{ij} \wedge \omega_j; \\ d\omega_{ij} = \sum_{k=1}^4 \omega_{ik} \wedge \omega_{kj}, \end{cases}$$

where d is exterior derivative.

It follows $\omega_3 = \omega_4 = 0$ from the fact $\langle dX, \mathbf{e}_3\rangle = \langle dX, \mathbf{e}_4\rangle = 0$. Therefore we have

$$\begin{cases} d\omega_3 = -\omega_{31} \wedge \omega_1 + \omega_{32} \wedge \omega_2 = 0; \\ d\omega_4 = -\omega_{41} \wedge \omega_1 + \omega_{42} \wedge \omega_2 = 0. \end{cases}$$

By Cartan's Lemma, we can write:

$$\begin{aligned} \omega_{13} &= a\omega_1 + b\omega_2, & \omega_{14} &= \bar{a}\omega_1 + \bar{b}\omega_2, \\ \omega_{23} &= b\omega_1 + c\omega_2, & \omega_{24} &= \bar{b}\omega_1 + \bar{c}\omega_2, \end{aligned}$$

for appropriate functions $a, b, c, \bar{a}, \bar{b}, \bar{c}$ in $C^\infty(M, \boldsymbol{R})$.

We define $\langle d^2X, \mathbf{e}_i\rangle = -\langle dX, d\mathbf{e}_i\rangle$ for $i = 3, 4$, then we can define a vector-valued quadratic form

$$\langle d^2X, \mathbf{e}_3\rangle\mathbf{e}_3 + \langle d^2X, \mathbf{e}_4\rangle\mathbf{e}_4 = (a\omega_1^2 + 2b\omega_1\omega_2 + c\omega_2^2)\mathbf{e}_3 + (\bar{a}\omega_1^2 + 2\bar{b}\omega_1\omega_2 + \bar{c}\omega_2^2)\mathbf{e}_4,$$

which is called *the second fundamental form* of timelike surface M.

For a given $\mathbf{v} = m\mathbf{e}_1 + n\mathbf{e}_2 \in T_pM$, we have $d\mathbf{v} = dm\mathbf{e}_1 + md\mathbf{e}_1 + dn\mathbf{e}_2 + nd\mathbf{e}_2$. Then

$$\langle d\mathbf{v}, \mathbf{e}_3\rangle \wedge \langle d\mathbf{v}, \mathbf{e}_4\rangle = \mathcal{K}_t(m, n)\omega_1 \wedge \omega_2,$$

where $\mathcal{K}_t(m, n) = (am + bn)(\bar{b}m + \bar{c}n) - (\bar{a}m + \bar{b}n)(bm + cn)$, which is called *tangent lightcone curvature*.

We now consider a matrix

$$A^\pm = \begin{bmatrix} a \pm b & b \pm c \\ \bar{a} \pm \bar{b} & \bar{b} \pm \bar{c} \end{bmatrix}.$$

Let $k_i^\pm, i = 1, 2$ be the eigenvalues of $A^\pm$, which are called *principal tangent lightcone curvature*.

On the other hand, we define

$$LC_p = \left\{ x \in \boldsymbol{R}^4_1 \,;\, -(x_1 - p_1)^2 + \sum_{i=2}^{4} (x_i - p_i)^2 = 0 \right\}$$

and

$$S^2_+ = \{ x = (x_1, x_2, x_3, x_4) \in LC := LC_0 \,;\, x_1 = 1 \} ,$$

where $p = (p_1, p_2, p_3, p_4) \in \boldsymbol{R}^4_1$. We call S^2_+ the *lightlike unit sphere* and $LC^*_p = LC_p \setminus \{p\}$ the *lightcone* at the vertex p. Given any lightlike vector $x = (x_1, x_2, x_3, x_4)$, we have $\tilde{x} = (1, x_2/x_1, x_3/x_1, x_4/x_1) \in S^2_+$.

On the other hand, we define three maps: two of them are maps

$$TL^{\pm}_M : M \to S^2_+$$

defined by $TL^{\pm}_M(p) = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x, y)$, where $p = X(x, y)$. Each one of these maps shall be called the *tangent lightcone map* of $X(U) = M$.

The last one is

$$\eta : \boldsymbol{R}^4_1 \to \boldsymbol{R}^4_1$$

defined by $\eta(v) = \sum_{i=1}^{4} k_i \mathbf{e}_{(i+2)\mathrm{mod}4}(x, y)$, which is called the *alternation map,* where $v = \sum_{i=1}^{4} k_i \mathbf{e}_i(x, y)$, $k_i \in \boldsymbol{R}$, $\mathbf{e}_0(x, y) = \mathbf{e}_4(x, y)$.

For a lightlike vector $v \in \boldsymbol{R}^4_1$ and a real number c, we define a *spacelike hyperplane* (associate with v) with normal vector $\eta(v)$ by

$$HP(v, c) = \{ x \in \boldsymbol{R}^4_1 \,;\, \langle x, \eta(v) \rangle = c \} .$$

2. Tangent lightcone height functions on timelike surface. Let $M \subset \boldsymbol{R}^4_1$ be a timelike surface. We define the function

$$H : M \times S^2_+ \to \boldsymbol{R}$$

by

$$H(X(x, y), v) = \left\langle X(x, y) - \left(0, 0, \frac{1}{2}ax^2 + \frac{1}{2}cy^2 + bxy, \frac{1}{2}\bar{a}x^2 + \frac{1}{2}\bar{c}y^2 + \bar{b}xy\right), \eta(v) \right\rangle + \frac{1}{2}k^{\sigma}_1 x^2 + \frac{1}{2}k^{\sigma}_2 y^2 ,$$

where $v \in S^2_+$, $\sigma = +$ or $\sigma = -$. We call H the *tangent lightcone height function* on timelike surface M.

For any fixed $v_0 \in S^2_+$, we denote $h_{v_0}(X(x, y)) = H(X(x, y), v_0)$. Then we have the following proposition:

PROPOSITION 2.1. *Let $M \subset \boldsymbol{R}^4_1$ be timelike surface, H be the tangent lightcone height function on M and $p_0 = X(x_0, y_0)$. Then the following assertions hold*:

(1) $\partial h_v / \partial x(p_0) = \partial h_v / \partial y(p_0) = 0$ *if and only if* $v = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x_0, y_0)$;

(2) $\partial h_v/\partial x(p_0)=\partial h_v/\partial y(p_0)=\det\mathbf{H}(h_v)(p_0)=0$ *if and only if* $v=\widetilde{\mathbf{e}_1\pm\mathbf{e}_2}(x_0,y_0)$ *and* $\mathcal{K}_{\mathrm{t}}(1,\pm 1)(p_0)=0$, *where* $\det\mathbf{H}(h_v)(p_0)$ *is the determinant of the Hessian matrix of* h_v *at* p_0.

