COMPOSITIO MATHEMATICA

# Singularities on normal varieties

Tommaso de Fernex and Christopher D. Hacon

Compositio Math. **145** (2009), 393–414.

doi:10.1112/S0010437X09003996

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

Compositio Math. 145 (2009) 393–414
doi:10.1112/S0010437X09003996

# Singularities on normal varieties

Tommaso de Fernex and Christopher D. Hacon

Abstract

In this paper we generalize the definitions of singularities of pairs and multiplier ideal sheaves to pairs on arbitrary normal varieties, without any assumption on the variety being $\mathbb{Q}$-Gorenstein or the pair being log $\mathbb{Q}$-Gorenstein. The main features of the theory extend to this setting in a natural way.

## 1. Introduction

The theory of singularities of pairs and multiplier ideal sheaves has become a core part of the study of higher-dimensional algebraic varieties (e.g., see [Kol97, KM98, Laz04b, EM06] for an overview of the theory and various applications). In fact, pairs naturally arise in a geometrically meaningful way in a variety of instances: as boundaries of open varieties, markings on varieties in moduli problems, discriminants and orbifold structures of morphisms, base schemes of rational maps, and inductive tools in higher dimensional geometry.

The main purpose of this paper is to investigate possible extensions of the theory to settings which are more general than the ones in which it has been introduced and studied. Our priority, naturally, is to perform this generalization in such a way that the essential features are preserved.

Given a $\mathbb{Q}$-Gorenstein variety $X$, several invariants have been defined via resolution of singularities. A key ingredient in their definition is the relative canonical divisor of a resolution $f\colon Y \to X$, that is, the exceptional $\mathbb{Q}$-divisor $K_{Y/X} := K_Y - f^*K_X$ (here we fix $K_Y$ so that $f_*K_Y = K_X$). The difficulty in extending the definitions of such invariants to arbitrary normal varieties arises as soon as $K_X$ is not $\mathbb{Q}$-Cartier, as it is unclear in this case what should be its pullback. One way around the problem is to perturb $K_X$ by adding a *boundary*, that is, an effective $\mathbb{Q}$-divisor $\Delta$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. This also gives rise to a pair $(X, \Delta)$, but the boundary itself may have no particular geometric meaning, and it is not clear *a priori* that there exists a natural choice for $\Delta$.

Our approach to the problem is different and more direct. We introduce a notion of pullback of (Weil) $\mathbb{Q}$-divisors which agrees with the usual one for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. In this way we are able to define *relative canonical divisors* $K_{Y/X} := K_Y + f^*(-K_X)$ and $K^-_{Y/X} := K_Y - f^*K_X$ for any proper birational morphism $f\colon Y \to X$ of normal varieties. These are exceptional $\mathbb{R}$-divisors that coincide when $K_X$ is $\mathbb{Q}$-Cartier, but may be different otherwise. We also define a suitable approximation of $K^-_{Y/X}$ via $\mathbb{Q}$-divisors $K_{m,Y/X}$ (for $m \geq 1$), that we call *limiting relative canonical divisors*. Using these notions, we generalize the definitions of multiplier ideals and

Received 19 May 2008, accepted in final form 21 October 2008, published online 25 March 2009.
*2000 Mathematics Subject Classification* 14B05 (primary), 14J17, 14E15 (secondary).
*Keywords:* divisorial valuation, relative canonical divisor, singularities of pairs, multiplier ideals.
The first author was partly supported by NSF research grant 0548325. The second author was partly supported by NSF research grant 0456363 and an AMS Centennial fellowship.

singularities of pairs to pairs of the form $(X,Z)$, where $X$ is an arbitrary normal variety and $Z=\sum b_k\cdot Z_k$ is an effective formal linear combination of proper closed subschemes of $X$.

The *multiplier ideal sheaf* $\mathcal{J}(X,Z)$ of $(X,Z)$ is defined, in our generality, as the unique maximal element in the collection of ideal sheaves

$$\{(f_m)_*\mathcal{O}_{Y_m}(\lceil K_{m,Y_m/X}-f_m^{-1}(Z)\rceil)\}_{m\geq 1},$$

where for every $m$ the morphism $f_m\colon Y_m\to X$ is a 'high enough' log resolution of $(X,Z)$ depending on $m$. The core result of the paper is that $\mathcal{J}(X,Z)$ can be realized as the multiplier ideal sheaf of a suitable log $\mathbb{Q}$-Gorenstein pair.

Theorem 1.1. *For any pair $(X,Z)$ as above, there is a boundary $\Delta$ on $X$ such that*

$$\mathcal{J}(X,Z)=\mathcal{J}((X,\Delta);Z).$$

In particular, we deduce the surprising fact that the set of ideal sheaves

$$\{\mathcal{J}((X,\Delta);Z)\mid\Delta\text{ is a boundary on }X\}$$

has a unique maximal element, namely $\mathcal{J}(X,Z)$. *A posteriori*, one can take this maximal element as the definition of $\mathcal{J}(X,Z)$. Using this result, all the main properties related to multiplier ideals, such as vanishing theorems, connectedness properties, and basic inversion of adjunction statements, extend immediately to the general setting.

In order to generalize the notions of *log terminal* and *log canonical singularities*, we impose log discrepancy conditions with respect to the limiting relative canonical divisors $K_{m,Y/X}$. In a similar vein, we have the following result.

Theorem 1.2. *A pair $(X,Z)$ is log terminal (respectively, log canonical) if and only if there is a boundary $\Delta$ on $X$ such that $((X,\Delta);Z)$ is log terminal (respectively, log canonical).*

We immediately deduce, for instance, that as in the $\mathbb{Q}$-Gorenstein case a normal variety with log terminal singularities (respectively, with Cohen–Macaulay log canonical singularities) has rational singularities (respectively, Du Bois singularities). Kawamata's subadjunction theorem is also generalized to our context. In fact, we observe that minimal log canonical centers (which in general are not known to be $\mathbb{Q}$-Gorenstein) are log terminal; this provides in particular a natural setting for the theory developed in this paper. Finally, we check that in dimension two our notions of log terminal and log canonical singularities agree with those of numerically log terminal and numerically log canonical singularities, which in particular implies that they are always $\mathbb{Q}$-Gorenstein.

By contrast, our definition of *terminal* and *canonical singularities* uses log discrepancy conditions with respect to the relative canonical divisor $K_{Y/X}$. When $Z$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, we extend to this setting the following characterization of canonical singularities.

Proposition 1.3. *If $Z$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then $(X,Z)$ is canonical if and only if for any sufficiently divisible $m\geq 1$ and for every sufficiently high log resolution $f\colon Y\to X$ there is an inclusion $\mathcal{O}_X(m(K_X+Z))\cdot\mathcal{O}_Y\subseteq\mathcal{O}_Y(m(K_Y+Z_Y))$ as sub-$\mathcal{O}_X$-modules of the constant sheaf of rational functions, where $Z_Y$ is the proper transform of $Z$.*

Using this property, the main features of canonical singularities, such as the deformation invariance properties of plurigenera (for singular varieties of general type), of canonical singularities and of numerical Kodaira dimension easily extend to the more general setting.

We expect that the larger freedom in defining these notions of singularities should have interesting applications. In the (log) $\mathbb{Q}$-Gorenstein setting many applications rely on multiplier ideals and their vanishing theorems, and it is encouraging that these powerful methods extend to our setting.

