SINGULARITIES OF THE CHERN-RICCI FLOW

QUANG-TUAN DANG

ABSTRACT. We study the nature of finite-time singularities for the Chern-Ricci flow, partially answering a question of Tosatti-Weinkove [53]. We show that a solution of degenerate parabolic complex Monge-Ampère equations starting from arbitrarily positive (1,1)-currents are smooth outside some analytic subset, generalizing works by Di Nezza-Lu [16]. We extend Guedj-Lu's recent approach to establish uniform a priori estimates for degenerate complex Monge-Ampère equations on compact Hermitian manifolds. We apply it to studying the Chern-Ricci flows on complex log terminal varieties starting from an arbitrary current.

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1. INTRODUCTION

Finding canonical metrics on complex varieties has been a central problem in complex geometry over the last few decades. Since Yau's solution to Calabi's conjecture, there have been a lot of developments in this direction. Cao [6] introduced a parabolic approach to provide an alternative proof of the existence of Kähler-Einstein metrics on manifolds with numerically trivial or ample canonical line bundle by the Kähler-Ricci flow. This flow is only Hamilton's Ricci flow evolving Kähler metrics. Motivated by the problem of the classification of complex varieties, Song-Tian [42, 43] have proposed an *Analytic Minimal Model Program* to classify algebraic varieties with mild singularities, using the Kähler-Ricci flow. It requires to a theory of weak solutions for degenerate parabolic complex Monge–Ampère equations starting from a rough initial data. Since then, there have been various achieved results in this direction. Song-Tian initiated the study of the Kähler-Ricci flow starting from an initial current with continuous potentials. While, Guedj-Zeriahi [30] (also [54]) showed that the Kähler-Ricci could be continued from an initial current with zero Lelong number. To the author's knowledge, the best results were, at least so far, obtained by DiNezza-Lu [16], where

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they succeeded in running the Kähler-Ricci flow from an initial current with positive Lelong number. There have been several related works in such singular settings, from a pluripotential theoretical point of view, and we refer to the recent works [27, 10] and the references therein.

Beyond the Kähler setting, there more recently has been interest in the study of geometric flows, in the context of non-Kähler manifolds. Unlike the Kähler case, Hamilton's Ricci flow will not, in general, preserve special Hermitian condition. It is natural to look for another geometric flow of Hermitian metrics, which somehow specializes in the Ricci flow in the Kähler context. Many parabolic flows on complex manifolds which do preserve the Hermitian property have been proposed by Streets-Tian [45, 44] and Liu-Yang [33]. The Anomaly flow of $(n-1, n-1)$-forms has been extensively studied by Phong-Picard-Zhang [36, 37].

This paper is devoted to the Chern-Ricci flow which is an evolution equation of Hermitian metrics on a complex manifold by their Chern-Ricci form, first introduced by Gill [21] in the setting of manifolds with vanishing first Bott-Chern class. Let (X, ω_0) be a compact n-dimensional Hermitian manifold. The *Chern-Ricci flow* $\omega = \omega(t)$ starting at ω_0 is an evolution equation of Hermitian metrics

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega), \quad \omega|_{t=0} = \omega_0, \tag{1.1}$$

where $\mathrm{Ric}(\omega)$ is the *Chern-Ricci form* of ω associated to the Hermitian metric $g = (g_{i\bar{j}})$, which in local coordinates is given by

$$\mathrm{Ric}(\omega) = -dd^c \log\det(g).$$

Here $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)/2$ are both real operators, so that $dd^c = i\partial\bar\partial$. In the Kähler setting, $\mathrm{Ric}(\omega) = iR_{j\bar{k}} dz_j \wedge d\bar{z}_k$, where $R_{j\bar{k}}$ is the usual Ricci curvature of ω. Thus if ω_0 is Kähler i.e., $d\omega_0 = 0$, (1.1) coincides with the Kähler-Ricci flow. For complex manifolds with $c_1^{\mathrm{BC}}(X) = 0$, Gill [21] proved the long time existence of the flow and smooth convergence of the flow to the unique Chern-Ricci-flat metric in the $\partial\bar\partial$-class of the initial metric. For general complex manifolds, Tosatti and Weinkove [52, Theorem 1.3] characterize the maximal existence time $T_{\max}$ of the flow as

$$T_{\max} := \sup\{t > 0 : \exists\, \psi \in \mathcal{C}^\infty(X) \text{ with } \omega_0 - t\mathrm{Ric}(\omega_0) + dd^c\psi > 0\}.$$

Finite time singularities. Suppose that the flows (1.1) exists on maximal interval $[0, T_{\max})$ with $T_{\max} < \infty$, so the flow develops a singularity at finite time. Tosatti-Weinkove [53, Question 6.1] ask the following question

Question 1.1. *Do singularities of the Chern-Ricci flow develop precisely along closed analytic subvarieties of X?*

In the Kähler setting, this question was posed by Feldman-Ilmanen-Knopf [19] and affirmatively answered by Collins-Tosatti [8]. When X is a compact complex surface and ω_0 is Gauduchon, i.e., $dd^c\omega_0 = 0$, the Chern-Ricci flow preserves Gauduchon (pluriclosed) condition, in particular, the limiting form $\alpha_{T_{\max}} = \omega_0 - T_{\max}\mathrm{Ric}(\omega_0)$ is Gauduchon. The answer is thus affirmative in this case, due to Gill-Smith [22] (also [51, 53]) where they proved singularities of the Chern-Ricci flow form a finite union of disjoint (-1)-curves. We partially answer this question when the limiting form $\alpha_{T_{\max}}$ is *uniformly non-collapsing*:

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^2 \geq c_0 > 0, \ \forall\, \psi \in \mathcal{C}^\infty(X). \tag{1.2}$$

We mention that when ω_0 is a Gauduchon metric on compact complex surface X the latter condition is equivalent to $\int_X \alpha_{T_{\max}}^2 > 0$. We say in such a case that the Chern-Ricci flow is *volume non-collapsing* at time $T_{\max}$, otherwise we say that the flow is *volume collapsing*; cf. [51]). As also mentioned in [53] the question is trivial when the flow is volume collapsing. We generalize the result to higher dimensional manifold X which admits a Hermitian metric ω_X such that $v_+(\omega_X) < +\infty$ (cf. Definition 2.4). The latter automatically holds in the case of compact complex surfaces.

Theorem A. *Let (X, ω_0) be a compact complex n-dimensional manifold with $v_+(\omega_0) < +\infty$. Assume that the Chern-Ricci flow (1.1) starting at ω_0 exists on the maximal interval $[0, T_{\max})$ with $T_{\max} < \infty$, and that the limiting form $\alpha_{T_{\max}}$ is uniformly non-collapsing, i.e.,*

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0, \ \forall\, \psi \in \mathcal{C}^\infty(X). \tag{1.3}$$

Then as $t \to T^-$ the metric ω_t converge to $\omega_{T_{\max}}$ in $\mathcal{C}^\infty(\Omega)$ for some Zariski open set $\Omega \subset X$.

The strategy of the proof is as follows. From the uniformly non-collapsing condition of $\alpha_{T_{\max}}$, we show that there exists a quasi-plurisubharmonic function ρ with analytic singularities such that $\alpha_{T_{\max}} + dd^c\rho$ dominates a hermitian metric. Such a form is called *big* (cf. Definition 2.6). Then Ω is the set in which ρ is smooth. In particular, it is Zariski open. We next establish several uniformly local estimates of ω near the maximal time $T_{\max}$, adapting the same as that of [8, 21]. The convergence immediately follows.

Degenerate parabolic complex Monge-Ampère equations. In the previous paragraph, we studied the behavior of the Chern-Ricci flow at finite singularity time. It is natural to ask whether the flow can pass through this singularity. To do this, we must define weak solutions of the Chern-Ricci flow starting from degenerate initial currents on a compact complex variety with mild singularities. Several geometric contexts are encountered in the minimal model program, which require us to treat the case of complex variety with Kawamata log terminal (klt) singularities. From an analytic point of view, the latter naturally leads one to deal with densities that are allowed to blow up while belonging to L^p for some exponent $p > 1$ whose size depends on the algebraic nature of the singularities.

On a compact complex n-manifold (X, ω_X), we consider the following degenerate parabolic complex Monge– Ampère equation

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right], \tag{1.4}$$

for $t \in (0, T_{\max})$, where $T_{\max} < \infty$ and

- $\theta_t = \theta + t\chi$ is an affine family of smooth semi-positive forms and there is a quasi-plurisubharmonic function ρ with analytic singularities such that
$$\theta + dd^c\rho \geq \delta\omega_X \text{ for some } \delta > 0;$$
- μ is a positive measure on X of the form
$$\mu = e^{\psi^+ - \psi^-}$$
with $\psi^\pm$ quasi-plurisubharmonic functions, being smooth on a given Zariski open subset $U \subset \{\rho > -\infty\}$ and $e^{-\psi^-} \in L^p$ for some $p > 1$;
- $\varphi : [0, T_{\max}] \times X \to \mathbb{R}$ is the unknown function, with $\varphi_t := \varphi(t, \cdot)$.

We define the weak solution of the Chern-Ricci flow:

Definition 1.2. A family of functions $\varphi_t : X \to \mathbb{R}$ for $t \in (0, T_{\max})$ is said to be a weak solution of the equation (1.4) starting with φ_0 if the following hold.

(1) for each t, φ_t is θ_t-plurisubharmonic on X;
(2) $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$;
(3) for each $\varepsilon > 0$ there exists a Zariski open set $\Omega_\varepsilon \subset X$ such that the function $(t,x) \mapsto \varphi(t,x) \in \mathcal{C}^\infty([\varepsilon, T_{\max} - \varepsilon] \times \Omega_\varepsilon)$. Furthermore, the equation (1.4) satisfies in the classical sense on $[\varepsilon, T_{\max}) \times \Omega_\varepsilon$.

Our first theorem establishes the existence for the complex Monge-Ampère flow starting with an initial function φ_0 with small Lelong numbers.

Theorem B. *Let φ_0 be an θ-plurisubharmonic function satisfying $p^*/2c(\varphi_0) < T_{\max}$ where p^* is the conjugate exponent of p. Then there exists a weak solution φ of the flow (1.4) starting at φ_0 for $t \in (0, T_{\max})$.*

Here $c(\varphi_0)$ denotes the integrability index of φ_0 which is the superemum of positive constant $c > 0$ such that $e^{-2c\varphi_0}$ is locally integrable. We note that $c(\varphi_0) = +\infty$ if and only if φ_0 have zero Lelong numbers at all points, as follows from Skoda's integrability theorem.

Let us briefly describe the strategy of the proof of Theorem B. We first approximate φ_0 by a decreasing sequence of smooth $(\theta + 2^{-j}\omega_X)$-plurisubharmonic functions $\varphi_{0,j}$ thanks to Demailly's regularization result. Similarly, $\psi^\pm$ are approximated by smooth quasi-plurisubharmonic functions. We consider the corresponding solution $\varphi_{t,j}$ to the equation (1.4) with $\theta_{t,j} = \theta_t + 2^{-j}\omega_X$. We aim to establish several a priori estimates allowing us to pass to the limit $j \to +\infty$. Precisely, we are going to prove that for any $\varepsilon > 0$, there is a Zariski open set $\Omega_\varepsilon \subset X$ such that for each $0 < T < T_{\max}$ fixed and $K \subset \Omega_\varepsilon$,

- $\|\varphi_{t,j}\|_{\mathcal{C}^0([\varepsilon,T]\times K)} \leq C_{\varepsilon,T,K}$;
- $\partial_t \varphi_{t,j}$ is uniformly bounded on $[\varepsilon, T] \times K$;
- $\Delta_{\omega_X} \varphi_{t,j}$ is uniformly bounded on $[\varepsilon, T] \times K$.

We then apply the parabolic Evans-Krylov theory and Schauder estimates to obtain more higher locally uniformly estimates for all derivatives of $\varphi_{t,j}$ (we can refer to [21] for a recent account in the Chern-Ricci flow context). We therefore can pass to the limit to show that

$$\varphi_{t,j} \to \varphi_t \in \mathcal{C}^\infty([\varepsilon, T] \times \Omega_\varepsilon)$$

as $j \to +\infty$. We automatically have the weak convergence $\varphi_t \to \varphi_0$ as $t \to 0^+$. More stronger convergence are discussed in Section 4.4 when φ_0 are less singular.

We also emphasize here that the mild assumption $p^*/2c(\varphi_0) < T_{\max}$ guarantees that the approximating flow is well-defined (not identically $-\infty$) and is crucial for the smoothing properties of the flow. As mentioned by DiNezza-Lu [16] in the Kähler context, without this assumption, the Kähler-Ricci flow can still run, but there is probably no regularization effect at all due to the presence of positive Lelong numbers. Also, as in this case, they mentioned that the main difficulty is establishing a priori $\mathcal{C}^0$-estimate. Their proof relies on Kolodziej's method by using their generalized Monge-Ampère capacity. The approach we use is recently developed by Guedj-Lu [24, 25], whose advantage is that it still can be applied in the case of degenerate (1,1) forms in non-Kähler context.

We finally apply the previous analysis to treat the case of mildly singular varieties. This allows us to define a good notion of the weak Chern-Ricci flow on complex compact varieties with log terminal singularities. We will discuss it in Section 6 and prove the following.

Theorem C. *Let Y be a compact complex variety with log terminal singularities. Assume that θ_0 is a Hermitian metric such that*

$$T_{\max} := \sup\{t > 0 : \exists\, \psi \in \mathcal{C}^\infty(Y) \text{ such that } \theta_0 - tRic(\theta_0) + dd^c\psi > 0\} > 0.$$

Assume that $S_0 = \theta_0 + dd^c\phi_0$ is a positive (1,1)-current with sufficiently small slopes. Then there exists a family $(\omega_t)_{t\in[0,T_{\max})}$ of positive (1,1) current on Y starting at S_0 such that

(1) *$\omega_t = \theta_0 - tRic(\theta_0) + dd^c\varphi_t$ are positive (1,1) currents;*
(2) *$\omega_t \to S_0$ weakly as $t \to 0^+$;*
(3) *for each $\varepsilon > 0$ there exists a Zariski open set Ω_ε such that on $[\varepsilon, T_{\max}) \times \Omega_\varepsilon$, ω is smooth and*

$$\frac{\partial\omega}{\partial t} = -Ric(\omega).$$

This generalizes previous results of Song-Tian [43], Guedj-Zeriahi [29], Tô [54], DiNezza-Lu [16], Guedj-Lu-Zeriahi [27] and the author [10] to the non-Kähler case, and of [55, 35] and the author [9] to more degenerate initial data.