PROOF. By applying a Lorentzian motion, we may assume that p_0 is the origin of $\boldsymbol{R}^4_1$. We can choose appropriate local coordinate such that X is given in the Monge form:

$$
\begin{aligned}
&X(x,y)=(x,y,f_1(x,y),f_2(x,y))\,,\\
&f_{1,x}(0,0)=f_{1,y}(0,0)=f_{2,x}(0,0)=f_{2,y}(0,0)=0\,,\\
&f_{1,xx}(0,0)=a,\quad f_{1,xy}(0,0)=b,\quad f_{1,yy}(0,0)=c\,,\\
&f_{2,xx}(0,0)=\bar a,\quad f_{2,xy}(0,0)=\bar b,\quad f_{2,yy}(0,0)=\bar c\,,\\
&\mathbf{e}_1(0,0)=(1,0,0,0),\quad \mathbf{e}_2(0,0)=(0,1,0,0)\,,\\
&\mathbf{e}_3(0,0)=(0,0,1,0),\quad \mathbf{e}_4(0,0)=(0,0,0,1)\,.
\end{aligned}
$$

By a straightforward calculation, we know $\partial h_v/\partial x(p_0)=\partial h_v/\partial y(p_0)=0$ if and only if

$$\langle(1,0,0,0),\eta(v)\rangle=\langle(0,1,0,0),\eta(v)\rangle=0\,.$$

It is also equivalent to the condition that v is in $T_{p_0}M\cap S^2_+$, which means

$$v=\mu(\mathbf{e}_1\pm\mathbf{e}_2)(x_0,y_0)=\widetilde{\mathbf{e}_1\pm\mathbf{e}_2}(x_0,y_0)\,.$$

On the other hand, since

$$
\begin{aligned}
&\det\mathbf{H}(h_v)(p_0)\\
&=\begin{vmatrix}\langle(0,0,f_{1,xx}-a,f_{2,xx}-\bar a),\eta(v_0)\rangle+k_1^\sigma & \langle(0,0,f_{1,xy}-b,f_{2,xy}-\bar b),\eta(v_0)\rangle\\ \langle(0,0,f_{1,xy}-b,f_{2,xy}-\bar b),\eta(v_0)\rangle & \langle(0,0,f_{1,yy}-c,f_{2,yy}-\bar c),\eta(v_0)\rangle+k_2^\sigma\end{vmatrix}\\
&=0\,,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\det\mathbf{H}(h_v)(p_0)\\
&=\begin{vmatrix}\langle(0,0,f_{1,xx}-a,f_{2,xx}-\bar a),\eta(v_0)\rangle+k_1^\sigma & \langle(0,0,f_{1,xy}-b,f_{2,xy}-\bar b),\eta(v_0)\rangle\\ \langle(0,0,f_{1,xy}-b,f_{2,xy}-\bar b),\eta(v_0)\rangle & \langle(0,0,f_{1,yy}-c,f_{2,yy}-\bar c),\eta(v_0)\rangle+k_2^\sigma\end{vmatrix}\\
&=k_1^\sigma k_2^\sigma=\mathcal{K}_{\mathrm{t}}(1,\sigma 1)(p_0)=0\,.
\end{aligned}
$$

□

THEOREM 2.2. *Let* $M\subset\boldsymbol{R}^4_1$ *be a timelike surface. We denote by* $\mathbf{H}(h_v)$ *the Hessian matrix of tangent lightcone height function* H *on* M, $TL^{\pm}_M$ *the tangent lightcone map of* M *and* $p=X(x,y)$. *Then the following conditions are equivalent*:

(1) *For a fixed* v *in* S^2_+, p *is a degenerate critical point of* h_v;

(2) p *is a singularity of the tangent lightcone map on* M *and* v *is equal to* $TL^{\pm}_M(p)$ *in* S^2_+;

(3) $\mathcal{K}_{\mathrm{t}}(1,\pm 1)(p)=0$.

PROOF. Consider the subset $\Sigma(H) = \{(p, v) \in M \times S^2_+ ; \partial h_v/\partial x(p) = \partial h_v/\partial y(p) = 0\}$, which is equal to $\{(p, v) \in M \times S^2_+ ; v = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x, y)\}$ by Proposition 2.1(1). We now observe that the restriction $\pi|_{\Sigma(H)}$ of the canonical projection $\pi : U \times S^2_+ \to S^2_+$ can be identified with the tangent lightcone map $TL^{\pm}_M$. Under this identification, we see easily that the condition (1) is equivalent to the condition (2).

Since p is a degenerate critical point of h_v, we have $\det \mathbf{H}(h_v(p)) = 0$. Hence, by Proposition 2.1(2), we have $\mathcal{K}_t(1, \pm 1)(p) = 0$, and the condition (1) is equivalent to the condition (3). $\square$

From this theorem, we know that the point at which the tangent lightcone curvature is 0 is the singularity of the tangent lightcone map, and is the point at which the Hessian matrix of the tangent lightcone height function H is 0.

3. The tangent lightcone pedal surface of a timelike surface. In this section, we consider a singular surface in the positive lightcone

$$LC^*_+ = \{x = (x_1, x_2, x_3, x_4) \in LC_0 ; x_1 > 0\}$$

associated to M whose singularities correspond to singularities of the tangent lightcone map of M. We now define a family of functions

$$\tilde{H} : M \times LC^*_+ \to \boldsymbol{R}$$

by

$$\begin{aligned}\tilde{H}(X(x, y), v) = \Big\langle X(x, y) - \Big(0, 0, \frac{1}{2}ax^2 + \frac{1}{2}cy^2 + bxy, \frac{1}{2}\bar{a}x^2 + \frac{1}{2}\bar{c}y^2 + \bar{b}xy\Big), \eta(\tilde{v})\Big\rangle \\ + \frac{1}{2}k^\sigma_1 x^2 + \frac{1}{2}k^\sigma_2 y^2 - v_1 ,\end{aligned}$$

where $v = (v_1, v_2, v_3, v_4)$. We call $\tilde{H}$ the *extended tangent lightcone height function* of $M = X(U)$. Denote $\tilde{h}_v(X(x, y)) = \tilde{H}(X(x, y), v)$. As an immediate consequence of Proposition 2.1, we have the following proposition.

PROPOSITION 3.1. *Let M be a timelike surface and $\tilde{H} : M \times LC^*_+ \to \boldsymbol{R}$ be the extended tangent lightcone height function of M. For $v_0 \in LC^*_+$, we have the following*:

(1) $\tilde{h}_{v_0}(p_0) = \partial\tilde{h}_{v_0}/\partial x(p_0) = \partial\tilde{h}_{v_0}/\partial y(p_0) = 0$ *if and only if* $\tilde{v}_0 = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x_0, y_0)$ *and*

$$\begin{aligned}v_1 = \Big\langle X(x_0, y_0) - \Big(0, 0, \frac{1}{2}ax_0^2 - \frac{1}{2}cy_0^2 - bx_0y_0, \frac{1}{2}\bar{a}x_0^2 - \frac{1}{2}\bar{c}y_0^2 - \bar{b}x_0y_0\Big), \eta(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})\Big\rangle \\ + \frac{1}{2}k^\sigma_1 x_0^2 + \frac{1}{2}k^\sigma_2 y_0^2 .\end{aligned}$$

(2) $\tilde{h}_{v_0}(p_0) = \partial\tilde{h}_{v_0}/\partial x(p_0) = \partial\tilde{h}_{v_0}/\partial y(p_0) = \det \mathbf{H}(\tilde{h}_{v_0})(p_0) = 0$ *if and only if* $\tilde{v}_0 = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x_0, y_0)$,

$$v_1 = \left\langle X(x_0, y_0) - \left(0, 0, \frac{1}{2}ax_0^2 - \frac{1}{2}cy_0^2 - bx_0y_0, \frac{1}{2}\bar{a}x_0^2 - \frac{1}{2}\bar{c}y_0^2 - \bar{b}x_0y_0\right), \eta(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})\right\rangle + \frac{1}{2}k_1^\sigma x_0^2 + \frac{1}{2}k_2^\sigma y_0^2\,,$$

and $\mathcal{K}_t(1, \pm 1)(p_0) = 0$. *Here*, $X(x, y) = (X_1(x, y), X_2(x, y), X_3(x, y), X_4(x, y))$.

The assertion of Proposition 3.1 means that the discriminant set of the extended tangent lightcone height function $\tilde{H}$ is given by

$$\mathcal{D}_{\tilde{H}} = \left\{ v\ ;\ v = \left(\langle X(x, y) - \alpha, \eta(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})\rangle + \frac{1}{2}k_1^\sigma x^2 + \frac{1}{2}k_2^\sigma y^2\right)(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})(x, y) \quad \text{for some } (x, y) \in U \right\},$$

where $\alpha = (0, 0, \frac{1}{2}ax^2 - \frac{1}{2}cy^2 - bxy, \frac{1}{2}\bar{a}x^2 - \frac{1}{2}\bar{c}y^2 - \bar{b}xy)$. Therefore we define a pair of singular surfaces in LC_+^* by

$$TP_M^\pm(p) = TP_M^\pm(x, y) = \left(\langle X(x, y) - \alpha, \eta(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})\rangle + \frac{1}{2}k_1^\sigma x^2 + \frac{1}{2}k_2^\sigma y^2\right)(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})(x, y)\,.$$

We call each $TP_M^\pm$ the *tangent lightcone pedal surface* of $X(U) = M$. Each singularity of the tangent lightcone pedal surface corresponds exactly to a singularity of the tangent lightcone map.