The original motivation of this research comes from a question posed by Valery Alexeev during the AIM Workshop [AIM06], which asks whether it is possible to generalize the definitions of singularities of pairs in a wider context than the usual one. The question itself was motivated by an example, due to Paul Hacking, of a flat family of pairs $(S_t, D_t)$, where $S_t$ is a smooth surface and $D_t$ is an effective divisor, that specializes to a pair $(S_0, D_0)$, where $S_0$ is a singular surface and the ideal sheaf of $D_0$ acquires an embedded prime at the singularity of $S_0$.

The example brings to light an important issue: namely that often, in the literature, pairs $(X, Z)$ have been intended in a combined way, both geometrically (as in one of the situations previously described) and as a correction to the possible failure of $K_X$ being $\mathbb{Q}$-Cartier; by incorporating a boundary $\Delta$ into $Z$, so to speak. We insist in this paper on keeping the two things separated.

The question of defining multiplier ideals in the generality treated in this paper arises naturally also in connection with the *generalized test ideal* introduced by Hara and Yoshida [HY03] (see also [HT04]) using the Frobenius action in positive characteristics, as the latter can be defined without any (log) $\mathbb{Q}$-Gorenstein assumption. In the (log) $\mathbb{Q}$-Gorenstein setting, multiplier ideals reduce, for sufficiently large characteristics, to the corresponding generalized test ideals (see [Smi00, Har01, HY03, Tak04]). It follows by independent results of Hara and Blickle (see [Bli04]) that, in the toric setting, the same happens without any (log) $\mathbb{Q}$-Gorenstein assumption for the multiplier ideals defined in this paper. It would be interesting to see if this property holds in general; this question was raised by Hara.

In the first two sections of the paper we work over an arbitrary field; starting from § 4 we will restrict the setting to varieties over an algebraically closed field of characteristic zero. A *divisor* on a normal variety $X$ will be understood to be a Weil divisor, unless otherwise specified.

## 2. Valuations of $\mathbb{Q}$-divisors

Let $X$ be a normal variety. A *divisorial valuation* $v$ on $X$ is a discrete valuation of the function field of $X$ of the form $v = q\,\mathrm{val}_F$ where $q \in \mathbb{Z}_+$ and $F$ is a prime divisor *over* $X$, that is, on a normal variety $X'$ with a given birational morphism $\mu\colon X' \to X$.

Throughout this section, we fix a divisorial valuation $v$ of $X$. If $D$ is a Cartier divisor on $X$, then the valuation $v(D)$ of $D$ is given by $q$ times the coefficient of $F$ in the divisor $\mu^*D$. The valuation $v(Z)$ of a proper closed subscheme $Z \subset X$ is given by

$$v(Z) = v(\mathcal{I}_Z) := \min\{v(\phi) \mid \phi \in \mathcal{I}_Z(U),\ U \cap c_X(v) \neq \emptyset\},$$

where $\mathcal{I}_Z \subseteq \mathcal{O}_X$ is the ideal sheaf of $Z$. This definition extends to formal $\mathbb{R}$-linear combinations $\sum a_k \cdot Z_k$ of proper closed subschemes $Z_k \subset X$ by setting $v(\sum a_k \cdot Z_k) := \sum a_k \cdot v(Z_k)$.

More generally, let $\mathcal{I} \subset \mathcal{K}$ be a finitely generated sub-$\mathcal{O}_X$-module of the constant sheaf of rational functions $\mathcal{K} = \mathcal{K}_X$ on $X$. For short, we will refer to $\mathcal{I}$ as a *(coherent) fractional ideal sheaf* on $X$.

DEFINITION 2.1. The *valuation* $v(\mathcal{I})$ of a non-zero fractional ideal sheaf $\mathcal{I} \subset \mathcal{K}$ along $v$ is given by

$$v(\mathcal{I}) := \min\{v(\phi) \mid \phi \in \mathcal{I}(U),\ U \cap c_X(v) \neq \emptyset\}.$$

The *valuation* $v(I)$ of a formal linear combination $I = \sum a_k \cdot \mathcal{I}_k$ of fractional ideal sheaves $\mathcal{I}_k \subset \mathcal{K}$ along $v$ is defined by $v(I) := \sum a_k \cdot v(\mathcal{I}_k)$.

If $\mathcal{I}$ and $\mathcal{I}'$ are fractional ideal sheaves on $X$ with $\mathcal{I} \subseteq \mathcal{I}'$, then $v(\mathcal{I}) \geq v(\mathcal{I}')$. In the case of ideal sheaves, this definition coincides with the one previously given, and if $D$ is a Cartier divisor, then $v(D) = v(\mathcal{O}_X(-D))$.

Consider now an arbitrary divisor $D$ on $X$.

DEFINITION 2.2. The $\natural$-*valuation* (or *natural valuation*) along $v$ of a divisor $D$ on $X$ is

$$v^\natural(D) := v(\mathcal{O}_X(-D)).$$

Clearly, we have $v^\natural(D) = v(D)$ for any Cartier divisor $D$. Note also that, as $\mathcal{O}_X(D) \cdot \mathcal{O}_X(-D) \subseteq \mathcal{O}_X$, we have $v^\natural(-D) + v^\natural(D) \geq 0$.

In general the $\natural$-valuation of divisors is not linear with respect to the group structure of $\mathrm{Div}(X)$, as the next example shows.

*Example* 2.3. Let $X = \{xy = z^2\} \subset \mathbb{C}^3$, and let $v = \mathrm{val}_E$, where $E$ is the exceptional divisor of the blow up of $X$ at the origin. Then, for any two lines $L, M \subset X$ (passing through the origin), we have $v^\natural(L) = v^\natural(M) = v^\natural(L+M) = 1$, and thus $v^\natural(L+M) \neq v^\natural(L) + v^\natural(M)$. In particular, $v^\natural(2L) = v^\natural(L)$. Note also that $v^\natural(-L) = 0$.

LEMMA 2.4. *Let $C$ be a Cartier divisor on $X$. Then $v^\natural(C + D) = v(C) + v^\natural(D)$ for every divisor $D$ on $X$.*

*Proof.* Since $\mathcal{O}_X(-C)$ is locally generated by one rational function, one can check that $\mathcal{O}_X(-C - D) = \mathcal{O}_X(-C) \cdot \mathcal{O}_X(-D)$, and the assertion follows. □

DEFINITION 2.5. To any non-trivial fractional ideal sheaf $\mathcal{I}$ on $X$, we associate the divisor

$$\mathrm{div}(\mathcal{I}) := \sum_{E \subset X} \mathrm{val}_E(\mathcal{I}) \cdot E,$$

where the sum is taken over all prime divisors $E$ on $X$. Equivalently, $\mathrm{div}(\mathcal{I})$ is the divisor on $X$ for which $\mathcal{O}_X(-\mathrm{div}(\mathcal{I})) = \mathcal{I}^{\vee\vee}$. In particular, $\mathrm{div}(\mathcal{O}_X(-D)) = D$ for any divisor $D$. We call $\mathrm{div}(\mathcal{I})$ the *divisorial part* of $\mathcal{I}$.

Consider now a birational morphism $f : Y \to X$ from a normal variety $Y$.

DEFINITION 2.6. For any divisor $D$ on $X$, the $\natural$-*pullback* (or *natural pullback*) of $D$ to $Y$ is given by

$$f^\natural D := \mathrm{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y).$$

In other words, $f^\natural D = \sum \mathrm{val}_E^\natural(D) \cdot E$, where the sum is taken over all prime divisors $E$ on $Y$. In particular, $\mathcal{O}_Y(-f^\natural D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}$.

