Singularities of the Reflected Riemann Matrix

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1. Introduction. This is a study of the mixed initial-boundary value problem in the quarter space $x_1 \geqq 0$, $t \geqq 0$, where propagation is governed by the first-order hyperbolic system of equations

$$Lu = \left(E\, \partial/\partial t - \sum_{k=1}^{N} A_k\, \partial/\partial x_k\right)u(x, t) = 0$$

with constant matrix coefficients. The fundamental solution or Riemann matrix is a distribution or generalized function, and the initial conditions play the role of "test functions" upon which this fundamental solution acts. The Riemann matrix can be divided into two parts, one corresponding to the incident system of wavefronts and the other corresponding to the system of wavefronts formed by reflection from the hyperplane $x_1 = 0$. These we shall call the *incident Riemann matrix* and the *reflected Riemann matrix*, respectively.

In section two the incident Riemann matrix for $Lu = 0$ in the neighborhood of the boundary is obtained in terms of plane waves propagating toward the boundary. In the third section these incident plane waves are reflected from the boundary (subject to the boundary conditions) to determine the reflected system of waves. Our purpose is to find the leading term in the asymptotic estimate of the reflected Riemann matrix, which is usually the most singular part. This is done in sections five through seven, using theory of fractional integrals. The necessary lemmas as well as some of the more important properties of fractional integrals are given in section four and the appendix. The distributional nature of the asymptotic estimate is given explicitly for the three principal types of waves formed upon reflection.

When an incident wavefront impinges on the boundary and the main reflected sheets form, it may happen that one or more of the outer reflected sheets become detached from the incident sheet on the boundary (*cf.* Fig. 1.1). If this happens, branch waves form which (in mathematical terms) are generated by branch points in the reflection coefficients. The real branch waves discussed in section six are conical wavefronts linking the outer detached sheets with all inner reflected surfaces. The most singular part of the reflected Riemann matrix due to

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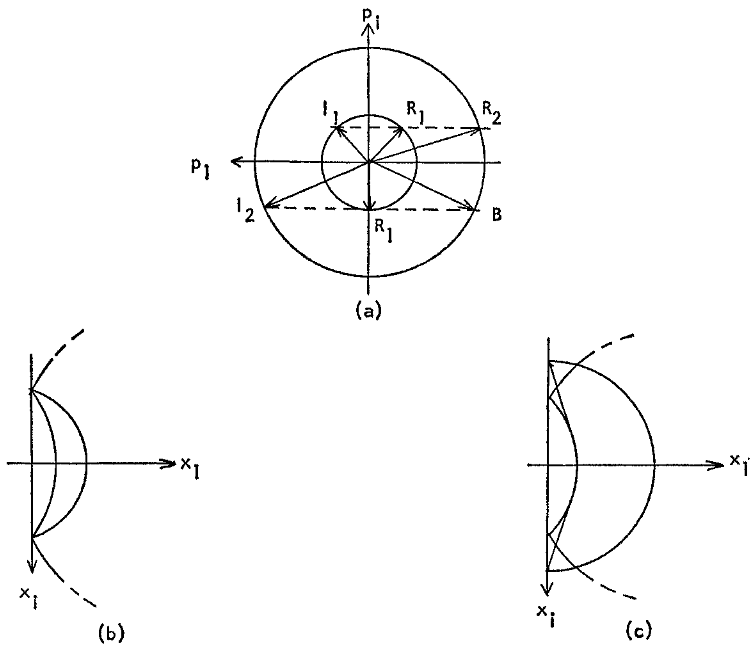

FIGURE 1.1 (a) Slowness (normal) surface defined by $\det(\sum A_k p_k - E) = 0$. I_1 indicates the slowness vector for an incident fast plane wave and R_1 and R_2 are the two possible reflected waves. (b) Reflected wave surface for the incident fast wavefront. The reflected wavefronts for the incident slow wavefront take this form initially (For details concerning the relationship between slowness and wave surfaces, *cf.* [3]). At I_2, R_1 has a branch point. (c) The fast reflected wave has become detached from the incident slow wavefront and it propagates with a constant phase velocity along the boundary. Real branch waves form which are envelopes of plane waves with slowness vectors B.

these real branch waves is one derivative less singular than that corresponding to the main reflected wavefronts. The nature of the singularity near the point of tangency is determined, and it appears to differ from the results in the literature. Complex branch waves, having no well-defined wavefronts, may also form. In this case the reflected Riemann matrix is singular only on the boundary.

There is another class of wave surfaces that may form upon reflection. These waves, which we call boundary waves, are generated by poles in the reflection coefficients. Real boundary waves are conical wavefronts resembling real branch waves and extending from the boundary to the point of tangency with the main reflected wavefronts (*cf.* Fig. 1.2). In this case, however, the most singular part of the reflected Riemann matrix is half a derivative more singular than that corresponding to the main reflected wavefronts. Complex boundary waves (Rayleigh-like waves) have no well defined wavefronts and in this case the reflected Riemann matrix is usually non-singular. We estimate the magnitude of the reflected Riemann matrix corresponding to complex boundary waves.

If boundary waves are present, the reflected Riemann matrix is not uniquely

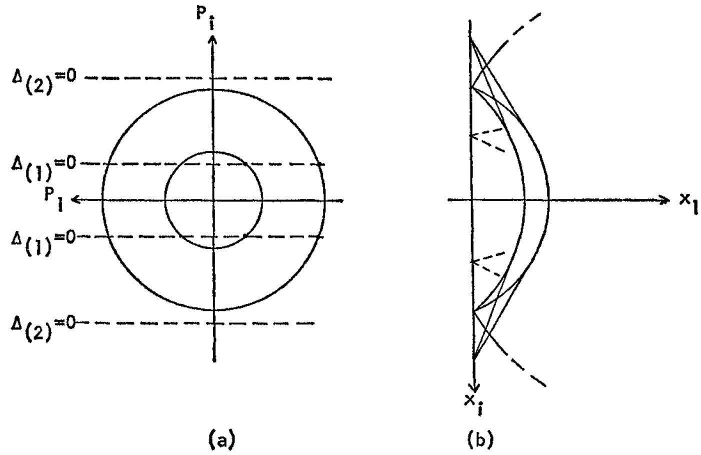

FIGURE 1.2 (a) Slowness surface. The reflection coefficients have a pole on the cylinders defined by $\Delta_{(1)} = \Delta_{(2)} = 0$. (b) Reflected wave surface for the incident fast wavefront. The real boundary waves are shown and the complex boundary waves, generated by $\Delta_{(2)} = 0$, are indicated by dotted lines. The slowness vectors for the boundary waves are defined by the intersection of the cylinders $\Delta_{(1)} = \Delta_{(2)} = 0$ with the slowness surface. The reflected wave surface for the incident slow wavefront has a similar configuration with the exceptions noted in Fig. 1.1c.

determined. Hersh [7] gives an example of this and proves that a necessary and sufficient condition for uniqueness of the initial-boundary value problem is, essentially, that boundary waves do not exist. We will show that if more conditions are imposed on the problem, uniqueness for the asymptotic estimate is guaranteed. The condition used here is based on the domain of support that one would expect for boundary waves. In section eight each point in the theory is illustrated by example.

The most singular part of the Riemann matrix for $Lu = 0$ was determined by Duff [4]. The more general case $Lu + Bu = 0$ has been considered by Ludwig [8]. Duff [5] has considered the asymptotic estimate of the reflected Riemann matrix for the initial-boundary value problem involving a single homogeneous differential equation of degree m. The present paper provides more details in connection with the branch and boundary waves. There are a number of monographs that consider the problem of reflection and refraction from a plane in liquid and elastic media. We mention in particular Brekhovskikh [1] and Cagniard [2].

This paper is a revision of a doctoral thesis submitted at the University of Toronto under the supervision of G. F. D. Duff in 1964.