For given two tangent lightcone pedal surfaces $f_1 : U_1 \to LC_+^*$ and $f_2 : U_2 \to LC_+^*$, write $f_1 \sim f_2$ provided there exists a neighborhood U of x such that $U \subset U_1 \cap U_2$, and the restriction $f_1|_U$ coincides with $f_2|_U$. These equivalence classes are called *tangent lightcone pedal surface germs*.

We now explain the reason why such a correspondence exists from the view point of symplectic and contact geometry. We consider a point $v = (v_1, v_2, v_3, v_4)$ in LC_+^*, then we have a relation $v_1 = \sqrt{v_2^2 + v_3^2 + v_4^2}$. We adopt the coordinate (v_2, v_3, v_4) of the manifold LC_+^*. We now consider the projective cotangent bundle $\pi : PT^*(LC_+^*) \to LC_+^*$ with the canonical contact structure. We review geometric properties of this space. Consider the tangent bundle $\tau : TPT^*(LC_+^*) \to PT^*(LC_+^*)$ and the differential map $d\pi : TPT^*(LC_+^*) \to TLC_+^*$ of π. For any X in $TPT^*(LC_+^*)$, there exists an element α in $T^*(LC_+^*)$ such that $\tau(X)$ is equal to $[\alpha]$. For an element V in $T_x(LC_+^*)$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(LC_+^*)$ by

$$K = \{X \in TPT^*(LC_+^*)\ ;\ \tau(X)(d\pi(X)) = 0\}\,.$$

Since we consider the coordinate (v_2, v_3, v_4), we have $PT^*(LC^*_+) \cong LC^*_+ \times P(\boldsymbol{R}^2)^*$. We call

$$((v_2, v_3, v_4), [\xi_2 : \xi_3 : \xi_4])$$

the homogeneous coordinate of $PT^*(LC^*_+)$, where $[\xi_2 : \xi_3 : \xi_4]$ is the homogeneous coordinate of the dual projective space $P(\boldsymbol{R}^2)^*$.

It is easy to show that X is in $K_{(x,[\xi])}$ if and only if $\sum_{i=2}^4 \mu_i \xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=2}^4 \mu_i \partial/\partial v_i$. An immersion $i : L \to PT^*(LC^*_+)$ is said to be *a Legendrian immersion* if $\dim L = 2$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ *the Legendrian map* and the set $W(i) = \operatorname{image} \pi \circ i$ *the wave front* of i. Moreover, i (or the image of i) is called *the Legendrian lift* of $W(i)$.

If j is a Lagrangian immersion, then the critical value $C(j)$ of $\bar{\pi} \circ j$ is called the corresponding *caustic* (see [1, p. 296]), where $\bar{\pi} : T(LC^*_+) \to LC^*_+$ is the canonical projection. Moreover, j (or the image of j) is called the *Lagrangian lift* of $C(j)$. If X is a Legendrian submanifold, Y is a Lagrangian submanifold and $f : X \to Y$ is a covering map, then f is called *Legendrian covering map*.

In order to study the tangent lightcone pedal surface, we give a quick survey on the Legendrian singularity theory mainly due to Arnol'd-Zakalyukin [1, 24]. Although the general theory has been described for general dimension, we only consider the 3-dimensional case for the purpose. Let $F : (\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0}) \to (\boldsymbol{R}, \boldsymbol{0})$ be a function germ. We say that F is *a Morse family* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k} \right) : (\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0}) \to (\boldsymbol{R} \times \boldsymbol{R}^k, \boldsymbol{0})$$

is non-singular, where $(q, x) = (q_1, \ldots, q_k, x_1, x_2, x_3) \in (\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0})$. In this case we have a smooth 2-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0}) \ ; \ F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\},$$

and the map germ $\Phi_F : (\Sigma_*(F), \boldsymbol{0}) \to PT^* \boldsymbol{R}^3$ defined by

$$\Phi_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \frac{\partial F}{\partial x_2}(q, x) : \frac{\partial F}{\partial x_3}(q, x) \right] \right)$$

is a Legendrian immersion. Then we have the following fundamental theorem of Arnol'd-Zakalyukin [1, 24].

PROPOSITION 3.2. *All Legendrian submanifold germs in $PT^* \boldsymbol{R}^3$ are constructed by the above method.*

We call F *a generating family* of Φ_F. Therefore the corresponding wave front is

$$W(\Phi_F) = \left\{ x \in \boldsymbol{R}^3 \ ; \ \text{there exists } q \in \boldsymbol{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$

By definition, we have $\mathcal{D}_F = W(\Phi_F)$. By the previous arguments, the tangent lightcone pedal surface $TP_M^{\pm}$ is the discriminant set of the extended tangent lightcone height function $\tilde{H}$. We have the following proposition.

PROPOSITION 3.3. *The extended tangent lightcone height function $\tilde{H}$ is a Morse family.*

PROOF. We define the function

$$\bar{H} : U \times S_+^2 \times \boldsymbol{R} \to \boldsymbol{R}$$

by $\bar{H}((x, y), w, r) = H(X(x, y), w) - r$. We consider a C^∞-diffeomorphism

$$\Phi : U \times S_+^2 \times \boldsymbol{R} \to LC_+^*$$

defined by $\Phi((x, y), w, r) = ((x, y), rw)$. Then we have $\tilde{H} \circ \Phi = \bar{H}$. It is enough to show that $\bar{H}$ is a Morse family. For any w in S_+^2 with $\eta(w) = (w_1, w_2, w_3, w_4)$, we have

$$\begin{aligned}\bar{H}((x, y), w, r) &= -X_1(x, y)w_1 + X_2(x, y)w_2 + \left(X_3(x, y) - \frac{1}{2}ax^2 - \frac{1}{2}cy^2 - bxy\right)w_3 \\ &\quad + \left(X_4(x, y) - \frac{1}{2}\bar{a}x^2 - \frac{1}{2}\bar{c}y^2 - \bar{b}xy\right)w_4 + \frac{1}{2}k_1^\sigma x^2 + \frac{1}{2}k_2^\sigma y^2 - r\,,\end{aligned}$$

where $X(x, y) = (X_1(x, y), X_2(x, y), X_3(x, y), X_4(x, y))$. We now prove that the mapping

$$\Delta^*\bar{H} = \left(\bar{H}, \frac{\partial \bar{H}}{\partial x}, \frac{\partial \bar{H}}{\partial y}\right)$$

is non-singular at any point. The Jacobian matrix of $\Delta^*\bar{H}$ is given as follows:

$$\left(\begin{array}{ccc} \langle X_x, \eta(w)\rangle - \langle I_x, \eta(w)\rangle + k_1^\sigma x & \langle X_y, \eta(w)\rangle - \langle I_y, \eta(w)\rangle + k_2^\sigma y & \\ \langle X_{xx}, \eta(w)\rangle - \langle I_{xx}, \eta(w)\rangle + k_1^\sigma & \langle X_{xy}, \eta(w)\rangle - \langle I_{xy}, \eta(w)\rangle & A \\ \langle X_{yx}, \eta(w)\rangle - \langle I_{yx}, \eta(w)\rangle & \langle X_{yy}, \eta(w)\rangle - \langle I_{yy}, \eta(w)\rangle + k_2^\sigma & \end{array}\right),$$

where

$$A = \begin{pmatrix} -X_1 & X_2 & X_3 - J & X_4 - K & -1 \\ -X_{1,x} & X_{2,x} & X_{3,x} - J_x & X_{4,x} - K_x & 0 \\ -X_{1,y} & X_{2,y} & X_{3,y} - J_y & X_{4,y} - K_y & 0 \end{pmatrix},$$

$$I = \left(0, 0, \frac{1}{2}ax^2 + \frac{1}{2}cy^2 + bxy, \frac{1}{2}\bar{a}x^2 + \frac{1}{2}\bar{c}y^2 + \bar{b}xy\right),$$

$$J = \frac{1}{2}ax^2 + \frac{1}{2}cy^2 + bxy\,, \quad K = \frac{1}{2}\bar{a}x^2 + \frac{1}{2}\bar{c}y^2 + \bar{b}xy\,.$$