LEMMA 2.7. *Let $f : Y \to X$ and $g : V \to Y$ be two birational morphisms of normal varieties. Then, for every divisor $D$ on $X$, the divisor $(fg)^\natural D - g^\natural(f^\natural D)$ is effective and $g$-exceptional. Moreover, if $\mathcal{O}_X(-D) \cdot \mathcal{O}_Y$ is an invertible sheaf, then $(fg)^\natural D = g^\natural(f^\natural D)$.*



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

*Proof.* By Lemma 2.4, we have

$$(fg)^\natural(C+D) - g^\natural(f^\natural(C+D)) = (fg)^\natural D - g^\natural(f^\natural D)$$

for every Cartier divisor $C$. Therefore, after restricting to an open quasi-projective subset and replacing $D$ with $C+D$ for some Cartier divisor $C \geq D$, we may assume without loss of generality that $D$ is effective. Then it suffices to observe that $\mathcal{O}_X(-D)\cdot\mathcal{O}_Y \subseteq \mathcal{O}_Y(-f^\natural D)$, with equality holding at the generic point of every codimension-one subvariety of $Y$. For the last assertion, we first remark that the condition that $\mathcal{O}_X(-D)\cdot\mathcal{O}_Y$ is an invertible sheaf remains unchanged if we multiply $\mathcal{O}_X(-D)$ by an invertible sheaf $\mathcal{O}_X(-C)$, and that $\mathcal{O}_X(-D)\cdot\mathcal{O}_Y = \mathcal{O}_Y(-f^\natural D)$ if $\mathcal{O}_X(-D)\cdot\mathcal{O}_Y$ is locally principal. □

By the homogeneity of valuations and pullbacks of Cartier divisors it is very natural to extend this definition by setting

$$v(D) := \frac{v(mD)}{m} \quad \text{and} \quad f^*D := \frac{f^*(mD)}{m} \tag{1}$$

for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on $X$, where $m$ is any non-zero integer such that $mD$ is a Cartier divisor. In general, however, $\natural$-valuations and $\natural$-pullbacks of arbitrary divisors do not enjoy a similar homogeneity property, as pointed out in Example 2.3. In fact, the case of a line on a quadric cone is an example of a $\mathbb{Q}$-Cartier divisor $D$ for which $v^\natural(D) \neq v(D)$, where the valuation on the right side is intended as defined above. This problem is resolved by giving relevance to the asymptotic nature of the definitions given in (1) for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors.

LEMMA 2.8. *For every divisor $D$ on $X$ and every $m \in \mathbb{Z}_+$, we have $m\cdot v^\natural(D) \geq v^\natural(mD)$, and*

$$\inf_{k\geq 1}\frac{v^\natural(kD)}{k} = \liminf_{k\to\infty}\frac{v^\natural(kD)}{k} = \lim_{k\to\infty}\frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

*Proof.* If $\phi_1,\ldots,\phi_m \in \mathcal{O}_X(-D)(U)$ for some open set $U \subseteq X$, then $\operatorname{div}(\phi_i) \geq D$ on $U$ for each $i$, and therefore $\operatorname{div}(\prod\phi_i) \geq mD$ on $U$, which means that $\prod\phi_i \in \mathcal{O}_X(-mD)(U)$. This implies that $\mathcal{O}_X(-D)^m \subseteq \mathcal{O}_X(-mD)$, and thus $m\cdot v(\mathcal{O}_X(-D)) \geq v(\mathcal{O}_X(-mD))$, which proves the first assertion of the lemma. Both equalities in the display of the lemma follow from this inequality, and the fact that the infimum is not $-\infty$ follows from the fact that, if $X$ is quasi-projective, then $D \geq C$ for some Cartier divisor $C$. In general, one reduces to the quasi-projective case by restricting to an affine open neighborhood of the generic point of the center of $v$ in $X$. □

DEFINITION 2.9. Let $D$ be a $\mathbb{Q}$-divisor on $X$. The *valuation* along $v$ of $D$ is

$$v(D) := \lim_{k\to\infty}\frac{v^\natural(k!D)}{k!} \in \mathbb{R}.$$

If $f\colon Y \to X$ is a birational morphism from a normal variety $Y$, then the *pullback* of $D$ to $Y$ is

$$f^*D := \sum \operatorname{val}_E(D)\cdot E,$$

where the sum is taken over all prime divisors $E$ on $Y$.

PROPOSITION 2.10. *Let $D$ be a $\mathbb{Q}$-divisor on $X$, let $v$ be a divisorial valuation on $X$, and let $f\colon Y \to X$ be a birational morphism from a normal variety $Y$. If $D$ is $\mathbb{Q}$-Cartier, then the definition of valuation $v(D)$ and of pullback $f^*D$ given in Definition 2.9 agrees with the one given in (1). In general, if $D$ is an arbitrary $\mathbb{Q}$-divisor, then*

$$v(C+D) = v(C) + v(D) \quad \text{and} \quad f^*(C+D) = f^*C + f^*D$$

*for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $C$ on $X$.*

*Proof.* For clarity, we momentarily denote by $v'(D)$ the valuation of a $\mathbb{Q}$-divisor $D$ as defined in Definition 2.9. Let $C$ and $D$ be $\mathbb{Q}$-divisors, with $C$ $\mathbb{Q}$-Cartier. We have $v^\natural(mC) = v(mC)$ for every $m \in \mathbb{Z}$ such that $mC$ is a Cartier divisor, and therefore, observing that $k!C$ is a Cartier divisor for every $k \gg 0$, we get

$$v'(C+D) = \lim_{k\to\infty} \frac{v^\natural(k!C + k!D)}{k!} = \lim_{k\to\infty} \frac{v(k!C) + v^\natural(k!D)}{k!} = v(C) + v'(D)$$

by Proposition 2.4. In particular, if $D$ is $\mathbb{Q}$-Cartier, then we have $v'(D) = v(D)$. The analogous statements on pullbacks follows from these. □

*Remark* 2.11. We have $v(D) \le v^\natural(D)$ and $f^*D \le f^\natural D$ for every divisor $D$ on $X$. Moreover, we have $f^\natural(-D) \ge -f^\natural D$, and hence $f^*(-D) \ge -f^*D$, for every $D$. The following example implies that, in general, the last inequality may be strict.

The following example was found in a conversation with Lawrence Ein.

*Example* 2.12. Let $Y \dashrightarrow Y^+$ be a flip and $f\colon Y \to X$ be the flipping contraction, with $X$ normal and affine. Let $v$ be any divisorial valuation on $X$ whose center $C$ in $Y$ is a positive dimensional subset of a fiber of $f$. Let $H \subset Y$ be a general hyperplane section, and let $D = f_*H$. Note that $D$ contains $f(C)$, but $H$ does not contain $C$. If $H^+$ is the proper transform of $H$ on $Y^+$ and $f^+\colon Y^+ \to X$ is the contraction induced on $Y^+$, then

$$\bigoplus_{m\ge 0} \mathcal{O}_X(-mD) = \bigoplus_{m\ge 0} f^+_*\mathcal{O}_{Y^+}(-mH^+)$$

is finitely generated as an $\mathcal{O}_X$-module, since $-H^+$ is $f^+$-ample. Therefore $v(D) = v(qD)/q$ for a sufficiently divisible $q \ge 1$. As the ideal sheaf $\mathcal{O}_X(-qD)\cdot\mathcal{O}_Y$ vanishes along $C$, it follows that $v(D) > 0$. On the other hand, since $\mathcal{O}_X(mD)\cdot\mathcal{O}_Y = \mathcal{O}_Y(mH)$ if $m \ge 1$, we have $v(-D) = 0$, and hence, in particular, $v(-D) \ne -v(D)$.