2. The incident Riemann matrix. Let x be a vector in N-dimensional Euclidean space E_N, and let t be the time coordinate. The hyperbolic system of m differential equations under consideration is

$$Lu = \left(E\,\partial/\partial t - \sum_{k=1}^{N} A_k\,\partial/\partial x_k\right)u(x, t) = 0, \tag{2.1}$$

where E is the identity matrix and A_k are constant square matrices. We assume that the displacement vector is $g(x)$ at $t = 0$. The hyperbolic nature of the system implies that $\Delta \equiv \det(\lambda E - \sum A_k y_k) = 0$ has real roots in λ for all real y_k (*cf.* [9]). From this equation ($\Delta = 0$) the slowness or normal surface is obtained by setting $p_k = y_k/\lambda$.

To determine the incident Riemann matrix for $Lu = 0$ we take the Fourier transform with respect to the spatial variables x_k and the Laplace transform with respect to time t. The Fourier transforms are

$$\tilde{u}(y, t) = \int_{E_N} e^{-ix\cdot y} u(x, t)\, dx, \qquad u(x, t) = (2\pi)^{-N} \int_{E_N} e^{ix\cdot y} \tilde{u}(y, t)\, dy$$

and the Laplace transforms are

$$\bar{u}(x, \tau) = \int_0^\infty u(x, t) e^{-\tau t}\, dt, \qquad u(x, t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \bar{u}(x, \tau) e^{\tau t}\, d\tau.$$

Transforming (2.1), we obtain

$$[\tau E - iA(y)]\bar{\tilde{u}}(y, \tau) = \tilde{u}(y, 0),$$

where $A(y) = \sum A_k y_k$. Using the inverse transforms and the fact that $\tilde{u}(y,0) = \tilde{g}(y)$, we obtain a solution of the form

$$u(x, t) = \int_{E_N} R(x - z, t) g(z)\, dz, \tag{2.2}$$

where the Riemann matrix $R(x - z, t)$ is given by the expression

$$\frac{1}{2\pi i (2\pi)^N} \int_{E_N} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{e^{[i(x-z)\cdot y + \tau t]}}{\tau E - iA(y)}\, d\tau\, dy.$$

Substituting $\tau = \epsilon + i\lambda$ and rearranging, $R(x - z, t)$ can be expressed as

$$\frac{1}{(2\pi i)(2\pi)^N} \int_{E_N} dy \int_0^\infty d\lambda \left[\frac{e^{i[(x-z)\cdot y + (\lambda - i\epsilon)t]}}{(\lambda - i\epsilon)E - A(y)} - \frac{e^{-i[(x-z)\cdot y + (\lambda + i\epsilon)t]}}{(\lambda + i\epsilon)E - A(y)} \right]. \tag{2.3}$$

We wish to express the incident Riemann matrix in terms of plane waves propagating toward the boundary $x_1 = 0$. In order to do this we perform the y_1-integration in (2.3), taking the first term as an example:

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\exp[i(x_1 - z_1) y_1]}{(\lambda - i\epsilon)E - A(y)}\, dy_1\,. \tag{2.4}$$

We make the substitution

$$[(\lambda - i\epsilon)E - A(y)]^{-1} = \frac{B(\lambda, y)}{P(\lambda - i\epsilon, y)}\,, \tag{2.5}$$

where $B(\lambda,y)$ is the adjoint matrix and $P(\lambda - i\epsilon,y)$ is the determinant of $(\lambda - i\epsilon)E - A(y)$. It is necessary to determine the nature of the roots of $P(\lambda - i\epsilon,y)$ with respect to y_1 before evaluating (2.4) by Cauchy's theorem.

It will then be shown that the columns of the adjoint matrix are the displacement vectors of plane waves propagating toward the boundary, thus establishing the principal result of this section.

Lemma 2.1. *Let A_k, $k = 1, \cdots, N$, be real or Hermitian where A_1 has rank r. Then the roots y_1^j, $j = 1, \cdots, r$, of $P(\lambda - i\epsilon, y) = 0$ with respect to y_1 are (i) not real for any real λ, $y_\nu = (y_2, \cdots, y_N)$; (ii) the complex conjugate of those of $P(\lambda + i\epsilon, y) = 0$.*

To prove part (*i*) we suppose that μ is a real root for y_1. Since $Lu = 0$ is a hyperbolic system then $\lambda - i\epsilon$ must be real in order to have $P(\lambda - i\epsilon, \mu, y_\nu) = 0$. This cannot be so since λ is real. The second part follows from the fact that $P(\lambda - i\epsilon, y_1, y_\nu)^* = P(\lambda + i\epsilon, y_1^*, y_\nu)$ where the symbol * denotes complex conjugacy.

Let y_1^j, $j = 1, \cdots, n$, be the roots with negative imaginary part. These are the incident roots which produce the contribution in (2.4) for $x_1 < z_1$. The remaining roots y_1^j, $j = n + 1, \cdots, r$, have positive imaginary parts and are the reflected roots that we use to form the reflected Riemann matrix. From these roots y_1^j the hyperbolic system $Lu = 0$ has two main classifications on the assumption that multiplicities, if they occur, are uniform. The system is said to be strongly hyperbolic if there are r linearly independent nullvectors for the matrix $(E\lambda - A_1 y_1^j - \sum' A_k y_k)$. Otherwise, the system is said to be weakly hyperbolic (*cf.* [8], p. 22; [9]). For the moment we assume that the roots y_1^j are distinct for all y_ν and λ which implies that the characteristic surfaces are simple and non-intersecting.

Evaluating (2.4) in a neighborhood of the boundary $x_1 < z_1$, by Cauchy's theorem, we have

$$\sum_{j=1}^{n} \frac{B_j \exp\,[i(x_1 - z_1)y_1^j]}{(\partial P/\partial y_1)_{y_1 = y_1^j}}, \tag{2.6}$$

where $B_j = (B)_{y_1 = y_1^j}$. Thus the first term in the integrand of (2.3) is

$$\sum_{j=1}^{n} F_j B_j e^{i\Xi_j}, \qquad \Xi_j = (x - z)\cdot y + (\lambda - i\epsilon)t, \tag{2.7}$$

where

$$F_j = (\partial P/\partial y_1)^{-1}_{y_1 = y_1^j}.$$

Lemma 2.2. *(i) The non-zero columns of B_j are proportional. (ii) The non-zero columns of each term in (2.7) are independent plane wave solutions of $Lu = 0$.*

To prove (*i*) we have from (2.5), $[(\lambda - i\epsilon)E - A(y)]_{y_1 = y_1^j} \cdot B_j = 0$. Since the roots y_1^j are distinct the matrix $[(\lambda - i\epsilon)E - A(y)]$ has one nullvector for each root. Part (*ii*) can be verified directly by substituting (2.7) into $Lu = 0$.

If the roots y_1^j, $j = 1, \cdots, n$, contain uniform multiplicities then it is possible to find n independent 'incident' plane solutions to replace (2.6). To illustrate this we suppose that $y_1^2 = y_1^1$ and that the rest of the roots are distinct. Evaluating (2.4) the first two terms in (2.6) are replaced by

$$\left\{\left[ix_1B_1 + \left(\frac{\partial B}{\partial y_1}\right)_{y_1=y_1^1}\right] + B_1\left[\frac{\partial}{\partial y_1^1}\left[\frac{1}{\left(\frac{\partial^2 P}{\partial {y_1}^2}\right)_{y_1=y_1^1}}\right] - iz_1\right]\right\}\frac{\exp\,[i(x_1 - z_1)y_1^1]}{\left(\frac{\partial^2 P}{\partial {y_1}^2}\right)_{y_1=y_1^1}}.$$

(2.8)

From the identity (2.5) it follows, by differentiation, that

$$([(\lambda - i\epsilon)E - A(y)]\ \partial B/\partial y_1)_{y_1=y_1^1} = A_1B_1\ .$$

Columns in $(\partial B/\partial y_1)_{y_1=y_1^1}$ that are not nullvectors of the matrix $[(\lambda - i\epsilon)E - A(y)]_{y_1=y_1^1}$ we refer to as generalized nullvectors.