Organization of the paper. We establish a priori estimates in Section 3, which will be used to prove Theorem B in Section 4. While, Theorem A will be proved in Section 5, studying the behavior of the Chern-Ricci flow at non-collapsing finite time singularities. In Section 6 we apply these tools to prove the existence for the weak Chern-Ricci flow with initial degenerate data on compact complex varieties with log terminal singularities, proving Theorem C.

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2. Preliminaries

2.1. **Recap on pluripotential theory.** Let X be a compact complex manifold of dimension n, equipped with a Hermitian metric ω_X. We fix θ a smooth semi-positive (1,1)-form on X.

2.1.1. *Quasi-plurisubharmonic functions and Lelong numbers.* A function is quasi-plurisubharmonic (quasi-psh for short) if it is locally given as the sum of a smooth and a plurisubharmonic (psh for short) function.

Definition 2.1. A quasi-psh function $\varphi : X \to [-\infty, +\infty)$ is called *θ-plurisubharmonic* (*θ-psh* for short) if it satisfies $\theta_\varphi := \theta + dd^c\varphi \geq 0$ in the weak sense of currents. We let $\mathrm{PSH}(X, \theta)$ denote the set of all θ-psh functions which are not identically $-\infty$.

The set $\mathrm{PSH}(X, \theta)$ isendowed with the $L^1(X)$-topology. By Hartogs' lemma $\varphi \mapsto \sup_X \varphi$ is continuous in this weak topology. Since the set of closed positive currents in a fixed dd^c-class is compact (in the weak topology), it follows that the set of $\varphi \in \mathrm{PSH}(X, \theta)$, with $\sup_X \varphi = 0$ is compact. We refer the reader to [13, 29] for basic properties of θ-psh functions.

Quasi-psh functions are in general singular, and a convenient way to measure their singularities is the Lelong numbers.

Definition 2.2. Let $x_0 \in X$. Fixing a holomorphic chart $x_0 \in V_{x_0} \subset X$, the *Lelong number* $\nu(\varphi, x_0)$ of a quasi-psh function φ at $x_0 \in X$ is defined as follows:

$$\nu(\varphi, x_0) := \sup\{\gamma \geq 0 : \varphi(z) \leq \gamma \log\|z - x_0\| + O(1), \text{ on } V_{x_0}\}.$$

We remark here that this definition does not depend on the choice of local charts. In particular, if $\varphi = \log|f|$ in a neighborhood V_{x_0} of x_0, for some holomorphic function f, then $\nu(\varphi, x_0)$ is equal to the vanishing order $\mathrm{ord}_{x_0}(f) := \sup\{k \in \mathbb{N} : D^\gamma f(x_0) = 0, \forall |\gamma| < k\}$.

In some contexts, it is more convenient to deal with the integrability index instead of the Lelong numbers. The *integrability index* of a quasi-psh function φ at a point $x \in X$ is defined by

$$c(\varphi, x) := \sup\{c > 0 : e^{-2c\varphi} \in L^1(V_x)\}$$

where V_x is some neighborhood around x. As above this definition does not depend on the choice of open neighborhood V_x. We denote by $c(\varphi)$ the infimum of $c(\varphi, x)$ for all $x \in X$. Since X is compact it follows that $c(\varphi) > 0$.

Skoda's integrability theorem states that one can get the following "optimal" relation between the Lelong number of a quasi-psh function φ at a point $x_0 \in X$ and the local integrability index of φ at x_0:

$$\frac{1}{\nu(\varphi, x_0)} \leq c(\varphi, x_0) \leq \frac{n}{\nu(\varphi, x_0)}. \tag{2.1}$$

In particular $c(\varphi) = +\infty$ if and only if $\nu(\varphi, x) = 0$ for all $x \in X$ (cf. [41] for Skoda's theorem or [56] for a uniform version).

2.1.2. *Monge-Ampère measures.* The complex Monge-Ampère measure $(\theta + dd^c u)^n$ is well-defined for any θ-psh function u which is bounded, as follows from Bedford-Taylor theory: if $\beta = dd^c\rho$ is a Kähler form such that $\beta > \theta$ in a local open chart $U \subset X$, the function u is β-psh hence the positive currents $(\beta + dd^c u)^j$ are well-defined for $1 \leq j \leq n$, one thus obtains

$$(\theta + dd^c u)^n := \sum_{j=0}^{n} \binom{n}{j} (\beta + dd^c u)^j \wedge (\theta - \beta)^{n-j}.$$

as a positive Radon measure on X. Indeed, by Demailly's regularization theorem we can approximate u be a decreasing sequence of smooth $(\theta + \varepsilon_j\omega_X)$-psh functions u_j.We obtain that $(\theta + dd^c u)^n$ is the limit of positive measures $(\theta + \varepsilon_j\omega_X + dd^c u_j)^n$, so is positive.

This definition does not depend on the choice of β by the same arguments. We refer to [17] for an adaptation of [2, 3] to the Hermitian context. We recall the following maximum principle:

Lemma 2.3. *Let φ, ψ are bounded θ-psh functions such that $\varphi \leq \psi$. Then*

$$\mathbf{1}_{\{\varphi=\psi\}}(\theta + dd^c\varphi)^n \leq \mathbf{1}_{\{\varphi=\psi\}}(\theta + dd^c\psi)^n.$$

Proof. This is a direct consequence of Bedford-Taylor's maximum principle; see [29, Theorem 3.23]. We refer the reader to [26, Lemma 1.2] for a brief proof. □

2.1.3. *Positivity assumptions.* For our purpose we need to assume slightly stronger positivity property of the form θ in the sense of [25].

Definition 2.4. We consider

$$v_-(\theta) := \inf\left\{\int_X (\theta + dd^c\varphi)^n : \varphi \in \mathrm{PSH}(X, \theta) \cap L^\infty(X)\right\}$$

and

$$v_+(\theta) := \sup\left\{\int_X (\theta + dd^c\varphi)^n : \varphi \in \mathrm{PSH}(X, \theta) \cap L^\infty(X)\right\}$$

We emphasize that when θ is Hermitian, the supremum and infimum in the definition of these quantities can be taken over $\mathrm{PSH}(X,\theta)\cap\mathcal{C}^\infty(X)$ due to Demailly's regularization theorem and Bedford-Taylor's convergence results.

Definition 2.5. We say that θ is *uniformly non-collapsing* if $v_-(\theta)\geq c_0>0$.

This condition is not obvious even when θ is Hermitian. We refer the reader to [1, Sect. 3] for several examples of uniformly non-collapsing Hermitian form.

Recall that a function ρ is said to have *analytic singularities* if if there exists a constant $c>0$ such that locally on X, $\rho=c\log\sum_{j=1}^N|f_j|^2+O(1)$ where the f_j's are holomorphic functions.

Definition 2.6. We say θ is *big* if there exists a θ-psh function with analytic singularities such that $\theta+dd^c\rho\geq\delta\omega_X$ for some $\delta>0$. We let Ω denote the Zariski open set where ρ is smooth.

Such a form appears in some contexts of complex differential geometry. For instance, if V is a compact complex space endowed with a hermitian form ω_Y and $\pi:X\to Y$ is a log resolution of singularities, then the form $\theta:=\pi^*\omega_Y$ is big; see [20, Proposition 3.2]. Moreover, we can find a θ-psh function ρ with analytic singularities such that $\theta+dd^c\rho\geq\delta\omega_X$, and

$$\Omega=\{\rho>-\infty\}=X\setminus\mathrm{Exc}(\pi)=\pi^{-1}(Y_{\mathrm{reg}})\simeq Y_{\mathrm{reg}}.$$

2.1.4. *Envelopes.*

Definition 2.7. Given a measurable function $h:X\to\mathbb{R}$, we define the *θ-psh envelope* of h by

$$P_\theta(h):=(\sup\{u\in\mathrm{PSH}(X,\theta):u\leq h\text{ on }X\})^*$$

where the star means that we take the upper semi-continuous regularization.

We have the following result which has been established in [26, Theorem 2.3].

Theorem 2.8. *If h is bounded from below, quasi l.s.c, and $P_\theta(h)<+\infty$ then*

(1) $P_\theta(h)$ is a bounded θ-psh function;
(2) $P_\theta(h)\leq h$ in $X\setminus P$, for some pluripolar set P;
(3) $(\theta+dd^cP_\theta(h))^n$ is concentrated on the contact set $\{P_\theta(h)=h\}$.

The following $\mathcal{C}^0$-estimate is crucial in the sequel.

Lemma 2.9. *Let θ be a smooth real semi-positive and big (1,1)-form. Assume $\varphi\in\mathrm{PSH}(X,\theta)\cap L^\infty(X)$ satisfies*

$$(\theta+dd^c\varphi)^n\leq e^{A\varphi-g}fdV_X,$$

where $A>0$ and f,g are measurable functions such that $e^{A\psi-g}f\in L^q(X)$ with $q>1$, for some $\psi\in\mathrm{PSH}(X,\delta\theta)$, with $\delta\in(0,1)$. Then we have the following estimate

$$\varphi\geq\psi-C$$

where C is a positive constant only depending on n, A, δ, θ, q and a upper bound for $\int_X e^{q(A\psi-g)}f^qdV_X$.

Proof. We apply the approach which has recently developed by Guedj-Lu [24, 25]. Set $u:=P_{(1-\delta)\theta}(\varphi-\psi)$. Since φ is bounded one has $u=P_{(1-\delta)\theta}(\varphi-\max(\psi,-t))$ for $t>0$ big enough, we can thus assume that ψ is also bounded. Since $\varphi-\psi$ is bounded and quasicontinuous. It follows from Theorem 2.8 that $((1-\delta)\theta+dd^cu)^n$ is supported on the contact set $D:=\{u+\psi=\varphi\}$. We observe that $u+\psi$ and φ are both θ-psh functions satisfying $u+\psi\leq\varphi$, it follows from Lemma 2.3 that

$$\mathbf{1}_D(\theta+dd^c(u+\psi))^n\leq\mathbf{1}_D(\theta+dd^c\varphi)^n.$$

From these, we have

$$\begin{aligned}((1-\delta)\theta + dd^c u)^n &= \mathbf{1}_D((1-\delta)\omega + dd^c u)^n \\ &\le \mathbf{1}_D(\theta + dd^c(u+\psi))^n \\ &\le \mathbf{1}_D(\theta + dd^c\varphi)^n \\ &\le \mathbf{1}_D e^{A\varphi - g} f dV_X \\ &\le \mathbf{1}_D e^{Au} e^{A\psi - g} f dV_X,\end{aligned}$$

with $e^{A\psi-g}f \in L^q(X)$. We argue the same as in the proof of [25, Theorem 3.4 (1)] to ensure that there exists a constant $C > 0$ only depending on n, q, A, θ, δ, and $\|e^{A\psi-g}f\|_{L^q}$, such that $u \ge -C$. This completes the proof. □

2.2. **Equisingular approximation.** Fix φ a θ-psh function on X. We aim at approximating φ by a decreasing sequence of quasi-psh functions which are less singular than φ and such that their singularities are somehow comparable to those of φ. This leads us to make use of Demailly's equisingular approximation theorem. For each $c > 0$, we define the *Lelong super-level sets*

$$E_c(\varphi) := \{x \in X : \nu(\varphi, x) \ge c\}.$$

We also use the notation $E_c(T)$ for a closed positive $(1,1)$-current T. A well-known result of Siu [40] asserts that the Lelong super-level sets $E_c(\varphi)$ are analytic subsets of X. We refer the reader to [11, Remark 3.2] for an alternative proof.

The following result of Demailly on the equisingular approximation of a quasi-psh function by quasi-psh functions with analytic singularities is crucial.

Theorem 2.10 (Demailly's equisingular approximation)**.** *Let φ be a θ-psh function on X. There exists a decreasing sequence of quasi-psh functions (φ_m) such that*

(1) (φ_m) converges pointwise and in $L^1(X)$ to φ as $m \to +\infty$,
(2) φ_m has the same singularities as $1/2m$ times a logarithm of a sum of squares of holomorphic functions,
(3) $dd^c\varphi_m \ge -\theta - \varepsilon_m\omega_X$, where $\varepsilon_m > 0$ decreases to 0 as $m \to +\infty$,
(4) $\int_X e^{2m(\varphi_m - \varphi)} dV < +\infty$,
(5) φ_m is smooth outside the analytic subset $E_{1/m}(\varphi)$.

Proof. We briefly sketch the idea for the convenience of the reader and which we believe is known to experts. We follow the proof of [11] by applying with the current $T = dd^c\varphi$ and the smooth real (1,1) form $\gamma = -\theta$. We also borrow notation from there.