Since X is embedding, the rank of the matrix

$$A = \begin{pmatrix} -X_{1,x} & X_{2,x} & X_{3,x} - J_x & X_{4,x} - K_x \\ -X_{1,y} & X_{2,y} & X_{3,y} - J_y & X_{4,y} - K_y \end{pmatrix}$$

is equal to 2 at $(0, 0)$, so $\bar{H}$ is a Morse family at $(0, 0)$. For any point (x_0, y_0) in U, consider the map

$$\tilde{\Phi} : U \times S_+^2 \times \boldsymbol{R} \to U \times S_+^2 \times \boldsymbol{R}$$

defined by $\tilde{\Phi}((x,y),w,r) = ((x-x_0, y-y_0), w, r)$. Then we have $\tilde{H} \circ \Phi \circ \tilde{\Phi} = \bar{H} \circ \tilde{\Phi}$, and we know that $\bar{H} \circ \tilde{\Phi}$ is a Morse family at (x_0, y_0). This complete the proof. □

By Proposition 3.3, we remark that the tangent lightcone pedal surfaces $TP_M^{\pm}$ are wave fronts and the extended tangent lightcone height function $\tilde{H}$ gives generating families of the Legendrian lifts of $TP_M^{\pm}$.

4. Contact with spacelike hyperplanes. In this section we consider the geometric meanings of the singularity of the tangent lightcone map and the tangent lightcone pedal surface of $X(U) = M$.

We consider the contact between timelike surface and spacelike hyperplane as the classical differential geometry. In the first place, we briefly review the theory of contact due to Montaldi [18]. Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\boldsymbol{R}^n$ with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that *the contact of X_1 and Y_1 at y_1 is of same type with that of X_2 and Y_2 at y_2* if there is a diffeomorphism germ $\Phi : (\boldsymbol{R}^n, y_1) \to (\boldsymbol{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that, in the definition, $\boldsymbol{R}^n$ can be replaced by any manifold. In his paper [18], Montaldi gives a characterization of contacts by using the terminology of singularity theory.

THEOREM 4.1. *Let X_i, Y_i $(i = 1, 2)$ be submanifolds of $\boldsymbol{R}^n$ with* $\dim X_1 = \dim X_2$ *and* $\dim Y_1 = \dim Y_2$. *Let $g_i : (X_i, x_i) \to (\boldsymbol{R}^n, y_i)$ be immersion germs and $f_i : (\boldsymbol{R}^n, y_i) \to (\boldsymbol{R}^p, \mathbf{0})$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then*

$$K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$$

if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{K}$-equivalent.

For any $p_0 = X(x_0, y_0)$, and v_0 in LC_+^*, we consider a function $\mathcal{H} : \boldsymbol{R}_1^4 \times LC_+^* \to \boldsymbol{R}$ defined by

$$\mathcal{H}(x,v) = \left\langle \left(x_1, x_2, x_3 - \frac{1}{2}ax_0^2 - \frac{1}{2}cy_0^2 - bx_0y_0, x_4 - \frac{1}{2}\bar{a}x_0^2 - \frac{1}{2}\bar{c}y_0^2 - \bar{b}x_0y_0 \right), \eta(\tilde{v}) \right\rangle + \frac{1}{2}k_1^{\sigma}x_0^2 + \frac{1}{2}k_2^{\sigma}y_0^2 - v_1 .$$

We denote $\mathfrak{h}_{v_0}(x) = \mathcal{H}(x, v_0)$ and we have a spacelike hyperplane

$$\mathfrak{h}_{v_0}^{-1}(0) = HP\left(\tilde{v}_0, \left\langle \left(0, 0, \frac{1}{2}ax_0^2 + \frac{1}{2}cy_0^2 + bx_0y_0, \frac{1}{2}\bar{a}x_0^2 + \frac{1}{2}\bar{c}y_0^2 + \bar{b}x_0y_0\right), \eta(\tilde{v}_0)\right\rangle + v_{0,1}\right).$$

We consider the lightlike vector $v_0^{\pm} = \mathbf{e}_1 \pm \mathbf{e}_2(x_0, y_0)$ and

$$c^{\pm} = \left\langle X(x_0, y_0) - \left(0, 0, \frac{1}{2}ax_0^2 - \frac{1}{2}cy_0^2 - bx_0y_0, \frac{1}{2}\bar{a}x_0^2 - \frac{1}{2}\bar{c}y_0^2 - \bar{b}x_0y_0\right), \eta(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}) \right\rangle + \frac{1}{2}k_1^{\sigma}x_0^2 + \frac{1}{2}k_2^{\sigma}y_0^2 .$$

Then we have

$$\mathfrak{h}_{v_0^\pm} \circ X(p_0) = \mathcal{H} \circ (X \times \mathrm{id}_{LC_0^+})(p_0, v_0^\pm) = H(p_0, \tilde{v}_0^\pm) - c^\pm = 0\,.$$

We also have relations

$$\frac{\partial \mathfrak{h}_{v_0^\pm} \circ X}{\partial x}(p_0) = \frac{\partial H}{\partial x}(p_0, \tilde{v}_0^\pm) = 0$$

and

$$\frac{\partial \mathfrak{h}_{v_0^\pm} \circ X}{\partial y}(p_0) = \frac{\partial H}{\partial y}((p_0), \tilde{v}_0^\pm) = 0\,.$$

This means that the spacelike hyperplane $\mathfrak{h}_{v_0^\pm}^{-1}(0) = HP(\tilde{v}_0^\pm, w^\pm)$ is tangent to $M = X(U)$ at p_0, where

$$w^\pm = \left\langle \left(0, 0, \frac{1}{2}ax_0^2 + \frac{1}{2}cy_0^2 + bx_0y_0, \frac{1}{2}\bar{a}x_0^2 + \frac{1}{2}\bar{c}y_0^2 + \bar{b}x_0y_0\right), \eta(\tilde{v}_0)\right\rangle + c^\pm\,.$$

In this case, we call each $HP(\tilde{v}_0^\pm, w^\pm)$ the *tangent spacelike hyperplane* of $M = X(U)$ at p_0. Moreover, the intersection

$$HP(\tilde{v}_0^+, w^+) \cap HP(\tilde{v}_0^-, w^-)$$

is the tangent plane of M at p_0. Let v_1 and v_2 be vectors. If v_1 and v_2 are linearly dependent, then corresponding hyperplanes $HP(v_1, c_1)$ and $HP(v_2, c_2)$ are parallel. Then we have the following simple lemma.

LEMMA 4.2. *Let $X : U \to \boldsymbol{R}_1^4$ be a timelike surface and $\sigma = \pm$. Consider two points $p_1 = X(x_1, y_1)$, $p_2 = X(x_2, y_2)$. Then we have the following*:

(1) $TL_M^\sigma(p_1) = TL_M^\sigma(p_2)$ *if and only if* $HP(v_1^\sigma, c_1^\sigma)$ *and* $HP(v_2^\sigma, c_2^\sigma)$ *are parallel.*

(2) $TP_M^\sigma(p_1) = TP_M^\sigma(p_2)$ *if and only if* $HP(v_1^\sigma, c_1^\sigma) = HP(v_2^\sigma, c_2^\sigma)$.