For a strongly hyperbolic system there are two independent nullvectors of the matrix $[(\lambda - i\epsilon)E - A(y)]_{y_1=y_1^1}$. This implies that $B_1 = 0$ and $(\partial B/\partial y_1)_{y_1=y_1^1}$ contains the two independent nullvectors. Thus (2.4) has the same form as (2.6).

If the system $Lu = 0$ is weakly hyperbolic then there is only one nullvector for $[(\lambda - i\epsilon)E - A(y)]_{y_1=y_1^1}$. Writing $(\partial B/\partial y_1)_{y_1=y_1^1}$ as $C_1 + C_2$ where C_1 and C_2 contain the nullvector and generalized nullvector, respectively, then the two independent displacement vectors of the plane wave solutions are the columns of $(ix_1B_1 + C_1)$ and C_2 or B_1.

3. The reflected Riemann matrix. In this section the reflected Riemann matrix is expressed in a distributional form from which the asymptotic estimate is determined in subsequent sections. This is done for the case in which the roots y_1^i are distinct. At the end of this section we outline the effect that multiple roots have on the singularity of the reflected Riemann matrix.

In the formation of the reflected system of waves we suppose for simplicity that there is no propagation through the boundary. Unless otherwise stated we take as boundary conditions $Bu = 0$, where the boundary operator is homogeneous in the derivatives that occur in each row. Also, we assume that B, whose elements may be complex, does not depend explicitly on x or t. If B is a $r - n$ by m matrix operator then the reflection coefficients can be determined, albeit, not always uniquely. The effect of non-uniqueness is considered in section 7.

In order to avoid considering the reflection of $B_j \exp(i\Xi_j)$ in the incident system (2.7) column by column, we define the columns of B_j^l as the nullvectors for the roots y_1^l, $l > n$, where any two columns have the same constant of proportionality as the corresponding columns in B_j. Thus to form the reflected system of waves generated by the j^{th} term in the incident system (2.7), we take as a trial solution of $Lu = 0$

$$u_j = B_je^{i\Xi_j} + \sum_{l=n+1}^{r} D_{lj}B_j^le^{i\Xi_{lj}}, \tag{3.1}$$

where $\Xi_{lj} = x_1y_1^l + (x_\nu - z_\nu)y_\nu - z_1y_1^j + (\lambda - i\epsilon)t$ and $D_{lj} = D_{lj}(\lambda - i\epsilon, y_\nu)$ are the reflection coefficients. The phase function Ξ_{lj} has the form indicated since the incident and reflected plane waves must have the same phase on the boundary $x_1 = 0$. From (3.1) the system of plane waves generated by the incident system (2.7) is

$$\sum_{l=n+1}^{r} \sum_{j=1}^{n} F_j D_{lj} B_j^l e^{i\Xi_{lj}}. \tag{3.2}$$

To construct the reflected Riemann matrix we replace the integrand of (2.3) with expressions of the form (3.2). The reflected Riemann matrix is

$$\sum_{l,j} \frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} dy_\nu \int_0^{\infty} d\lambda [T_{lj} e^{i\Xi_{lj}} + T_{lj}^* e^{-i\Xi^*_{lj}}] \tag{3.3}$$

where $T_{lj} = F_j D_{lj} B_j^l$. The symbol * has the following meaning. $T_{lj}^* = T_{lj}(\lambda + i\epsilon, y_\nu)$ where $T_{lj} = T_{lj}(\lambda - i\epsilon, y_\nu)$. Also, it follows from Lemma 2.1 that Ξ_{lj}^* is the complex conjugate of Ξ_{lj}. If A_k and the boundary operator are real then the symbol * denotes complex conjugacy.

To express (3.3) in a form suitable for determining the asymptotic estimate we let ϵ approach zero, substitute $y_\nu = \lambda p_\nu$, and then perform the λ-integration. The terms in (3.3) have the following dependence on λ: $T_{lj}(\lambda - i0, y_\nu) = T_{lj}(1 - i0, p_\nu)$; $y_1^k(\lambda - i0, y_\nu) = \lambda p_1^k(1 - i0, p_\nu)$; $\Xi_{lj}(\lambda - i\epsilon, y_\nu) = \lambda(\Xi_{lj}(p_1^l, p_1^j, p_\nu) + i0)$. Subsequently, we use the same symbol Ξ_{lj} to represent the phase function in terms of p_ν. Thus the integrals in λ that appear in (3.3) are

$$\int_0^{\infty} \lambda^{N-1} \exp [\pm i(\Xi_{lj} \pm i0)\lambda] \, d\lambda$$

which are evaluated in appendix–3. Hence the reflected Riemann matrix for the system $Lu = 0$ is

$$\sum_{l,j} -\frac{1}{(2\pi i)^N} \int_{-\infty}^{\infty} dp_\nu [T_{lj} \log^{(N)} (\Xi_{lj} + i0) + (-1)^N T_{lj}^* \log^{(N)} (\Xi_{lj}^* - i0)], \tag{3.4}$$

where $\Xi_{lj} = x_1 p_1^l + (x_\nu - z_\nu)p_\nu - z_1 p_1^j + t$.

We outline briefly the result for multiple characteristic sheets. The trial solution can be expressed in the form (3.1) with each term consisting of distinct nullvectors or generalized nullvectors. It turns out that if the system is strongly hyperbolic then the reflected Riemann matrix has the form (3.4). However, for weakly hyperbolic systems T_{lj} is a rational function of λ in (3.3). The following result can then be proved. In general the most singular part of the reflected and incident Riemann matrix, corresponding to a system of reflected wavefronts generated by an incident wavefront, have the same order. If σ is the number of generalized nullvectors associated with the incident wavefront then the singularity is enhanced by σ derivatives (*cf.* [8]).

If the boundary operator is not homogeneous in the sense indicated at the beginning of this section, T_{lj} is a rational function of λ. Then the λ-integration in (3.3) can be performed using the result in appendix–4.

4. Asymptotic estimates. In this section we present several lemmas involving the leading term in the asymptotic estimate, or most singular part, of integrals of distributions. The singularity of an integral of certain distributions may be represented as a series of fractional integrals of increasing order (*cf.* [8], p. 47; [4]).

Fractional integrals are defined, for some a > 0, by

$$I_\pm^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_0^a u^{\alpha-1} f(s \mp u)\, du. \tag{4.1}$$

Some of the common properties of these integrals are:

$$I_\pm^\alpha I_\pm^\beta = I_\pm^{\alpha+\beta}, \qquad \frac{d}{ds} I_\pm^\alpha = \pm I_\pm^{\alpha-1}, \qquad \frac{d}{ds} I_\pm^1 f(s) = \pm f(s).$$

We formally extend the limit of integration in (4.1) so that the singular part of fractional integrals of f are then defined by (4.1) with a = ∞.

Lemma 4.1. Let f (Ξ) be a distribution singular only at the origin. Let Ξ and g be C^∞ functions in their arguments. Suppose that the phase function Ξ, which is real, has leading terms in the power series expansion of the form

$$s + pv + \frac{1}{2} \sum_{i,j=1}^{N-1} c_{ij} u_i u_j\ , \tag{4.2}$$

where $\det(c_{ij}) \neq 0$. *Then the integral*

$$\int_0^a f(\Xi) h(u)\, dv\, du \tag{4.3}$$

has an asymptotic estimate whose leading term is

$$\frac{h(0)(\pi/2)^{(N-1)/2}\,|p|^{-1}}{|\det(c_{ij})|^{1/2}}\, I_\pm^1 I_-^{(N-1-\alpha)/2} I_+^{\alpha/2} f(s), \tag{4.4}$$

where α denotes the number of negative eigenvalues of (c_{ij}). Also, the sign of the fractional integral $I_\pm^1$ is equal to $-\operatorname{sgn}(p)$ *and $h(0)$ is the value of $h(u)$ at $u = 0$.*

If $h(0)$ vanishes in (4.3) then a higher term in the power series expansion of $h(u)$ must be taken to determine the asymptotic estimate. In most cases that we consider $f(s)$ is the logarithmic distribution $\log^{(N)}(s \pm i0)$. The singular part of fractional integrals in (4.4) involving the logartihmic distribution are listed in the appendix.