*Remark* 2.13. If $f\colon Y \to X$ and $g\colon V \to Y$ are birational morphisms of normal varieties, then it follows from Lemma 2.7 that $(fg)^*D - g^*(f^*D)$ is effective and $g$-exceptional for every $\mathbb{Q}$-divisor $D$ on $X$.

## 3. Relative canonical divisors

We recall that a *canonical divisor* $K_X$ on a normal variety $X$ is, by definition, the (componentwise) closure of any canonical divisor of the regular locus of $X$. We also recall that $X$ is said to be $\mathbb{Q}$-*Gorenstein* if some (equivalently, every) canonical divisor $K_X$ is $\mathbb{Q}$-Cartier.

We consider a proper birational morphism $f\colon Y \to X$ of normal varieties. Push-forward along $f$ gives a bijection between the canonical divisors of $Y$ and those of $X$. Moreover, if $K_Y$ and $K'_Y$ are two canonical divisors on $Y$, then $f_*K_Y - f_*K'_Y$ is a principal divisor and $K_Y - K'_Y = f^*(f_*K_Y - f_*K'_Y)$.

Throughout the section, we fix a canonical divisor $K_Y$ on $Y$, and let $K_X = f_*K_Y$. The standard notion of relative canonical $\mathbb{Q}$-divisor (given in the case when $K_X$ is $\mathbb{Q}$-Cartier) admits two generalizations to non $\mathbb{Q}$-Gorenstein varieties, corresponding to whether one pulls back $K_X$ or $-K_X$. As we keep into consideration what the main features of the theory of singularities of pairs are, and wish to preserve these in our generalization, it turns out that there are two different sides of the theory, each of which requires a different approach, a phenomenon that disappears in the $\mathbb{Q}$-Gorenstein case (cf. Remarks 3.3 and 8.5).



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

When dealing with the generalization of multiplier and adjoint ideal sheaves, as well as of log canonical and log terminal singularities, we will rely on the following notion.

Definition 3.1. For every $m \geq 1$, the *mth limiting relative canonical* $\mathbb{Q}$*-divisor* $K_{m,Y/X}$ of $Y$ over $X$ is

$$K_{m,Y/X} := K_Y - \frac{1}{m} \cdot f^{\natural}(mK_X).$$

On the contrary, in order to extend the definitions of canonical and terminal singularities, we consider the following definition.

Definition 3.2. The *relative canonical* $\mathbb{R}$*-divisor* $K_{Y/X}$ of $Y$ over $X$ is

$$K_{Y/X} := K_Y + f^*(-K_X).$$

Note that the definitions of $K_{m,Y/X}$ and $K_{Y/X}$ do not depend on the choice of $K_Y$. Moreover, if $X$ is $\mathbb{Q}$-Gorenstein, then $K_{Y/X}$ is the usual relative canonical $\mathbb{Q}$-divisor, and it is equal to $K_{m,Y/X}$ for every $m \geq 1$ such that $mK_X$ is Cartier.

*Remark* 3.3. It follows by Lemma 2.8 and Remark 2.11 that $K_{m,Y/X} \leq K_{mq,Y/X} \leq K_{Y/X}$ for all $m, q \geq 1$. In particular, taking the limsup of the coefficients of the components of the $\mathbb{Q}$-divisors $K_{m,Y/X}$, one obtains the $\mathbb{R}$-divisor $K^-_{Y/X} := K_Y - f^*K_X$, which satisfies $K^-_{Y/X} \leq K_{Y/X}$. Clearly the two divisors coincide if $X$ is $\mathbb{Q}$-Gorenstein, but in general they may be different, as the following example shows.

*Example* 3.4. With the same notation as in Example 2.12, suppose that $-K_Y$ is $f$-ample. Then a positive multiple $-mK_Y$ of $-K_Y$ is linearly equivalent to a general hyperplane section $H$ of $Y$. We fix $K_X = f_*K_Y$, and let $D = f_*H$. Note that $B := D - mK_X$ is a principal divisor, and hence it is Cartier. Then for every birational morphism $g \colon X' \to X$ factoring through $Y$ and extracting a divisor with center in $Y$ equal to $C$ (cf. Example 2.12), we have

$$g^*(-mK_X) = g^*B + g^*(-D) \neq g^*B - g^*D = -g^*(mK_X)$$

by Proposition 2.10 and Example 2.12. This implies that $g^*(-K_X) \neq -g^*K_X$, since the pullback is by definition homogeneous (with respect to positive multiples) on all divisors. In particular, in this example we have $K_{X'/X} \neq K^-_{X'/X}$.

Lemma 3.5. *Let $m$ be a positive integer, and let $f \colon Y \to X$ be a proper birational morphism from a normal variety $Y$ such that $mK_Y$ is Cartier and $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y$ is invertible. Then for every proper birational morphism $g \colon V \to Y$ from a normal variety $V$ we have*

$$K_{m,V/X} = K_{m,V/Y} + g^*K_{m,Y/X}.$$

*Proof.* Note that $f^{\natural}(mK_X)$ is a Cartier divisor on $Y$, and thus

$$K_{m,V/X} = K_V - \frac{1}{m} \cdot (fg)^{\natural}(mK_X) = K_V - \frac{1}{m} \cdot g^{\natural}(f^{\natural}(mK_X)) = K_V - \frac{1}{m} \cdot g^*(f^{\natural}(mK_X))$$

by Lemma 2.7. Since $mK_Y$ is Cartier, we also have

$$K_{m,V/Y} = K_V - \frac{1}{m} \cdot g^{\natural}(mK_Y) = K_V - \frac{1}{m} \cdot g^*(mK_Y) = K_V - g^*K_Y,$$

and moreover

$$g^*K_{m,Y/X} = g^*(K_Y - \frac{1}{m} \cdot f^{\natural}(mK_X)) = g^*K_Y - \frac{1}{m} \cdot g^*(f^{\natural}(mK_X)).$$

The lemma follows. □

*Remark* 3.6. Similarly, Remark 2.13 implies that given proper birational morphisms of normal varieties $f\colon Y \to X$ and $g\colon V \to Y$ with $Y$ $\mathbb{Q}$-Gorenstein, we have $K_{V/X} \geq K_{V/Y} + g^* K_{Y/X}$ and the difference is $g$-exceptional.

A different approach to deal with varieties that are not $\mathbb{Q}$-Gorenstein, largely followed in recent decades, is to introduce a 'boundary'. The trick is to 'perturb' a canonical divisor $K_X$ of $X$ to make it $\mathbb{Q}$-Cartier.

DEFINITION 3.7. An effective $\mathbb{Q}$-divisor $\Delta$ is a *boundary* on $X$ if $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor for some (equivalently, for any) canonical divisor $K_X$ of $X$. If $\Delta$ is a boundary, then we refer to the pair $(X, \Delta)$ as a *log variety* (or *variety with boundary*).

