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# Singularities of Toric Varieties Associated with Finite Distributive Lattices

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**Abstract.** With each finite lattice $L$ we associate a projectively embedded scheme $V(L)$; as Hibi has shown, the lattice $D$ is distributive if and only if $V(D)$ is irreducible, in which case it is a toric variety. We first apply Birkhoff's structure theorem for finite distributive lattices to show that the orbit decomposition of $V(D)$ gives a lattice isomorphic to the lattice of contractions of the bounded poset of join-irreducibles $\hat{P}$ of $D$. Then we describe the singular locus of $V(D)$ by applying some general theory of toric varieties to the fan dual to the order polytope of $P$: $V(D)$ is nonsingular along an orbit closure if and only if each fibre of the corresponding contraction is a tree. Finally, we examine the local rings and associated graded rings of orbit closures in $V(D)$. This leads to a second (self-contained) proof that the singular locus is as described, and a similar combinatorial criterion for the normal link of an orbit closure to be irreducible.

**Keywords:** toric variety, distributive lattice, singular locus, associated graded ring

## 0. Introduction

With a finite lattice $L$ and a field k we associate a graded Noetherian k-algebra as follows. Let $\mathbf{X} := \{X_\alpha : \alpha \in L\}$ be commuting indeterminates over k, and in the polynomial ring k[**X**] define the ideal

$$I(L) := (X_\alpha X_\beta - X_{\alpha\wedge\beta} X_{\alpha\vee\beta} : \alpha, \beta \in L).$$

Finally, we let $\mathsf{k}[L] := \mathsf{k}[\mathbf{X}]/I(L)$. Since $I(L)$ is a homogeneous ideal of k[**X**], the algebra $\mathsf{k}[L]$ inherits the standard grading and may be regarded as the projective coordinate ring of a projectively embedded scheme $V(L) := \mathrm{Proj}\ \mathsf{k}[L]$. An interesting question thus arises: to what extent can the geometric properties of $V(L)$ be related to the combinatorial properties of the lattice $L$, independently of the choice of field k?

As Hibi [7] has observed, the following are equivalent: $L$ is distributive; $V(L)$ is irreducible; $V(L)$ is a toric variety. This follows from Birkhoff's structure theorem and excluded sublattice theorem for finite distributive lattices [6, 14]. This structure theorem is a contravariant equivalence between the category of finite distributive lattices and lattice homomorphisms and the category of finite bounded posets and bounded poset morphisms (we review it in Section 1). Thus, for distributive lattices $D$ we can hope to apply Birkhoff's

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structure theorem to describe the geometry of $V(D)$; this is the project we begin here. A more general structure theorem for modular lattices based on the one due to Benson and Conway [1] (which considers only objects) could provide the basis for a similar description in that case.

Over the past decades the theory of toric varieties has seen many developments, partly because it provides an extensive collection of nontrivial but tractable examples of algebraic-geometric phenomena, and partly because of its connections with convex geometry and combinatorics. Much (but not all) of what we do in Sections 2 and 3 can be derived from the general theory by passing from a distributive lattice $D$ to its bounded poset of join-irreducibles $\hat{P}$, from that to the order polytope $\mathcal{P}$ of $P$, and thence to the fan $\Delta$ dual to $\mathcal{P}$. However, since the structure of $\hat{P}$ determines those of $D$, $\mathcal{P}$, and $\Delta$, it is more efficient simply to establish our results directly. Further motivation for this approach is the hope that it can be generalized to the case of modular lattices, in which the schemes $V(L)$ are reducible. (Note, however, that even for arbitrary lattices the results of Eisenbud and Sturmfels [3] do apply.)

Section 1 reviews some known results which are required: first, Birkhoff's structure theorem for distributive lattices together with a correspondence for properties of morphisms in the two categories (which is partially new); second, the isomorphism of three lattices: the lattice of faces of the order polytope $\mathcal{P}$, the lattice of contractions of $\hat{P}$, and the lattice of embedded sublattices of $D$; and finally, a relation between the Ehrhart polynomial of $\mathcal{P}$ and a normal form for order-reversing functions $\sigma\colon \hat{P} \to \mathbb{N}$.

In Section 2 we apply the results of Section 1 to describe the geometry of $V(D)$. We first show that $\mathsf{k}[D]$ is a normal affine semigroup ring, describe the action of the torus $(\mathsf{k}^{\times})^{P}$ on $V(D)$, and show that the orbit decomposition of this action on $V(D)$ gives a lattice isomorphic to the three in Section 1. We then describe the fan $\Delta$ in terms of the poset $\hat{P}$, and apply the general theorem on singularities of toric varieties. If $L$ is an embedded sublattice of $D$ then the closure of the corresponding orbit of $V(D)$ is isomorphic to $V(L)$, and $V(D)$ is nonsingular along $V(L)$ if and only if each fibre of the corresponding contraction is a tree.

In Section 3 we examine the structure of the local rings $\mathcal{O}_{L,D}$ of $V(D)$ along $V(L)$, and of the associated graded rings $\mathrm{gr}_L(D)$. In particular, we obtain a combinatorial interpretation for the embedding dimension of $V(L)$ in $V(D)$ which leads to a self-contained proof that the singular locus of $V(D)$ is as described in Section 2. Finally, we give a similar criterion for the normal link of the subvariety $V(L)$ in $V(D)$ to be irreducible: this happens if and only if each fibre of the corresponding contraction is a poset which satisfies a certain linear-algebraic condition, which we call a "valuable" poset. This criterion for irreducibility seems not to be derivable at present from a general theorem for toric varieties. Characterization of the class of valuable posets remains as an interesting open problem.

We intend to examine in a future paper the structure of operational Chow homology and cohomology, computation of the Todd class, and application of singular Riemann-Roch to the varieties $V(D)$.

## 1. Distributive lattices and bounded posets

We assume familiarity with the theory of posets and distributive lattices as developed in [6, 14] for instance. All posets we consider are finite. A poset is *bounded* if it has a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$; it is *proper* if $\hat{0} \neq \hat{1}$. Given any poset $P$, we let $\hat{P} := \{\hat{0}\} \oplus P \oplus \{\hat{1}\}$, where $\oplus$ denotes ordinal sum of posets. Given a proper bounded poset $P$ we let $P^\circ := P \backslash \{\hat{0}, \hat{1}\}$. Up to unique isomorphism there is only one bounded poset $O$ with $\hat{0} = \hat{1}$. A *bounded poset morphism* is an order-preserving function $f: P \to Q$ such that $f(\hat{0}) = \hat{0}$ and $f(\hat{1}) = \hat{1}$. We denote the set of bounded poset morphisms from $P$ to $Q$ by $\hom(P, Q)$. Notice that $\hom(O, Q) = \emptyset$ unless $Q = O$; in all other cases $\hom(P, Q)$ inherits a partial order from the product order on $Q^P$, and is a bounded poset. The opposite of a poset $P$ is the same set but with the order reversed, and is denoted by $P^{\mathrm{op}}$.

All distributive lattices $D$ we consider are finite; thus, if $D \neq \emptyset$ then $\hat{0}$ and $\hat{1}$ exist in $D$. We do not require lattice morphisms to preserve $\hat{0}$ and $\hat{1}$, so that although they are order-preserving they need not be bounded poset morphisms.

Given a bounded poset $P$, let $J(P) := \hom(P, \{0, 1\}^{\mathrm{op}})^{\mathrm{op}}$; thus $J(P)$ is the set of order-reversing $\{0, 1\}$-valued functions $\sigma$ on $P$ such that $\sigma(\hat{0}) = 1$ and $\sigma(\hat{1}) = 0$. Two such functions $\sigma$ and $\tau$ are such that $\sigma \leq \tau$ in $J(P)$ if and only if $\sigma^{-1}(1) \subseteq \tau^{-1}(1)$. Since $\{0, 1\}^{\mathrm{op}}$ is a lattice, the set $J(P)$ inherits a lattice structure by coordinatewise operations. In fact, as is easily checked, $J(P)$ is a distributive lattice, and $J(O) = \emptyset$. Moreover, if $f: P \to Q$ is a bounded poset morphism, then $J(f): J(Q) \to J(P)$ defined by $J(f)(\alpha) := \alpha \circ f$ is a lattice morphism. It follows that $J(\cdot)$ is a contravariant functor from the category of bounded posets and bounded poset morphisms to the category of finite distributive lattices and lattice morphisms.