Lemma 4.2. Integrals of the type (4.3) *with limits of integration* $-a$ *to* a *and a C^∞ function $h(u)$ are not singular if the phase function is not stationary.*

To prove this we need only consider an integral of the form

$$\int_{-a}^a u^\beta f(s + qu)\, du, \qquad q > 0,$$

where β is a positive integer. In terms of its singular part, this integral is equal to

$$|q|^{1-\beta}\, [I_-^{\beta+1} f(s) + (-1)^\beta I_+^{\beta+1} f(s)]$$

which vanishes. This follows if we express $f(s) = (d/ds)^{\beta+1} g(s)$ and then simplify. This lemma is not true if the condition on $h(u)$ is removed.

In the case of complex branch and boundary waves the phase function Ξ_{lj} in (3.4) is complex. Thus the integrals may not be singular, however, their magnitude can be quite large. Integrals that we consider in sections six and seven are of the type

$$\int_{E_{N-1}} h(u, v) \log^{(N)} \left[s + \rho v + \frac{1}{2} \sum_{i,j=1}^{N-2} c_{ij} u_i u_j \right] du\, dv, \tag{4.5}$$

where $h(u,v)$ has a branch point or pole at $v = 0$. Also, s and ρ are complex and c_{ij} are real. The asymptotic estimate of (4.5) in the u-variables follows the pattern indicated in lemma 4.1. Fractional integrals for s complex, as well as the estimates of (4.5) in the v-variable, are given in appendix—6.

5. The main reflected wavefronts. We describe the formation of the reflected wavefronts in a chronological sequence assuming that the slowness surface has n distinct ovals (*cf.* Fig. 1.1). When the first incident wavefront impinges on the boundary, n reflected wavefronts form corresponding to the number of possible modes of propagation into the medium ($x_1 > 0$). Subsequently, for this incident sheet no other wavefronts form. For the second incident wavefront there are initially n possible reflected sheets. But at some instant the outermost reflected sheet breaks away from the incident sheet on the boundary and propagates with a constant phase velocity along the boundary. At the moment of separation $n - 1$ conical wavefronts form which link the outer sheet on the boundary to the points of tangency with all the inner reflected sheets. For the third incident sheet two reflected sheets become detached from the incident sheet. The second outermost reflected wavefront generates $n - 2$ real branch waves and one complex branch wave. Other complex branch waves may be present which we describe in section six.

We determine the most singular part of (3.4) corresponding to the lth main reflected wavefront generated by the jth incident wavefront. For some x and z there is some value of p_i, say p_i°, for which the first derivatives of the phase function Ξ_{lj} vanish. The leading terms in the power series expansion for the phase function in (3.4) are

$$\Xi_{lj} \approx s + \tfrac{1}{2} \sum c_{ij} \xi_i \xi_j , \tag{5.1}$$

where $s = \Xi\,(x, p_i^\circ)$ and $\xi_\nu = p_\nu - p_\nu^\circ$.

Applying lemma 4.1 to (3.4), we have

$$\sum_{l,j} \frac{-i^{\alpha-N}}{(2\pi)^{(N+1)/2} Q^{1/2}} I_-^{(N-1)/2} [(-1)^\alpha T_{lj}^\circ \log^{(N)} (s + i0) + (-1)^N T_{lj}^{*\circ} \log^{(N)} (s - i0)], \tag{5.2}$$

where α is the number of negative eigenvalues of (c_{ij}); $Q = |\det(c_{ij})|$; and the symbol '°' implies that the coefficients are evaluated at $\xi = 0$. Since the first derivatives of the phase function vanish p_i° can be expressed in terms of x_1, $x_\nu - z_\nu$, z_1. Then p_i° can be eliminated in T_{li}°, Q, and s.

Using the properties of fractional integrals listed in appendix—1, we have

Theorem 5.1. *The most singular part of the reflected Riemann matrix corresponding to the main reflected wavefronts is, from* (5.2),

$$(5.3) \quad \sum_{l,i} \frac{i^\alpha \Gamma((N+1)/2)}{Q^{1/2}(2\pi)^{(N+1)/2}} \cdot [i^N((-1)^\alpha T_{li}^\circ + T_{li}^{*\circ})s_+^{-(N+1)/2} - i((-1)^\alpha T_{li}^\circ - T_{li}^{*\circ})s_-^{-(N+1)/2}]$$

for N even and

$$(5.4) \quad \sum_{l,i} \frac{i^\alpha}{Q^{1/2}(2\pi)^{(N+1)/2}} \left[\pi((-1)^\alpha T_{li}^\circ + T_{li}^{*\circ})\, \delta^{(N-1)/2}(s) + i^N((-1)^\alpha T_{li}^\circ - T_{li}^{*\circ})\Gamma\left(\frac{N+1}{2}\right)s^{-(N+1)/2}\right]$$

for N odd.

For the definition of the generalized functions in these expressions we refer to [6], p. 51. These distributions act upon the datum function $g(z)$ in (2.2) to produce part of the displacement vector $u(x,t)$.

The amplitude of (5.3) and (5.4), for fixed s, behave as $o(|x|^{-(N-1)/2})$ where $|x|$ is the magnitude of x. Also $T_{li} = T_{li}^*$ for those regions of the reflected wave surfaces that form before any of the sheets separate from the incident sheet.

The support for the second terms in (5.3) and (5.4) is not indicated in (5.2) and must be determined by other means (s is negative for points outside the lth sheet).

6. Real and complex branch waves. In this section the most singular part of the reflected Riemann matrix due to branch waves is determined. Initially, we consider the nature of the functions T_{li} and Ξ_{li} in (3.4) in a neighborhood of a branch point.

The branch points in the reflection coefficients are due to branch points in the normal component p_1^k of the slowness vector. To illustrate this take an arbitrary incident slowness vector p and draw a line L perpendicular to the plane $p_1 = 0$ as indicated in Fig. 1.1a. The intersection of this line with the appropriate slowness sheets defines the reflected slowness vectors. At the points of tangency of this line with a slowness sheet the normal component p_1^k has a branch point. Let $\xi_1, \cdots, \xi_{N-1}$ be a local orthogonal coordinate system in the p_ν-plane where $\xi_1 = 0$ defines the branch points of p_1^k. If the slowness sheets are ovals then p_1^k has the form

$$(6.1) \quad p_1^k = a^k(\xi) - (\xi_1 - i0)^{1/2}b^k(\xi), \qquad b^k(\xi) > 0$$

in some neighborhood of $\xi = 0$ where a^k and b^k are C^∞ functions. The distributions $(\xi_1 \pm i0)^{1/2}$ are equal to $(\xi_1)_+^{1/2} \pm i\,(\xi_1)_-^{1/2}$ where $(\xi_1)_\pm^\alpha$ is defined in appendix-1 (*cf.* [6], p. 60). p_1^k has the form (6.1) for two reasons. The imaginary part of p_1^k cannot be negative in order to have a bounded trial solution in (5.1). Also, for $\xi_1 > 0$, (6.1) implies that 'energy' is propagating into the medium ($x_1 > 0$). If the slowness sheets are not ovals then p_1^k may differ from (6.1). Consider the plane containing the line L and the normal to the slowness sheet at $\xi = 0$. If the curve of intersection of this plane with the slowness sheet is locally concave at $\xi = 0$ then p_1^k has the form (6.1). If it is locally convex then b^k is replaced by $-ic^k$ where $c^k > 0$ and is a C^∞ function.