DEFINITION 3.8. Let $\Delta$ be a boundary on $X$, and let $\Delta_Y$ be the proper transform of $\Delta$ on $Y$. The *log relative canonical* $\mathbb{Q}$*-divisor* of $(Y, \Delta_Y)$ over $(X, \Delta)$ is given by

$$K^{\Delta}_{Y/X} := K_Y + \Delta_Y - f^*(K_X + \Delta) = K_Y + \Delta_Y + f^*(-K_X - \Delta).$$

If $X$ is $\mathbb{Q}$-Gorenstein and $\Delta = 0$, then $K^0_{Y/X} = K_{Y/X} = K_{m,Y/X}$ for all sufficiently divisible $m \geq 1$, and $K^{\Delta}_{Y/X}$ depends on $\Delta$ but not on the choice of $K_X$.

*Remark* 3.9. For every boundary $\Delta$ on $X$ and every $m \geq 1$ such that $m(K_X + \Delta)$ is Cartier, we have

$$K_{m,Y/X} = K^{\Delta}_{Y/X} - \frac{1}{m} \cdot f^{\natural}(-m\Delta) - \Delta_Y \quad \text{and} \quad K_{Y/X} = K^{\Delta}_{Y/X} + f^*\Delta - \Delta_Y.$$

Indeed we have $f^*(-K_X) = f^*(-K_X - \Delta + \Delta) = -f^*(K_X + \Delta) + f^*\Delta$ by Proposition 2.10, and similarly $f^{\natural}(mK_X) = f^{\natural}(m(K_X + \Delta) - m\Delta) = m \cdot f^*(K_X + \Delta) + f^{\natural}(-m\Delta)$ by Lemma 2.4. In particular, if $m$ is any positive integer such that $m(K_X + \Delta)$ is a Cartier divisor, then $K^{\Delta}_{Y/X} \leq K_{m,Y/X}$.

## 4. Multiplier ideal sheaves

For the reminder of this paper, we work over an algebraically closed field of characteristic zero. We consider *pairs* of the form $(X, I)$, where $X$ is a normal quasi-projective variety and $I = \sum a_k \cdot \mathcal{I}_k$ is a formal $\mathbb{R}$-linear combination of non-zero fractional ideal sheaves on $X$. If each $\mathcal{I}_k$ is an ideal sheaf, $Z_k \subset X$ the subscheme defined by $\mathcal{I}_k$, and $Z = \sum a_k \cdot Z_k$ is the corresponding formal linear combination, then we identify the pairs $(X, I)$ and $(X, Z)$. More generally, we allow hybrid notation by considering pairs $(X, W + J)$, where $W$ is a formal linear combination of proper closed subschemes and $J$ is a formal linear combination of fractional ideal sheaves. If $\Delta$ is a boundary on $X$, then we consider *log pairs* of the form $((X, \Delta); I)$ (or more generally of the form $((X, \Delta); W + J))$.

Given a formal linear combination $Z = \sum a_k \cdot Z_k$ on $X$, if $f\colon Y \to X$ is a morphism such that the scheme theoretic inverse image $f^{-1}(Z_k)$ is a Cartier divisor for every $k$, then for short we denote $f^{-1}(Z) := \sum a_k \cdot f^{-1}(Z_k)$.

DEFINITION 4.1. Consider a pair $(X, I)$ as above. A *log resolution* of $(X, I)$ is a proper birational morphism $f\colon Y \to X$ from a smooth variety $Y$ such that for every $k$ the sheaf $\mathcal{I}_k \cdot \mathcal{O}_Y$ is the invertible sheaf of a divisor $E_k$ on $Y$, the exceptional locus $\mathrm{Ex}(f)$ of $f$ is also a divisor, and $\mathrm{Ex}(f) \cup E$ has simple normal crossing, where $E := \bigcup \mathrm{Supp}(E_k)$. If $\Delta$ is a boundary on $X$, then a *log resolution* of the log pair $((X, \Delta); I)$ is given by a log resolution $f\colon Y \to X$ of $(X, I)$ such that $\mathrm{Ex}(f) \cup E \cup \mathrm{Supp}(f^*(K_X + \Delta))$ has simple normal crossings.



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

Theorem 4.2. [Hir64a, Hir64b] *Let $(X, I)$ be a pair as above, where $X$ is a normal quasi-projective variety defined over an algebraically closed field of characteristic zero. Then there exists a log resolution of $(X, I)$. If $\Delta$ is a boundary on $X$, then there exists a log resolution of singularities of $((X, \Delta); I)$.*

*Proof.* Let $C$ be a Cartier divisor on $X$ such that $\mathcal{O}_X(C) \cdot \mathcal{I}_k \subseteq \mathcal{O}_X$ for every $k$, and let $W = \sum a_k \cdot W_k$, where $W_k \subset X$ is the subscheme defined by the ideal sheaf $\mathcal{O}_X(C) \cdot \mathcal{I}_k$. Then any log resolution of the pair $(X, W + \operatorname{Supp}(C))$ (respectively, of $((X, \Delta); W + \operatorname{Supp}(C)))$ is a log resolution of $(X, I)$ (respectively, of $((X, \Delta); I))$. This reduces the theorem to the original version due to Hironaka. □

Definition 4.3. We say that $Z$, or $(X, Z)$, is *effective* if $a_k \geq 0$ for all $k$. If $\Delta$ is a boundary on $X$, then we say that the log pair $((X, \Delta); Z)$ is *effective* if so is $Z$.

Consider now an arbitrary effective pair $(X, Z)$. Because of the possible failure of functoriality for composition of pullback of arbitrary $\mathbb{Q}$-divisors (cf. Lemma 2.7), the definition of multiplier ideal sheaf of $(X, Z)$ requires some preparation.

We fix a canonical divisor $K_X$ on $X$. For any fixed integer $m \geq 1$, we consider a log resolution $f: Y \to X$ of the pair $(X, Z + \mathcal{O}_X(-mK_X))$, and define

$$\mathcal{J}_m(X, Z) := f_* \mathcal{O}_Y(\lceil K_{m,Y/X} - f^{-1}(Z) \rceil).$$

When $Z = 0$, we denote this sheaf by $\mathcal{J}_m(X)$.

The proof of the next proposition is similar to the proof of the analogous properties for multiplier ideals in the $\mathbb{Q}$-Gorenstein case; we give it for completeness.

Proposition 4.4. *The sheaf $\mathcal{J}_m(X, Z)$ is a (coherent) sheaf of ideals on $X$, and its definition is independent of the choice of $f$.*

We start with two lemmas.