Here, $v_i^\pm = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x_i, y_i)$ *and*

$$c_i^\pm = \langle X(x_i, y_i), \eta(\widetilde{\mathbf{e}_1 \pm \mathbf{e}_2})\rangle + \frac{1}{2}k_1^\sigma x_i^2 + \frac{1}{2}k_2^\sigma y_i^2$$

for $i = 1, 2$.

On the other hand, for any map $f : N \to P$, we denote $\Sigma(f)$ the set of singular points of f and $D(f) = f(\Sigma(f))$. In this case, we call $f|_{\Sigma(f)} : \Sigma(f) \to D(f)$ *the critical part of* the mapping f. For any Morse family $F : (\boldsymbol{R}^k \times \boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}, \mathbf{0})$, $(F^{-1}(0), \mathbf{0})$ is a smooth hypersurface. Hence we define a smooth map germ $\pi_F : (F^{-1}(0), \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ by $\pi_F(q, x) = x$. We can easily show that $\Sigma_*(F)$ is equal to $\Sigma(\pi_F)$. Therefore, the corresponding Legendrian map $\pi \circ \Phi_F$ is the critical part of π_F.

Now we introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\boldsymbol{R}^3, p)$ and $i' : (L', p') \subset (PT^*\boldsymbol{R}^3, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\boldsymbol{R}^3, p) \to (PT^*\boldsymbol{R}^3, p')$ such that H preserves fibers of π and $H(L) = L'$. A Legendrian immersion germ into $PT^*\boldsymbol{R}^3$ at a point is said to be *Legendrian stable* if, for every map with the given germ, there is a neighborhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighborhood of the original point such that each Legendrian immersion belonging to the first neighborhood has a point in the second neighborhood at which the germ is Legendrian equivalent to the original one.

Since the Legendrian lift $i : (L, p) \subset (PT^*\boldsymbol{R}^3, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs.

PROPOSITION 4.3. *Let $i : (L, p) \subset (PT^*\boldsymbol{R}^3, p)$ and $i' : (L', p') \subset (PT^*\boldsymbol{R}^3, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$ and $\pi \circ i'$ are dense, respectively. Then i and i' are Legendrian equivalent if and only if the wave front sets $W(i)$ and $W(i')$ are diffeomorphic as set germs.*

This result has been pointed out first by Zakalyukin [25]. The assumption in the above Proposition is a generic condition for i and i'. Especially, if i and i' are Legendrian stable, then they satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\boldsymbol{R}^n, \boldsymbol{0}) \to \boldsymbol{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \ ; \ h(0) = 0\}$. Let $F, G : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \to (\boldsymbol{R}, \boldsymbol{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0}) \to (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\boldsymbol{R}^k \times \boldsymbol{R}^n, \boldsymbol{0})$ such that $\Psi^*(\langle F\rangle_{\mathcal{E}_{k+n}}) = \langle G\rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \to \mathcal{E}_{k+n}$ is the pull-back $\boldsymbol{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0}) \to (\boldsymbol{R}, \boldsymbol{0})$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation of $f = F|_{\boldsymbol{R}^k \times \{\boldsymbol{0}\}}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \left.\frac{\partial F}{\partial x_1}\right|_{\boldsymbol{R}^k \times \{\boldsymbol{0}\}}, \left.\frac{\partial F}{\partial x_2}\right|_{\boldsymbol{R}^k \times \{\boldsymbol{0}\}}, \left.\frac{\partial F}{\partial x_3}\right|_{\boldsymbol{R}^k \times \{\boldsymbol{0}\}} \right\rangle_{\boldsymbol{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}$$

(see [14]).

The main result in Arnol'd-Zakalyukin's theory [1, 24] is the following:

THEOREM 4.4. *Let $F, G : (\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0}) \to (\boldsymbol{R}, \boldsymbol{0})$ be Morse families. Then*

(1) *Φ_F and Φ_G are Legendrian equivalent if and only if F and G are P-$\mathcal{K}$-equivalent.*

(2) *Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $F|_{\boldsymbol{R}^k \times \{\boldsymbol{0}\}}$.*

Since F and G are function germs on the common space germ $(\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0})$, we do not need the notion of stably P-$\mathcal{K}$-equivalences under this situation (cf. [1]). By the uniqueness of the $\mathcal{K}$-versal deformation of a function germ, and by Proposition 4.3 and Theorem 4.4, we have the following classification of Legendrian stable germs (cf. [10]). For any map germ $f : (\boldsymbol{R}^n, \boldsymbol{0}) \to (\boldsymbol{R}^p, \boldsymbol{0})$, we define *the local ring of f* by $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p)\mathcal{E}_n$.

PROPOSITION 4.5. *Let $F, G : (\boldsymbol{R}^k \times \boldsymbol{R}^3, \boldsymbol{0}) \to (\boldsymbol{R}, \boldsymbol{0})$ be Morse families. Suppose that Φ_F and Φ_G are Legendrian stable. Then the following conditions are equivalent.*

(1) $(W(\Phi_F), \boldsymbol{0})$ *and* $(W(\Phi_G), \boldsymbol{0})$ *are diffeomorphic as germs.*

(2) Φ_F *and* Φ_G *are Legendrian equivalent.*

(3) $Q(f)$ *and* $Q(g)$ *are isomorphic as* $\boldsymbol{R}$*-algebras, where* $f = F|_{\boldsymbol{R}^k \times \{\boldsymbol{0}\}}$, $g = G|_{\boldsymbol{R}^k \times \{\boldsymbol{0}\}}$.

PROOF. Since Φ_F and Φ_G are Legendrian stable, they satisfy the generic condition of Proposition 4.3. Hence the conditions (1) and (2) are equivalent. The condition (3) implies that f and g are $\mathcal{K}$-equivalent [14, 16]. By the uniqueness of the $\mathcal{K}$-versal deformation of a function germ, F and G are P-$\mathcal{K}$-equivalent. This means that the condition (2) holds. By Theorem 4.4, the condition (2) implies the condition (3). □

Now we have tools for the study of the contact between timelike surfaces and spacelike hyperplanes. Let $TP^\sigma_{M,i} : (U, (x_i, y_i)) \to (LC^*_+, v^\sigma_i)$ $(i = 1, 2)$ be two tangent lightcone pedal surface germs of timelike surface germs $X_i : (U, (x_i, y_i)) \to (\boldsymbol{R}^4_1, p_i)$, where $\sigma = \pm$. We say that $TP^\sigma_{M,1}$ and $TP^\sigma_{M,2}$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism germs $\phi : (U, (x_1, y_1)) \to (U, (x_2, y_2))$ and $\Phi : (LC^*_+, v^\sigma_1) \to (LC^*_+, v^\sigma_2)$ such that $\Phi \circ TP^\sigma_{M,1} = TP^\sigma_{M,2} \circ \phi$. If both of the regular sets of $TP^\sigma_{M,i}$ are dense in $(U, (x_i, y_i))$, it follows from Proposition 4.5 that $TP^\sigma_{M,1}$ and $TP^\sigma_{M,2}$ are $\mathcal{A}$-equivalent if and only if the corresponding Legendrian lift germs are Legendrian equivalent. This condition is also equivalent to the condition that two generating families $\tilde{H}_1$ and $\tilde{H}_2$ are P-$\mathcal{K}$-equivalent by Theorem 4.4. Here, $\tilde{H}_i : (U \times LC^*_+, ((x_i, y_i), v^\sigma_i)) \to \boldsymbol{R}$ is the extended tangent lightcone height function germ of X_i.