Since p_1^k appears in the coefficients T_{lj} in (3.4) then T_{lj} has the same form as p_1^k in a neighborhood of $\xi_1 = 0$, *viz*:

$$T_{lj} = A_{ljk}(\xi) - (\xi_1 - i0)^{1/2} B_{ljk}(\xi), \tag{6.2}$$

where A_{ljk} and B_{ljk} are C^∞ functions.

The next step is to determine the leading terms in the power series expansion of the phase function, and we consider, initially, the case in which Ξ_{lj} is real. Thus,

$$\Xi_{lj} \approx s + q\xi_1 + \sum_{i=2}^{N-1} q_i\xi_i + \tfrac{1}{2} \sum c_{i\nu}\xi_i\xi_\nu \,. \tag{6.3}$$

For some x and z there is a point p_ν° on the slowness surface such that $q_i = 0$. Also, the $\xi_1\,\xi_i$ terms are second order ones compared with $\xi_1 = 0$, so that we have

$$\Xi_{lj} \approx s + q\xi_1 + \frac{1}{2} \sum_{i,\nu=2}^{N-1} c_{i\nu}\xi_i\xi_\nu \equiv \Xi_{lj}^1 + q\xi_1 \,. \tag{6.4}$$

For points between the boundary and the point of tangency $q > 0$. This can be deduced most readily from particular forms for Ξ_{lj}, and the example in section 8 illustrates this point.

To find the asymptotic estimate of (3.4) we substitute (6.2) and (6.4) into (3.4). The integral with A_{ljk} in the integrand cannot be singular by lemma 4.2. The asymptotic estimate of (3.4) in the ξ_1 variable is more difficult than the others so we consider it separately. In (3.4) we have an integral in the ξ_1 variable of the form

$$-\int_{-a}^{a} (\xi_1 - i0)^{1/2} \log^{(N)} (\Xi_{lj}^1 + q\xi_1 + i0)\, d\xi_1 \tag{6.5}$$

from the first term and

$$-\int_{-a}^{a} (\xi_1 + i0)^{1/2} \log^{(N)} (\Xi_{lj}^1 + q\xi_1 - i0)\, d\xi_1 \tag{6.6}$$

from the second term. Using the relations in appendix-2 the singular part of these integrals is, respectively,

$$-\pi^{1/2} q_+^{-3/2} I_-^{3/2} \log^{(N)} (\Xi_{lj}^1 + i0), \qquad -\pi^{1/2} q_+^{-3/2} I_-^{3/2} \log^{(N)} (\Xi_{lj}^1 - i0). \tag{6.7}$$

We now show that the distributional interpretation of $q^{-3/2}$ in (6.7) is correct. In (2.2) let q be the independent variable and suppose that the test function in q is $\phi(q)$. We write (2.2) as $\langle R, \phi(q)\rangle$ and consider only the q and ξ_1 variables explicitly in what follows. Thus one term in $\langle R, \phi\rangle$ from (6.5) and (3.4) is

$$-\frac{1}{(2\pi i)^N}\langle\langle(\xi_1 - i0)^{1/2}\log^{(N)}(\Xi^1_{lj}(q) + q\xi_1 + i0), B_{ljk}(q)\rangle, \phi(q)\rangle.$$

Taking the asymptotic estimate in the ξ_1 variable then we have

$$-\frac{\pi^{1/2}}{(2\pi i)^N}\langle q_+^{-3/2}, B^\circ_{ljk}(q)\, I_-^{3/2}\log^{(N)}(\Xi^1_{lj}(q) + i0)\phi(q)\rangle$$

$$+\frac{1}{(2\pi i)^N}\langle\langle(\xi_1 - i0)^{1/2}\log^{(N)}(\Xi^1_{lj}(0) + q\xi_1 + i0), B^\circ_{ljk}(0)\rangle, \phi(0)\rangle.$$

Evaluating the last term in the q-variable it follows that there is no contribution from it to (6.7).

The leading terms in the power series expansion of Ξ_{lj} may differ from (6.4). Regarding (3.4) as a distribution in z for fixed x, then $q = 0$, $q_i = 0$ in (6.3) on an $(N - 1)$-dimensional surface in z-space. As a result there is no contribution to the displacement vector (2.2) from the singularity of (3.4) in this case.

The asymptotic estimate of (3.4) with respect to the other variables ξ_i, $i \neq 1$, follows directly from lemma 4.1. The asymptotic estimate of (3.4) is

$$\text{(6.8)} \qquad \sum \frac{q_+^{-3/2} i^{\alpha - N}}{2^{1/2}(2\pi)^{(N+1)/2}Q^{1/2}} I_-^{(N+1)/2}[(-1)^\alpha B^\circ_{ljk}\log^{(N)}(s + i0) + (-1)^N B^{*\circ}_{ljk}\log^{(N)}(s - i0)]$$

where α is the number of negative eigenvalues of $(c_{i\nu})$ in (6.4); $Q = |\det(c_{i\nu})|$; and the symbol '$\circ$' implies that the coefficients are evaluated at $\xi = 0$.

Theorem 6.1. *The most singular part of the reflected Riemann matrix corresponding to the real branch waves is, from* (6.8),

$$\text{(6.9)} \qquad \sum \frac{q_+^{-3/2}\Gamma\left(\frac{N-1}{2}\right)i^\alpha}{(2\pi)^{(N+1)/2}2^{1/2}Q^{1/2}}[-i^N((-1)^\alpha B^\circ_{ljk} + B^{*\circ}_{ljk})s_+^{-(N-1)/2} - i((-1)^\alpha B^\circ_{ljk} - B^{*\circ}_{ljk})s_-^{-(N-1)/2}],$$

$$\text{(6.10)} \qquad \sum \frac{q_+^{-3/2} i^\alpha}{2^{1/2}(2\pi)^{(N+1)/2}Q^{1/2}}\left[\pi((-1)^\alpha B^\circ_{ljk} + B^{*\circ}_{ljk})\,\delta^{(N-3)/2}(s) - i^N((-1)^\alpha B^\circ_{ljk} - B^{*\circ}_{ljk})\Gamma\left(\frac{N-1}{2}\right)s^{-(N-1)/2}\right]$$

for N even and odd, respectively.

For the first set of branch waves that form from each incident sheet $B^\circ_{ljk} =$

$B_{lik}^{*\circ}$. The amplitude of (6.9) and (6.10) for fixed s behave as $o(|x|^{-(N+1)/2})$ where $|x|$ is the magnitude of x. Brekhovskikh ([1], p. 281) shows that the rate of decay with respect to x in a particular case is three-quarters of a degree less rapid than indicated in theorem 6.1 for small q.

If p_1^l and/or p_1^j are complex in a neighborhood of $\xi_1 = 0$, where $p_1^l \neq p_1^k$ and $p_1^j \neq p_1^k$ in (6.1) for $\xi_1 = 0$, then we have complex branch waves. We suppose the phase function Ξ_{lj} to be such that the imaginary part is a function of ξ_1 only. To indicate the reason for this consider the power series expansion (6.3). In order to have a major contribution the first derivatives q_i, $i > 2$, must vanish so that there are $N - 2$ equations to solve for the $N - 2$ independent unknowns q_ν° on the curve defined by $\xi_1 = 0$. Now suppose that the imaginary part Ξ_{lj} contains ξ_j, $j \neq 1$. Then q_j is complex and so we have a system of $N - 1$ equations to solve. In general one would not expect a consistent solution for p_ν°, so that there will not be a major contribution for the reflected Riemann matrix in this case. If the imaginary part of the phase function is a function of ξ_1 only, then the phase function has a form indicated in (4.5).