Lemma 4.5. *Let $f: Y \to X$ be a proper birational morphism from a smooth variety $Y$ to a normal variety $X$, let $P$ and $N$ be effective divisors on $Y$ without common components, and suppose that $P$ is $f$-exceptional. Then $f_* \mathcal{O}_Y(P - N) = f_* \mathcal{O}_Y(-N)$.*

*Proof.* The assertion follows from a lemma of Fujita, which gives the vanishing $f_* \mathcal{O}_P(P) = 0$ (see [KMM87, Lemma 1-3-2]), and hence $f_* \mathcal{O}_P(P - N) = 0$. □

Lemma 4.6. *Let $g: Y' \to Y$ be a proper birational morphism of smooth varieties, and let $D$ be an effective $\mathbb{R}$-divisor on $Y$ with simple normal crossing support. Then*

$$g_* \mathcal{O}_{Y'}(K_{Y'/Y} + \lceil g^* D \rceil) = \mathcal{O}_Y(\lceil D \rceil).$$

*Proof.* See [Laz04b, Lemma 9.2.19 and Remark 9.2.10]. □

*Proof of Proposition 4.4.* Let $f': Y' \to X$ be another log resolution of $(X, Z + \mathcal{O}_X(-mK_X))$. Since we can always compare $f$ and $f'$ with a common resolution, we may assume, without loss of generality, that $f'$ factors through $f$ and a morphism $g: Y' \to Y$. By Lemma 3.5, we have $K_{m,Y'/X} = K_{Y'/Y} + g^* K_{m,Y/X}$, and hence

$$(fg)_* \mathcal{O}_{Y'}(\lceil K_{m,Y'/X} - (fg)^{-1}(Z) \rceil) = f_*(g_* \mathcal{O}_{Y'}(K_{Y'/Y} + \lceil g^*(K_{m,Y/X} - f^{-1}(Z)) \rceil)).$$

Therefore, by Lemma 4.6, we obtain

$$(fg)_*\mathcal{O}_{Y'}(\lceil K_{m,Y'/X} - (fg)^{-1}(Z)\rceil) = f_*\mathcal{O}_Y(\lceil K_{m,Y/X} - f^{-1}(Z)\rceil).$$

This proves the independence of the definition from the choice of $f$. The fact that $\mathcal{J}_m(X, Z)$ is a sheaf of ideals follows from Lemma 4.5. □

Proposition 4.7. *The set of ideal sheaves $\{\mathcal{J}_m(X, Z)\}_{m\geq 1}$ has a unique maximal element.*

*Proof.* We have $\mathcal{J}_m(X, Z) \subseteq \mathcal{J}_{mq}(X, Z)$ for all $m, q \geq 1$. Indeed, by Proposition 4.4, this inclusion can be verified for any choice of $m$ and $q$ by taking a log resolution $f\colon Y \to X$ of $(X, Z + \mathcal{O}_X(-mK_X) + \mathcal{O}_X(-mqK_X))$ and applying the first formula in Remark 3.3. Therefore, by the Noetherian property of $X$, the set of ideal sheaves $\{\mathcal{J}_m(X, Z)\}_{m\geq 1}$ has a unique maximal element. □

Definition 4.8. Let $(X, Z)$ be an effective pair. The unique maximal element of $\{\mathcal{J}_m(X, Z)\}_{m\geq 1}$ is called the *multiplier ideal sheaf* of $(X, Z)$, and is denoted by $\mathcal{J}(X, Z)$. If $Z$ is trivial, then we denote the corresponding multiplier ideal sheaf by $\mathcal{J}(X)$.

Note that $\mathcal{J}(X, Z) = \mathcal{J}_m(X, Z)$ for all sufficiently divisible $m \geq 1$. If $X$ is $\mathbb{Q}$-Gorenstein, then this definition of multiplier ideal sheaf agrees with the usual one.

We close this section with some basic properties of multiplier ideals.

Proposition 4.9. *Let $Z = \sum b_k \cdot Z_k$ be an effective linear combination of proper closed subschemes of a normal variety $X$.*

(a) *If $Z' = \sum b'_k \cdot Z'_k$ with $b'_k \geq b_k$ and $\mathcal{I}_{Z'_k} \subseteq \mathcal{I}_{Z_k}$ for all $k$, then $\mathcal{J}(X, Z') \subseteq \mathcal{J}(X, Z)$.*

(b) *There is an $\epsilon > 0$ such that $\mathcal{J}(X, (1 + t)Z) = \mathcal{J}(X, Z)$ for all $0 < t \leq \epsilon$.*

*Proof.* We can fix $m$ such that $\mathcal{J}(X, Z) = \mathcal{J}_m(X, Z)$ and $\mathcal{J}(X, Z') = \mathcal{J}_m(X, Z')$. Property (a) is then immediate from the definition of these ideal sheaves. Regarding property (b), we observe that $\mathcal{J}_m(X, Z) = \mathcal{J}_m(X, (1 + t)Z)$ for all $0 < t \ll 1$. Thus property (b) follows by the chain of inclusions

$$\mathcal{J}_m(X, (1 + t)Z) \subseteq \mathcal{J}(X, (1 + t)Z) \subseteq \mathcal{J}(X, Z) = \mathcal{J}_m(X, Z),$$

the second of which holding by part (a). □

*Remark* 4.10. One can define the jumping numbers of an effective pair $(X, Z)$ in a similar fashion as in the $\mathbb{Q}$-Gorenstein case, by declaring that a number $\mu > 0$ is a *jumping number* of an effective pair $(X, Z)$ if $\mathcal{J}(X, \lambda Z) \neq \mathcal{J}(X, \mu Z)$ for all $0 \leq \lambda < \mu$. It would be interesting to study the properties of these numbers. For instance, is the set of jumping numbers of an effective pair a discrete set of rational numbers?

## 5. First properties and applications

As we will see below, it turns out that multiplier ideals (as defined in § 4) can actually be realized as multiplier ideals of suitable log pairs. Using this fact, we will see that the main features of the theory automatically extend to our setting.

Definition 5.1. Let $(X, Z)$ be an effective pair, and fix an integer $m \geq 2$. Given a log resolution $f\colon Y \to X$ of $(X, Z + \mathcal{O}_X(-mK_X))$, a boundary $\Delta$ on $X$ is said to be *m-compatible* for $(X, Z)$

with respect to $f$ if the following hold;

(i) $m\Delta$ is integral and $\lfloor \Delta \rfloor = 0$;

(ii) no component of $\Delta$ is contained in the support of $Z$;

(iii) $f$ is a log resolution for the log pair $((X,\Delta); Z + \mathcal{O}_X(-mK_X))$; and

(iv) $K^{\Delta}_{Y/X} = K_{m,Y/X}$.

The pair $(X,Z)$ is said to *admit $m$-compatible boundaries* if there are $m$-compatible boundaries with respect to any sufficiently high log resolution of $(X, \mathcal{O}_X(-mK_X) + Z)$.

This definition is motivated by the following useful property.

PROPOSITION 5.2. *If $(X,Z)$ is an effective pair and $m \geq 2$ is such that $\mathcal{J}(X,Z) = \mathcal{J}_m(X,Z)$, then we have*

$$\mathcal{J}(X,Z) = \mathcal{J}((X,\Delta); Z)$$

*for any $m$-compatible boundary $\Delta$, where $\mathcal{J}((X,\Delta); Z)$ is the multiplier ideal sheaf of the log pair $((X,\Delta); Z)$ as defined in [Laz04b, Definition 9.3.56].*

*Proof.* It suffices to observe that if $\Delta$ has no common components with $Z$ and $\lfloor \Delta \rfloor = 0$, then

$$\mathcal{J}((X,\Delta); Z) = f_*\mathcal{O}_Y(\lceil K_Y - g^*(K_X + \Delta) - f^{-1}(Z)\rceil) = f_*\mathcal{O}_Y(\lceil K^{\Delta}_{Y/X} - f^{-1}(Z)\rceil)$$

for any log resolution $f$ of $((X,\Delta); Z)$. $\square$

*Remark* 5.3. In general, if $(X,Z)$ is an effective pair and $\Delta$ is a boundary on $X$, then $\mathcal{J}((X,\Delta); Z) \subseteq \mathcal{J}(X,Z)$. Indeed, if $m \geq 1$ such that $m(K_X + \Delta)$ is a Cartier divisor and $\mathcal{J}(X,Z) = \mathcal{J}_m(X,Z)$, then the inclusion follows from the last formula in Remark 3.9.