For $\delta > 0$ small, let us cover X by $N = N(\delta)$ geodesic balls $B_{2r}(a_j)$ with respect to ω_X such that $X = \cup_j B_r(a_j)$ and in terms of coordinates $z^j = (z_1^j, \dots, z_n^j)$,

$$\sum_{l=1}^n \lambda_l^j i dz_l^j \wedge d\bar{z}_l^j \le \gamma|_{B_{2r}(a_j)} \le \sum_{l=1}^n (\lambda_l^j + \delta) i dz_l^j \wedge d\bar{z}_l^j$$

where we have diagonalized $\gamma(a_j)$ at the center a_j. Here N and r are taken to be uniform. Set $\varphi^j := \varphi|_{B_{2r}(a_j)} - \sum_{l=1}^n \lambda_l^j |z_j^l|^2$. On each $B_{2r}(a_j)$, we define

$$\varphi_{j,\delta,m} := \frac{1}{2m} \log \sum_{k\in\mathbb{N}} |f_{j,m,k}|^2,$$

where $(f_{j,m,k})_{k\in\mathbb{N}}$ is an orthogonal basis of the Hilbert space $\mathcal{H}_{B_{2r}(a_j)}(m\varphi^j)$ of holomorphic functions on $B_{2r}(a_j)$ with finite L^2 norm $\|u\| = \int_{B_{2r}(a_j)} |u|^2 e^{-2m\varphi^j} dV(z^j)$. Note that since $dd^c\varphi \ge \gamma$ it follows that $\varphi - \sum_{l=1}^n \lambda_l^j |z_j^l|^2$ is psh on $B_{2r}(a_j)$. The

Bergman kernel process applied on each ball $B_{2r}(a_j)$ have provided approximations $\varphi_{j,\delta,m}$ of $\varphi^j = \varphi|_{B_{2r}(a_j)} - \sum_{l=1}^n \lambda_l^j |z_j^l|^2$, it thus remains to glue these functions into a function $\varphi_{\delta,m}$ globally defined on X. For this, we set

$$\varphi_{\delta,m}(x) = \frac{1}{2m} \log \left(\sum_j \theta_j(x)^2 \exp \left(2m \left(\varphi_{j,\delta,m} + \sum_l (\lambda_l^j - \delta)|z_l^j|^2 \right) \right) \right)$$

where $(\theta_j)_{1\le j\le N}$ is the partition of unity subordinate to the $B_r(a_j)$'s. Now we take $\delta = \delta_m \searrow 0$ slowly and $\varphi_m = \varphi_{\delta_m,m}$ the same computations as in [11, p. 16] ensure that

$$dd^c\varphi_m \ge \gamma - \varepsilon(\delta_m)\omega_X$$

for $m \ge m_0$ sufficiently large and $\varepsilon_m = \varepsilon(\delta_m) \searrow 0$ as $m \to +\infty$. By the construction the properties (1), (2), (3) and (5) are satisfied.

The property (4) is crucial for later use whose proof should be provided. The argument originated from [15, Theorem 2.3, Step 2] using local uniform convergence and the strong Noetherian property. By the properties of functions φ_m it suffices to show that on each ball $B_j = B_r(a_j)$,

$$\int_{B_j} e^{2m\varphi_m - 2m\varphi} dV = \int_{B_j} \left(\sum_{k\in\mathbb{N}} |f_{j,m,k}|^2 \right) e^{-2m\varphi} dV(z^j) < +\infty.$$

We let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \mathcal{F}_k \subset \dots \subset \mathcal{O}(B_{2r}(a_j) \times B_{2r}(a_j))$ denote the sequence of ideal coherent sheaves generated by the holomorphic functions $\left(f_{j,m,l}(z)\overline{f_{j,m,l}(\bar{w})}\right)_{l\le k}$ on $B_{2r}(a_j) \times B_{2r}(a_j)$. By the strong Noetherian property (see e.g. [13, C. II, 3.22]) the sequence $(\mathcal{F}_k)$ is stationary on a compact subset $B_j \times B_j \subset\subset B_{2r}(a_j) \times B_{2r}(a_j)$ at a index k_0 large enough. Using the Cauchy-Schwarz inequality we have that the sum of the series $U(z,w) = \sum_{k\in\mathbb{N}} f_{j,m,k}(z)\overline{f_{j,m,k}(\bar{w})}$ is bounded from above by

$$\left(\sum_{k\in\mathbb{N}} |f_{j,m,k}(z)|^2 \sum_{k\in\mathbb{N}} |f_{j,m,k}(\bar{w})|^2 \right)^{\frac{1}{2}}$$

hence uniformly convergent on every compact subset of $B_{2r}(a_j) \times B_{2r}(a_j)$. Since the space of sections of a coherent ideal sheaf is closed under the topology of uniform convergence on compact subsets, the Noetherian property grantees that $U(z,w) \in \mathcal{F}_{k_0}(B_j \times B_j)$. Hence, by restricting to the conjugate diagonal $w = \bar{z}$, we obtain

$$\sum_{k\in\mathbb{N}} |f_{j,m,k}(z)|^2 \le C_0 \left(\sum_{k\le k_0} |f_{j,m,k}(z)|^2 \right)$$

on B_j. Since all terms $f_{j,m,k}$ have L^2-norm equal to 1 with respect to the weight $e^{-2m\varphi}$ this completes the proof. □

Using this one obtains the following lemma which is slightly more general to the one in [16].

Lemma 2.11. *Let θ be a big form. Assume $\varphi \in \mathrm{PSH}(X,\theta)$. Then for each $\varepsilon > 0$ there exist $c(\varepsilon) > 0$ and $\psi_\varepsilon \in \mathrm{PSH}(X,\theta) \cap \mathcal{C}^\infty \left(X \setminus (\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi)) \right)$ such that*

$$\int_X e^{\frac{2}{\varepsilon}(\psi_\varepsilon - \varphi)} dV_X < +\infty. \tag{2.2}$$

Proof. The proof is quite close to that of [16, Lemma 2.7]. Recall that the bigness of θ implies that there exists ρ an θ-psh function with singularities and $\sup_X \rho = 0$ such that

$$\theta + dd^c\rho \geq 3\delta_0\omega_X \quad \text{for a fixed constant } \delta_0 > 0.$$

Let $c(\varphi)$ be the integrability index of φ. We can assume that $c(\varphi) < +\infty$, otherwise we are done. By Theorem 2.10, we can find (φ_m) a Demailly's equisinglar approximant of φ. We have that φ_m is smooth in the complement of the analytic subset $E_{1/m}(\varphi)$ and

$$\theta + dd^c\varphi_m \geq -\varepsilon_m\delta_0\omega_X$$

for $\varepsilon_m > 0$ decreasing to zero as m goes to $+\infty$. We notice here that the errors $\varepsilon_m > 0$ appear in the gluing process; see [11] for more details. We choose $m = m(\varepsilon)$ to be the smallest positive integer such that

$$m > \frac{2}{\varepsilon(1+\varepsilon_m)}, \quad \frac{2\varepsilon_m}{\varepsilon(1+\varepsilon_m)} < c(\varphi).$$

We now set

$$\psi_\varepsilon := \frac{\varphi_m}{1+\varepsilon_m} + \frac{\varepsilon_m}{1+\varepsilon_m}\rho. \tag{2.3}$$

Thus, we have

$$\theta + dd^c\psi_\varepsilon \geq \frac{\varepsilon_m}{1+\varepsilon_m}2\delta_0\omega_X := 2\kappa\omega_X.$$

Holder's inequality ensures that (2.2) holds, noticing that $\rho \leq 0$. We easily see that ψ_ε is smooth in the complement of $\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi)$ with $c(\varepsilon) = m(\varepsilon)^{-1}$. □

3. A priori estimates

3.1. **Notation.** We use some notation as in [16, Sect. 3.1]. Until further notice, X denotes a compact complex manifold of dimension n endowed with a reference Hermitian form ω_X. Following the strategy in the introductory part, we assume in this section that $\theta_t = \theta + t\chi$ with $t \in [0, T_{\max})$ are Hermitian forms and φ_0 is a smooth strictly θ-psh function. We denote by $\mu = f dV_X$ a positive measure with density $\|f\|_{L^p} \leq C$ uniformly, for some $p > 1$. For more higher estimates, we assume moreover that

$$f = e^{\psi^+ - \psi^-}$$

where $\psi^\pm$ are smooth quasi-psh functions. Recall that ρ is a θ-psh function with analytic singularities such that $\theta + dd^c\rho$ dominates a Hermitian form. We may assume that $\sup_X \rho = 0$.

We consider φ_t a smooth solution of the following parabolic complex Monge-Ampère equation

$$\frac{\partial\varphi_t}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right], \ \varphi|_{t=0} = \varphi_0 \tag{3.1}$$

on $[0, T_{\max})$; see [52]. We should keep in mind that φ_t plays a role of its approximants $\varphi_{t,j}$ in establishing a priori estimates. For brevity, we supress the index j.

We fix T and S such that

$$\frac{p^*}{2c(\varphi_0)} < T < S < T_{\max}.$$

where p^* is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$. Since we are interested in the behavior of the flow (3.1) near zero, we can assume that

$$\theta_S \geq (1-a)\theta, \quad \text{for } a \in [0, 1/2).$$

It is truly natural in some several geometric context, for instance, θ_t are the pull back of a Hermitian forms. Thus for each $t \in [0, S]$ we have

$$\theta_t = \frac{t\theta_S}{S} + \frac{S-t}{S}\theta \geq \left(1 - \frac{at}{S}\right)\theta.$$

During the proof, we use the notation $\omega_t := \theta_t + dd^c\varphi_t$ for the smooth path of Hermitian forms and denote $\Delta_t = \mathrm{tr}_{\omega_t} dd^c$ the corresponding time-dependent Laplacian operator on functions.

We fix $\varepsilon_0 > 0$ small enough, and let denote by $\psi_0 := \psi_{\varepsilon_0}$ established in Lemma 2.11. By construction, ψ_0 is smooth outside an analytic subset $\{\rho = -\infty\} \cup E_{c(\varepsilon)}(\varphi_0)$ and satisfies

$$\theta + dd^c\psi_0 \geq 2\kappa\omega_X. \tag{3.2}$$

We let E_1, E_1 denotes the following quantities

$$E_1 := \int_X e^{\frac{2(\psi_0 - \varphi_0)}{\varepsilon_0}} dV_X < +\infty,$$

$$E_2 := \int_X e^{-\frac{p^*\psi_0}{T}} dV_X < +\infty.$$

Observe that E_1 is finite thanks to Lemma 2.11, while E_2 is finite since $p^*/(2c(\varphi_0)) < T$ and that ψ_0 is less singular than φ_0. One should emphasize that φ_0 in this a priori estimate section plays a role of its approximating sequence $\varphi_{0,j}$ (which are smooth strictly θ-psh functions decreasing to φ_0). The corresponding sequence E_1^j are uniformly bounded from above in j, hence we can pass to the limit.

In what follows we use C for a positive constant whose value may change from line to line but be uniformly controlled.

3.2. **Uniform estimate.** We first look for a upper a priori bound for φ_t. We recall that

$$\frac{1}{2}\theta \leq \theta_t \leq A\omega_X, \ \forall\, t \in [0, T],$$

for $A > 0$ sufficiently large. It follows from [25, Theorem 3.4] (see also [31]) that there exists a constant c and a bounded $A\omega_X$-psh function ϕ normalized by $\inf_X \phi = 0$ such that

$$(A\omega_X + dd^c\phi)^n = e^c f dV_X.$$

Proposition 3.1. *For any $(t, x) \in [0, T] \times X$, there exists a uniform constant $C > 0$ such that*

$$\varphi_t(x) \leq C.$$

Proof. For any $(t, x) \in [0, T] \times X$, we set $v(t, x) = \phi(x) + ct + \sup_X \varphi_0$. Then we can check that

$$\frac{\partial v}{\partial t} = \log\left[\frac{(A\omega_X + dd^c v_t)^n}{\mu}\right], \text{ while } \frac{\partial \varphi}{\partial t} \leq \log\left[\frac{(A\omega_X + dd^c\varphi_t)^n}{\mu}\right],$$

and $v_0 \geq \varphi_0$. Hence, it follows from the classical maximum principle that $v(t, x) \geq \varphi(t, x)$ for $(t, x) \in [0, T] \times X$. Therefore, one gets a upper bound for $\varphi(t, x)$ by

$$\sup_X |\phi| + \max(c, 0)T + \sup_X \varphi_0.$$

$\square$

We fix two positive constants α, β such that

$$\frac{p^*}{2c(T_0)} < \frac{1}{\alpha} < \frac{1}{\alpha - \beta} < T_{\max},$$

hence

$$\theta + (\alpha - \beta)\chi \geq 0.$$

We observe that the density $e^{-\alpha\varphi_0} f$ belongs to L^q for $q > 1$. Indeed, for any $\delta > 0$ we choose $q > 1$ so that $\frac{1}{q} = \frac{1}{p} + \frac{1}{p^*+\delta}$. Hölder's inequality and Skoda's theorem yield

$$\int_X e^{-\alpha q\varphi_0} f^q dV \leq \|f\|_{L^p}^q \left(\int_X e^{-\alpha(p^*+\delta)\varphi_0} dV \right)^{q/p^*+\delta} < +\infty.$$

It thus follows from [25] that there exists a bounded θ-psh function u such that

$$\beta^n(\theta + dd^c u)^n = e^{\beta u - \alpha\varphi_0} f dV.$$

Proposition 3.2. *For $t \in (0, \alpha^{-1})$,*

$$(1 - \alpha t)\varphi_0 + \beta t u + n(t\log t - t) \leq \varphi_t.$$

In particular, there exists a uniform constant $C > 0$ such that

$$\varphi_0 - C(t - t\log t) \leq \varphi_t, \quad \forall t \in (1, \alpha^{-1}).$$

Proof. The proof is identical to that of [30, Lemma 2.9]. Set $u_t := (1-\alpha t)\varphi_0 + \beta t u + n(t\log t - t)$. We observe that

$$\theta_t + dd^c u_t = (1-\alpha t)\omega_0 + \beta t\theta_u + t[(\alpha-\beta)\theta + \chi] \geq 0$$

by the choice of α, β. Moreover, we can check that

$$(\theta_t + dd^c u_t)^n \geq \beta^n t^n \theta_u^n = e^{\dot{u}_t}\mu$$

so u_t is a subsolution of (3.1). Together with $u_0 = \varphi_0$ the conclusion thus follows from the maximum principle. □

Before finding a lower bound for solution φ_t, we prove the following upper bound for $\dot{\varphi}_t := \frac{\partial\varphi}{\partial t}$.