Using the results in appendix-6 we have

Theorem 6.2. *The leading term in the asymptotic estimate of the magnitude of the reflected Riemann matrix corresponding to complex branch waves is*

$$\sum \frac{(-1)^{\alpha+N}\Gamma\left(\frac{N-1}{2}\right)(\operatorname{sgn}\operatorname{Re} s^{1/2}/q^{1/2})H(\operatorname{Im} s/q)}{2^{1/2}(2\pi)^{(N+1)/2}Q^{1/2}} \cdot\left[\frac{(-1)^N B_{lik}^{\circ}}{q^{3/2}} s^{-(N-1)/2} + \frac{B_{lik}^{*\circ}}{q^{*3/2}} s^{*-(N-1)/2}\right],$$

where α is equal to the number of negative eigenvalues of (c_{ij}) in (4.5). This asymptotic estimate is meaningful if the magnitude of s/Q and s/q are sufficiently small (i.e., x is sufficiently far from the source and close to the boundary).

The remaining cases to be considered are: $p_1^j = p_1^k$ or $p_1^l = p_1^k$ on $\xi_1 = 0$. In these cases the distribution $(\xi_1 - i0)^{1/2}$ appears in the phase function. Let the phase function in (6.5) be

$$\Xi_{lj} = f(\xi_1)(\xi_1 - i0)^{1/2} + g(\xi),$$

where f and g are C^∞ functions. Then (6.5) can be expressed as

$$-2 \oint_L w^2\, dw \log^{(N)} [f(w^2)w + g(w^2)],$$

where the contour L goes from $i\infty$ to ∞ along the coordinate axes. Since the first derivative of the phase function is real and does not vanish at the origin, this integral is non-singular by lemma 4.2.

7. Real and complex boundary waves. As stated in the Introduction the question of uniqueness of the solution arises when there are poles in the reflection coefficients. In this section the nature of the solution generated by these poles is shown as well as the form that the non-uniqueness can take. However, uniqueness in the asymptotic estimate is obtained by imposing an additional condition on the solution. We take this condition to be: The reflected Riemann matrix must vanish for sufficiently small time t.

We assume that the reflection coefficients $D_{lj}(1 - i0, p_\nu)$ have a simple pole on a hypercurve H in the p_ν -plane (*cf.* Fig. 1.2). Complex poles in D_{lj} will not be considered. Since the denominator of D_{lj} is a homogeneous function in p_ν and $(1 - i0)$ then upon changing to spherical polars with a radial variable r, the denominator is a homogeneous function in r and $(1 - i0)$. Thus, the behavior of D_{lj} in a neighborhood of a pole is the reciprocal of $r - r_0\,(1 - i0)$. We change to the local orthogonal coordinate system ξ employed in the case of branch waves where $\xi_1 = 0$ defines the curve H and ξ_1 is positive for points within H. Then,

$$T_{lj} = C_{lj}/\xi_1 - i0 = C_{lj}(\xi_1^{-1} + i\pi\delta(\xi)) \tag{7.1}$$

in a neighborhood of H where C_{lj} is a C^∞ function. The poles in T_{lj} and T_{lj}^* may not coincide and so the indices 1 and 2 will refer to the first and second terms, respectively, in (3.4).

In the case of real boundary waves the expansion of the phase function about p_ν° on H in terms of the coordinate system ξ has the form (6.4). Using the results in appendix-5 we have, upon performing the ξ_1 -integration in (3.4),

$$\begin{aligned} -\frac{1}{(2\pi i)^{N-1}} \int_{-\infty}^{\infty} [C_{lj}^{1\circ} H(q_1) \log^{(N)} (\Xi_{lj}^1 + i0) \\ + (-1)^{N+1} C_{lj}^{*2\circ} H(q_2) \log^{(N)} (\Xi_{lj}^{*2} - i0)]\, d\xi_\nu \,, \end{aligned} \tag{7.2}$$

where the symbol '$\circ$' implies that the coefficients are evaluated at $\xi = 0$.

Theorem 7.1. *The most singular part of the reflecied Riemann matrix corresponding to the real boundary waves is, from* (7.2),

$$\begin{aligned} \frac{-i^{\alpha-N+1}}{(2\pi)^{N/2}} I_-^{(N-2)/2} \Bigg[\frac{(-1)^\alpha C_{lj}^{1\circ}}{Q_1^{1/2}} H(q_1) \log^{(N)} (s_1 + i0) \\ + \frac{(-1)^{N+1} C_{lj}^{*2\circ}}{Q_2^{1/2}} H(q_2) \log^{(N)} (s_2 - i0) \Bigg], \end{aligned} \tag{7.3}$$

where α is equal to the number of negative eigenvalues of $(c_{i\nu})$ in (6.4).

For the case in which all p_1^l and p_1^j are real for p_ν on the singular locus H then $q_1 = q_2$, $Q_1 = Q_2^*$, etc.. In this case with $\alpha = 0$, (7.3) reduces to

$$\frac{H(q) C_{lj}^\circ \delta^{N/2}(s)}{Q^{1/2}(2\pi)^{(N-2)/2}}, \qquad \frac{H(q) C_{lj}^\circ s_+^{-(N+2)/2}}{(2\pi)^{(N-2)/2} Q^{1/2} \Gamma(-N/2)}$$

for N even and odd, respectively.

If p_1^l or p_1^j, or both, in the phase function Ξ_{lj} are complex on H then we have complex boundary waves. In this case the phase function has the same form as that of the complex branch waves (*cf.* (4.5); sec. 6). To determine the asymptotic estimate of (3.4) in ξ variables we use the results in appendix-6 and lemma 4.1.

Theorem 7.2. *The asymptotic estimate of the reflected Riemann matrix corresponding to the complex boundary waves is*

$$\frac{(-i)^{N+\alpha+1}\Gamma\left(\frac{N+2}{2}\right)}{(2\pi)^{N/2}}\left[(-1)^{N+1}\frac{H(\rho_1)C_{lj}^{1\circ}}{Q_1^{1/2}}s_1^{-(N+2)/2}+\frac{H(\rho_2)C_{lj}^{*2\circ}}{Q_2^{1/2}}s_2^{*-(N+2)/2}\right],$$

where $\rho_i = Im\ (s_i/q_i)$.

Since the reflection coefficients are not uniquely determined for p_ν on the hypercurve H we can append terms to (3.1) which vanish for p_ν not on H. If the appendage were not a delta-distribution then it would not contribute to the displacement vector (2.2). Therefore, we suppose that the non-uniqueness in (3.4) is of the form

$$(7.5)\qquad \sum_{l,j}\int_{-\infty}^{\infty}\delta(\xi_1)[C_{lj}^m(\xi)\log^{(m)}(\Xi_{lj}+i0) + (-1)^{N+1}M_{lj}^m(\xi)\log^{(m)}(\Xi_{lj}-i0)]\,d\xi,$$

for some C_{lj}^m, M_{lj}^m, and integer m. However, in the case of real boundary waves the support for the asymptotic estimate of (7.5) extends out from the boundary with no cut-off point. Thus, the coefficients in (7.5) must be identically zero by the principle stated at the beginning of this section. There are greater difficulties in the case of complex boundary waves. If, as seems likely, the support for (3.4) corresponding to complex boundary waves is bounded by the outermost reflected wavefront for a given incident wavefront, then this case is analogous to the previous one. That is, (7.5), although it is not singular except possibly on the boundary, must vanish in some neighborhood of $t = 0$ and so (7.5) must vanish identically.

We note that if there are poles in the reflection coefficients then these poles appear in the leading term of the reflected Riemann matrix (5.3) and (5.4). In this context it is understood that T_{lj}° be regarded as distributions.