On the other hand, if we denote $\tilde{h}_{i,v^\sigma_i}(u) = \tilde{H}_i(u, v^\sigma_i)$, then we have $\tilde{h}_{i,v^\pm_i}(u) = \mathfrak{h}_{v^\pm_i} \circ X_i(u)$. By Theorem 4.1, $K(X_1(U), HP(\tilde{v}^\sigma_1, w_1), v^\sigma_1) = K(X_2(U), HP(\tilde{v}^\sigma_2, w_2), v^\sigma_2)$ if and only if $\tilde{h}_{1,v_1}$ and $\tilde{h}_{2,v_2}$ are $\mathcal{K}$-equivalent, where

$$w_i = \left\langle \left(0, 0, \frac{1}{2}ax_i^2 + \frac{1}{2}cy_i^2 + bx_iy_i, \frac{1}{2}\bar{a}x_i^2 + \frac{1}{2}\bar{c}y_i^2 + \bar{b}x_iy_i\right), \eta(\tilde{v}_i)\right\rangle + v_{i,1}\,.$$

Therefore, we can apply the previous arguments to our situation. We denote by $Q^\sigma(X, (x_0, y_0))$ the local ring of the function germ $\tilde{h}_{v^\sigma_0} : (U, (x_0, y_0)) \to \boldsymbol{R}$, where $v^\sigma_0 = TP^\sigma_M(x_0, y_0)$. We remark that we can explicitly write the local ring as follows:

$$Q^{\pm}(X, (x_0, y_0)) = \frac{C^\infty_{(x_0,y_0)}(U)}{\langle \langle X(x, y), \eta\widetilde{(\mathbf{e}_1 \pm \mathbf{e}_2)}(x_0, y_0)\rangle - 1\rangle_{C^\infty_{(x_0,y_0)}(U)}}\,,$$

where $C^\infty_{(x_0,y_0)}(U)$ is the local ring of function germs at (x_0, y_0) with the unique maximal ideal $\mathfrak{M}_{(x_0,y_0)}(U)$.

THEOREM 4.6. *Let $X_i : (U, (x_i, y_i)) \to (\boldsymbol{R}^4_1, X_i(x_i, y_i))$ $(i = 1, 2)$ be timelike surface germs such that the corresponding Legendrian lift germs are Legendrian stable and $\sigma = \pm$. Then the following conditions are equivalent*:

(1) *Tangent lightcone pedal surface germs $TP^\sigma_{M,1}$ and $TP^\sigma_{M,2}$ are $\mathcal{A}$-equivalent.*

(2) *$\tilde{H}_1$ and $\tilde{H}_2$ are P-$\mathcal{K}$-equivalent.*

(3) *$\tilde{h}_{1,v_1}$ and $\tilde{h}_{2,v_2}$ are $\mathcal{K}$-equivalent.*

(4) *$K(X_1(U), HP(\tilde{v}_1^\sigma, w_1), v_1^\sigma) = K(X_2(U), HP(\tilde{v}_2^\sigma, w_2), v_2^\sigma)$, where $w_i (i = 1, 2)$ are defined as above.*

(5) *$Q^\sigma(X_1, (x_1, y_1))$ and $Q^\sigma(X_2, (x_2, y_2))$ are isomorphic as $\boldsymbol{R}$-algebras.*

PROOF. By the previous arguments (mainly by Theorem 4.1), it has been already shown that conditions (3) and (4) are equivalent. Other assertions follow from Proposition 4.5. $\square$

For a timelike surface germ $X : (U, (x_0, y_0)) \to (\boldsymbol{R}^4_1, X(x_0, y_0))$, we call each set

$$(X^{-1}(HP(v^\pm, w^\pm)), (x_0, y_0))$$

a *tangent spacelike hyperplane indicatrix germ* of X, where $v^\pm = \widetilde{\mathbf{e}_1 \pm \mathbf{e}_2}(x_0, y_0)$ and

$$w^\pm = \langle X(x_0, y_0), \eta(v^\pm)\rangle + \frac{1}{2}k_1^\sigma x_0^2 + \frac{1}{2}k_2^\sigma y_0^2 .$$

Moreover, by the above results, we can borrow some basic invariants from the singularity theory on function germs. We need $\mathcal{K}$-invariants for function germ. The local ring of a function germ is a complete $\mathcal{K}$-invariant for generic function germs. It is, however, not a numerical invariant. The $\mathcal{K}$-codimension (or, Tyurina number) of a function germ is a numerical $\mathcal{K}$-invariant of function germs [14]. We denote

$$\text{L-ord}^\pm(X, (x_0, y_0)) = \dim_{\boldsymbol{R}} \frac{C^\infty_{(x_0,y_0)}(U)}{\langle \tilde{h}_{v_0^\pm}(x, y), \tilde{h}_{v_0^\pm,x}(x, y), \tilde{h}_{v_0^\pm,y}(x, y)\rangle} .$$

Usually $\text{L-ord}^\sigma(x, u_0)$ is called *the $\mathcal{K}$-codimension of $\tilde{h}_{v_0^\sigma}$*, where $\sigma = \pm$. However, we call it the *order of contact with the tangent spacelike hyperplane* at $X(x_0, y_0)$. We also have the notion of corank of function germs.

$$\text{L-corank}^\sigma(X, (x_0, y_0)) = 2 - \operatorname{rank} \operatorname{Hess}(\tilde{h}_{v_0^\sigma}(x_0, y_0)) ,$$

where $v_0^\pm = \mathbf{e}_1 \pm \mathbf{e}_2(x_0, y_0)$.

By Proposition 3.1, $X(x_0, y_0)$ is an L^σ-parabolic point if and only if

$$\text{L-corank}^\sigma(X, (x_0, y_0)) \geq 1 .$$

Moreover $X(x_0, y_0)$ is a lightlike umbilic point if and only if $\text{L-corank}^\sigma(X, (x_0, y_0)) = 2$.

On the other hand, a function germ $f : (\boldsymbol{R}^{n-1}, a) \to \boldsymbol{R}$ has the A_k-type singularity if f is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\text{L-corank}^\sigma(X, (x_0, y_0)) = 1$, the extended tangent lightcone height function $\tilde{h}_{v_0^\sigma}$ has generically an A_k-type singularity at (x_0, y_0). In this case we have $\text{L-ord}^\sigma(x, u_0) = k$. This number k is equal to the order of contact in the classical sense (cf. [5]). This is the reason why we call $\text{L-ord}^\sigma(X, (x_0, y_0))$ the order of contact with the tangent spacelike hyperplane at $X(x_0, y_0)$.

5. Classification of singularities of tangent lightcone map and tangent lightcone pedal surface. In this section we consider generic singularities of tangent lightcone map and tangent lightcone pedal surface. We consider the space of timelike embeddings $\text{Emb}_t(U, \boldsymbol{R}^4_1)$

with Whitney C^∞-topology, where $U \subset \boldsymbol{R}^2$ is an open subset. We have the following theorem.

THEOREM 5.1. *There exists an open dense subset $\mathcal{O}$ of* $\mathrm{Emb_t}\,(U, \boldsymbol{R}^4_1)$ *such that, for any X in $\mathcal{O}$, the following conditions hold*:

(1) *Each lightlike parabolic set $\mathcal{K}_t(1, \sigma 1)^{-1}(0)$ is a regular curve. We call such a curve the lightlike parabolic curve.*

(2) *The tangent lightcone pedal surface TP^σ_M along the lightlike parabolic curve is a cuspidal edge except at some isolated points. At these points TP^σ_M is a swallowtail. Here, $\sigma = \pm$ and a map germ $f : (\boldsymbol{R}^2, a) \to (\boldsymbol{R}^3, b)$ is called a cuspidal edge if it is $\mathcal{A}$-equivalent to the germ (u_1, u_2^2, u_2^3) at the origin (cf. Fig. 1) and a swallowtail if it is $\mathcal{A}$-equivalent to the germ $(3u_1^4 + u_1^2 u_2, 4u_1^3 + 2u_1 u_2, u_2)$ at the origin (cf. Fig. 1).*

For the proof of Theorem 5.1, we consider the function $\mathcal{H} : \boldsymbol{R}^4_1 \times LC^*_+ \to \boldsymbol{R}$ which is given in §4. We claim that $\mathcal{H}_v$ is a submersion for any $v \in LC^*_+$, where $\mathcal{H}_v(x) = \mathcal{H}(x, v)$. For any X in $\mathrm{Emb_t}\,(U, \boldsymbol{R}^4_1)$, we have $\tilde{H} = \mathcal{H} \circ (X \times \mathrm{id}_{LC^*_+})$. We also have the l-jet extension

$$j^l_1 \tilde{H} : U \times LC^*_+ \to J^l(U, \boldsymbol{R})$$

defined by $j^l_1 \tilde{H}((x, y), v) = j^l \tilde{h}_v(x, y)$. We consider the trivialization $J^l(U, \boldsymbol{R}) \equiv U \times \boldsymbol{R} \times J^l(2, 1)$. For any submanifold $Q \subset J^l(2, 1)$, we denote $\tilde{Q} = U \times \{\mathbf{0}\} \times Q$. Then we have the following proposition as a corollary of Wassermann [21, Lemma 6] (see also Montaldi [19] and Looijenga [13]).