Proposition 3.3. *For all $(t, x) \in (0, T] \times X$,*

$$\dot{\varphi}_t(x) \leq \frac{\varphi_t(x) - \varphi_0(x)}{t} + n. \tag{3.3}$$

Proof. We argue the same as in [30] (also in [27]). We consider the function

$$H(t, x) := t\dot{\varphi}_t(x) - (\varphi_t - \varphi_0)(x) - nt.$$

Since $\dot{\varphi}_t = \log(\omega_t^n/\mu)$ hence

$$\frac{\partial H}{\partial t} = t\Delta_t\dot{\varphi}_t + t\,\mathrm{tr}_{\omega_t}\chi - n.$$

On the other hand, we compute

$$\Delta_t H = t\Delta_t\dot{\varphi}_t - \Delta_t(\varphi_t - \varphi_0) = t\Delta_t\dot{\varphi}_t - [n - t\,\mathrm{tr}_{\omega_t}(\chi) - \mathrm{tr}_{\omega_t}(\theta + dd^c\varphi_0)].$$

Therefore

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) H = -\,\mathrm{tr}_{\omega_t}(\theta + dd^c\varphi_0) \leq 0.$$

By the maximum principle, H achieves its maximum along $(t = 0)$. Since $H(0, x) \equiv 0$ hence the desired inequality follows. □

We use the same arguments as in [16] to establish the following uniform estimate for the complex parabolic Monge–Ampère equation.

Theorem 3.4. *Fix $\varepsilon > p^*\varepsilon_0$. For $t \in [\varepsilon, T]$ we have the following estimate*

$$\varphi_t \geq \left(1 - \frac{bt}{T}\right)\psi_0 - C,$$

for some uniform constant $C > 0$.

Proof. Fixing $t \in [\varepsilon, T]$, it follows from Proposition 3.3 that

$$(\theta_t + dd^c\varphi_t)^n = e^{\dot{\varphi}_t} \leq e^{n + \frac{\varphi_t - \varphi_0}{t}} f dV.$$

We set

$$\psi_t := \left(1 - \frac{bt}{T}\right)\psi_0,$$

for $b \in (a, 1/2)$ close to a. We recall that

$$\theta_t \geq \left(1 - \frac{at}{S}\right)\theta,$$

it then follows that ψ_t is $\delta\theta_t$-psh with $\delta \in (0,1)$ only depending on ε_0, a, b, T, S (more precisely, $\delta = \frac{TS - bS\varepsilon_0}{TS - aT\varepsilon_0}$). Using the same arguments as in the proof of [16, Theorem 3.2], we can bound the following quantity

$$\int_X e^{\frac{q(\psi_t - \varphi_0)}{t}} f^q dV < +\infty, \tag{3.4}$$

for some $q > 1$, in terms of $\|f\|_{L^p}$, E_1 and E_2. Indeed, fixing $\gamma > 0$ small enough, we choose $q > 1$ so that

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{2p^* + \gamma} + \frac{1}{2p^* + \gamma}.$$

Hölder's inequality thus ensures that

$$\int_X e^{\frac{q(\psi_t - \varphi_0)}{t}} f^q dV \leq \|f\|_{L^p}^q \left(\int_X e^{\frac{(2p^*+\gamma)(\psi_0 - \varphi_0)}{t}} dV\right)^{\frac{q}{2p^*+\gamma}} \left(\int_X e^{-\frac{(2p^*+\gamma)b\psi_0}{T}} dV\right)^{\frac{q}{2p^*+\gamma}}$$

The second term on the RHS is finite due to the construction of ψ_0 in Lemma 2.11. Also, since ψ_0 is less singular than φ_0 hence the third term is finite.

From (3.4), we can apply Lemma 2.9 with $A = 1/t$ and $g = \varphi_0/t - n$ to get the desired estimate. Note that our $\mathcal{C}^0$-estimate only depends on n, θ, q, our fixed parameters ε_0, ε, T, S, and a upper bound for E_1 and E_2. □

Remark 3.5. When φ_0 is bounded or more general has zero Lelong numbers, it was shown in [55] (generalizing the result of [30] in Kähler context) that the estimate (3.3) ensures a lower bound for φ_t using Kolodziej-Nguyen's theorem [31]. Unfortunately, this method can not apply in more general case, namely when φ_0 are more singular, for instance, it has a positive Lelong numbers. So, in order to analyze the singularities of the initial potential φ_0, Guedj-Lu's approach [25] could help.

3.3. **Laplacian estimate.** We recall the following classical inequality.

Lemma 3.6. *Let α, β be two positive (1,1)-forms. Then*

$$n\left(\frac{\alpha^n}{\beta^n}\right)^{\frac{1}{n}} \leq tr_\beta(\alpha) \leq n\left(\frac{\alpha^n}{\beta^n}\right)(tr_\alpha(\beta))^{n-1}.$$

We define

$$\Psi_t := \left(1 - \frac{bt}{S}\right)\psi_0,$$

where ψ_0 is defined in Lemma 2.11 with $\varepsilon_0 > 0$ fixed.

In order to establish the $\mathcal{C}^2$ estimate it is necessary a lower bound for $\dot{\varphi}_t = \frac{\partial \varphi}{\partial t}$.

Proposition 3.7. *For $(t, x) \in (\varepsilon, T] \times X$,*

$$\dot{\varphi}_t(x) \geq n \log(t - \varepsilon) + A(\Psi_t - \varphi_t) - C$$

where $A, C > 0$ are positive constants only depending on ε, T, $\|f\|_{L^p}$ and a upper bound of E_1 and E_2.

Proof. The proof is almost identical to that of [16, Proposition 3.5]. The only difference is that we use Theorem 3.4 instead of the corresponding one in [16]. We include the proof for the convenience of readers.

By [25, Theorem 3.4], there exist a bounded θ-psh function ϕ_1 and a constant c_1 such that

$$(\theta + dd^c\phi_1)^n = e^{c_1} d\mu, \quad \sup_X \phi_1 = 0.$$

We set

$$G(t, x) := \dot{\varphi}_t(x) + A(\varphi_t - \Psi_t) - \phi_1 - n\log(t - \varepsilon)$$

for a constant $A > 0$ to be determined hereafter. We see that G attains its minimum at a point $(t_0, x_0) \in (\varepsilon, T] \times (X \backslash \{\psi_0 = -\infty\})$. In the sequel, all our computations will take place at this point. We compute

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) G = A\dot{\varphi}_t - \frac{n}{t-\varepsilon} + A\frac{b\psi_0}{S} - nA + A\,\mathrm{tr}_{\omega_t}(\theta + dd^c\Psi_t) + \mathrm{tr}_{\omega_t}(\chi + dd^c\phi_1).$$

We observe that

$$\begin{aligned}\theta_t + dd^c\Psi_t &= \frac{t(b-a)}{S}\theta + \left(1 - \frac{bt}{S}\right)(\theta + dd^c\psi_0)\\ &\geq \frac{\varepsilon(b-a)}{S}\theta + \frac{1}{2}2\kappa\omega_X.\end{aligned}$$

We now choose $A > 0$ so big that

$$A(\theta_t + dd^c\Psi_t) + \chi \geq \theta.$$

Therefore

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) G \geq A\dot{\varphi}_t - \frac{n}{t-\varepsilon} + A\frac{b\psi_0}{S} - nA + \mathrm{tr}_{\omega_t}(\theta + dd^c\phi_1). \tag{3.5}$$

On the other hand, Lemma 3.6 ensures that

$$\mathrm{tr}_{\omega_t}(\theta + dd^c\phi_1) \geq n\left(\frac{(\theta + dd^c\phi_1)^n}{\omega_t^n}\right)^{1/n} = ne^{\frac{-\dot{\varphi}_t}{n}}.$$

Using the elementary inequality $\gamma x - \log x \geq -C_\gamma$ for each small constant $\gamma > 0$, $x > 0$, we have that

$$A\dot{\varphi}_t + ne^{\frac{-\dot{\varphi}_t}{n}} \geq e^{\frac{-\dot{\varphi}_t}{n} - C_1} - C_2.$$

Plugging this into (3.5) it follows from the minimum principle that at (t_0, x_0),

$$\dot{\varphi}_t \geq -n\log\left(C_2 + \frac{n}{t-\varepsilon} - \frac{Ab\psi_0}{S} + nA\right) - nC_1,$$

hence

$$G(t_0, x_0) \geq -C_3 - n\log\left(C_2(t_0 - \varepsilon) + n - \frac{Ab(t_0 - \varepsilon)\psi_0}{S}\right) - \frac{Abt_0(S-T)}{ST}\psi_0$$

where we have used Theorem 3.4. We thus obtain a uniform lower bound for $G(t_0, x_0)$ again by $\gamma x - \log x \geq -C_\gamma$ for $x > 0$. The desired lower bound follows. □

We are now in position to establish the C^2-estimate. We follow the computations of [52, Lemma 4.1] (see also [55, Lemma 6.4]) in which they have used the technical trick due to Phong and Sturm [38]. Recall that the measure μ is of the form

$$\mu = e^{\psi^+ - \psi^-} dV$$

where $\psi^\pm$ are smooth $K\omega_X$-psh function on X for uniform constant $K > 0$. For simplicity, we assume $K = 1$ and normalize $\sup_X \psi^\pm = 0$.

Theorem 3.8. *Fix $\varepsilon > p^*\varepsilon_0$. For $(t, x) \in [\varepsilon, T] \times X$ we have the following bound*

$$(t-\varepsilon) \log tr_{\omega_X}(\omega_t) \leq -B\psi_0 - C\psi^- + C$$

where A, C are positive constants only depending on ε, T, $\|e^{-\psi^-}\|_{L^p}$ and a upper bound of E_1, E_2.

Proof. We follow the computations of [21, 55] (which are due to the trick of Phong and Sturm [38]) with a twist in order to deal with unbounded functions. The constant C denotes various uniform constants which may be different.

Consider

$$H := (t-\varepsilon) \log \mathrm{tr}_{\omega_X}(\omega_t) - \gamma(u), \ (t, x) \in [\varepsilon, T] \times X,$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth concave increasing function such that $\lim_{t\to+\infty} \gamma(t) = +\infty$, and

$$u(t, x) := \varphi_t(x) - \Psi_t(x) - \kappa\psi^- + 1 \geq 1,$$

as follows from Theorem 3.4 and $\psi_0, \psi^- \leq 0$. We are are going to show that H is uniformly bounded from above for an appropriate choice of γ.

We let g denote the Riemann metric associated to ω_X and $\tilde{g}$ the one associated to $\omega_t := \theta_t + dd^c\varphi_t$. Since H goes to $-\infty$ on the boundary of $X_0 := \{x \in X : \psi_0(x) > -\infty\}$, H attains its maximum at some point $(t_0, x_0) \in [\varepsilon, T] \times X_0$. If $t_0 = \varepsilon$ we are done. Assume that $t_0 > \varepsilon$. At this maximum point we use the following local coordinate systems due to Guan and Li [23, Lemma 2.1, (2.19)]:

$$g_{i\bar{j}} = \delta_{ij}, \ \frac{\partial g_{i\bar{i}}}{\partial z_j} = 0 \text{ and } \tilde{g}_{i\bar{j}} \text{ is diagonal.}$$

Following the computations in [55, Eq. (3.20)], we have

$$\Delta_t \, \mathrm{tr}_{\omega_X}(\omega_t) \geq \sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}j} \tilde{g}_{j\bar{i}\bar{j}} - \mathrm{tr}_{\omega_X} \mathrm{Ric}(\omega_t) - C_1 \, \mathrm{tr}_{\omega_X}(\omega_t) \, \mathrm{tr}_{\omega_t}(\omega_X). \tag{3.6}$$

From standard arguments as in [25, Eq. (4.5), p. 29], we obtain

$$\begin{aligned} \frac{|\partial \, \mathrm{tr}_{\omega_X}(\omega_t)|^2_{\tilde{\omega}_t}}{(\mathrm{tr}_{\omega_X}(\omega_t))^2} &\leq \frac{1}{\mathrm{tr}_{\omega_X}(\omega_t)} \left(\sum_{i,j} \tilde{g}^{i\bar{i}} \tilde{g}^{j\bar{j}} \tilde{g}_{i\bar{j}j} \tilde{g}_{j\bar{i}\bar{j}} \right) + C \frac{\mathrm{tr}_{\omega_t}(\omega_X)}{(\mathrm{tr}_{\omega_X}(\omega_t))^2} \\ &\quad + \frac{2}{(\mathrm{tr}_{\omega_X}(\omega_t))^2} \mathrm{Re} \sum_{i,j,k} \tilde{g}^{i\bar{i}} T_{ij\bar{j}} \tilde{g}_{k\bar{i}\bar{k}}, \end{aligned} \tag{3.7}$$

where $T_{ij\bar{j}} := \tilde{g}_{j\bar{j}i} - \tilde{g}_{i\bar{j}j}$ is the torsion term corresponding to ω_t which is under control: $|T_{ij\bar{j}}| \leq C$. Now at the point (t_0, x_0), we have $\partial_{\bar{i}} H = 0$, hence

$$(t-\varepsilon) \sum_k \tilde{g}_{k\bar{k}\bar{i}} = \mathrm{tr}_{\omega_X}(\omega_t) \gamma'(u) u_{\bar{i}}.$$