8. Example. In this section we present an example to illustrate the theory in the preceding sections. In order to simplify the numerical work and at the same time have branch and boundary waves we take the case in which there is transmission through the boundary. The differential equation governing propagation in the region $x_1 > 0$ is

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\partial/\partial t - \begin{pmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{pmatrix}\partial/\partial x_1 - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\partial/\partial x_2\right]u(x,t) = 0,$$

and in the region $x_1 < 0$,

$$\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \partial/\partial t - \begin{pmatrix}0 & 2\\ 2 & 0\end{pmatrix} \partial/\partial x_1 - \begin{pmatrix}0 & -2i\\ 2i & 0\end{pmatrix} \partial/\partial x_2\right] u(x,\, t) = 0.$$

For boundary conditions we take

$$\begin{pmatrix}\partial/\partial x_1 + i\,\partial/\partial x_2 & 0\\ 0 & 1\end{pmatrix} u(x,\, t) = 0.$$

The slowness surfaces in the two media are defined by $(p_1)^2 + 4(p_2)^2 = 4$ and $4(p_1)^2 + 4(p_2)^2 = 1$ for $x_1 > 0$ and $x_1 < 0$, respectively (*cf.* Introduction). Indicated in Fig. 8.1a is half of each slowness surface corresponding to the set of reflected ($p_1 < 0$) and transmitted ($p_1 > 0$) slowness vectors. The dotted lines in Fig. 8.1a refer to the presence of poles in the reflection coefficients. The system of wavefronts and the complex boundary waves (dotted lines) are illustrated in Fig. 8.1b.

In (2.3),

$$\Lambda \equiv (\lambda - i\epsilon)E - A(y) = \begin{bmatrix}\lambda - i\epsilon & -y_1/2 + iy_2\\ -y_1/2 - iy_2 & \lambda - i\epsilon\end{bmatrix}.$$

Then $\det \Lambda = -\frac{1}{4}\,(y_1 - y_1^1)\,(y_1 - y_1^2)$ where $y_1^1 = 2[(\lambda - i\epsilon)^2 - (y_2)^2]^{1/2}$ and $y_1^2 = -\,y_1^1$. The incident root is y_1^1 and the reflected root is y_1^2 . The adjoint matrix B of Λ is Λ. The integral (2.4) can be evaluated readily and so (2.7), representing the incident system of waves, is

$$[(\lambda - i\epsilon)^2 - (y_2)^2]^{-1/2}\begin{pmatrix}\lambda - i\epsilon & -y_1^1 + iy_2\\ -y_1^1/2 - iy_2 & \lambda - i\epsilon\end{pmatrix} e^{i\Xi_1},$$

where $\Xi_1 = (x_1 - z_1)\, y_1^1 + (x_2 - z_2)\, y_2 + (\lambda - i\epsilon)\, t$.

To form the reflected system of waves (3.2), substitute y_1^2 into B and take the first column as the reflected nullvector. Since the second column in $B_1 = (B)_{y_1 = y_1^2}$,

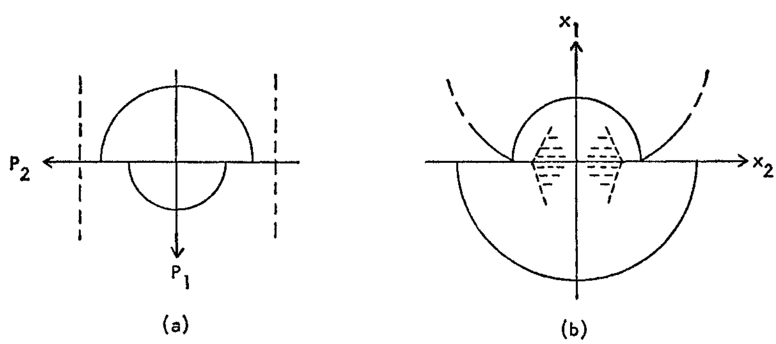

FIGURE 8.1

is $(\lambda - i\epsilon)^{-1} (- y^1_{1/2} + iy_2)^{-1}$, (call this k), times the first we take B^2_1 as

$$\begin{pmatrix} \lambda - i\epsilon & k(\lambda - i\epsilon) \\ -y^2_1/2 - iy_2 & k(-y^2_1/2 - iy_2) \end{pmatrix}$$

so that the columns of B^2_1 have the same constant of proportionality as the corresponding columns in B_1. Similarly, in the case of transmission we take B^t_1 as

$$\begin{pmatrix} \lambda - i\epsilon & k(\lambda - i\epsilon) \\ -2y^t_1 - 2iy_2 & k(-2y^t_1 - 2iy_2) \end{pmatrix},$$

where the transmitted root

$$y^t_1 = \left[\left(\frac{\lambda - i\epsilon}{2}\right)^2 - (y_2)^2\right]^{1/2}$$

for which $\operatorname{Im} y^t_1 < 0$. Thus, the trial solution in (3.1) is

$$u = B_1 e^{i\Xi_1} + D_{21} B^2_1 e^{i\Xi_{21}} + D_{t1} B^t_1 e^{i\Xi_{t1}},$$

where $\Xi_{t1} = x_1 y^t_1 + (x_2 - z_2)\, y_2 - z_1\, y^1_1 + (\lambda - i\epsilon)\, t$.

To formulate (3.4) we let ϵ approach zero and then let $y_2 = \lambda p$. Thus, (3.4) is

$$\frac{1}{4\pi^2} \int_{-\infty}^{\infty} dp [T_{21} \log^{(2)} (\Xi_{21} + i0) + T^*_{21} \log^{(2)} (\Xi^*_{21} - i0)], \tag{8.1}$$

where $\Xi_{21} = x_1\, p^2_1 + (x_2 - z_2)\, p + t - z_1 p^1_1$, $p^2_1 = -2[(1 - i0)^2 - p^2]^{1/2}$, $p^1_1 = -p^2_1$, and $T_{21} = D_{21}\, B^2_1\, [(1 - i0)^2 - p^2]^{-1/2}$. The transmitted root that appears in T_{21} is $p^t_1 = [(1 - i0/2)^2 - p^2]^{1/2}$.

Applying the method of stationary phase we expand the phase function Ξ_{21} in a Taylor series about the point p° for which the first derivative vanishes. Thus, in (5.3)

$$Q = [4(x_1 + z_1)^2 + (x_2 - z_2)^2]^{3/2}/4(x_1 + z_1)^2,$$
$$s = -[4(x_1 + z_1)^2 + (x_2 - z_2)^2]^{1/2} + t.$$

From (5.3) the leading term in the asymptotic estimate of (8.1) corresponding to that part of the main reflected wavefront not preceded by a branch wave is $(\alpha = 0)$

$$\frac{D^\circ_{21} B^{\circ 2}_1 s_+^{-3/2}}{8^{1/2}\pi[4(x_1 + z_1)^2 + (x_2 - z_2)^2]^{1/4}}.$$

For $p = \pm \frac{1}{2}$, p^t_1 has a branch point. We define the coordinate ξ_1 as $\frac{1}{2} - p$ and $p - \frac{1}{2}$ in a neighborhood of $p = \frac{1}{2}$ and $p = -\frac{1}{2}$, respectively. The expansion of the phase function (6.4) is $\Xi_{21} \approx s + q\, \xi_1$ where

$$s = -(x_1 + z_1)3^{1/2} \pm (x_2 - z_2)/2 + t$$
$$q = -(x_1 + z_1)/3^{1/2} \mp (x_2 - z_2)/2.$$