Given a nonempty distributive lattice $D$, an element $\alpha \in D$ is *join-irreducible* if $\alpha \neq \hat{0}$ and whenever $\alpha = \beta \vee \gamma$ in $D$ then either $\alpha = \beta$ or $\alpha = \gamma$. The set $R(D)$ of join-irreducible elements of $D$ inherits an order relation from $D$, so that $\hat{R}(D)$ is a bounded poset; we also make the convention that $\hat{R}(\emptyset) := O$. Moreover, if $L$ is distributive and $g: L \to D$ is a lattice morphism then we define a bounded poset morphism $\hat{R}(g): \hat{R}(D) \to \hat{R}(L)$ as follows: for $x \in R(D)$ we put

$$\hat{R}(g)(x) := \wedge \{\alpha \in L: x \leq g(\alpha)\},$$

with the conventions that $\wedge\emptyset := \hat{1}_{\hat{R}(L)}$ and $\hat{0}_L = \hat{0}_{\hat{R}(L)}$. We must check that if $\hat{R}(g)(x) \neq \hat{1}$ and $\hat{R}(g)(x) \neq \hat{0}$ then $\hat{R}(g)(x)$ is join-irreducible in $L$. Accordingly, assume that $A := \{\alpha \in L: x \leq g(\alpha)\} \neq \emptyset$, and let $\xi := \wedge A \neq \hat{0}$. Suppose that $\xi = \beta \vee \gamma$ with $\beta < \xi$ and $\gamma < \xi$ in $L$. Then, since $\xi = \wedge A$, we have $g(\beta) \wedge x < x$ and $g(\gamma) \wedge x < x$ in $D$. Since $x$ is join-irreducible in $D$ we have

$$g(\xi) \wedge x = g(\beta \vee \gamma) \wedge x = (g(\beta) \wedge x) \vee (g(\gamma) \wedge x) < x.$$

On the other hand,

$$g(\xi) \wedge x = \wedge g(A) \wedge x = \wedge\{g(\alpha) \wedge x: \alpha \in A\} = x.$$

This contradiction shows that $\xi \in R(L)$, as required. It is easily verified that $\hat{R}(g)$ is order-preserving, and it follows that $\hat{R}(g)$ is a bounded poset morphism from $\hat{R}(D)$ to $\hat{R}(L)$. A direct calculation also establishes that $\hat{R}(\cdot)$ is a contravariant functor. (This construction is adapted from p. 348 of Dilworth [2], where a slightly different definition of lattice homomorphism is used.)

In fact, the functors $J(\cdot)$ and $\hat{R}(\cdot)$ provide a contravariant equivalence of categories, as we now show. For a proper bounded poset $P$ and $x \in P^\circ$ we define $\delta_x$ and $\delta'_x$ in $J(P)$ by saying that for each $y \in P^\circ$: $\delta_x(y) := 1$ if and only if $y \le x$ in $P$, and $\delta'_x(y) := 1$ if and only if $y < x$ in $P$. For a bounded poset $P$, let $P' := \hat{R}(J(P))$ and define a bounded poset morphism $\tau_P: P \to P'$ by $\tau_P(x) := \delta_x$ for each $x \in P^\circ$ when $P$ is proper; also, there is a unique morphism $\tau_O: O \to O'$. Then for all $P$, $\tau_P$ is a natural isomorphism of bounded posets, and for any bounded poset morphism $f: P \to Q$ the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \tau_P \downarrow & & \downarrow \tau_Q \\ P' & \xrightarrow{f'} & Q' \end{array}$$

is commutative, where $f' := \hat{R}(J(f))$. On the other hand, given a distributive lattice $D$, let $D' := J(\hat{R}(D))$ and define a function $\nu_D: D \to D'$ as follows: if $D = \emptyset$ then $D' = \emptyset$ and $\nu_\emptyset: \emptyset \to \emptyset$ is conventional; if $D \ne \emptyset$ then for $\alpha \in D$ put $\nu_D(\alpha) := \vee\{\delta_x : x \in R(D) \text{ and } x \le \alpha \text{ in } D\}$. Then $\nu_D$ is a natural isomorphism of distributive lattices, and for any lattice morphism $g: L \to D$ the diagram

$$\begin{array}{ccc} L & \xrightarrow{g} & D \\ \nu_L \downarrow & & \downarrow \nu_D \\ L' & \xrightarrow{g'} & D' \end{array}$$

is commutative, where $g' := J(\hat{R}(g))$. Thus we have a contravariant equivalence of categories, as claimed.

We now describe some conditions on morphisms in these two categories, and relations among them. A poset $P$ is *disconnected* if either $P = \emptyset$ or we may write $P = X \cup Y$ with $X$ and $Y$ disjoint and nonempty and such that every $x \in X$ and $y \in Y$ are incomparable in $P$. If $P$ is not disconnected then $P$ is *connected*; a *component* of $P$ is a maximal connected subset of $P$. A bounded poset morphism $f: P \to Q$ is *fibre-connected* if for each $q \in Q$, the fibre $f^{-1}(q)$ is either empty or connected; it is *tight* when for each covering relation $q_1 \lessdot q_2$ of $f(P)$ there exists a covering relation $p_1 \lessdot p_2$ in $P$ such that $f(p_1) = q_1$ and $f(p_2) = q_2$. (Note that a covering relation in $f(P)$ need not be a covering relation in $Q$.) Finally, a *contraction* is a surjective tight fibre-connected morphism.

A lattice morphism $g: L \to D$ is *generous* when it satisfies the following condition: for all $\alpha, \beta \in D$, if $\alpha \wedge \beta \in g(L)$ and $\alpha \vee \beta \in g(L)$ then $\alpha \in g(L)$ and $\beta \in g(L)$. If $g$ is injective and generous then it is called an *embedding*.

**Theorem 1.1** *Let $g: L \to D$ be a morphism of nonempty distributive lattices, and denote the corresponding bounded poset morphism $\hat{R}(g): \hat{R}(D) \to \hat{R}(L)$ by $f: \hat{P} \to \hat{Q}$.*
(a) $g(\hat{0}) = \hat{0}$ *if and only if* $f^{-1}(\hat{0}) = \{\hat{0}\}$.
(b) $g(\hat{1}) = \hat{1}$ *if and only if* $f^{-1}(\hat{1}) = \{\hat{1}\}$.
(c) *$g$ is injective if and only if $f$ is surjective.*
(d) *$g$ is surjective if and only if $f$ is injective and tight.*

(e) *$g$ is generous if and only if $f$ is fibre-connected and tight.*
(f) *$g$ is an embedding if and only if $f$ is a contraction.*

**Proof:** By equivalence of categories we may assume instead that $D = J(\hat{P})$, $L = J(\hat{Q})$, and $g = J(f)\colon J(\hat{Q}) \to J(\hat{P})$.

For (a) simply recognize that for $p \in \hat{P}$, $g(\hat{0}_L)(p) = \hat{0}_L \circ f(p) = 1$ if and only if $f(p) = \hat{0}_Q$. Thus $g(\hat{0}_L) = \hat{0}_D$ if and only if $f^{-1}(\hat{0}_Q) = \{\hat{0}_P\}$. Part (b) is similar.

For (c), let $\alpha, \beta \in L$ be such that $g(\alpha) = g(\beta)$, so that $\alpha \circ f = \beta \circ f$ in $D$. Thus, if $f$ is surjective then $\alpha = \beta$, so that $g$ is injective. Conversely, if $q \in Q \backslash f(P)$ then $\delta_q \neq \delta'_q$ in $L$ but $g(\delta_q) = g(\delta'_q)$ in $D$, so that $g$ is not injective.

For (d), if $f$ is not injective then let $q \in \hat{Q}$ be such that $\#f^{-1}(q) \geq 2$ and let $p \in \hat{P}$ be a minimal element of $f^{-1}(q)$. Then $\delta_p$ is not constant on $f^{-1}(q)$, so that $\delta_p \notin g(L)$. Hence $g$ is not surjective. If $f$ is injective but not tight then let $f(p_1) = q_1$ and $f(p_2) = q_2$ with $q_1 \lessdot q_2$ in $f(\hat{P})$ but $p_1$ and $p_2$ incomparable in $\hat{P}$. Then $\delta_{p_2}$ takes the values 0 at $p_1$ and 1 at $p_2$, and so it is not in $g(L)$. Thus $g$ is not surjective.

Conversely, assume that $f$ is injective and tight, so that $f$ is an isomorphism from $P$ to $f(P)$. For each $\alpha \in D$, let $\beta \in L$ be the join of all $\delta_q \in L$ such that $q = f(p)$ for some $p \in P$ with $\alpha(p) = 1$. One easily checks that $g(\beta) = \alpha$, so $g$ is surjective.

For (e), assume first that $f$ is not fibre-connected, and let $q \in \hat{Q}$ be such that $f^{-1}(q) \neq \emptyset$ is disconnected. Let $f^{-1}(q) = X \cup Y$ with $X$ and $Y$ disjoint, nonempty, and such that each $x \in X$ and $y \in Y$ are incomparable in $\hat{P}$. Let $\xi := g(\delta'_q) \vee \vee\{\delta_x : x \in X\}$ and $\eta := g(\delta'_q) \vee \vee\{\delta_y : y \in Y\}$ in $D$. Notice that $g(\delta'_q) < \xi < g(\delta_q)$ and $g(\delta'_q) < \eta < g(\delta_q)$ in $D$, and that $\xi \wedge \eta = g(\delta'_q)$ and $\xi \vee \eta = g(\delta_q)$ in $D$. Since neither $\xi$ nor $\eta$ is constant on $f^{-1}(q)$, we have $\xi \notin g(L)$ and $\eta \notin g(L)$. Therefore $g$ is not generous.

Now assume that $f$ is not tight, and let $q_1 \lessdot q_2$ be a covering relation in $f(\hat{P})$ such that every $x \in f^{-1}(q_1)$ and $y \in f^{-1}(q_2)$ are incomparable in $\hat{P}$. Let $\xi := g(\delta_{q_1})$ and $\eta := \vee\{\delta_y : f(y) \leq q_2 \text{ and } f(y) \neq q_1\}$ in $D$. Then $\xi \wedge \eta = g(\delta'_{q_1})$ and $\xi \vee \eta = g(\delta_{q_2})$ but $\eta \notin g(L)$, so that $g$ is not generous.