PROPOSITION 5.2. *Let Q be a submanifold of $J^l(2, 1)$. Then the set*

$$T_Q = \{X \in \mathrm{Emb_t}\,(U, \boldsymbol{R}^4_1)\ ;\ j^l_1 \tilde{H} \text{ is transversal to } \tilde{Q}\}$$

is a residual subset of $\mathrm{Emb_t}\,(U, \boldsymbol{R}^4_1)$. *If Q is a closed subset, then T_Q is open.*

On the other hand, we denote by $\mathscr{K}^l(z)$ the $\mathscr{K}$-orbit through $z = j^l \tilde{h}_{v_0}(\mathbf{0})$ in $j^l(2, 1)$ (cf. [14]). If $\tilde{h}_{v_0}(q)$ is l-determined relative to $\mathscr{K}$, then $\tilde{H}$ is a $\mathscr{K}$-versal deformation of $\tilde{h}_{v_0}$ if and only if $j^l_1 \tilde{H}$ is transversal to $U \times \{\mathbf{0}\} \times \mathscr{K}^l(z)$ (cf. [14, p. 149]).

We now consider the stratification of the l-jet space $J^l(U, R)$ such that $\mathscr{K}$-versal deformations are transversal to the stratification and the pull-back stratification in the parameter space corresponds to the canonical stratification of the discriminant set. By Theorem 4.4, such

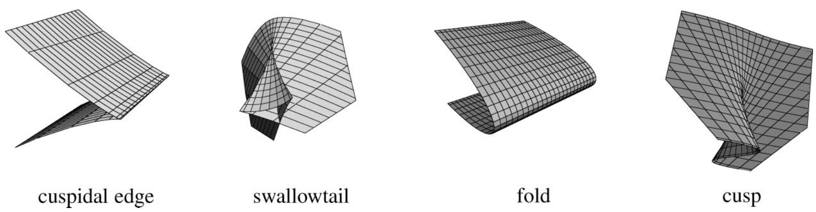

FIGURE 1.

a stratification should be $\mathscr{K}$-invariant, where we have the $\mathscr{K}$-action on $J^l(2,1)$ (cf. [14, 16]). For this reason, we use Mather's canonical stratification here [7, 15], [17, §7 and §8]. Let $\mathscr{A}^l(2,1)$ be the finitely many canonical stratification of $J^l(2,1)\backslash W^l(2,1)$, where

$$W^l(2,1) = \{j^l\tilde{h}_{v_0}(\mathbf{0})\,;\,\dim_{\boldsymbol{R}}\mathscr{E}_k/((T_e\mathscr{K})(\tilde{h}_{v_0}) + \mathfrak{M}_k^l) \geq l\}\,.$$

If l is sufficiently large, then $\operatorname{codim} W^l(2,1) \geq 3$ holds (see [7, Chapter 3]). We now define the stratification $\mathscr{A}_0^l(U,\boldsymbol{R})$ of $J^l(U,\boldsymbol{R})\backslash W^l(U,\boldsymbol{R})$ by

$$U \times (\boldsymbol{R}\backslash\{\mathbf{0}\}) \times (J^l(2,1)\backslash W^l(2,1)),\ U \times \{\mathbf{0}\} \times \mathscr{A}^l(2,1)\,,$$

where

$$W^l(U,\boldsymbol{R}) \equiv U \times \boldsymbol{R} \times W^l(2,1)\,.$$

In [21, Theorem 2], Wan has shown that if $j_1^l\tilde{H}(\mathbf{0})$ is not in $W^l(U,\boldsymbol{R})$ and $j_1^l\tilde{H}$ is transversal to $\mathscr{A}_0^l(U,\boldsymbol{R})$, then $\pi_{\tilde{H}} : (\tilde{H}^{-1}(0),\mathbf{0}) \to (LC_+^*,\mathbf{0})$ is an MT-stable map germ (see also [7, Chapters 3 and 4], [8, Theorem 1.2]). Here, we call a map germ *MT-stable* if it is transversal to the canonical stratification of a jet space which is introduced in [14, 16]. The main assertion of Mather's topological stability theorem is that an MT-stable map germ is a topological stable map germ. Moreover, the critical value set of an MT-stable map germ is canonically stratified. For some notions, see [15]. As an application of Theorem in [6], we have the following proposition.

PROPOSITION 5.3. *Let $\tilde{H}_1, \tilde{H}_2 : (U \times LC_+^*) \to (\boldsymbol{R},\mathbf{0})$ be Morse families such that $\pi_{\tilde{H}_1}$ and $\pi_{\tilde{H}_2}$ are MT-stable map germs. If $Q(\tilde{h}_{1,v_0})$ and $Q(\tilde{h}_{2,v_0})$ are isomorphic as $\boldsymbol{R}$-algebras, then $\pi_{\tilde{H}_1}$ and $\pi_{\tilde{H}_2}$ are topologically equivalent. Moreover, in this case, $\mathcal{D}_{\tilde{H}_1}$ and $\mathcal{D}_{\tilde{H}_2}$ are stratified equivalent.*

By the Propositions 5.2, 5.3 and above discussion, we have the following theorem.

THEOREM 5.4. *There exists an open dense subset $\mathcal{O}$ of* $\mathrm{Emb_t}(U,\boldsymbol{R}_1^4)$ *such that, for any X in $\mathcal{O}$, the germ of the corresponding tangent lightcone pedal surface $TP_M^{\pm}$ at each point is the critical part of an MT-stable map germ. Here, $TP_M^{\pm}(U)$ is the critical part of $\pi|_{\tilde{H}^{-1}(0)} : M \times LC_+^* \to LC_+^*$.*

As an application of Legendrian singularity theory [1, Chapters 20 and 21], we have Theorem 5.1. The assertion of Theorem 5.1 can be interpreted that the Legendrian lift of the tangent lightcone pedal surface $TP_M^{\pm}$ of X in $\mathcal{O}$ is Legendrian stable at each point. Since the Legendrian lift of $TP_M^{\pm}$ is the Legendrian covering of the Lagrangian lift of $TL_M^{\pm}$ [1, p. 323, Proposition], it has been known that the corresponding singularities of $TL_M^{\pm}$ are folds or cusps [1]. Hence, we have the following corollary.