Cauchy-Schwarz's inequality yields

$$\left|\frac{2}{(\mathrm{tr}_{\omega_X}(\omega_t))^2}\mathrm{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{k}\bar{i}}\right| \le C\frac{\gamma'(u)(t_0-\varepsilon)}{-\gamma''(u)}\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{(\mathrm{tr}_{\omega_X}(\omega_t))^2}+\frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t},$$

hence

$$\left|\frac{2}{(\mathrm{tr}_{\omega_X}(\omega_t))^2}\mathrm{Re}\sum_{i,j,k}\tilde{g}^{i\bar{i}}T_{ij\bar{j}}\tilde{g}_{k\bar{i}\bar{k}}\right| \le C\left(\frac{\gamma'(u)T}{-\gamma''(u)}+1\right)\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{(\mathrm{tr}_{\omega_X}(\omega_t))^2}+\frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t},$$

using that $|g_{k\bar{k}\bar{i}}-g_{k\bar{i}\bar{k}}|\le C$. From this, the inequality (3.7) becomes

$$\begin{aligned}\frac{|\partial\,\mathrm{tr}_{\omega_X}(\omega_t)|^2_{\tilde{\omega}_t}}{(\mathrm{tr}_{\omega_X}(\omega_t))^2} &\le \frac{1}{\mathrm{tr}_{\omega_X}(\omega_t)}\left(\sum_{i,j}\tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{j}}\tilde{g}_{i\bar{j}j}\tilde{g}_{j\bar{i}\bar{j}}\right)\\ &\quad + C\left(\frac{\gamma'(u)T}{-\gamma''(u)}+2\right)\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{(\mathrm{tr}_{\omega_X}(\omega_t))^2}+\frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t}.\end{aligned} \tag{3.8}$$

Set $\alpha := \mathrm{tr}_{\omega_X}(\omega_t)$. We compute

$$\begin{aligned}\dot{\alpha} &= \mathrm{tr}_{\omega_X}(\chi+dd^c\dot{\varphi}) = \mathrm{tr}_{\omega_X}(\chi)-\mathrm{tr}_{\omega_X}\mathrm{Ric}(\omega_t)-\mathrm{tr}_{\omega_X}dd^c(\psi^+-\psi^-)+\mathrm{tr}_{\omega_X}(dV)\\ &\le \mathrm{tr}_{\omega_X}(C_1\omega_X+dd^c\psi^-)-\mathrm{tr}_{\omega_X}\mathrm{Ric}(\omega_t)\end{aligned}$$

where we have use the fact that $\mathrm{tr}_\omega(\chi)$ is bounded from above together with the trivial inequality $n\le \mathrm{tr}_{\omega_X}(\omega_t)\,\mathrm{tr}_{\omega_t}(\omega_X)$. Combining this together with (3.6) and (3.8), we infer that

$$\begin{aligned}\frac{\dot{\alpha}}{\alpha}-\Delta_t\log\alpha &= \frac{\dot{\alpha}}{\alpha}-\frac{\Delta_{\omega_t}\alpha}{\alpha}+\frac{|\partial\alpha|^2_{\omega_t}}{\alpha^2}\\ &\le \frac{\mathrm{tr}_{\omega_t}(C_1\omega_X+dd^c\psi^-)}{\alpha}+C\left(\frac{\gamma'(u)T}{-\gamma''(u)}+2\right)\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{\alpha^2}+\frac{-\gamma''(u)}{t_0-\varepsilon}|\partial u|^2_{\omega_t}\end{aligned} \tag{3.9}$$

From this, at the maximum point (t_0,x_0),

$$\begin{aligned}0 \le \left(\frac{\partial}{\partial t}-\Delta_t\right)H &= \log\alpha+(t-\varepsilon)\left(\frac{\dot{\alpha}}{\alpha}-\Delta_t\log\alpha\right)\\ &\quad -\gamma'(u)\dot{u}+\gamma'(u)\Delta_t u+\gamma''(u)|\partial u|^2_{\omega_t}\\ &\le \log\alpha+\frac{C_3\,\mathrm{tr}_{\omega_t}(\omega_X+dd^c\psi^-)}{\alpha}+C_4\left(\frac{\gamma'(u)T}{-\gamma''(u)}+2\right)\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{\alpha^2}\\ &\quad -\gamma'(u)\dot{\varphi}_t+\gamma'(u)\dot{\Psi}_t+\gamma'(u)\Delta_{\omega_t}(\varphi_t-\Psi_t-\kappa\psi^-),\end{aligned} \tag{3.10}$$

with $C_3, C_4>0$ under control. Moreover, since $\theta_t \ge \left(1-\frac{t}{3S}\right)\theta$ hence

$$\theta_t+dd^c\Psi_t \ge \left(1-\frac{bt}{s}\right)2\kappa\omega_X.$$

Thus we obtain

$$\Delta_t(\varphi_t-\Psi_t)\le n-\kappa\,\mathrm{tr}_{\omega_t}(\omega_X). \tag{3.11}$$

Plugging (3.11) into (3.10) we thus arrive at

$$\begin{aligned}0 &\le \log\alpha+\frac{C_3\,\mathrm{tr}_{\omega_t}(\omega_X+dd^c\psi^-)}{\alpha}-\gamma'(u)(n-\kappa\,\mathrm{tr}_{\omega_t}(\omega_X+dd^c\psi^-))\\ &\quad -\gamma'(u)\dot{\varphi}_t-\gamma'(u)\frac{\psi_0}{2S}+C_4\left(\frac{\gamma'(u)T}{-\gamma''(u)}+2\right)\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{(\mathrm{tr}_{\omega_X}(\omega_t))^2}+C_5.\end{aligned}$$

We now choose the function γ to obtain a simplified formulation. We set

$$\gamma(u) := \frac{C_3+3}{\min(\kappa,1)}u + \ln(u).$$

Since $u \geq 1$ we have

$$\frac{C_3+3}{\min(\kappa,1)} \leq \gamma'(u) \leq 1 + \frac{C_3+3}{\min(\kappa,1)}, \qquad \frac{\gamma'(u)T}{-\gamma''(u)} + 2 \leq C_5 u^2.$$

Using $\mathrm{tr}_{\omega_X}(\omega_X + dd^c\psi^-) \leq \mathrm{tr}_{\omega_t}(\omega_X + dd^c\psi^-)\,\mathrm{tr}_{\omega_X}(\omega_t)$ we obtain

$$0 \leq \log\alpha - \gamma'(u)\dot{\varphi}_t - \gamma'(u)\frac{\psi_0}{2S} - 3\,\mathrm{tr}_{\omega_t}(\omega_X) + C_6(u^2+1)\frac{\mathrm{tr}_{\omega_t}(\omega_X)}{\alpha^2}. \tag{3.12}$$

If at the point (t_0, x_0), we have $\alpha^2 \leq C_6(u^2+1)$ then

$$H(t_0,x_0) \leq T\log\sqrt{C_6(u^2+1)} - \gamma(u) \leq C_7,$$

we are done. Otherwise, we assume that at (t_0,x_0), $\alpha^2 \geq C_6(u^2+1)$. Applying Lemma 3.6 we obtain

$$\log\alpha = \log\mathrm{tr}_{\omega_X}(\omega_t) \leq (n-1)\log\mathrm{tr}_{\omega_t}(\omega_X) + \log n + \dot{\varphi}_t - \psi^-$$

using that $\sup_X \psi^+ = 0$. Plugging this into (3.12) we obtain

$$0 \leq C_5 + (n-1)\log\mathrm{tr}_{\omega_t}(\omega_X) - 2\,\mathrm{tr}_{\omega_t}(\omega_X) - (\gamma'(u)-1)\dot{\varphi}_t - \gamma'(u)\frac{\psi_0}{2S} - \psi^-,$$

or, equivalently

$$2\,\mathrm{tr}_{\omega_t}(\omega_X) \leq C_8 - (\gamma'(u)-1)\dot{\varphi}_t - \gamma'(u)\frac{\psi_0}{2S} - \psi^- \tag{3.13}$$

since $(n-1)\log y - 2y \leq -y + O(1)$ for $y > 0$. In particular, we have

$$\dot{\varphi}_t \leq \frac{C_5}{\gamma'(u)-1} - \frac{\gamma'(u)}{\gamma'(u)-1}\frac{\psi_0}{2S} \leq \frac{C_5}{A-1} - \frac{A\psi_0}{(A-1)2S} - \frac{\psi^-}{A-1} \tag{3.14}$$

at (t_0,x_0) since $\mathrm{tr}_{\omega_t}(\omega_X) \geq 0$ and $A \leq \gamma'(u) \leq A+1$ with $A =: \frac{C_3+3}{\min(\kappa,1)}$. It follows from Lemma 3.6 that

$$\mathrm{tr}_{\omega_t}(\omega_X) \geq n e^{\frac{-\dot{\varphi}_t+\psi^-}{n}}.$$

Plugging this into (3.13) we obtain

$$\mathrm{tr}_{\omega_t}(\omega_X) \leq C_9 - \gamma'(u)\frac{\psi_0}{2S} - \gamma'(u)\psi^- \leq C_9 - \frac{(A+1)\psi_0}{2S} - (A+1)\psi^-$$

with $C_9 > 0$ under control, since $be^y - By \geq -C(b,B)$ for $y \in \mathbb{R}$. Again Lemma 3.6 yields

$$\log\alpha \leq (n-1)\log\left(C_9 - \frac{(A+1)\psi_0}{2S} - (A+1)\psi^-\right) + \log n + \dot{\varphi}_t - \psi^-.$$

Combining this together with (3.14) we have at (t_0,x_0)

$$H \leq C_{10} - A\left[\varphi_t - \left(1 - \frac{t}{2S} - \frac{t-\varepsilon}{2(A-1)S}\right)\psi_0\right] + \left(A\kappa - 1 - \frac{1}{A-1}\right)\psi^-$$
$$+ (t-\varepsilon)(n-1)\log\left(C_9 - \frac{(A+1)\psi_0}{2S} - (A+1)\psi^-\right).$$

Up to increasing $A > 0$ if necessary, so that

$$\delta := \frac{\varepsilon}{2T} - \frac{\varepsilon}{2S} - \frac{T}{2(A-1)S} > 0,$$

since $\psi_0 \le 0$ we obtain, at (t_0, x_0),

$$H \le C_{10} - A\left[\varphi_t - \left(1 - \frac{t}{2T}\right)\psi_0\right] + A\delta\psi_0 + A\kappa/2\psi^-$$
$$+ (t-\varepsilon)(n-1)\log\left(C_9 - \frac{(A+1)\psi_0}{2S} - (A+1)\psi^-\right).$$

The second term is uniformly bounded from above thanks to Theorem 3.4. Since $-by + \log y$ is bounded from above for $y > 0$, we obtain that H attains a uniform bound at (t_0, x_0). This finishes the proof. $\square$

3.4. **More estimates.** Recall that there exists ρ an θ-psh function with analytic singularities such that $\sup_X \rho = 0$ and

$$\theta + dd^c\rho \ge 3\delta_0\omega_X$$

for some $\delta_0 > 0$. By [25, Theorem 3.4], there exist a bounded θ-psh function ϕ_1 and a constant c_1 such that

$$(\theta + dd^c\phi_1)^n = e^{c_1}d\mu, \quad \sup_X \phi_1 = 0.$$

Proposition 3.9. *Assume that ψ_1, ψ_2 are two smooth ω_X-psh functions satisfying*

$$\dot\varphi_0 \ge C_1\psi_1, \quad \varphi_0 \ge \frac{1}{2}(\rho + \delta_0\psi_2)$$

for some constants $C_1 > 0$. Fix $T_1 \in (0, T_{\max})$ such that $\theta_t > \frac{1}{2}\theta$ for all $t \in [0, T_1]$. Then there exists a uniform constant $C_2 > 0$ only depending on C_1, δ_0, T_1 and $\sup_X |\phi_1|$ such that

$$\dot\varphi_t \ge C_2(\rho + \delta_0\psi_2 + 1) + C_1\psi_1,\ \forall\, t \in [0, T_1].$$

Proof. The proof is identical to that of Proposition 3.7. We consider

$$H(t,x) := \dot\varphi_t - C_1\psi_1 + A\left(\varphi_t - \frac{1}{2}(\rho + \delta_0\psi_2)\right) - \phi_1,$$

for $A > 0$ to be chosen hereafter. We observe that H attains its minimum at some point $(t_0, x_0) \in [0, T_1] \times X$. If $t_0 = 0$ we are done by assumptions. Otherwise, by the minimum principle we have at (t_0, x_0),

$$0 \ge \left(\frac{\partial}{\partial t} - \Delta_t\right)H \ge -An + A\dot\varphi_t + (-C_1 + A\delta_0)\,\mathrm{tr}_{\omega_t}(\omega_X) + \mathrm{tr}_{\omega_t}(dd^c\phi_1)$$

where are have used that $\theta_t + dd^c\frac{1}{2}(\rho + \delta_0\psi_2) \ge \delta_0\omega_X$. Now, we choose $A = \delta_0(C_1 + 1)$ thus

$$\mathrm{tr}_{\omega_t}(\omega_X + dd^c\phi_1) \ge n\left(\frac{(\theta + dd^c\phi_1)^n}{\omega_t^n}\right)^{1/n} = ne^{\frac{-\dot\varphi_t}{n}}.$$

using Lemma 3.6. Together with the inequality $e^y \ge By - C_B$ we obtain a uniform lower bound for $\dot\varphi_t$ at (t_0, x_0). On the other hand, by Proposition 3.2 we see that $\varphi_t \ge \varphi_0 - c(t)$ so $\varphi_t \ge \frac{1}{2}(\rho + \delta_0\psi_2) - c(t)$ where $c(t) \to 0$ as $t \to 0$. The lower bound for $H(t_0, x_0)$ thus follows, this completes the proof. $\square$

Proposition 3.10. *Assume that ψ_1, ψ_2 are two smooth ω_X-psh functions satisfying*

$$\Delta_{\omega_X}\varphi_0 \le e^{-C_1\psi_1}, \quad \varphi_0 \ge \frac{1}{2}(\rho + \delta_0\psi_2)$$

for some constants $C_1 > 0$. Fix $T_1 \in (0, T_{\max})$ such that $\theta_t > \frac{1}{2}\theta$ for all $t \in [0, T_1]$. Then there exists uniform constants $C_2 > 0$, $C_3 > 0$ only depending on C_1, δ_0 and T_1 such that

$$tr_{\omega_X}(\omega_t) \le C_3 e^{-C_1\psi_1 - C_2(\rho + \delta_0\psi_2 + \delta_0\psi^-)}),\ \forall\, t \in [0, T_1].$$