From this it follows that $q > 0$ for points between the boundary and the point of tangency. In a neighborhood of $p = \pm \frac{1}{2}$,

$$p_1^t = (\xi_1)_+^{1/2} - i(\xi_1)_-^{1/2}$$

and so T_{21} must have a similar form (*cf.* (6.2)). The asymptotic estimate (6.9) of the reflected Riemann matrix corresponding to the branch wave is ($\alpha = 0$, $Q = 1$)

$$\pi^{-1} q_+^{-3/2} B_{21t}^{\circ} s_+^{-1/2}.$$

For the complex boundary wave we need only determine the nature of the denominator Δ of D_{21} in a neighborhood of the pole. We have

$$\Delta = \det \begin{vmatrix} -2[(1 - i0)^2 - p^2]^{1/2} + ip, & \left[\left(\frac{1 - i0}{2}\right)^2 - p^2\right]^{1/2} + ip \\ [(1 - i0)^2 - p^2]^{1/2} - ip, & -2\left[\left(\frac{1 - i0}{2}\right)^2 - p^2\right]^{1/2} - 2ip \end{vmatrix}$$

which vanishes when $p = -3(1 - i0)/8^{1/2}$. It follows that if $\xi_1 = p + 3/8^{1/2}$ then Δ is proportional to $\xi_1 - i0$ as indicated in (7.1). The pole in D_{21}^* occurs at $p = 3(1 + i0)/8^{1/2}$ and we define $\xi_1 = 3/8^{1/2} - p$ in this case. The phase function Ξ_{21} in (8.1) has the expansion (6.4) where

$$s_1 = (x_1 + z_1)i/2^{1/2} - (x_2 - z_2)3/8^{1/2} + t,$$
$$q_1 = -6i(x_1 + z_1) - (x_2 - z_2).$$

Thus, in the asymptotic estimate (7.4) we have

$$H(\rho_1) = H(t - 5(x_2 - z_2)/3.2^{1/2}),$$
$$H(\rho_2) = H(t + 5(x_2 - z_2)/3.2^{1/2}),$$
$$s_2^* = -(x_1 + z_1)i/2^{1/2} + (x_2 - z_2)3/8^{1/2} + t,$$

and so (7.4) is equal to ($\alpha = 0$, $Q_1 = Q_2 = 1$)

$$\frac{1}{2\pi i} [H(\rho_1) C_{21}^{1\circ} s_1^{-2} - H(\rho_2) C_{21}^{*2\circ} s_2^{*-2}].$$

Appendix. We present in this appendix some of the properties of fractional integrals (*cf.* [8], p. 45). In the sequel we deal only with the singular parts of integrals.

1. We define

$$s_+^\alpha = \begin{cases} s^\alpha & s > 0 \\ 0 & s < 0 \end{cases} \quad \text{and} \quad s_-^\alpha = \begin{cases} 0 & s > 0 \\ (-s)^\alpha & s < 0. \end{cases}$$

For $\alpha \neq$ integer

$$I_+^\alpha \, \delta(s) = \frac{s_+^{\alpha - 1}}{\Gamma(\alpha)}, \qquad I_-^\alpha \, \delta(s) = \frac{s_-^{\alpha-1}}{\Gamma(\alpha)}.$$

For $0 < \alpha < 1$,

$$I_+^\alpha \log^{(1)}(s) = \Gamma(1-\alpha)\cos(\alpha\pi)s_+^{\alpha-1} - \Gamma(1-\alpha)s_-^{\alpha-1},$$

$$I_-^\alpha \log^{(1)}(s) = \Gamma(1-\alpha)s_+^{\alpha-1} - \Gamma(1-\alpha)\cos(\alpha\pi)s_-^{\alpha-1},$$

$$I_-^{1/2}\log^{(1)}(s) = \pi I_+^{1/2}\,\delta(s), \qquad I_+^{1/2}\log^{(1)}(s) = -\pi I_-^{1/2}\,\delta(s),$$

$$I_\pm^\beta f^{(N)}(s) = (\pm 1)^\beta f^{(N-\beta)}(s) = (\pm 1)^N I_\pm^{\beta-N} f(s),$$

where β is a positive integer.

2. Since

$$I_+^{1/2}\log^{(1)}(s \pm i0) = \mp i\sqrt{\pi}\,s_+^{-1/2} - \sqrt{\pi}\,s_-^{-1/2},$$

$$I_-^{1/2}\log^{(1)}(s \pm i0) = \sqrt{\pi}\,s_+^{-1/2} \mp i\sqrt{\pi}\,s_-^{-1/2}$$

then it follows that

$$I_+^{1/2}\log^{(1)}(s \pm i0) = \mp i I_-^{1/2}\log^{(1)}(s \pm i0),$$

$$I_+^{3/2}\log^{(N)}(s \pm i0) = \pm i I_-^{3/2}\log^{(N)}(s \pm i0),$$

$$I_+^{\beta/2}\log^{(1)}(s \pm i0) = (\mp i)^\beta I_-^{1/2}\log^{(1)}(s \pm i0),$$

where β is a positive integer.

3.

$$\int_0^\infty \lambda^{N-1} e^{\pm i(\xi \pm i0)\lambda}\,d\lambda = \begin{cases} -(\pm i)^{-N}\log^{(N)}(\xi \pm i0) & \xi \text{ real} \\ -(\pm i)^{-N}\log^{(N)}(\xi) & \xi \text{ complex,} \end{cases}$$

$$\log^{(N)}(\xi \pm i0) = \left(\frac{d}{d\xi}\right)^{N-1}\log^{(1)}(\xi \pm i0) = \left(\frac{d}{d\xi}\right)^{N-1}\left[\frac{1}{\xi} \mp i\pi\delta(\xi)\right]$$

To prove this we change $i0$ to $i\epsilon(\epsilon > 0)$, perform the integration and then let ϵ approach zero. For details on the logarithmic distribution we refer to [6].

4.

$$\text{P.V.}\int_0^\infty \frac{e^{i(\xi+i0)\lambda}\,d\lambda}{\lambda + d} = -\sum_{j=0}^{\infty}(-id)^j \log^{(-j)}(\xi + i0).$$

To determine this asymptotic expansion we replace $i0$ by $i\epsilon$ $(\epsilon > 0)$ and show that

$$\frac{\partial}{\partial\xi}\text{P.V.}\int_0^\infty \frac{e^{i(\xi+i\epsilon)(\lambda+d)}}{\lambda + d}\,d\lambda = -\frac{e^{i(\xi+i\epsilon)d}}{\xi + i\epsilon}.$$

Upon integration by parts the result follows, where the negative superscript on the logarithmic distribution implies that we must integrate the distribution j times. The same procedure is applicable if the integrand is a more complicated rational function in λ.

5.

$$\int_{-\infty}^{\infty} \frac{du}{u} \log^{(N)} (s + \rho u \pm i0) = \pm \pi i \operatorname{sgn} \rho \log^{(N)} (s \pm i0).$$

To prove this we consider

$$\int_0^{\infty} \lambda^{N-1}\, d\lambda \int_{-\infty}^{\infty} e^{\pm i(s+\rho u \pm i0)\lambda} \frac{du}{u}.$$

The result follows by evaluating the u-integration and then the λ-integration, and equating this to the integral evaluating only the λ-integration.

6. In the following s and q are complex.

$$\int_{-\infty}^{\infty} \frac{\log^{(1)} (s + qu)\, du}{u} = \pi i \operatorname{sgn} [\operatorname{Im} (s/q)] \log^{(1)} (s),$$

$$\int_{-\infty}^{\infty} \frac{(-u_+^{1/2} + iu_-^{1/2})\, du}{(s + qu)^2} = \pi^{1/2} q^{-3/2} \operatorname{sgn} \operatorname{Re} (s^{1/2}/q^{1/2}) H[\operatorname{Im} (s/q)] I_-^{3/2} \log^{(2)} (s)$$

$$- \int_{-\infty}^{\infty} \frac{(u_+^{1/2} + iu_-^{1/2})\, du}{(s^* + q^*u)^2} = \pi^{1/2} q^{*-3/2} \operatorname{sgn} \operatorname{Re} (s^{1/2}/q^{1/2}) H[\operatorname{Im} (s/q)] I_-^{3/2} \log^{(2)} (s^*),$$

$$I_-^{1/2} \log^{(1)} (s) = \pi^{1/2} s^{-1/2}, \qquad I_+^{1/2} \log^{(1)} (s) = -iI_-^{1/2} \log^{(1)} (s),$$

$$I_-^{1/2} I_-^{1/2} \log^{(2)} (s) = I_-^1 \log^{(2)} (s) = -s^{-1}.$$

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