THEOREM 5.4. *Every effective pair $(X,Z)$ admits $m$-compatible boundaries for any $m \geq 2$.*

*Proof.* Let $D$ be an effective divisor such that $K_X - D$ is Cartier, and let $f\colon Y \to X$ be a log resolution of $(X, \mathcal{O}_X(-mK_X) + \mathcal{O}_X(-mD))$, and let $E = f^{\natural}(mD)$, so that $\mathcal{O}_X(-mD) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$. Since

$$f^{\natural}(mK_X) = f^{\natural}(m(K_X - D) + mD) = m \cdot f^*(K_X - D) + f^{\natural}(mD),$$

we have

$$K_{m,Y/X} = K_Y - f^*(K_X - D) - \frac{1}{m}E.$$

Let $\mathcal{L}$ be an invertible sheaf on $X$ such that $\mathcal{L} \otimes \mathcal{O}_X(-mD)$ is globally generated, and let $G$ be a general element in the linear system $\{L \in |\mathcal{L}| \mid L - mD \geq 0\}$. Then $G = M + mD$ and $f^*G = M_Y + E$, where $M$ is an effective divisor and $M_Y$ is its proper transform. As $G$ varies, $M_Y$ moves in a base-point-free linear system. In particular, we can assume that $M$ is a reduced divisor with no common components with $D$ or $Z$. We let

$$\Delta := \frac{1}{m}M.$$

Note that $m\Delta$ is integral, $\lfloor \Delta \rfloor = 0$, and $K_X + \Delta = K_X - D + (1/m)G$ is $\mathbb{Q}$-Cartier. Moreover, by choosing $G$ general, we can also assume that $f$ is a log resolution for $((X,\Delta); \mathcal{O}_X(-mK_X))$.

The fact that $\Delta$ is $m$-compatible follows then by the computation

$$\begin{aligned} K^{\Delta}_{Y/X} &= K_Y + \Delta_Y - f^*(K_X + \Delta) \\ &= K_Y + \Delta_Y - f^*(K_X + \Delta - \frac{1}{m} G) - \frac{1}{m} f^* G \\ &= K_Y - f^*(K_X - D) - \frac{1}{m} E. \end{aligned}$$

□

We deduce the following corollary. *A posteriori*, one can use this corollary as the definition of $\mathcal{J}(X, Z)$.

Corollary 5.5. *For any effective pair $(X, Z)$, the set of ideal sheaves*

$$\{\mathcal{J}((X, \Delta); Z) \mid \Delta \text{ is a boundary on } X\}$$

*has a unique maximal element, namely $\mathcal{J}(X, Z)$.*

Using $m$-compatible boundaries, we obtain the following generic restriction result.

Proposition 5.6. *Let $(X, Z)$ be an effective pair. If $H \subset X$ is a general hyperplane section, then $\mathcal{J}(X, Z) \cdot \mathcal{O}_H = \mathcal{J}(H, Z|_H)$.*

*Proof.* If $\Delta$ is a boundary on $X$ with no common components with $H$, then $\Delta|_H$ is a boundary on $H$, and we have $\mathcal{J}((X, \Delta), Z) \cdot \mathcal{O}_H = \mathcal{J}((H, \Delta|_H), Z|_H)$ (cf. [Laz04b, Example 9.5.9]). Suppose that $\Delta$ is a $m$-compatible boundary on $X$ for some $m$ sufficiently divisible. By Remark 5.3 applied on $H$, we see immediately that $\mathcal{J}(X, Z) \cdot \mathcal{O}_H \subseteq \mathcal{J}(H, Z|_H)$. To get an equality, we need to show that the restriction $\Delta|_H$ of $\Delta$ to $H$ is also $m$-compatible, if $H$ is sufficiently general.

To this end, we fix a canonical divisor $K_0$ on $X$. Working locally on $X$, we may assume that $K_0$ is effective. Assume that $H$ is general with respect to $Z$, $\Delta$ and $K_0$. Then we replace $K_0$ by $K_X := K_0 - H + H_0$, where $H_0$ is another general hyperplane section linearly equivalent to $H$. Note that $K_H := (K_X + H)|_H$ is a canonical divisor on $X$.

We claim that

$$\mathcal{O}_X(-m(K_X + H)) \cdot \mathcal{O}_H = \mathcal{O}_H(-mK_H) \quad \text{for all } \ m \geq 0. \tag{2}$$

For short, let $B := m(K_X + H) = mK_0 + mH_0$. Note that $H$ has been chosen generally with respect to $B$. Let $g\colon X' \to X$ be a resolution of singularities of $X$. Let $B'$ be the proper transform of $B$. We can assume that the pullback of $H$ to $X'$ coincides with its proper transform $H'$, and moreover that $B'|_{H'}$ is the proper transform of $B|_H$. Note also that, since $B$ is effective, $g_*\mathcal{O}_{X'}(-B') = \mathcal{O}_X(-B)$, and similarly, $g_*\mathcal{O}_{H'}(-B'|_{H'}) = \mathcal{O}_H(-B|_H)$. On $X'$ we have the exact sequence

$$0 \to \mathcal{O}_{X'}(-B') \otimes g^*\mathcal{O}_X(-H) \to \mathcal{O}_{X'}(-B') \to \mathcal{O}_{H'}(-B'|_{H'}) \to 0.$$

Since $H$ is generic, the map $R^1 g_*\mathcal{O}_{X'}(-B') \otimes \mathcal{O}_X(-H) \to R^1 g_*\mathcal{O}_{X'}(-B')$ is injective. Therefore, taking direct images, we obtain a surjection $\mathcal{O}_X(-B) \to \mathcal{O}_H(-B|_H)$, which shows that (2) holds.

We take a log resolution $f\colon Y \to X$ of $(X, Z + \mathcal{O}_X(-mK_X) + H)$. Let $\widetilde{H}$ be the proper transform of $H$. Since $\Delta$ is a $m$-compatible boundary, we can assume that $K^{\Delta}_{Y/X} = K_{m,Y/X}$. On the other hand, using (2) we see that

$$K_{m,Y/X}|_{\widetilde{H}} = K_{m,\widetilde{H}/H}.$$

Since adjunction in the log $\mathbb{Q}$-Gorenstein case implies that

$$K^{\Delta}_{Y/X}|_{\widetilde{H}} = K^{\Delta|_H}_{\widetilde{H}/H},$$

we conclude that $\Delta|_H$ is a $m$-compatible boundary for $(H, Z|_H)$ (the other defining conditions of $m$-compatible being easily verified). This concludes the proof of the proposition. $\square$

The existence of $m$-compatible boundaries allows us to deduce immediately many other properties of multiplier ideal sheaves. We start with Skoda's theorem, which extends to our setting in a straightforward manner.

Corollary 5.7. *If $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on an $n$-dimensional normal variety $X$, then for every integer $m \geq n$*

$$\mathcal{J}(X, \mathfrak{a}^m) = \mathfrak{a}^{m+1-n} \cdot \mathcal{J}(X, \mathfrak{a}^{n-1}).$$

*Proof.* Fix an $m$-compatible boundary. The result then follows from [Laz04b, Variation 9.6.39]. $\square$

The main application, however, is the following extension of Nadel's vanishing theorem ([EV84, Nad89, Nad90]; see also [Laz04b, § 9.4]).