Proof. Consider

$$H(t,\cdot) = \log \operatorname{tr}_{\omega_X}(\omega_t) + C_1\psi_1 - \gamma(u)$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ is a smooth concave increasing function such that $\lim_{t\to+\infty} \gamma(t) = +\infty$, and

$$u(t,x) := \varphi_t(x) - \frac{1}{2}(\rho(x) + \delta_0\psi_2(x)) + \delta_0\psi^-(x) + 1.$$

We observe that H attains its maximum at a point $(t_0, x_0) \in [0, T_1] \times \{\rho > -\infty\}$. If $t_0 = 0$ then $H(0,\cdot) \leq \log n - \gamma(1)$. Otherwise, assume $t_0 > 0$. We compute from now on at this point. By the maximum principle and the arguments in Theorem 3.8 we have

(3.15)

$$\begin{aligned}0 \leq \left(\frac{\partial}{\partial t} - \Delta_t\right) H \leq \frac{C \operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)}{\operatorname{tr}_{\omega_X}(\omega_t)} - \gamma'(u)(n - \delta_0 \operatorname{tr}_{\omega_t}(\omega_X + dd^c\psi^-)) + C \\ - C_1 \operatorname{tr}_{\omega_t}(dd^c\psi_1) - \gamma'(u)\dot{\varphi}_t + C\left(\frac{\gamma'(u)}{-\gamma''(u)} + 2\right)\frac{\operatorname{tr}_{\omega_t}(\omega_X)}{(\operatorname{tr}_{\omega_X}(\omega_t))^2}.\end{aligned}$$

Here we use that $\theta_t + dd^c\frac{1}{2}(\rho + \delta_0\psi_2) \geq \delta_0\omega_X$. We set

$$\gamma(u) := \frac{C + C_1 + 3}{\min(\kappa, 1)}u + \ln(u).$$

We proceed the same as in the proof of Theorem 3.8 to obtain the uniform upper bound for $H(t_0, x_0)$. This finishes the proof. □

4. Degenerate Monge-Ampère flows

4.1. **Proof of Theorem B.** By Demailly's regularization theorem (Theorem 2.10) we can find two sequences $\psi_j^\pm \in \mathcal{C}^\infty(X)$ such that

- $\psi_j^\pm$ decreases pointwise to $\psi^\pm$ on X and the convergence is in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$;
- $dd^c\psi^\pm \geq -\omega_X$.

We note that $|\sup_X \psi_j^\pm|$ is uniformly bounded and for all j,

$$\|e^{-\psi_j^-}\|_{L^p} \leq \|e^{-\psi^-}\|_{L^p}.$$

Thanks to Demailly's regularization theorem again, we can find a smooth sequence $(\varphi_{0,j})$ of strictly $\theta + 2^{-j}\omega_X$-psh functions decreasing towards φ_0. We set $\theta_{t,j} = \theta_t + 2^{-j}\omega_X$ and $\mu_j = e^{\psi_j^+ - \psi_j^-}$. It follows from [52, Theorem 1.2] (see also [55]) that there exists a unique function $\varphi_j \in \mathcal{C}^\infty([0,T[\times X)$ such that

$$\begin{cases} \dfrac{\partial\varphi_{t,j}}{\partial t} = \log\left[\dfrac{(\theta_{t,j} + dd^c\varphi_{t,j})^n}{\mu_j}\right] \\ \varphi_j|_{t=0} = \varphi_{0,j}. \end{cases} \tag{4.1}$$

It follows from the maximum principle that the sequence $\varphi_{t,j}$ is decreasing with respect to j. Moreover, Proposition 3.1 yields that $\sup_X \varphi_{t,j}$ is uniformly bounded from above. It follows from Proposition 3.2 that $j \to +\infty$, the family $\varphi_{t,j}$ decreases to φ_t which is a well-defined θ_t-psh function on X. Following the same arguments in [55, Sect. 4.1], we get that $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0^+$.

We next study the partial regularity of φ_t for small t. We fix $\varepsilon_0 > 0$ and $\varepsilon > p^*\varepsilon_0$. Let T and S as in Section 3.1. Let ρ be a θ-psh function with analytic singularities along D such that $\theta + dd^c\rho$ dominates a Hermitian form. Thanks to Lemma 2.11 we can find a function $\psi_0 \in \mathrm{PSH}(X,\theta) \cap \mathcal{C}^\infty(X \setminus (D \cup E_c(\varphi_0)))$ (the constant $c = c(\varepsilon_0)$ is defined as in Lemma 2.11) such that

$$\int_X e^{\frac{2(\psi_0 - \varphi_0)}{\varepsilon_0}} dV_X < +\infty,$$

We assume w.l.o.g that $\psi_0 \leq 0$. Since $\frac{p^*}{2c(\varphi_0)} < T$ and ψ_0 is less singular than φ_0 we also have

$$\int_X e^{\frac{-p^*\psi_0}{T}} dV_X < +\infty.$$

We mention that since φ_0 is a decreasing limit of a smooth sequence $\varphi_{0,j}$, the corresponding constant for $\varphi_{0,j}$ are uniformly bounded (in j) and we can pass the limit when $j \to +\infty$.

Recall that $\psi^\pm$ are smooth (merely locally bounded) in a Zariski open set $U \subset X \backslash D$. We are going to show that φ_t is smooth on $U \setminus (D \cup E_c(\varphi_0))$ for each $t > \varepsilon$. Let K be an arbitrarily compact subset of $U \setminus (D \cup E_c(\varphi_0))$. It follows from Proposition 3.1, Theorem 3.4 and the remark above that

$$\sup_{[\varepsilon,T]\times K} |\varphi_j| \leq C(\varepsilon, T, K).$$

Next, Proposition 3.7 yields

$$\sup_{[\varepsilon,T]\times K} |\dot{\varphi}_j| \leq C(\varepsilon, T, K).$$

Moreover, thanks to Theorem 3.8 we also have a uniform bound for $\Delta\varphi_t^j$:

$$\sup_{[\varepsilon,T]\times K} |\Delta\varphi_j| \leq C(\varepsilon, T, K).$$

Using the complex parabolic Evans-Krylov theory together with parabolic Schauder's estimates (see e.g. [5, Theorem 4.1.4]), we then obtain higher order estimates for φ_j on $[\varepsilon, T] \times K$:

$$\|\varphi_j\|_{\mathcal{C}^k([\varepsilon,T]\times K)} \leq C(\varepsilon, T, K, k).$$

This shows the smoothness of φ_t on $U \setminus (D \cup E_c(\varphi_0))$ for each $t > \varepsilon$ since K was taken arbitrarily. Passing to the limit in (4.1) and we deduce that φ satisfies (1.4) in the classical sense on $[\varepsilon, T] \times \Omega_\varepsilon$ with $\Omega_\varepsilon = X \setminus (D \cup E_{c(\varepsilon)}(\varphi_0))$.

4.2. **Uniqueness.** We now follow the argument in [30] to prove that the solution φ to the equation (1.4) constructed in previous part is the unique maximal solution in the sense of the following:

Proposition 4.1. *Let ψ_t be a weak solution to the equation* (1.4) *with initial data φ_0. Then $\psi_t \leq \varphi_t$ for all $t \in (0, T_{\max})$.*

Proof. By construction in previous paragraph, $\varphi_{t,j}$ are smooth $(\theta_t + 2^{-j}\omega_X)$-psh functions decreasing pointwise to φ_t. It thus suffices to show that $\psi_t \leq \varphi_{t,j}$ for all fixed j.

Fix $0 < T < T_{\max}$ and $2^{-j} > \varepsilon > \delta > 0$. We let $U_\varepsilon \subset X$ denote the Zariski open set in which $\psi_{t+\varepsilon}$ is smooth. We can find a ω_X-psh function ϕ with analytic singularities along $X \setminus U_\varepsilon$; see e.g. [14]. We apply the maximum principle to the function $H := \psi_{t+\varepsilon} - \varphi_{t+\varepsilon,j} + \delta\phi$. Suppose that H attains its maximum on $[0, T - \varepsilon] \times X$ at $(t_\varepsilon, x_\varepsilon)$ with $t_\varepsilon > 0$. Note that $x_\varepsilon \in U_\varepsilon$. We thus have

$$0 \leq \frac{\partial}{\partial t} H \leq \log \left[\frac{(\theta_{t+\varepsilon} + dd^c\varphi_{t+\varepsilon,j} - \delta dd^c\phi)^n}{(\theta_{t+\varepsilon} + 2^{-j}\omega_X + dd^c\varphi_{t+\varepsilon,j})^n} \right] < 0$$

using that $-dd^c\phi \le \omega_X$, which is a contradiction. Thus we have by letting $\delta \searrow 0$,

$$\psi_{t+\varepsilon}(x) - \varphi_{t+\varepsilon,j}(x) \le \sup_X(\psi_\varepsilon - \varphi_{\varepsilon,j}).$$

Moreover, since $(\varepsilon, x) \mapsto \varphi_{\varepsilon,j}(x)$ is continuous it follows from Hartogs' lemma that

$$\sup_X(\psi_\varepsilon - \varphi_{\varepsilon,j}) \xrightarrow{\varepsilon\to 0} \sup_X(\varphi_0 - \varphi_{0,j})$$

The proof is complete. $\square$

The uniqueness property we have just shown is called "maximally stretched" by P. Topping in Riemann surface; cf. [47, Remark 1.9].

4.3. **Short time behavior.** In this subsection we study the behavior of the solution to the degenerate Monge-Ampère flow in short time. We show that the solution φ_t to the equation starting from a current with positive Lelong numbers also has positive Lelong numbers for very short time. This follows almost verbatim from the Kähler case [16, Sect. 4.2].

Theorem 4.2. *If φ_0 has positive Lelong number, then*

$$E_c(\varphi_0) \subset E_{c(t)}(\varphi_t), \qquad c(t) = c - 2nt.$$

In particular, the maximal solution φ_t has positive Lelong numbers for any $t < 1/2nc(\varphi_0)$.

Proof. This is identical to that of [16, Theorem 4.5]. We give a sketch of proof here. Fixing $x_0 \in E_c(\varphi_0)$ we can find a cut-off function $\chi \in \mathcal{C}^\infty(X)$ with support near x_0 and being identical to 1 in a neighborhood of x_0. Thus $\phi := \chi(x)c\log\|x - x_0\|$ is $B\omega_X$-psh and $e^{2\phi/c} \in \mathcal{C}^\infty(X)$. Since $x_0 \in E_c(\varphi_0)$ we can choose ϕ so that $\phi \ge \varphi_0$ by adding a positive constant. Lemma 4.3 yields

$$\varphi_t \le (1 - 2nt/c)\phi + Ct,$$

hence $\nu(\varphi_t, x_0) \ge c - 2nt$. If $t < 1/2nc(\varphi_0)$ then by Skoda's integrability theorem, $e^{-2\varphi_0/c}$ is not integrable for $2nt < c < 1/c(\varphi_0)$. Thus $E_c(\varphi_0)$ is not empty, neither is $E_{c(t)}(\varphi_t)$ for $t > 0$ sufficiently small. $\square$

Lemma 4.3. *Assume that $\phi \in \mathrm{PSH}(X, \omega_X)$ satisfies $e^{\gamma\phi} \in \mathcal{C}^\infty(X)$ for some constant $\gamma > 0$ and $0 \ge \psi^\pm \ge \phi \ge \varphi_0$. Then there exists a positive constant C depending on an upper bound for $dd^c e^{\gamma\phi}$ such that*

$$\varphi(t) \le (1 - n\gamma t)\phi + Ct, \quad \forall\, t \in [0, 1/n\gamma].$$

Proof. Assume that $\theta_t \le \omega_X$ for $t \in [0, 1/(n\gamma+1)]$. As argued in [16, Lemma 4.4] we can assume that ϕ is smooth and work with approximants $\varphi_{t,j}$ instead. We choose $C > 0$ depending only on a upper bound of $dd^c e^{\gamma\phi}$ such that $dd^c\phi \le Ce^{-\gamma\phi}\omega_X$. Consider

$$\phi_t := (1 - (n\gamma+1)t)\phi + t\log(2^n C^n).$$

We observe that

$$0 \le \omega_X + dd^c\phi \le 2Ce^{-\gamma\phi}\omega_X,$$

hence

$$(\omega_X + dd^c\phi_t)^n \le (2C)^n e^{-n\gamma\phi}\omega_X^n \le e^{\dot\phi_t + \psi^+ - \psi^-}\omega_X^n.$$

Therefore ϕ_t is a supersolution to the parabolic equation

$$(\omega_X + dd^c u_t)^n = e^{\dot u_t + \psi^+ - \psi^-}\omega_X^n$$

while $\varphi_{t,j}$ is a subsolution. The classical maximum principle thus yields that $\phi_t \le \varphi_{t,j}$ for any fixed j. This finishes the proof. $\square$

4.4. **Convergence at time zero.** We study in this part the convergence at zero of the degenerate complex Monge-Ampère flow.

We recall the quasi-monotone convergence in the sense of Guedj-Trusiani [28]: $\varphi_j \to \varphi$ quasi-monotonically if $P_\theta(\inf_{l\geq j}\varphi_j)$ a sequence of θ-psh functions that increases to φ.