Conversely, assume that $g$ is not generous, and let $\xi, \eta \in D$ be such that either $\xi \notin g(L)$ or $\eta \notin g(L)$, but both $\xi \wedge \eta \in g(L)$ and $\xi \vee \eta \in g(L)$; we may assume that $\xi \notin g(L)$. We must show that either $f$ is not tight or $f$ is not fibre-connected. We may suppose that $f$ is tight, so that $\alpha \in D$ is in $g(L)$ if and only if $\alpha$ is constant on each fibre of $f$. Thus there is a $q \in \hat{Q}$ such that $\xi$ is not constant on $f^{-1}(q)$. Since $\xi \wedge \eta$ and $\xi \vee \eta$ are both constant on $f^{-1}(q)$ we must have $\xi(p) = 0$ if and only if $\eta(p) = 1$, for each $p \in f^{-1}(q)$. Since both $\xi$ and $\eta$ are order-reversing the sets $X := \{p \in f^{-1}(q) : \xi(p) = 0\}$ and $Y := \{p \in f^{-1}(q) : \xi(p) = 1\}$ show that $f^{-1}(q)$ is not connected. Thus $f$ is not fibre-connected.

Clearly (f) follows from (c) and (e). □

Given a distributive lattice $D$, a sublattice $L \subseteq D$ is *embedded* if the inclusion $L \hookrightarrow D$ is an embedding. The set $\mathcal{L}(D)$ of all embedded sublattices, partially ordered by inclusion, is a bounded poset. (We consider $\emptyset \subseteq D$ to be embedded.)

Let $P$ be a bounded poset. A contraction $f_1\colon P \to Q_1$ *dominates* a contraction $f_2\colon P \to Q_2$ if there is a morphism $f'\colon Q_1 \to Q_2$ such that $f_2 = f' \circ f_1$; in this case $f'$ is also a contraction. If $f_1$ dominates $f_2$ and $f_2$ dominates $f_1$ then $f_1$ and $f_2$ are *equivalent*. The set

$\mathcal{C}(P)$ of equivalence classes of contractions of $P$ is finite. We say that $[f_1] \leq [f_2]$ in $\mathcal{C}(P)$ if $f_2$ dominates $f_1$; this makes $\mathcal{C}(P)$ into a bounded poset.

Let $P$ be a poset. As in Stanley [13], the *order polytope* of $P$ is the set $\mathcal{P}(P)$ of vectors $v \in \mathbb{R}^P$ satisfying the inequalities $1 \geq v(x) \geq 0$ for all $x \in P$, and $v(x) \geq v(y)$ for all $x \lessdot y$ in $P$. Clearly this is a compact convex polytope; the set $\mathcal{F}(P)$ of faces of $\mathcal{P}(P)$, ordered by inclusion, is a bounded poset.

**Theorem 1.2** *For a nonempty distributive lattice $D$, with $P = \hat{R}(D)$, the bounded posets $\mathcal{L}(D)$, $\mathcal{C}(P)$, and $\mathcal{F}(P^\circ)$ are isomorphic lattices.*

**Proof:** The intersection of two embedded sublattices of $D$ is an embedded sublattice; since $\mathcal{L}(D)$ has a unique maximal element (namely $D$) it follows that $\mathcal{L}(D)$ is a lattice. Theorem 1.1(f) and equivalence of categories shows that $f \mapsto J(f)$ induces a bounded poset isomorphism from $\mathcal{C}(P)$ to $\mathcal{L}(D)$; hence these are also isomorphic as lattices. The proof that $\mathcal{F}(P^\circ)$ is isomorphic with $\mathcal{C}(P)$ is due (with a different terminology) to Geissinger [5]; see also Theorem 1.2 of Stanley [13]. □

Finally, we review an enumerative result. Recall that for a bounded poset $P$, the elements of $J(P)$ are $\mathbb{N}$-valued functions on $P$. We let $\Sigma^+(P)$ denote the additive semigroup generated by $J(P)$ in $\mathbb{N}^P$, and put $\Sigma(P) := \Sigma^+(P) \cup \{0_P\}$, where $0_P$ is the neutral element of $\mathbb{N}^P$ (which takes the value 0 everywhere).

Given a proper bounded poset $P$, a *chain* is a sequence $\hat{0} < p_r < \cdots < p_1 < \hat{1}$ in $P$; the *size* of this chain is $r$. The *order complex* of $P$ is the set $\Gamma(P)$ of all chains of $P$; it is partially ordered by inclusion and forms an *abstract simplicial complex*, that is, a downset in the set of all subsets of $P^\circ$. The *f-vector* of $\Gamma(P)$ is $f(\Gamma) := (f_0, f_1, \ldots, f_d)$ where $f_i$ is the number of chains in $P$ of size $i$, for each $0 \leq i \leq d$, and $f_d \neq 0$. The *h-vector* $h(\Gamma) := (h_0, h_1, \ldots, h_s)$ of $\Gamma(P)$ is defined by

$$\sum_{i=0}^{d} f_i x^i = \sum_{i=0}^{s} h_i x^i (1+x)^{d-i}.$$

Let $\mathcal{Q}$ be a compact convex polytope in $\mathbb{R}^n$ with integer vertices. For each positive integer $m$ let $i(\mathcal{Q}; m) := \#(\mathbb{Z}^n \cap m\mathcal{Q})$ be the number of $v \in \mathbb{Z}^n$ such that $v/m \in \mathcal{Q}$. This is a polynomial function of $m$, called the *Ehrhart polynomial* of $\mathcal{Q}$. See p. 235 of Stanley [14] or Part Two of Hibi [9] for further details.

**Theorem 1.3** *Let $\hat{P}$ be a proper bounded poset, let $D := J(\hat{P})$ and $\Gamma := \Gamma(D)$, and let the h-vector of $\Gamma$ be $h(\Gamma) = (h_0, h_1, \ldots, h_s)$. Let $\mathcal{P}$ be the order polytope of $P$.*

(a) *A function $\sigma: \hat{P} \to \mathbb{N}$ is in $\Sigma(\mathbb{P})$ if and only if it is order-reversing and $\sigma(\hat{1}) = 0$.*

(b) *Any $\sigma \in \Sigma(\hat{P})$ may be written uniquely as $\sigma = \alpha_m + \alpha_{m-1} + \cdots + \alpha_1$ with $m \geq 0$ and $\alpha_m \leq \alpha_{m-1} \leq \cdots \leq \alpha_1$ in $D$.*

(c)

$$\sum_{m=0}^{\infty} i(\mathcal{P}; m)\lambda^m = \sum_{\sigma \in \Sigma(\hat{P})} \lambda^{\sigma(\hat{0})} = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^{1+\#P}}.$$

**Proof:** Clearly every $\sigma \in \Sigma(\hat{P})$ is an order-reversing function such that $\sigma(\hat{1}) = 0$. Conversely, any order-reversing function $\sigma: \hat{P} \to \mathbb{N}$ with $\sigma(\hat{1}) = 0$ is a sum $\alpha_m + \alpha_{m-1} + \cdots + \alpha_1$ where $m := \sigma(\hat{0})$ and for $1 \le i \le m$ and $p \in \hat{P}$:

$$\alpha_i(p) := \begin{cases} 1 & \text{if } \sigma(p) \ge i, \\ 0 & \text{otherwise.} \end{cases}$$

This is the unique expression as in (b), so (a) and (b) are proved.

For (c) we identify $\mathcal{P}$ with the maximal face $\mathcal{P}'$ of $\mathcal{P}(\hat{P})$ satisfying $v(\hat{0}) = 1$ and $v(\hat{1}) = 0$ for all $v \in \mathcal{P}'$. A point in $\mathbb{Z}^{\hat{P}} \cap m\mathcal{P}'$ is an order-reversing function $\sigma: \hat{P} \to \mathbb{N}$ with $\sigma(\hat{0}) = m$ and $\sigma(\hat{1}) = 0$, which implies the first equality. For each chain $\hat{0} < \alpha_j < \alpha_{j-1} < \cdots < \alpha_1 < \hat{1}$ of size $j$ in $D$ the sum of $\lambda^{\sigma(\hat{0})}$ over all $\sigma \in \Sigma(\hat{P})$ for which the unique expression as in (b) is of the form

$$\sigma = b_0\hat{0}_D + c_j\alpha_j + c_{j-1}\alpha_{j-1} + \cdots + c_1\alpha_1 + b_1\hat{1}_D$$

where $b_0 \ge 0$, $b_1 \ge 0$, and each $c_i \ge 1$ gives a contribution of $\lambda^j(1-\lambda)^{-2-j}$. As each $\sigma \in \Sigma(\hat{P})$ is supported on exactly one chain of $D$ in this way, we get

$$\sum_{\sigma \in \Sigma(\hat{P})} \lambda^{\sigma(\hat{1})} = \sum_{j=0}^{\#P-1} f_j(\Gamma)\frac{\lambda^j}{(1-\lambda)^{2+j}}.$$

The result follows from the definition of $h(\Gamma)$. □

## 2. Orbit decomposition and singularities

For the rest of the paper $f: \hat{P} \to \hat{Q}$ denotes a contraction of proper bounded posets corresponding to the embedding $L \subseteq D$, where $L := J(\hat{Q})$ and $D := J(\hat{P})$.