COROLLARY 5.5. *Let $\mathcal{O} \subset$* $\mathrm{Emb_t}(U,\boldsymbol{R}_1^4)$ *be the same open dense subset as in Theorem* 5.1. *For any X in $\mathcal{O}$, the following assertions hold*:

(1) *A lightlike parabolic point (x_0, y_0) in U is a fold of the tangent lightcone map TL_M^{σ} if and only if it is a cuspidal edge of the tangent lightcone pedal surface TP_M^{σ}.*

(2) *A lightlike parabolic point* (x_0, y_0) *in* U *is a cusp of the tangent lightcone map* TL_M^σ *if and only if it is a swallowtail of the tangent lightcone pedal surface* TP_M^σ. *Here, a map germ* $f : (\boldsymbol{R}^2, a) \to (\boldsymbol{R}^2, b)$ *is called a fold if it is* $\mathcal{A}$*-equivalent to the germ* (u_1, u_2^2) *at the origin, and a cusp if it is* $\mathcal{A}$*-equivalent to the germ* $(u_1, u_2^3 + u_1u_2)$ *at the origin.*

Following the terminology of Whitney [23], we say that a surface $X : U \to \boldsymbol{R}_1^4$ has *the excellent tangent lightcone pedal surface* TP_M^σ if the Legendrian lift of TP_M^σ is a stable Legendrian immersion at each point. In this case, the tangent lightcone pedal surface TP_M^σ has only cuspidal edge and swallowtail as singularities. Theorem 5.1 asserts that a timelike surface with an excellent tangent lightcone pedal surface is generic in the space of all timelike surfaces in $\boldsymbol{R}_1^4$. We now consider the geometric meanings of cuspidal edge and swallowtail of the tangent lightcone pedal surface. We have the following results analogous to the results of Banchoff et al. [2].

THEOREM 5.6. *Let* $TP_M^\sigma : (U, (x_0, y_0)) \to (\boldsymbol{R}_1^4, p_0)$ *be the excellent tangent lightcone pedal surface of a timelike surface* X *and* $\tilde{h}_{v_0^\sigma} : (U, (x_0, y_0)) \to \boldsymbol{R}$ *be the extended tangent lightcone height function germ at* $v_0^\pm = \mathbf{e}_1 \pm \mathbf{e}_2(x_0, y_0)$, *where* $\sigma = \pm$. *Then we have the following*:

(1) (x_0, y_0) *is a lightlike parabolic point of* X *if and only if* $\text{L-corank}^\sigma(X, (x_0, y_0)) = 1$.

(2) *If* (x_0, y_0) *is a lightlike parabolic point of* X, *then* $\tilde{h}_{v_0^\sigma}$ *has the* A_k*-type singularity for* $k = 2, 3$.

(3) *Suppose that* (x_0, y_0) *is a lightlike parabolic point of* X, *then the following conditions are equivalent*:

(a) TP_M^σ *has a cuspidal edge at* (x_0, y_0).

(b) $\tilde{h}_{v_0^\sigma}$ *has an* A_2*-type singularity.*

(c) $\text{L-ord}^\sigma(X, (x_0, y_0)) = 2$.

(d) *The tangent spacelike hyperplane indicatrix is an ordinary cusp, where a curve* $C \subset \boldsymbol{R}^2$ *is called an ordinary cusp if it is diffeomorphic to the curve given by* $\{(u_1, u_2) \,;\, u_1^2 - u_2^3 = 0\}$.

(e) *For each* $\varepsilon > 0$, *there exist two distinct points* (x_i, y_i) *in* U $(i = 1, 2)$ *such that*

$$\|(x_0, y_0) - (x_i, y_i)\| < \varepsilon$$

for $i = 1, 2$, *both of these two points are not lightlike parabolic points, and the tangent spacelike hyperplanes to* $M = X(U)$ *at these points are parallel.*

(4) *Suppose that* (x_0, y_0) *is a lightlike parabolic point of* X, *then the following conditions are equivalent*:

(a) TP_M^σ *has a swallowtail at* (x_0, y_0).

(b) $\tilde{h}_{v_0^\sigma}$ *has an* A_3*-type singularity.*

(c) $\text{L-ord}^\sigma(X, (x_0, y_0)) = 3$.

(d) *The tangent spacelike hyperplane indicatrix is a point or a tachnodal, where a curve* $C \subset \boldsymbol{R}^2$ *is called a tachnodal if it is diffeomorphic to the curve given by* $\{(u_1, u_2) \,;\, u_1^2 - u_2^4 = 0\}$.

(e) *For each* $\varepsilon > 0$, *there exist three distinct points* (x_i, y_i) $(i = 1, 2, 3)$ *in* U *such that*

$$\|(x_0, y_0) - (x_i, y_i)\| < \varepsilon$$

for $i = 1, 2, 3$, *and the tangent spacelike hyperplanes to* $M = X(U)$ *at* (x_i, y_i) *are parallel.*

(f) *For each* $\varepsilon > 0$, *there exist two distinct points* (x_i, y_i) $(i = 1, 2)$ *in* U *such that*

$$\|(x_0, y_0) - (x_i, y_i)\| < \varepsilon$$

for $i = 1, 2$, *and the tangent spacelike hyperplanes to* $M = X(U)$ *at* (x_i, y_i) *are equal.*

PROOF. We have shown that (x_0, y_0) is a lightlike parabolic point if and only if

$$\text{L-corank}^{\sigma}(X, (x_0, y_0)) \geq 1\,.$$

We also have $\text{L-corank}^{\sigma}(X, (x_0, y_0)) \leq 2$. Since the extended tangent lightcone height function germ $\tilde{H} : (U \times LC^*_+, ((x_0, y_0), v_0)) \to \boldsymbol{R}$ can be considered as a generating family of the Legendrian lift of TP^{σ}_M, $\tilde{h}_{v_0^{\sigma}}$ has only A_k-type singularities for $k = 1, 2, 3$. This means that the corank of the Hessian matrix of $\tilde{h}_{v_0^{\sigma}}$ at a lightlike parabolic point is 1. The assertion (2) also follows. By the same reason, the conditions (a), (b) and (c) of (3); (resp. (a), (b) and (c) of (4)) are equivalent. If the extended tangent lightcone height function germ $\tilde{h}_{v_0^{\sigma}}$ has an A_2-type singularity, then it is $\mathcal{K}$-equivalent to the germ $\pm u_1^2 + u_2^3$. Since the $\mathcal{K}$-equivalence preserve the diffeomorphism type of zero level sets, the tangent lightlike hyperplane indicatrix is diffeomorphic to the curve given by $\pm u_1^2 + u_2^3 = 0$. This is an ordinary cusp. The normal form for the A_3-type singularity is given by $\pm u_1^2 + u_2^4$, so that the tangent spacelike hyperplane indicatrix is diffeomorphic to the curve $\pm u_1^2 + u_2^4 = 0$. This means that the condition (3), (d) (resp. (4), (d)) is also equivalent to the other conditions.

Suppose that (x_0, y_0) is a lightlike parabolic point, then the tangent lightcone map has only folds or cusps. If the point (x_0, y_0) is a fold point, there is a neighborhood of (x_0, y_0) on which the tangent lightcone map is 2 to 1 except at the lightlike parabolic curve (i.e, fold curve). By Lemma 4.2, the condition (3), (e) is satisfied. If the point (x_0, y_0) is a cusp, the critical value set is an ordinary cusp. By the normal form, we can understand that the tangent lightcone map is 3 to 1 inside the region of the critical value. Moreover, the point (x_0, y_0) is in the closure of the region. This means that the condition (4), (e) holds. We can also observe that, near the cusp, there are 2 to 1 points which approach to (x_0, y_0). However, one of those points are always lightlike parabolic points. Since other singularities do not appear in this case, the condition (3), (e) (resp. (4), (e)) characterizes a fold (resp. a cusp).

If we consider the tangent lightcone pedal surface instead of the tangent lightcone map, the only singularities are cuspidal edge and swallowtail. For the swallowtail point (x_0, y_0), there is a self intersection curve approaching to (x_0, y_0). On this curve, there are two distinct points (x_i, y_i) $(i = 1, 2)$ such that $TP^{\sigma}_M(x_1, y_1) = TP^{\sigma}_M(x_2, y_2)$. By Lemma 4.2, this means that tangent spacelike hyperplane to $M = X(U)$ at (x_i, y_i) are equal. Since there are no other singularities in this case, the condition (4), (f) characterizes a swallowtail point of TP^{σ}_M. This completes the proof. □

We can study more detailed properties of timelike surfaces in Minkowski 4-space. These will be discussed elsewhere.

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SCHOOL OF MATHEMATICS AND STATISTICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN 130024
P. R. CHINA

E-mail addresses: kongll111@nenu.edu.cn
peidh340@nenu.edu.cn