Theorem 4.4. *The flow φ_t converges quasi-monotonically to φ_0 as $t \to 0^+$.*

Proof. By Proposition 3.2, we have for t small

$$\varphi_t \geq \varphi_0 - C(t - t\log t)$$

for t small. It follows that

$$P_\theta\left(\inf_{0<s\leq t}\varphi_s\right) \geq \varphi_0 - C(t - t\log t),$$

finishing the proof. $\square$

Theorem 4.5. *Assume that φ_0 is continuous in an open set $U \subset X$. Then φ_t converges to φ_0 in $L^\infty_{\mathrm{loc}}(U)$.*

Proof. The proof is almost verbatim from the Kähler case [16]. We assume without loss of generality that $\varphi_t \leq 0$. By Proposition 3.2, there is a uniform constant $C > 0$ and small time t_0 such that

$$\varphi_s - C(t-s)\log(t-s) - C(t-s) \leq \varphi_t$$

for $0 \leq s < t \leq t_0$ small. Set $u_t := \varphi_t + (C + \log 4)t - Ct\log t$. Substituting $s = t/2$ we infer that $u_t \geq u_{t/2}$, hence the sequence $u_{t_0 2^{-j}}$ decreases to $u_0 = \varphi_0$. The conclusion therefore follows from Dini's theorem. $\square$

We also have the same result as in the Kähler case [16, Theorem 6.3]. We assume that θ is a big form and that $f = e^{\psi^+ - \psi^-} \in L^p$, $p > 1$ and $\psi^\pm$ are quasi-psh functions. Assume moreover that $\psi^- \in L^\infty_{\mathrm{loc}}(X \setminus D)$ for some closed set $D \subset X$. It follows from [25, Theorem 4.1] that there exists a bounded θ-psh function φ_0 such that $\sup_X \varphi_0 = 0$ and

$$(\theta + dd^c\varphi_0)^n = cfdV.$$

We recall that there is $\rho \in \mathrm{PSH}(X,\theta)$ with analytic singularities along a closed subset E such that $\theta + dd^c\rho \geq 2\delta\omega_X$ for some $\delta > 0$. Set $U := X \setminus (D \cup E)$.

Theorem 4.6. *Assume φ_0 is as above. Let φ_t be the weak solution of the equation* (1.4) *with initial data φ_0. Then φ_t converges to φ_0 in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$.*

Proof. The proof is identical to that of [16, Theorem 6.3]. We sketch the proof here for convenience's readers. We first approximates $\psi^\pm$ by smooth their smooth approximants $\psi_j^\pm$, thanks to [11]. We next apply Tosatti-Weinkove's theorem [49] to obtain smooth $(\theta + 2^{-j}\omega_X)$-psh functions such that $\sup_X \varphi_j = 0$ and

$$(\theta + 2^{-j}\omega_X + dd^c\varphi_{0,j})^n = c_j e^{\psi_j^+ - \psi_j^-} dV.$$

Note here that $f_j = e^{\psi_j^+ - \psi_j^-}$ have uniform L^p-norms. The same arguments in [25, Theorem 4.1] shows that

- $c_j \to c > 0$;
- for any $\varepsilon > 0$, $\varphi_j \geq \varepsilon(\rho + \delta\psi^-) - C(\varepsilon)$;
- $\Delta_{\omega_X}\varphi_{0,j} \leq e^{-C(\varepsilon)(\rho+\delta\psi^-)}$.

Let $\varphi_{t,j}$ be a smooth solution to the equation (1.4) with initial data $\varphi_{0,j}$. The sequence $\varphi_{t,j}$ converges to the unique weak solution φ_t. We use Proposition 3.9 and Proposition 3.10 together with boostrapping arguments to obtain locally uniform estimates of all derivatives of $\varphi_{t,j}$. This implies the convergence in $\mathcal{C}^\infty_{\mathrm{loc}}(U)$. $\square$

5. Finite time singularities

In this section we study finite time singularities of the Chern–Ricci flow, and provide the proof of Theorem A.

We consider a family of Hermitian metrics $\omega(t)$ evolving under the Chern -Ricci flow (1.1) with initial Hermitian metrics ω_0. Suppose that the maximal existence time of the flow $T_{\max} < \infty$. The form $\alpha_{T_{\max}} := \omega_0 - T_{\max}\mathrm{Ric}(\omega_0)$ is nef in the sense of [26], i.e. for each $\varepsilon > 0$ there exists $\psi_\varepsilon \in \mathcal{C}^\infty(X)$ such that $\alpha_{T_{\max}} + dd^c\psi_\varepsilon \geq -\varepsilon\omega_0$. Indeed, for $\varepsilon > 0$,

$$\alpha_{T_{\max}} + \varepsilon\omega_0 = (1+\varepsilon)\left(\omega_0 - \frac{T_{\max}}{1+\varepsilon}\mathrm{Ric}(\omega_0)\right)$$

and since $\frac{T_{\max}}{1+\varepsilon} < T_{\max}$ we have $\omega_0 - \frac{T_{\max}}{1+\varepsilon}\mathrm{Ric}(\omega_0) + dd^c\psi > 0$ for some smooth function ψ. We assume that $\alpha_{T_{\max}}$ is *uniformly non-collapsing*, i.e.,

$$\int_X (\alpha_{T_{\max}} + dd^c\psi)^n \geq c_0 > 0, \quad \forall\, \psi \in \mathcal{C}^\infty(X). \tag{5.1}$$

This condition implies that the volume of $(X, \omega(t))$ does not collapse to zero as $t \to T_{\max}^-$.

Theorem 5.1. *Let α be a nef (1,1) form satisfying the uniformly non-collapsing condition (5.1). If X admits a Hermitian metric ω_X such that $v_+(\omega_X) < +\infty$ then α is big.*

Conversely, if α is big and $v_-(\omega_X) > 0$ then α is uniformly non-collapsing.

When α is semi-positive or closed the result was proved by Guedj-Lu [26, Theorem 4.6, Theorem 4.9], answering the transcendental Grauert-Riemenschneider conjecture [14, Conjecture 0.8]. For our purpose, we would like to extend it in the case that α is no longer closed.

Proof. The proof is almost identical to that of [26, Theorem 4.6] which follows the idea of Chiose [7]. We give the details here for reader's convenience. By the Hahn-Banach theorem as in [32, Lemme 3.3], the bigness of α, i.e., $\exists\,\rho \in \mathrm{PSH}(X,\alpha)$ with analytic singularities such that $\alpha + dd^c\rho \geq \delta\omega_X$ with some $\delta > 0$, is equivalent to

$$\int_X \alpha \wedge \eta^{n-1} \geq \delta \int_X \omega_X \wedge \eta^{n-1}$$

for all Gauduchon metrics η. Suppose by contradiction that for each $\varepsilon > 0$ there exists Gauduchon metrics η_ε such that

$$\int_X \alpha \wedge \eta_\varepsilon^{n-1} \leq \varepsilon \int_X \omega_X \wedge \eta_\varepsilon^{n-1}.$$

We can normalize η_ε so that $\int_X \omega_X \wedge \eta_\varepsilon^{n-1} = 1$. We fix a function $\psi_\varepsilon \in \mathcal{C}^\infty(X)$ such that $\alpha_\varepsilon := \alpha + \varepsilon\omega_X + dd^c\psi_\varepsilon$ is a Hermitian form. By the main result of [49] there exist $c_\varepsilon > 0$ and $\varphi_\varepsilon \in \mathrm{PSH}(X,\alpha_\varepsilon) \cap \mathcal{C}^\infty(X)$ such that $\sup_X \varphi_\varepsilon = 0$ and

$$(\alpha_\varepsilon + dd^c u_\varepsilon)^n = c_\varepsilon \omega_X \wedge \eta_\varepsilon^{n-1}.$$

By normalization we have

$$c_\varepsilon = \int_X (\alpha_\varepsilon + dd^c u_\varepsilon)^n \geq \int_X (\alpha + dd^c(\psi_\varepsilon + u_\varepsilon))^n \geq c_0 > 0.$$

We apply [26, Lemma 4.13] which reformulates the one in [39, Lemma 3.1] to obtain

$$\int_X (\alpha_\varepsilon + dd^c u_\varepsilon) \wedge \eta_\varepsilon^{n-1} \times \int_X (\alpha_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega_X \geq \frac{c_\varepsilon}{n}. \tag{5.2}$$

The first term on the left-hand side can be written as $\int_X (\alpha + \varepsilon\omega_X) \wedge \eta_\varepsilon^{n-1}$ since η_ε is Gauduchon and by assumption,

$$\int_X (\alpha + \varepsilon\omega_X) \wedge \eta_\varepsilon^{n-1} \leq 2\varepsilon.$$

For the second term, it follows by assumption that

$$\int_X (\alpha_\varepsilon + dd^c u_\varepsilon)^{n-1} \wedge \omega_X \leq v_+(\omega_X) \tag{5.3}$$

is bounded from above. Therefore we obtain

$$2\varepsilon v_+(\omega_X) \geq \frac{c_0}{n}$$

which is a contradiction as $\varepsilon \to 0$.

The proof of the last statement follows the same lines as in [26, Theorem 4.6] which we omit here. □

Remark 5.2. When ω_0 is closed or, more generally, is a Guan-Li metric, i.e., $dd^c\omega_0 = dd^c\omega_0^2 = 0$, the condition (5.1) is simply written as $\int_X \alpha_{T_{\max}}^n > 0$. The assumption $v_+(\omega_X) < \infty$ or $v_-(\omega_X) > 0$ is independent of the choice of the Hermitian ω_X due to [26, Proposition 3.2]. We refer the reader to [1] for some examples of such X. In particular, X is arbitrarily compact complex surface.

This result is a slight generalization of the one [34, Theorem 4.3] when α is closed semi-positive and X admits a pluriclosed metric, i.e., $dd^c\omega_X = 0$. Indeed, in this case, the LHS of (5.3) is equal to

$$\int_X \alpha_\varepsilon^{n-1} \wedge \omega_X < \infty.$$

As a consequence of Theorem 5.1, we give a slight improvement of the main result of [50] (see also [34, Theorem 4.1]) which extends the one of Demailly [12] to the non-Kähler setting.

Theorem 5.3. *Let X be a compact complex n-manifold equipped with a Hermitian metric ω_X satisfying $v_+(\omega_X) < \infty$. Let α be a nef (1,1) form. Assume that $x_1, \ldots, x_N \in X$ are any fixed points and positive constants $\tau_1, \ldots, \tau_N$ such that*

$$0 < \sum_{j=1}^N \tau_j^n < \int_X (\alpha + dd^c\psi)^n, \ \forall\, \psi \in \mathrm{PSH}(X, \alpha) \cap \mathcal{C}^\infty(X).$$

Then there exists an α-psh function φ with logarithmic poles

$$\varphi(z - x_j) \leq \tau_j \log \|z - x_j\| + O(1)$$

in local coordinates near x_j, for all $j = 1, \ldots, N$.

Proof. By Theorem 5.1 we know that α is big. The rest of proof exactly follows the same as that of [48, Theorem 1.3]. □

We go back to the Chern-Ricci flow. Observe that one can deduce the Chern–Ricci flow (1.1) to a parabolic complex Monge–Ampère equation

$$\frac{\partial \varphi_t}{\partial t} = \log \left[\frac{(\alpha_t + dd^c\varphi_t)^n}{\omega_0^n} \right], \quad \alpha_t + dd^c\varphi > 0, \ \varphi(0) = 0$$

where $\alpha_t := \omega_0 - t\mathrm{Ric}(\omega_0)$. We assume that the form $\alpha_{T_{\max}}$ is uniformly non-collapsing. By Theorem 5.1, there exists a function ρ with analytic singularities such that

$$\alpha_{T_{\max}} + dd^c\rho \geq 2\delta_0\omega_0$$

for some $\delta_0 > 0$. We observe that

$$\begin{aligned}\alpha_t + dd^c\rho &= \frac{1}{T_{\max}}\left((T_{\max}-t)(\omega_0 + dd^c\rho) + t(\alpha_{T_{\max}} + dd^c\rho)\right) \\ &\geq \delta_0\omega_0\end{aligned} \tag{5.4}$$

for $t \in [T_{\max}-\varepsilon, T_{\max}]$ with sufficiently small $\varepsilon > 0$. Set

$$\Omega := X \setminus \{\rho = -\infty\}$$

We establish uniform C^∞_{loc} estimates on Ω.

Lemma 5.4. *There is a uniform constant $C_0 > 0$ such that on $[0, T_{\max}) \times X$ we have*

(i) $\varphi \leq C_0$;
(ii) $\dot\varphi \leq C_0$;
(iii) $\varphi \geq \rho - C_0$;
(iv) $\dot\varphi \geq C_0\rho - C_0$

Proof. The proofs of (i) and (ii) directly follow from the classical maximum principle; see e.g. [52, Lemma 4.1] (which follow almost verbatim from the Kähler case [46]).

For (iii), we set $\psi := \varphi - \rho$. Note that the function $\psi + At \geq -C$ holds on $[0, T_{\max} - \varepsilon]$ with ε as above. Fix $T_{\max} - \varepsilon < T' < T_{\max}$, assume that $\psi + At$ attains its minimum at $(t_0, x_0) \in [0, T'] \times X$. Note that $x_0 \in \Omega$. We compute at this minimum point,

$$\begin{aligned}\frac{\partial\psi}{\partial t} &= \log\frac{(\alpha_t + dd^c\rho + dd^c\psi)^n}{\omega_0^n} - A \\ &\geq \log\frac{(\delta_0\omega_0)^n}{\omega_0^n} - A \geq -C + A\end{aligned}$$

where we have used the estimate (5.4). If we choose $A > C$ then t_0 must be zero. This implies the lower bound for ψ, hence we are done.