Let $\mathbf{Z} := \{Z_p: p \in \hat{P}\}$ be commuting indeterminates over k, and for $\sigma: \hat{P} \to \mathbb{N}$ let $\mathbf{Z}^\sigma := \prod_{p \in \hat{P}} Z_p^{\sigma(p)}$. Define a k-algebra homomorphism $\varphi: \mathsf{k}[\mathbf{X}] \to \mathsf{k}[\mathbf{Z}]$ by $\varphi(X_\alpha) := \mathbf{Z}^\alpha$ for each $\alpha \in D$ and k-linear and multiplicative extension. Regard $\mathsf{k}[\mathbf{Z}]$ as a graded algebra by putting $\deg(\mathbf{Z}^\sigma) := \sigma(\hat{0})$. Then since $\alpha(\hat{0}) = 1$ for all $\alpha \in D$, it follows that $\varphi$ is a homogeneous homomorphism of degree 0. Theorem 2.1(a,b) appears on p. 99 of Hibi [7]; part (c) is an immediate consequence.

**Theorem 2.1** *Let $D = J(\hat{P})$ be a nonempty distributive lattice, and consider the homomorphism $\varphi: \mathsf{k}[\mathbf{X}] \to \mathsf{k}[\mathbf{Z}]$.*
(a) $\ker(\varphi) = I(D)$, *so that* $\mathsf{k}[D]$ *may be regarded as a subalgebra of* $\mathsf{k}[\mathbf{Z}]$.
(b) $\operatorname{im}(\phi) = \mathsf{k}\Sigma(\hat{P})$, *the semigroup algebra of* $\Sigma(\hat{P})$.
(c) *The Hilbert function of* $\mathsf{k}[D]$ *is* $i(\mathcal{P}(P); m)$.

**Proof:** For part (a) it is clear that $I(D) \subseteq \ker(\varphi)$. Conversely, suppose that $u \in \ker(\varphi)$. By applying the generators of $I(D)$ we see that modulo $I(D)$ each monomial

$c(\alpha, \beta, \ldots, \gamma)X_\alpha X_\beta \cdots X_\gamma$ which appears in $u$ with coefficient $c(\alpha, \beta, \ldots, \gamma) \in \mathsf{k}$ has $\alpha \leq \beta \leq \cdots \leq \gamma$ in $D$. For this monomial we have $\varphi(X_\alpha X_\beta \cdots X_\gamma) = \mathbf{Z}^\sigma$ where $\sigma = \alpha + \beta + \cdots + \gamma$. By Theorem 1.3(b), the coefficient of $\mathbf{Z}^\sigma$ in $\varphi(u)$ is $c(\alpha, \beta, \ldots, \gamma)$; since $\varphi(u) = 0$ this coefficient is zero. Hence $u \in I(D)$.

Part (b) follows from Theorem 1.3(a,b), and part (c) from Theorem 1.3(c). □

Theorem 2.1 shows that $\mathsf{k}[D] \simeq \mathsf{k}\Sigma(\hat{P})$ is an affine semigroup algebra, and from Hochster [10] it follows that $\mathsf{k}[D]$ is an integrally closed Cohen-Macaulay domain. As shown in [8, 15], this implies numerous inequalities among the entries of the $h$-vector $h(\Gamma)$ appearing Theorem 1.3, which can be given a direct combinatorial interpretation (see Theorem 4.5.14 of Stanley [14]). On page 105 of [7] Hibi shows that $\mathbf{k}[D]$ is Gorenstein if and only if all maximal chains of $\hat{P}$ have the same size; see also Theorem 5.4 of Stanley [12]. Theorem 2.9 of Hibi [8] gives a combinatorial criterion for $\mathbf{k}[D]$ to be level.

Let $A$ be an integral $\mathsf{k}$-algebra, and consider an $A$-valued point $\xi \in V(D)$ with homogeneous coordinates $[\xi(\alpha): \alpha \in D]$. Since $\xi$ satisfies every relation in $I(D)$, we have $\xi(\alpha)\xi(\beta) = \xi(\alpha \wedge \beta)\xi(\alpha \vee \beta)$ for all $\alpha, \beta \in D$. Since $A$ is a domain it follows that $\mathrm{supp}(\xi) := \{\alpha \in D: \xi(\alpha) \neq 0\}$ is a nonempty embedded sublattice of $D$. Given a nonempty embedded sublattice $L \subseteq D$, let $U(L) := \{\xi \in V(D)\text{: } \mathrm{supp}(\xi) = L\}$. Notice that $U(L)$ with the operation of coordinatewise multiplication is an abelian group. We now describe the action of the algebraic torus $(\mathsf{k}^\times)^P$ on the $\mathsf{k}$-valued points of $V(D)$.

**Theorem 2.2** *Let $L \subseteq D$ and $f: \hat{P} \to \hat{Q}$ be as above. Consider the* $\mathsf{k}$*-valued points of* $V(D)$.

(a) *The group $U(L)$ is isomorphic with $(\mathsf{k}^\times)^Q$.*
(b) *There is a group epimorphism $f': U(D) \to U(L)$ corresponding to $f: \hat{P} \to \hat{Q}$. If $M \subseteq L$ is an embedded sublattice corresponding to $g: \hat{Q} \to \hat{S}$, then $(g \circ f)' = g' \circ f'$. The orbits of $U(D)$ acting on $V(D)$ are the sets $U(L)$ for all nonempty embedded sublattices $L$ of $D$.*
(c) *The closure $\overline{U(L)}$ of $U(L)$ in $V(D)$ is isomorphic with $V(L)$, and if $V(L)$ is not a single point then $U(L)$ is open in $\overline{U(L)}$.*
(d) *The set of $U(D)$-orbits of $V(D)$, with $O_1 \leq O_2$ iff $O_1 \subseteq \bar{O}_2$, is isomorphic with $\mathcal{L}(D)\backslash\{\emptyset\}$.*

**Proof:** For part (a), let $i: U(L) \to (\mathsf{k}^\times)^L$ be the group monomorphism $\xi \mapsto \xi|_L$ defined by restriction to $L$. Also, let $j: (\mathsf{k}^\times)^Q \to (\mathsf{k}^\times)^L$ be the group monomorphism defined by $q \mapsto \delta_q$ and extension by 1: for $\theta: Q \to \mathsf{k}^\times$ and $\alpha \in L$, let $j(\theta)(\alpha) := \theta(q)$ if $\alpha = \delta_q$ for some $q \in Q$, and let $j(\theta)(\alpha) := 1$ otherwise. Define a function $h: (\mathsf{k}^\times)^L \to (\mathsf{k}^\times)^L$ as follows: for $\xi: L \to \mathsf{k}^\times$ and $\alpha \in L$ let

$$h(\xi)(\alpha) := \prod_{\beta \leq \alpha} \xi(\beta).$$

Then $h$ is a group automorphism, and

$$h^{-1}(\xi)(\alpha) = \prod_{\beta \leq \alpha} \xi(\beta)^{\mu(\beta,\alpha)},$$

where $\mu(\cdot,\cdot)$ is the Möbius function of $L$. It is easy to check that the image of the composite $h \circ j$ is contained in the image of $i$. Conversely, let $\xi$ be in the image of $i$, and let $\alpha \in L$ cover $\beta_1, \dots, \beta_m$, where $m \geq 2$. For nonempty $I \subseteq \{1, \dots, m\}$ let $\beta_I := \wedge\{\beta_i\colon i \in I\}$, and let $\beta_\emptyset := \alpha$. Then the interval $[\beta_{\{1,\dots,m\}}, \alpha]$ in $L$ is a Boolean algebra, and the Möbius function of $L$ is such that $\mu(\gamma, \alpha) = 0$ unless $\beta_{\{1,\dots,m\}} \leq \gamma \leq \alpha$, and for $I \subseteq \{1, \dots, m\}$ we have $\mu(\beta_I, \alpha) = (-1)^{\#I}$. Now for each $I \subseteq \{3, \dots, m\}$ we have $\xi(\beta_I)\xi(\beta_I \wedge \beta_1 \wedge \beta_2) = \xi(\beta_I \wedge \beta_1)\xi(\beta_I \wedge \beta_2)$ since $\xi$ is in the image of $i$. The contribution of these four factors to $h^{-1}(\xi)(\alpha)$ is either $\xi(\beta_I)\xi(\beta_I \wedge \beta_1 \wedge \beta_2)/\xi(\beta_I \wedge \beta_1)\xi(\beta_I \wedge \beta_2)$ or its reciprocal, and so is just 1. Thus $h^{-1}(\xi)(\alpha) = 1$ if $\alpha$ is not join-irreducible, so that the image of $h^{-1} \circ i$ is contained in the image of $j$. Hence $U(L)$ is isomorphic to $(\mathrm{k}^\times)^Q$.

For part (b), given a contraction $f\colon \hat{P} \to \hat{Q}$ define $f''\colon (\mathrm{k}^\times)^P \to (\mathrm{k}^\times)^Q$ as follows. For $\theta\colon P \to \mathrm{k}^\times$ and $q \in Q$ let

$$f''(\theta)(q) := \prod_{p \in f^{-1}(q)} \theta(p).$$

This is clearly a group epimorphism, since $f$ is surjective. Composing $f''$ with the appropriate isomorphisms from part (a) gives $f'\colon U(D) \to U(L)$. The fact that $(g \circ f)' = g' \circ f'$ is immediate from the construction. The claim describing the orbits of $U(D)$ on $V(D)$ follows, since the $f'$ are epimorphisms.