For (iv), we apply the minimum principle to

$$Q = \dot\varphi + A\psi + Bt$$

where A and B are large constants to be chosen later. Our goal is to show that $Q \geq -C$ on $X \times [0, T_{\max})$. As above, we observe that $Q \geq -C$ on $[0, T_{\max} - \varepsilon] \times X$. It thus suffices to show that given any $T_{\max} - \varepsilon < T' < T_{\max}$ the minimum of Q on $[0, T'] \times X$ is attained on $[0, T_{\max} - \varepsilon]$. Let (x_0, t_0) be the point in $(T_{\max} - \varepsilon, T'] \times X$ where Q attains its minimum. Note that $x_0 \in \Omega$. At this point we have

$$\begin{aligned}0 \geq \left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q &= -\operatorname{tr}_\omega \operatorname{Ric}(\omega_0) + A\dot\varphi - An + A\operatorname{tr}_\omega(\alpha_t + dd^c\rho) + B \\ &\geq \delta_0 \operatorname{tr}_\omega \omega_0 + A\log\frac{\omega^n}{\omega_0^n} + \operatorname{tr}_\omega\omega_0 - An + B\end{aligned}$$

where A is chosen so large that

$$(A-1)(\alpha_t + dd^c\rho) + \chi \geq \omega_0$$

for $t \in [T_{\max} - \varepsilon, T_{\max}]$. But since $A\log y - \delta_0 y^{1/n}$ is bounded from above for $y > 0$ the arithmetic-geometric inequality yields

$$\delta_0 \operatorname{tr}_\omega \omega_0 + A\log\frac{\omega^n}{\omega_0^n} \geq \delta\left(\frac{\omega_0^n}{\omega^n}\right)^{1/n} + A\log\frac{\omega^n}{\omega_0^n} \geq -C_1$$

for uniform constant $C_1 > 0$. If we choose $B = C_1 + An$ we obtain

$$0 \geq \left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q \geq \operatorname{tr}_\omega \omega_0 > 0$$

a contradiction. The desired estimate follows. $\square$

Lemma 5.5. *There is a uniform constant $C > 0$ such that on $[0, T_{\max}) \times X$ we have*

$$tr_{\omega_0} \omega(t) \leq Ce^{-C\rho}.$$

Proof. Set $\psi = \varphi - \rho + C_0 \geq 0$. We apply the maximum principle to

$$Q = \log \operatorname{tr}_{\omega_0} \omega - A\psi + e^{-\psi}$$

for $A > 0$ to be determined hereafter. The idea of making use of the last term in Q is due to Phong and Sturn [38] and was used in the context of Chern–Ricci flow [51, 52, 55]. Note that $e^{-\psi} \in (0, 1]$.

It suffices to show that Q is uniformly bounded from above. Again, it follows from the definition of Q that $Q \leq C$ on $[0, T_{\max} - \varepsilon] \times X$ for a uniform $C > 0$. Fixing , $T - \varepsilon < T' < T_{\max}$, suppose that Q attains its maximum at some point $(t_0, x_0) \in [0, T'] \times X$ with $t \in [T - \varepsilon, T']$. In what follows, we compute at this point. From [52, Prop. 3.1] (also [52, (4.2)]) we have

$$\left(\frac{\partial}{\partial t} - \Delta_\omega\right) \log \operatorname{tr}_{\omega_0} \omega \leq \frac{2}{(\operatorname{tr}_{\omega_0} \omega)^2} \operatorname{Re}(g^{\bar{q}k} (T_0)^p_{kp} \partial_{\bar{q}} \operatorname{tr}_{\omega_0} \omega) + C \operatorname{tr}_\omega \omega_0,$$

where $(T_0)^p_{kp}$ denote the torsion terms of ω_0. At the maximum point (x_0, t_0) of Q we have $\partial_i Q = 0$ hence

$$\frac{1}{\operatorname{tr}_{\omega_0} \omega} \partial_i \operatorname{tr}_{\omega_0} \omega - A\partial_i \psi - e^{-\psi} \partial_i \psi = 0.$$

Therefore, the Cauchy-Schwarz inequality yields

$$\begin{aligned}\left| \frac{2}{(\operatorname{tr}_{\omega_0} \omega)^2} \operatorname{Re}(g^{\bar{q}k} (T_0)^p_{kp} \partial_{\bar{q}} \operatorname{tr}_{\omega_0} \omega) \right| &\leq \left| \frac{2}{(\operatorname{tr}_{\omega_0} \omega)^2} \operatorname{Re}((A + e^{-\psi}) g^{\bar{q}k} (T_0)^p_{kp} \partial_{\bar{q}} \psi \right| \\ &\leq e^{-\psi} |\partial \psi|^2_\omega + C(A+1)^2 e^\psi \frac{tr_\omega \omega_0}{(\operatorname{tr}_{\omega_0} \omega)^2}.\end{aligned}$$

for uniform $C > 0$ only depending on the torsion term. It thus follows that, at the point (x_0, t_0),

(5.5)

$$\begin{aligned}0 \leq \left(\frac{\partial}{\partial t} - \Delta_\omega\right) Q &\leq C(A+1)^2 e^\psi \frac{tr_\omega \omega_0}{(\operatorname{tr}_{\omega_0} \omega)^2} + C \operatorname{tr}_\omega \omega_0 \\ &\quad - (A + e^{-\psi})\dot{\varphi} + (A + e^{-\psi}) \operatorname{tr}_\omega(\omega - (\alpha_t + dd^c \rho)) \\ &\leq C(A+1)^2 \frac{tr_\omega \omega_0}{(\operatorname{tr}_{\omega_0} \omega)^2} + (C - (A+1)\delta_0) \operatorname{tr}_\omega \omega_0 + (A+1) \log \frac{\omega_0^n}{\omega^n}\end{aligned}$$

where we have used $\alpha_t + dd^c \rho \geq \delta_0 \omega_0$. If at (x_0, t_0), $(\operatorname{tr}_{\omega_0} \omega)^2 \leq C(A+1)^2$ we are done. Otherwise, we choose $A = \delta^{-1}(C + 2)$. Hence, from (5.5) one gets

$$\operatorname{tr}_\omega \omega_0 \leq C \log \frac{\omega_0^n}{\omega^n} + C.$$

By Lemma 3.6 we obtain

$$\operatorname{tr}_{\omega_0} \omega \leq (\operatorname{tr}_\omega \omega_0)^{n-1} \frac{\omega^n}{\omega_0^n} \leq C \frac{\omega^n}{\omega_0^n} \left(\log \frac{\omega_0^n}{\omega^n}\right)^{n-1} + C \leq C'$$

since $\omega^n / \omega_0^n \leq C_0$ by Lemma 5.4 and $y \mapsto y |\log y|^{n-1}$ is bounded from above as $y \to 0$. Thanks to Lemma 5.4 (iii), Q is bounded from above at its maximum, this finishes the proof. $\square$

Proof of Theorem A. The existence a Hermitian metric ω_X satisfying $v_+(\omega_X) < \infty$ holds when $\dim X = 2$, we can generalize to the case that any n-manifold X admitting such a metric. Let $K \subset \Omega$ be any compact set. It follows from Lemma 5.4 and Lemma 5.5 that on $K \times [0, T_{\max})$,

$$C_K^{-1}\omega_0 \leq \omega(t) \leq C_K\omega_0.$$

Applying the local higher order estimates of Gill [21, Sect. 4], we obtain uniform $\mathcal{C}^\infty$ estimates for $\omega(t)$ on compact subsets of Ω. We exactly proceed the same as in [52, Theorem 1.6] to obtain the convergence. This finishes the proof. □

6. The Chern-Ricci flow on varieties with log terminal singularities

In this section we extend our previous analysis to the case of compact complex varieties with *mild singularities*. We refer the reader to [18, Sect. 5] for a brief introduction to complex analysis on mildly singular varieties.

We assume here that Y is a $\mathbb{Q}$-Gorenstein variety, i.e., Y is a normal complex space such that its canonical divisor K_Y is $\mathbb{Q}$-Cartier. We denote the singular set of Y by Y_{sing} and let $Y_{\text{reg}} := Y \setminus Y_{\text{sing}}$. Given a log resolution of singularities $\pi : X \to Y$ (which may and will always be chosen to be an isomorphism over Y_{reg}), there exists a unique (exceptional) $\mathbb{Q}$-divisor $\sum a_i E_i$ with simple normal crossings (snc for short) such that

$$K_X = \pi^* K_Y + \sum_i a_i E_i,$$

The coefficients $a_i \in \mathbb{Q}$ are called *discrepancies* of Y along E_i.

Definition 6.1. We say that X has *log terminal* (*lt* for short) singularities if and only if $a_i > -1$ for all i.

The following definition of *adapted measure* which is introduced in [18, Sect. 6]:

Definition 6.2. Let h be a smooth hermitian metric on the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(K_Y)$. The corresponding adapted measure $\mu_{Y,h}$ on Y_{reg} is locally defined by choosing a nowhere vanishing section σ of mK_Y over a small open set U and setting

$$\mu_{Y,h} := \frac{(i^{mn^2}\sigma \wedge \bar{\sigma})^{1/m}}{|\sigma|_{h^m}^{2/m}}.$$

The point of the definition is that the measure $\mu_{Y,h}$ does not depend on the choice of σ, so is globally defined. The arguments above show that Y has lt singularities if and only if $\mu_{Y,h}$ has finite total mass on Y, in which case we can consider it as a Radon measure on the whole of Y. Then $\chi = dd^c \log \mu_{Y,h}$ is well-defined smooth closed $(1,1)$-form on Y.

Given a Hermitian form ω_Y on Y, there exists a unique hermitian metric $h = h(\omega_Y)$ of K_Y such that

$$\omega_Y^n = \mu_{Y,h}.$$

We have the following definition.

Definition 6.3. The *Ricci curvature form* of ω_Y is $\mathrm{Ric}(\omega_Y) := -dd^c \log h$.

We recall the *slope* of a quasi-psh function ϕ at y in the sense of [4]. Choosing local generators (f_j) of the maximal ideal $\mathfrak{m}_y$ of $\mathcal{O}_{Y,y}$, we define

$$s(\phi, y) = \sup\{s \geq 0 : \varphi \leq s \log \sum |f_j| + O(1)\}.$$

Note that this definition is independent of the choice of (f_j). By [4, Theorem A.2] there is $C > 0$ such that for any log resolution $\pi : X \to Y$,

$$\nu(\phi \circ \pi, E) \leq Cs(\phi, y)$$

with E a prime divisor lying above y. In particular, the Lelong numbers of $\phi \circ \pi$ is sufficiently small if the $s(\phi, y)$ is also sufficiently small at all points $y \in Y$.

Applying the analysis in the previous section, we have the existence for the Chern-Ricci flow on log terminal singularities. This extends the one of the author [9, Theorem E].

Theorem 6.4. *Let Y be a compact complex variety with log terminal singularities. Assume that θ_0 is a Hermitian metric such that*

$$T_{\max} := \sup\{t > 0 : \exists\, \psi \in \mathcal{C}^\infty(Y) \text{ such that } \theta_0 - tRic(\theta_0) + dd^c\psi > 0\} > 0.$$

Assume that $S_0 = \theta_0 + dd^c\phi_0$ is a positive (1,1)-current with small slopes. Then there exists a family $(\omega_t)_{t\in[0,T_{\max})}$ of positive (1,1) current on Y starting at S_0 such that

(1) $\omega_t = \theta_0 - tRic(\theta_0) + dd^c\varphi_t$ are positive (1,1) currents;
(2) $\omega_t \to S_0$ weakly as $t \to 0^+$;
(3) for each $\varepsilon > 0$ there exists a Zariski open set Ω_ε such that on $[\varepsilon, T_{\max}) \times \Omega_\varepsilon)$, ω is smooth and

$$\frac{\partial\omega}{\partial t} = -Ric(\omega).$$

Proof. It is classical that solving the (weak) Chern-Ricci flow is equivalent to solving a complex Monge-Ampère equation flow. Let χ be a closed smooth (1,1) form that represents $c_1^{\mathrm{BC}}(K_Y)$. Given $T \in (0, T_{\max})$, there is $\psi_T \in \mathcal{C}^\infty(Y)$ such that $\theta_0 - t\mathrm{Ric}(\theta_0) + dd^c\psi_T > 0$ we set for $t \in [0, T]$

$$\hat{\theta}_t := \theta_0 + t\chi, \text{ with } \chi = -\mathrm{Ric}(\theta_0) + dd^c\frac{\psi_T}{T}$$

which defines an affine path of Hermitian forms. Since χ is a smooth representative of $c_1^{\mathrm{BC}}(K_Y)$, one can find a smooth metric h on the $\mathbb{Q}$-line bundle $\mathcal{O}_Y(K_Y)$ with curvature form χ. We obtain $\mu_{Y,h}$ the adapted measure corresponding to h. The Chern-Ricci flow is equivalent to the following complex Monge-Ampère flow

$$(\hat{\theta}_t + dd^c\phi_t)^n = e^{\partial_t\phi}\mu_{Y,h}. \tag{6.1}$$

Now let $\pi : X \to Y$ be a log resolution of singularities. We have seen that the measure

$$\mu := \pi^*\mu_{Y,h} = f dV \quad \text{where } f = \prod_i |s_i|^{2a_i}$$

has poles (corresponding to $a_i < 0$) or zeroes (corresponding to $a_i > 0$) along the exceptional divisors $E_i = (s_i = 0)$, dV is a smooth volume form. Passing to the resolution, the flow (6.1) becomes

$$\frac{\partial\varphi}{\partial t} = \log\left[\frac{(\theta_t + dd^c\varphi_t)^n}{\mu}\right] \tag{6.2}$$

where $\theta_t := \pi^*\hat{\theta}_t$ and $\varphi := \pi^*\phi$. Since $(\hat{\theta}_t)_{t\in[0,T]}$ is a smooth family of Hermitian forms, it follows that the family of semi-positive forms $[0, T] \ni t \mapsto \theta_t$ satisfies all our requirements. We also have that $\theta := \pi^*\theta_0$, the latter is smooth, semi-positive and big, but no longer hermitian. We fix a θ-psh function ρ with analytic singularities along a divisor $E = \pi^{-1}(Y_{\mathrm{sing}})$ such that $\theta + dd^c\rho \geq 2\delta\omega_X$ with $\delta > 0$. If we set $\psi^+ = \sum_{a_i>0} 2a_i \log|s_i|$, $\psi^- = \sum_{a_i<0} -2a_i \log|s_i|$, we observe that $\psi^\pm$ are quasi-psh functions with logarithmic poles along the exceptional divisors, smooth on $X \setminus \mathrm{Exc}(\pi) = \pi^{-1}(Y_{\mathrm{reg}})$, and $e^{-\psi^-} \in L^p(dV)$ for some $p > 1$. We observe that since the Lelong numberw $\nu(\varphi_0, x)$ are sufficiently small, so we have the assumption $p^*/2c(\varphi_0) < T_{\max}$ by Skoda's integrability theorem. The result therefore follows from Theorem B. □

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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS (ICTP), STR. COSTIERA, 11, 34151 TRIESTE, TS, ITALY

Email address: qdang@ictp.it