For (c), notice that $\overline{U(L)} = \cup_M U(M)$ where the union is over nonempty embedded sublattices $M \subseteq L$. This is the subvariety of $V(D)$ defined by $\{X_\alpha = 0\colon \alpha \in D \backslash L\}$; clearly it is isomorphic with $V(L)$. Now $V(L)$ is a single point if and only if $L$ is a minimal nonempty embedded sublattice of $D$. If this is not the case then

$$\overline{U(L)} \backslash U(L) = \bigcup_M \overline{U(M)}$$

where the union is over nonempty embedded sublattices strictly contained in $L$. This is a union of proper closed subvarieties of $\overline{U(L)}$, so that $U(L)$ is open in $\overline{U(L)}$.

Part (d) follows immediately from parts (b) and (c). □

To connect with the general theory of toric varieties we describe the fan $\Delta$ dual to $\mathcal{P}$, the order polytope of $P$. The maximal proper faces of $\mathcal{P}$ correspond to the equalities $v(x) = v(y)$ for each covering relation $x \lessdot y$ in $\hat{P}$, with the understanding that $v(\hat{0}) = 1$ and $v(\hat{1}) = 0$. Let $\{e_x\colon x \in P\}$ be the standard basis of $M := \mathbb{Z}^P$, and let $\{\varepsilon_x\colon x \in P\}$ be the dual basis of $N := M^*$. From p. 26 of Fulton [4], for example, one sees that the minimal generators of the rays of the fan $\Delta$ are of three types: $-\varepsilon_x$ for each minimal $x \in P$, $\varepsilon_x - \varepsilon_y$ for each $x \lessdot y$ in $P$, and $\varepsilon_x$ for each maximal $x \in P$. Moreover, a proper subset $C$ of these minimal generators spans a cone in $\Delta$ if and only if there is a contraction $f\colon \hat{P} \to \hat{Q}$ such that the following three conditions hold: for minimal $x \in P$, $f(x) = \hat{0}$ if and only if $-\varepsilon_x \in C$; for $x \lessdot y \in P$, $f(x) = f(y)$ if and only if $\varepsilon_x - \varepsilon_y \in C$; and for maximal $x \in P$, $f(x) = \hat{0}$ if and only if $\varepsilon_x \in C$.

The *Hasse graph* of a poset $S$ has vertex-set $S$ and edges $x \sim y$ for each covering relation $x \lessdot y$ in $S$. A poset $S$ is a *tree* if its Hasse graph is connected and contains no cycles.

**Theorem 2.3** *Let $L \subseteq D$ and $f: \hat{P} \to \hat{Q}$ be as above. Then $V(D)$ is nonsingular along $V(L)$ if and only if for each $q \in \hat{Q}$ the fibre $f^{-1}(q)$ is a tree.*

**Proof:** Let $C$ be the set of minimal generators of rays of $\Delta$ which corresponds to the contraction $f$. Then $V(D)$ is nonsingular along $V(L)$ if and only if $C \subseteq B$ for some basis $B$ of the free abelian group $N \simeq \mathbb{Z}^P$ (cf. p. 29 of Fulton [4]). Each fibre of $f$ is connected, since $f$ is a contraction. Suppose that $f^{-1}(q)$ contains a cycle, for some $q \in \hat{Q}$. Let $C' \subseteq C$ be the subset of $C$ corresponding to covering relations, minimal elements, and maximal elements of $P$ which are in $f^{-1}(q)$. Then $C'$ is linearly dependent, so $C$ is not contained in any basis of $N$. Conversely, suppose that each fibre of $f$ is a tree. For each $q \in Q$ choose any $x(q) \in f^{-1}(q)$. We claim that $B := C \cup \{\varepsilon_{x(q)}: q \in Q\}$ is a $\mathbb{Z}$-basis for $N$. Let $A$ be the matrix with rows indexed by $P$ and with columns indexed by $B$, in which the columns are the coordinate vectors of the members of $B$ with respect to the basis $\{\varepsilon_x: x \in P\}$ of $N$. Then $A$ can be partitioned into block-diagonal form $A = A_1 \oplus \cdots \oplus A_r$ where $r := \#Q$ and each $A_i$ is the incidence matrix of a tree (with arbitrarily directed edges) with an extra column indicating a distinguished (root) vertex. Such a matrix $A_i$ is well-known (and easily seen) to have $|\det A_i| = 1$, and hence $B$ is a $\mathbb{Z}$-basis for $N$. □

A subset $S$ of a poset $P$ is *convex* if whenever $a \leq b \leq c$ in $P$ and $a \in S$ and $c \in S$, then $b \in S$.

**Corollary 2.4** *Let $\hat{P}$ be a proper bounded poset, and $D := J(\hat{P})$. Then $V(D)$ is nonsingular if and only if $P$ is a disjoint union of chains (in which case $V(D)$ is the Segre embedding of a product of projective spaces). If $P$ is not a disjoint union of chains, then let $g(\hat{P})$ be the minimum cardinality of a convex subset of $\hat{P}$ which contains a cycle in the Hasse graph of $\hat{P}$ not passing through both $\hat{0}$ and $\hat{1}$. In this case the codimension of the singular locus of $V(D)$ is $g(\hat{P}) - 1$.*

**Proof:** The claim when $P$ is a disjoint union of chains is clear. Otherwise, if $S$ is a subset of $\hat{P}$ as in the statement then there is a contraction of $\hat{P}$ which identifies the elements of $S$ and maps the other elements of $\hat{P}$ to distinct points. □

So the codimension of the singular locus of $V(D)$ is always at least three, which is one more than we could expect merely from the fact that $V(D)$ is normal.

## 3. Local rings and normal links

We now examine the local rings $\mathcal{O}_{L,D}$ and associated graded rings $\mathrm{gr}_L(D)$, where $L \subseteq D$ and $f: \hat{P} \to \hat{Q}$ are as in Section 2. The projective coordinate ring of $V(D)$ is $\mathsf{k}[D]$; the ideal corresponding to the subvariety $V(L)$ is $(X_\alpha : \alpha \in D \backslash L) + I(D)$. The local ring $\mathcal{O}_{L,D}$ of $V(D)$ along $V(L)$ is obtained by inverting homogeneous elements of $\mathsf{k}[D]$ not in the ideal of $V(L)$, and taking the subring of homogeneous elements of degree zero. Since $f$ is a contraction, an element $\alpha \in D$ is in $L$ if and only if $\alpha$ is constant on each fibre of $f$. Now $\mathsf{k}[D]$ is isomorphic with the affine semigroup ring $\mathsf{k}\Sigma(\hat{P})$. The ideal of $V(L)$ in this view is

$(\mathbf{Z}^\alpha : \alpha \in D\backslash L)$. Identifying $\mathcal{O}_{L,D}$ with its image under the induced isomorphism, a typical element of $\mathcal{O}_{L,D}$ is of the form $t/u$, where $t$ and $u$ are homogeneous k-linear combinations of $\{\mathbf{Z}^\sigma : \sigma \in \Sigma(\hat{P})\}$ of the same degree (where $\deg(\mathbf{Z}^\sigma) := \sigma(\hat{0})$), $u \neq 0$, and at least one monomial $\mathbf{Z}^\sigma$ occurring in $u$ has $\sigma$ constant on each fibre of $f$.

We may identify $\mathbf{Z}^\alpha/\mathbf{Z}^\beta$ with $\mathbf{Z}^{\alpha-\beta}/1$ and assume that each element $t/u$ of $\mathcal{O}_{L,D}$ has the following form:

(N): the numerator $t \in \mathsf{k}[Z_p, Z_p^{-1} : p \in \hat{P}]$ is such that each monomial $\mathbf{Z}^\sigma$ occurring in $t$ satisfies $\sigma(\hat{1}) = \sigma(\hat{0}) = 0$ and $\sigma|_{f^{-1}(q)}$ is an order-reversing function for each $q \in \hat{Q}$;

(D): the denominator $0 \neq u \in \mathsf{k}[Z_p, Z_p^{-1} : p \in \hat{P}]$ satisfies the condition (N) and is such that at least one monomial $\mathbf{Z}^\sigma$ occurring in $u$ has $\sigma$ constant on each fibre of $f$.

The maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{L,D}$ consists of 0 and those elements $t/u$ such that every monomial $\mathbf{Z}^\sigma$ occurring in $t$ has some $q \in \hat{Q}$ such that $\sigma|_{f^{-1}(q)}$ is not constant. Thus the ideal $\mathfrak{m}$ is generated by the elements $\{\mathbf{Z}^{\alpha-\omega}/1 : \alpha \in D\backslash L\}$ where $\omega := \hat{0}_L \circ f$. (A minimal set of generators for $\mathfrak{m}$ is described in Corollary 3.3.)

To describe the residue field $\mathsf{K} := \mathcal{O}_{L,D}/\mathfrak{m}$ let $t_q := (\mathbf{Z}^{(\delta_q - \delta_q')\circ f}/1) + \mathfrak{m}$ for each $q \in Q$. It is clear that $\mathsf{k}(t_q : q \in Q) \subseteq \mathsf{K}$. Conversely, given $(t/u) + \mathfrak{m}$ in $\mathsf{K}$ we may assume that $t$ and $u$ are k-linear combinations of terms $(Z^\sigma/1) + \mathfrak{m}$ such that $\sigma(\hat{1}) = \sigma(\hat{0}) = 0$ and $\sigma$ is constant on each fibre of $f$. Denoting the common value of $\sigma$ on $f^{-1}(q)$ by $e(q)$ for each $q \in \hat{Q}$ it is clear that $(\mathbf{Z}^\sigma/1) + \mathfrak{m} = \prod_{q\in\hat{Q}} t_q^{e(q)}$ in $\mathsf{K}$. Hence $\mathsf{K} = \mathsf{k}(t_q : q \in Q)$. The elements $\{t_q : q \in Q\}$ are clearly algebraically independent over k, corresponding to the fact that $V(L)$ is birationally equivalent to $\mathsf{k}^{\#Q}$.

To describe the associated graded ring $\mathrm{gr}_L(D) := \coprod_{j\geq 0} \mathfrak{m}^j/\mathfrak{m}^{j+1}$ of $\mathcal{O}_{L,D}$ we need some more combinatorics. A *split* of a connected poset $S$ is an ordered pair $(S', S'')$ of subsets of $S$ such that $S' \cap S'' = \emptyset$ and $S' \cup S'' = S$, $S'$ is a connected downset (downward-closed subset) of $S$, and $S''$ is a connected upset (upward-closed subset) of $S$. Let $\Xi(S)$ denote the set of order-reversing functions $\sigma : S \to \mathbb{Z}$. Define an equivalence relation $\approx$ on $\Xi(S)$ by saying that $\sigma \approx \tau$ if and only $\sigma - \tau$ is constant. An element $\alpha \in \Xi(S)$ is a *split element* if and only if $\alpha(S) = \{0, 1\}$ and $(\alpha^{-1}(1), \alpha^{-1}(0))$ is a split of $S$.

**Proposition 3.1** *If $S$ is a connected poset then every $\sigma \in \Xi(S)$ is equivalent to a sum of split elements.*

**Proof:** Clearly each element of $\Xi(S)$ is equivalent to one of the form $\sigma : S \to \mathbb{N}$ with $\sigma^{-1}(0) \neq \emptyset$, so we need only consider this case. Let $m := \max\{\sigma(x) : x \in S\}$, and for $1 \leq i \leq m$ define $\beta_i \in \Xi(S)$ by

$$\beta_i(x) := \begin{cases} 1 & \text{if } \sigma(x) \geq i, \\ 0 & \text{if } \sigma(x) < i. \end{cases}$$

Then $\sigma = \beta_1 + \beta_2 + \cdots + \beta_m$, so it suffices to show that each $\{0, 1\}$-valued $\beta \in \Xi(S)$ is equivalent to a sum of split elements.

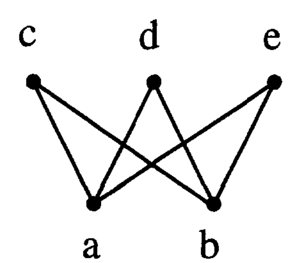
If $\beta$ is constant then this is trivial. Otherwise, first consider the case in which $\beta^{-1}(0)$ is connected, and let the components of $\beta^{-1}(1)$ be $C_1, \ldots, C_m$. For $1 \leq i \leq m$ define $\alpha_i: S \to \{0, 1\}$ by $\alpha_i(x) := 1$ if and only if $x \in C_i$. Since $S$ is connected, it follows that each $\alpha_i$ is a split element of $\Xi(S)$. Clearly $\beta = \alpha_1 + \cdots + \alpha_m$, as was to be shown. Now consider the case in which $\beta^{-1}(0)$ is not connected, and let the components of $\beta^{-1}(0)$ be $C_1, \ldots, C_m$. For $1 \leq i \leq m$ define $\alpha_i: S \to \{0, 1\}$ by $\alpha_i(x) := 0$ if and only if $x \in C_i$. Clearly $\beta + (m-1)1_S = \alpha_1 + \cdots + \alpha_m$ and each $\alpha_i$ is such that $\alpha_i^{-1}(0)$ is connected. Thus we have reduced to the first case, finishing the proof. □

For any $\sigma \in \Xi(S)$, define the *order of* $\sigma$ to be the supremum over all $r \in \mathbb{N}$ for which there exist split elements $\alpha_1, \ldots, \alpha_r$ of $\Xi(S)$ such that $\sigma \approx \alpha_1 + \cdots + \alpha_r$; the order of $\sigma$ is denoted by $\mathrm{ord}(\sigma)$. Thus $\mathrm{ord}(\sigma) = 0$ if and only if $\sigma$ is constant, and $\mathrm{ord}(\sigma) = 1$ if and only if $\sigma$ is equivalent to a split element. Given $\sigma \in \Xi(S)$, it is easy to see that $\mathrm{ord}(\sigma) \leq \sum_{x<y}(\sigma(x) - \sigma(y))$, the sum being over all covering relations of $S$, so that each $\sigma \in \Xi(S)$ has a finite order. The inequality $\mathrm{ord}(\sigma) + \mathrm{ord}(\tau) \leq \mathrm{ord}(\sigma + \tau)$ is immediate from the definition. This inequality may be strict, as the example in Figure 1 shows. A minimax formula for calculating $\mathrm{ord}(\sigma)$ has been derived by Vidyasankar [16] and generalized by Lucchesi and Younger [11].

Given a connected poset $S$ and a field $\mathsf{k}$ we define a graded $\mathsf{k}$-algebra $G_{\mathsf{k}}(S)$ as follows. It has as a $\mathsf{k}$-basis the set $\mathbf{T} := \{T_\sigma : \sigma \in \Xi(S)\}$, with grading given by $\deg(T_\sigma) := \mathrm{ord}(\sigma)$. The multiplicative structure is determined by putting

$$T_\sigma T_\tau := \begin{cases} T_{\sigma+\tau} & \text{if } \mathrm{ord}(\sigma) + \mathrm{ord}(\tau) = \mathrm{ord}(\sigma + \tau), \\ 0 & \text{otherwise,} \end{cases}$$

and extending $\mathsf{k}$-bilinearly. If $S$ has a unique minimal element $\hat{0}$, then let $\Xi'(S) \subset \Xi(S)$ consist of those $\sigma \in \Xi(S)$ such that $\sigma(\hat{0}) = 0$. Thus $\Xi'(S)$ is a submonoid of $\Xi(S)$, and so the $\mathsf{k}$-span of $\{T_\sigma : \sigma \in \Xi'(S)\}$ in $G_{\mathsf{k}}(S)$ is a $\mathsf{k}$-subalgebra of $G_{\mathsf{k}}(S)$ which we denote by




| | $\alpha_1$ | $\alpha_2$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
|---|---|---|---|---|---|
| a | 1 | 0 | 1 | 1 | 1 |
| b | 0 | 1 | 1 | 1 | 1 |
| c | 0 | 0 | 0 | 1 | 1 |
| d | 0 | 0 | 1 | 0 | 1 |
| e | 0 | 0 | 1 | 1 | 0 |

*Figure 1.* $\alpha_1 + \alpha_2 \approx \beta_1 + \beta_2 + \beta_3$.

$G'_{\mathsf{k}}(S)$. Similarly, if $S$ has a unique maximal element $\hat{1}$ we let $\Xi''(S)$ be the submonoid of $\Xi(S)$ consisting of those $\sigma \in \Xi(S)$ with $\sigma(\hat{1}) = 0$. Then $G''_{\mathsf{k}}(S) := \operatorname{span}_{\mathsf{k}}\{T_\sigma : \sigma \in \Xi''(S)\}$ is a k-subalgebra of $G_{\mathsf{k}}(S)$. Notice that $G_{\mathsf{k}}(S)$ is a free $\mathsf{k}[T, T^{-1}]$-algebra, where $T := T_{1_S}$; a free basis of $G_{\mathsf{k}}(S)$ over $\mathsf{k}[T, T^{-1}]$ may be constructed by choosing any $x \in S$ and taking the set of $T_\sigma$ for which $\sigma \in \Xi(S)$ is such that $\sigma(x) = 0$. For any $\mathsf{k}[T, T^{-1}]$-algebra $A$ we let $G_A(S) := A \otimes_{\mathsf{k}[T,T^{-1}]} G_{\mathsf{k}}(S)$, and for any k-algebra $B$ we let $G'_B(S) := B \otimes_{\mathsf{k}} G'_{\mathsf{k}}(S)$ and $G''_B(S) := B \otimes_{\mathsf{k}} G''_{\mathsf{k}}(S)$.

Now we can describe the associated graded ring $\mathrm{gr}_L(D)$. For each $q \in Q$ let $T_q$ be the basis element of $G_{\mathsf{k}}(f^{-1}(q))$ corresponding to $1_{f^{-1}(q)} \in \Xi(f^{-1}(q))$. So for each $q \in Q$, K is a $\mathsf{k}[T_q, T_q^{-1}]$-algebra via the monomorphism given by $T_q \mapsto t_q$ and algebraic extension (here $t_q := (\mathbf{Z}^{(\delta_q - \delta'_q)\circ f}/1) + \mathfrak{m}$, as above).

**Theorem 3.2** *Let $L \subseteq D$ and $f: \hat{P} \to \hat{Q}$ be as above. Then the associated graded ring $gr_L(D)$ of $V(L)$ in $V(D)$ is naturally isomorphic to the* K*-tensor product*

$$gr_L(D) \simeq G'_{\mathsf{K}}(f^{-1}(\hat{0})) \otimes \left(\bigotimes_{q\in Q} G_{\mathsf{K}}(f^{-1}(q))\right) \otimes G''_{\mathsf{K}}(f^{-1}(\hat{1})).$$

**Proof:** Let $z_q := \mathbf{Z}^{(\delta_q - \delta'_q)\circ f}/1$ in $\mathcal{O}_{L,D}$ for each $q \in Q$, and let $\mathsf{F} := \mathsf{k}(z_q : q \in Q)$. Then $\mathcal{O}_{L,D}$ is an F-algebra, and a spanning set for $\mathcal{O}_{L,D}$ over F may be constructed as follows. Choose an element $x(q) \in f^{-1}(q)$ for each $q \in \hat{Q}$; we insist that $x(\hat{0}) = \hat{0}$ and $x(\hat{1}) = \hat{1}$. The spanning set consists of those elements $\mathbf{Z}^\sigma/u$ for which $u$ satisfies condition (D) above and $\mathbf{Z}^\sigma$ satisfies condition (N) above and $\sigma(x(q)) = 0$ for each $q \in \hat{Q}$; we denote the set of such $\sigma$ by $\mathcal{B}$.

For any $t/u$ in $\mathcal{O}_{L,D}$ let $o(t/u)$ denote the largest $j \geq 0$ such that $t/u \in \mathfrak{m}^j$; for $\sigma \in \mathcal{B}$ we write $o(\sigma)$ instead of $o(\mathbf{Z}^\sigma/1)$. Since the elements $\mathbf{Z}^\sigma/u$ for $\sigma \in \mathcal{B}$ span $\mathcal{O}_{L,D}$ as an F-vector space, the residues $Y_\sigma := (\mathbf{Z}^\sigma/1) + \mathfrak{m}^{1+o(\sigma)} \in \mathfrak{m}^{o(\sigma)}/\mathfrak{m}^{1+o(\sigma)}$ for $\sigma \in \mathcal{B}$ span $\mathrm{gr}_L(D)$ as a K-vector space; we claim that in fact they form a basis. To see this, assume that $\sum_{\sigma\in\mathcal{B}} c_\sigma Y_\sigma = 0$ is a K-linear dependence in $\mathrm{gr}_L(D)$ among $\{Y_\sigma : \sigma \in \mathcal{B}\}$. If this dependence is not trivial, then let $j := \min\{o(\sigma) : \sigma \in \mathcal{B} \text{ and } c_\sigma \neq 0\}$, and let $\mathcal{B}(j)$ be the set of $\sigma \in \mathcal{B}$ with $o(\sigma) = j$. For each $\sigma \in \mathcal{B}(j)$ let $c_\sigma = (t_\sigma + \mathfrak{m})/(u_\sigma + \mathfrak{m})$ where each monomial $\mathbf{Z}^\beta$ occurring in $t_\sigma$ or $u_\sigma$ has $\beta$ constant on each fibre of $f$ (note that $u_\sigma \neq 0$). Then we have $\sum_{\sigma\in\mathcal{B}(j)} (t_\sigma + \mathfrak{m})Y_\sigma/(u_\sigma + \mathfrak{m}) \subseteq \mathfrak{m}^{j+1}$. This implies that $\sum_{\sigma\in\mathcal{B}(j)} t_\sigma Z^\sigma/1 \in \mathfrak{m}^{j+1}$, from which it follows that $t_\sigma \in \mathfrak{m}$ for each $\sigma \in \mathcal{B}(j)$. This contradicts our choice of $j$, so the dependence must have been trivial.

Now we claim that for any $\sigma \in \mathcal{B}$, $o(\sigma) = \sum_{q\in\hat{Q}} \mathrm{ord}(\sigma|_{f^{-1}(q)})$. Certainly, $o(\sigma)$ is no less than the right-hand side, since each $\sigma|_{f^{-1}(q)}$ can be expressed as a sum of $\mathrm{ord}(\sigma|_{f^{-1}(q)})$ split elements and an integer multiple of $1_{f^{-1}(q)}$; each of these split elements $\alpha$ determines an element of $\mathfrak{m}$: if $q \neq \hat{0}$ we have $\mathbf{Z}^\alpha/1 \in \mathfrak{m}$ and if $q = \hat{0}$ we have $\mathbf{Z}^{\alpha-\omega}/1 \in \mathfrak{m}$ (where $\omega := \hat{0}_L \circ f$, and we have extended $\alpha$ by 0 to obtain a function on $\hat{P}$). The product of these elements of $\mathfrak{m}$ is equal to $\mathbf{Z}^\sigma/1$ times a scalar from F, proving the inequality.

Conversely, if $o(\sigma) = p > \sum_{q\in\hat{Q}} \mathrm{ord}(\sigma|_{f^{-1}(q)})$ then let $\mathbf{Z}^\sigma/1 = (g_1 g_2 \cdots g_p)/u$ in $\hat{\mathcal{O}}_{L,D}$, where each $g_i \in \mathfrak{m}$; in other words $u\mathbf{Z}^\sigma = g_1 g_2 \cdots g_p$ (since $\mathsf{k}\Sigma(\hat{P})$ has no zero-divisors).

Since $u$ satisfies condition (D), there is a term $\mathbf{Z}^{\tau}$ occurring in $u\mathbf{Z}^{\sigma}$ such that $\sigma - \tau$ is contant on each fibre of $f$. We must be able to produce $\mathbf{Z}^{\tau}$ by choosing a monomial $\mathbf{Z}^{\beta_i}$ occurring in $g_i$ for each $1 \leq i \leq p$ and taking their product: that is, $\mathbf{Z}^{\tau} = \mathbf{Z}^{\beta_1+\cdots+\beta_p}$. Thus, for each $q \in \hat{Q}$, $\sigma|_{f^{-1}(q)} \approx (\beta_1 + \cdots + \beta_p)|_{f^{-1}(q)}$ in $\Xi(f^{-1}(q))$. For each $1 \leq i \leq p$, since $g_i \in \mathfrak{m}$ and $\mathbf{Z}^{\beta_i}$ is a term in $g_i$, there is some $q_i \in \hat{Q}$ such that $\beta_i$ is not constant on $f^{-1}(q_i)$. By our assumption that $p = o(\sigma)$ is large, there is some $q \in \hat{Q}$ such that $q = q_i$ for at least $p' + 1$ indices $i$, where $p' := \mathrm{ord}(\sigma|_{f^{-1}(q)})$. But now $\mathrm{ord}(\sigma|_{f^{-1}(q)}) \geq \sum_{i=1}^{p} \mathrm{ord}(\beta_i|_{f^{-1}(q)}) \geq p'+1$, a contradiction. This proves the equality.

From the equality just established and the superadditivity of $\mathrm{ord}(\cdot)$, it follows that for any $\sigma, \tau \in \mathcal{B}$, $o(\sigma + \tau) = o(\sigma) + o(\tau)$ if and only if $\mathrm{ord}((\sigma + \tau)|_{f^{-1}(q)}) = \mathrm{ord}(\sigma|_{f^{-1}(q)}) + \mathrm{ord}(\sigma|_{f^{-1}(q)})$ for each $q \in \hat{Q}$. Since in $\mathrm{gr}_L(D)$ we have

$$Y_\sigma Y_\tau = \begin{cases} Y_{\sigma+\tau} & \text{if } o(\sigma) + o(\tau) = o(\sigma + \tau), \\ 0 & \text{otherwise,} \end{cases}$$

it readily follows that the map defined by $Y_\sigma \mapsto \bigotimes_{q\in\hat{Q}} T_{\sigma|_{f^{-1}(q)}}$ and K-linear extension is a K-algebra isomorphism as in the statement of the theorem. □

With this description of $\mathrm{gr}_L(D)$ we can improve upon Theorem 2.3. The *embedding dimension* of $V(L)$ in $V(D)$ is $\mathrm{edim}_D(L) := \dim_{\mathsf{K}} \mathfrak{m}/\mathfrak{m}^2$, where K is the residue field and $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{L,D}$. The *split number* of a connected poset $S$ is $\mathrm{sn}(S)$, the number of splits of $S$.

**Corollary 3.3** *Let $L \subseteq D$ and $f: \hat{P} \to \hat{Q}$ be as above. Then*

$$\mathrm{edim}_D(L) = \sum_{q\in\hat{Q}} \mathrm{sn}(f^{-1}(q)).$$

**Proof:** With the notation in the proof of Theorem 3.2, the elements $Y_\sigma$ with $\sigma \in \mathcal{B}$ and $o(\sigma) = 1$ form a K-basis for $\mathfrak{m}/\mathfrak{m}^2$. By the formula for $o(\sigma)$ in that proof, we see that such a $\sigma$ has $\sigma|_{f^{-1}(q)}$ nonzero for exactly one $q \in \hat{Q}$, and on that fibre $\sigma|_{f^{-1}(q)}$ is equivalent to a split element of $\Xi(f^{-1}(q))$. This implies the result. □

**Theorem 3.4** *If $S$ is a connected poset then* $\mathrm{sn}(S) \geq \#S - 1$, *and equality holds if and only if $S$ is a tree.*

**Proof:** We proceed by induction on $\#S$, the basis $\#S \leq 2$ being clear. Let $x \in S$ be minimal, and let the connected components of $S\backslash x$ be $R_1, \ldots, R_m$. For each $1 \leq i \leq m$ and split $(R_i', R_i'')$ of $R_i$, we construct a split of $S$ as follows. If $x < y$ for some $y \in R_i'$ then $(\bigcup_{j\neq i} R_j \cup R_i' \cup \{x\}, R_i'')$ is a split of $S$; otherwise $(R_i', \bigcup_{j\neq i} R_j \cup R_i'' \cup \{x\})$ is a split of $S$. Also, for each $1 \leq i \leq m$ the pair $(\bigcup_{j\neq i} R_j \cup \{x\}, R_i)$ is a split of $S$. These splits are pairwise distinct, so that by induction $\mathrm{sn}(S) \geq \sum_{i=1}^{m} (\mathrm{sn}(R_i) + 1) \geq \sum_{i=1}^{m} \#R_i = \#S - 1$.

If equality holds then $\mathrm{sn}(R_i) = \#R_i - 1$ for all $1 \leq i \leq m$, so that by induction each $R_i$ is a tree. Suppose that for some $1 \leq j \leq m$, there are two elements $u$ and $v$ of $R_j$ which cover $x$ in $S$. Then $u$ and $v$ are incomparable in $R_j$. Since $R_j$ is a tree, there is a unique path $L$ in the Hasse graph of $R_j$ from $u$ to $v$, and since $u$ and $v$ are not comparable in $R_j$, there is an internal vertex $z$ of $L$ which is either minimal or maximal in $L$.

Suppose that we can find $u$ and $v$ such that $z$ is maximal in $L$. Let $R'_j$ be the union of the two components of $R_j\backslash z$ which contain $u$ and $v$, and let $R''_j := R_j\backslash R'_j$. Then $R''_j$ is a connected upset of $R_j$ and $R'_j$ is an downset of $R_j$ with two components. Thus $(R'_j, R''_j)$ is not a split of $R_j$, but $(\{x\} \cup R'_j \cup \bigcup_{i\neq j} R_i, R''_j)$ is a split of $S$. This implies that $\text{sn}(S) > \#S - 1$, a contradiction.

In the remaining case, for any $u$ and $v$ in $R_j$ which cover $x \in S$, the path in $R_j$ between $u$ and $v$ has a unique minimal element, and this is neither $u$ nor $v$. Choose any $u$ and $v$ in $R_j$ which cover $x \in S$, and let $z$ be the unique minimal element of the path between them. Let $R'_j$ be the component of $R_j\backslash C$ which contains $z$, where $C$ is the set of elements of $R_j$ which cover $x \in S$, and let $R''_j := R_j\backslash R'_j$. Then $R'_j$ is a connected downset of $R_j$, and $R''_j$ is an upset of $R_j$ with at least two components, each of which contains an element of $C$. Thus $(R'_j, R''_j)$ is not a split of $R_j$, but $(R'_j, \{x\} \cup R''_j \cup \bigcup_{i\neq j} R_i)$ is a split of $S$. This implies that $\text{sn}(S) > \#S - 1$, another contradiction.

It follows that for each $1 \le i \le m$ there is exactly one $q_i \in R_i$ which covers $x$ in $S$. Therefore, $S$ is a tree.

Conversely, if $S$ is a tree then there is an obvious bijection between the splits of $S$ and the covering relations of $S$. Hence, $\text{sn}(S) = \#S - 1$. □

Theorem 2.3 follows directly from Corollary 3.3 and Theorem 3.4, since (by the Zariski smoothness criterion) $V(D)$ is nonsingular along $V(L)$ if and only if $\text{edim}_D(L) = \dim(V(D)) - \dim(V(L))$, and from Theorem 2.2 the codimension of $V(L)$ in $V(D)$ is $\#P - \#Q$.

Finally, we consider the question of irreducibility of the normal link of $V(L)$ in $V(D)$. Since the normal link is by definition Proj $\text{gr}_L(D)$, this is equivalent to the condition that $\text{gr}_L(D)$ is an integral domain. For this it is clearly neccesary that $\text{ord}(\cdot)$ be additive on $\Xi(f^{-1}(q))$ for each $q \in \hat{Q}$; this turns out to be sufficient as well. Moreover, for a connected poset $S$, there is a nice linear-algebraic criterion for $\text{ord}(\cdot)$ to be additive on $\Xi(S)$ which leads to a result similar in form to Theorem 2.3. Given $S$, let $\langle\cdot,\cdot\rangle$ be the inner product on $\mathbb{Q}^S$ which makes the indicator functions of the vertices of $S$ into an orthonormal basis. A *valuation* of $S$ is a function $\varphi\colon S \to \mathbb{Q}$ such that $\langle\varphi, 1_S\rangle = 0$ and $\langle\varphi, \alpha\rangle = 1$ for each split element $\alpha \in \Xi(S)$. If $S$ has a valuation then $S$ is *valuable*.

**Theorem 3.5** *Given a connected poset $S$,* $\text{ord}(\cdot)$ *is additive on $\Xi(S)$ if and only if $S$ is valuable.*

**Proof:** First, assume that $\varphi$ is a valuation of $S$. Given any $\sigma \approx \alpha_1 + \cdots + \alpha_m$ in $\Xi(S)$, with each $\alpha_i$ split, we have $\langle\varphi,\sigma\rangle = m$, so that $\text{ord}(\sigma) = \langle\varphi,\sigma\rangle$ is additive on $\Xi(S)$.

For the converse, first note that $\text{ord}(\cdot)$ is additive on $\Xi(S)$ if and only if whenever $\sigma \approx \alpha_1 + \cdots + \alpha_m$ in $\Xi(S)$, with each $\alpha_i$ split, we have $m = \text{ord}(\sigma)$. Certainly this condition implies additivity of $\text{ord}(\cdot)$. For the other direction, suppose that the condition fails, and consider an equivalence $\sigma \approx \alpha_1 + \cdots + \alpha_m \approx \beta_1 + \cdots + \beta_n$ with each $\alpha_i$ and

$\beta_j$ split, with $m < n$, and with $m$ as small as possible. Then $m \geq 2$, and $\mathrm{ord}(\alpha_1) = 1$ and $\mathrm{ord}(a_2 + \cdots + \alpha_m) = m - 1$ and $\mathrm{ord}(\sigma) \geq n > m$, so that $\mathrm{ord}(\cdot)$ is not additive.

Now let $M$ be the matrix with $1 + \mathrm{sn}(S)$ rows and $\#S$ columns, with first row all 1's and the other rows being the split elements of $\Xi(S)$. Let $\mathbf{u}$ be the column vector of length $1 + \mathrm{sn}(S)$, with first entry 0 and all other entries 1. By definition, $S$ has a valuation if and only if $\mathbf{u}$ is in the column space of $M$. If $\mathbf{u}$ is not in the column space of $M$ then let $\mathbf{c}$ be a row vector such that $\mathbf{cu} \neq 0$ and $\mathbf{c}M = 0$. Since all the coefficients are rational, we can find such a $\mathbf{c}$ with integer entries. That is, if $\alpha_1, \ldots, \alpha_s$ are all the splits of $S$, we have integers $c_0, c_1, \ldots, c_s$ such that $c_0 1_S + c_1\alpha_1 + \cdots + c_s\alpha_s = 0_S$ and $c_1 + \cdots + c_s \neq 0$. Taking the terms with $c_i < 0$ to the right side we find an element of $\Xi(S)$ which is equivalent to sums of splits of two different lengths. Hence $\mathrm{ord}(\cdot)$ is not additive on $\Xi(S)$. $\square$

**Theorem 3.6** *Let $L \subseteq D$ and $f: \hat{P} \to \hat{Q}$ be as above. Then $gr_L(D)$ is a normal integral* K*-algebra if and only if for each $q \in \hat{Q}$ the fibre $f^{-1}(q)$ is valuable.*

**Proof:** From Theorems 3.2 and 3.5 it is clear that if some fibre of $f$ is not valuable then $\mathrm{gr}_L(D)$ contains zero-divisors.

Conversely, assume that each fibre of $f$ is valuable. For each $q \in \hat{Q}$, choose $x(q) \in f^{-1}(q)$ subject only to $x(\hat{0}) = \hat{0}$ and $x(\hat{1}) = \hat{1}$. Then, by the proof of Theorem 3.2, $\mathrm{gr}_L(D)$ has as a K-basis the set $\{Y_\sigma : \sigma \in \mathcal{B}\}$, in which $\mathcal{B}$ is the set of $\sigma: \hat{P} \to \mathbb{Z}$ such that for each $q \in \hat{Q}$, $\sigma|_{f^{-1}(q)}$ is order-reversing and $\sigma(x(q)) = 0$. By Theorem 3.5 and the proof of Theorem 3.2, the assignment $Y_\sigma \mapsto \mathbf{Z}^\sigma$ for $\sigma \in \mathcal{B}$, extended K-linearly, identifies $\mathrm{gr}_L(D)$ with the semigroup algebra of $\mathcal{B}$ (considered as an additive semigroup) over K. This semigroup algebra K$\mathcal{B}$ is a subalgebra of $\mathsf{K}[Z_p, Z_p^{-1}: p \in \hat{P}\backslash X]$ where $X := \{x(q): q \in \hat{Q}\}$, and hence is an integral domain. The semigroup $\mathcal{B}$ is easily seen to satisfy the definition (p. 320 of Hochster [10]) of a normal semigroup: if $\sigma, \sigma', \sigma'' \in \mathcal{B}$ and integer $m > 0$ are such that $\sigma + m\sigma' = m\sigma''$ then there is a $\tau \in \mathcal{B}$ such that $\sigma = m\tau$. By Proposition 1 of [10] it follows that $\mathrm{gr}_L(D)$ is normal. $\square$

As the example in Figure 1 shows, not all connected posets are valuable. It is easy to see that all bounded posets are valuable, but there are many valuable posets which are not bounded. For example, since the normal link of a regular point is irreducible, Theorems 2.3 and 3.6 give an indirect proof that all trees are valuable. The problem of structurally characterizing valuable posets is addressed in [17]. In particular, it is proved there that every orbit-closure of $V(D)$ has an irreducible normal link if and only if $\hat{P}$ is a dismantlable lattice.

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