Corollary 5.8. *Let $(X, Z)$ be an effective pair, where $X$ is a projective normal variety and $Z = \sum a_k \cdot Z_k$. Let $m \geq 2$ be an integer such that $\mathcal{J}(X, Z) = \mathcal{J}_m(X, Z)$, and let $\Delta$ be an $m$-compatible boundary for $(X, Z)$. For each $k$, let $B_k$ be a Cartier divisor such that $\mathcal{O}_X(B_k) \otimes \mathcal{I}_{Z_k}$ is globally generated, and suppose that $L$ is a Cartier divisor such that $L - (K_X + \Delta + \sum a_k B_k)$ is numerically effective and big. Then*

$$H^i(\mathcal{O}_X(L) \otimes \mathcal{J}(X, Z)) = 0 \quad \text{for } i > 0.$$

*Proof.* The proof follows by Proposition 5.2 and [Laz04b, Theorem 9.4.17]. $\square$

As in the log $\mathbb{Q}$-Gorenstein case (cf. [Laz04b, § 9.4.E]), one obtains the following corollary.

Corollary 5.9. *With the same notation and assumptions as in Corollary 5.8, let $A$ be a very ample Cartier divisor on $X$. Then the sheaf $\mathcal{O}_X(L + kA) \otimes \mathcal{J}(X, Z)$ is globally generated for every integer $k \geq \dim X + 1$.*

The existence of $m$-compatible boundaries also implies the following relative vanishing.

Corollary 5.10. *Let $X$ be a normal quasi-projective variety. Then for any integer $m \geq 2$ and every sufficiently high log resolution $f : Y \to X$ of the pair $(X, \mathcal{O}_X(-mK_X))$ we have*

$$R^i f_* \mathcal{O}_Y(\lceil K_{m,Y/X} \rceil) = 0 \quad \text{for } i > 0.$$

We close this section with a generalization of Shokurov–Kollár's connectedness lemma [Kol92, Sho92].

Corollary 5.11. *With the same notation and assumptions as in Corollary 5.8, let*

$$K_{m,Y/X} - f^{-1}(Z) = \sum e_i E_i = A - B \quad \text{where } A = \sum_{e_i > -1} e_i E_i.$$

*Assume that $\lceil A \rceil$ is exceptional (i.e., that all divisorial components of $Z$ appear with coefficient less than 1). Then $\mathrm{Supp}(B)$ is connected in a neighborhood of any fiber of $f$. If moreover $B$ is irreducible and reduced, then $f(B)$ is normal.*



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

*Proof.* By Theorem 5.4 and Proposition 5.2, we reduce to the log $\mathbb{Q}$-Gorenstein case, where the result is well known (cf. [Kol97, Theorem 7.4] or [Kaw97, Theorem 1.6]). □

## 6. Asymptotic constructions and adjoint ideal sheaves

This section is devoted to a discussion of asymptotic multiplier ideal sheaves and adjoint ideal sheaves. As in the last two sections, we work over an algebraically closed field of characteristic zero.

Let $D$ be a divisor on a normal variety $X$, and for every $n \geq 1$ let $B_n \subset X$ denote the base scheme of the linear system $|nD|$. We suppose that $|n_0 D| \neq \emptyset$ for some $n_0 \geq 1$, and let $N = n_0 \cdot \mathbb{Z}_+$. As in the usual case (cf. [Laz04b, ch. 11]), for any given $c > 0$ the set of multiplier ideal sheaves $\{\mathcal{J}(X, c/n \cdot B_n)\}_{n \in N}$ has a unique maximal element (which does not depend on the choice of $n_0$).

Definition 6.1. The unique maximal element of $\{\mathcal{J}(X, c/n \cdot B_n)\}_{n \in N}$ is denoted by $\mathcal{J}(X, c \cdot \|D\|)$, and is called the *asymptotic multiplier ideal sheaf* of $D$ with weight $c$.

Note that if $m$ is sufficiently divisible and $\Delta$ is an $m$-compatible boundary for $(X, B_{n_0})$, then $\mathcal{J}(X, \|D\|) = \mathcal{J}((X, \Delta); \|D\|)$. We deduce the following property (cf. [Laz04b, Theorem 11.1.8 and Remark 11.1.13]).

Proposition 6.2. *With the above notation, suppose that $\mathcal{J}(X) = \mathcal{O}_X$ (i.e., $X$ is 'log terminal', see Definition 7.1 below). Then $\mathcal{J}(X, \|D\|)$ contains the ideal sheaf of the base scheme of $|D|$. In particular, we obtain*

$$H^0(\mathcal{O}_X(D) \cdot \mathcal{J}(X, \|D\|)) \cong H^0(\mathcal{O}_X(D)).$$

We next define the adjoint ideal sheaf of an effective pair $(X, Z)$ along an effective Cartier divisor $H$. We fix a log resolution $f \colon Y \to X$ of $(X, Z + \mathcal{O}_X(-mK_X))$ such that all components of the proper transform $H_Y$ of $H$ on $Y$ are disconnected; if $\Delta$ is a given boundary on $X$, then we also suppose that $f$ is a log resolution of the log pair $((X, \Delta); Z + \mathcal{O}_X(-mH))$. Then we consider the ideal sheaf

$$\mathrm{adj}_{m,H}(X, Z) := f_* \mathcal{O}_Y(\lceil K_{m,Y/X} - f^{-1}(Z) - f^* H + H_Y \rceil).$$

Again, one can check that $\mathrm{adj}_{m,H}(X, Z)$ is a (coherent) sheaf of ideals on $X$, that its definition is independent of the choice of $f$ and that the set of ideal sheaves $\{\mathrm{adj}_{m,H}(X, Z)\}_{m \geq 1}$ has a unique maximal element.

Definition 6.3. The maximal element of $\{\mathrm{adj}_{m,H}(X, Z)\}_{m \geq 1}$ is called the *adjoint ideal sheaf* of the pair $(X, Z)$ along $H$, and is denoted by $\mathrm{adj}_H(X, Z)$.

*Remark* 6.4. If $\Delta$ is an $m$-compatible boundary for some $m$ sufficiently divisible, then $\mathrm{adj}_H(X, Z) = \mathrm{adj}_H((X, \Delta); Z)$.

Proposition 6.5. *Suppose that $H$ is a normal Cartier divisor on $X$ with no components contained in the support of $Z$, and let $\Delta$ be an $m$-compatible boundary for $(X, Z + H)$ for a sufficiently divisible $m$. Then the adjoint ideal $\mathrm{adj}_H(X, Z)$ sits in the exact sequence*

$$0 \to \mathcal{J}(X, Z) \otimes \mathcal{O}_X(-H) \to \mathrm{adj}_H(X, Z) \to \mathcal{J}((H, \Delta|_H); Z|_H) \to 0.$$

*Proof.* If $\Delta$ is an $m$-compatible boundary for $(X, Z+H)$ for a sufficiently divisible $m$, then $\mathcal{J}(X,Z)=\mathcal{J}((X,\Delta);Z)$ and $\mathrm{adj}_H(X,Z)=\mathrm{adj}_H((X,\Delta);Z)$. Therefore the result follows from the log $\mathbb{Q}$-Gorenstein case, in which case it is well known (see, for example, the arguments in [Tak06, Proposition 2.4]). □

*Remark* 6.6. One can try to apply adjunction directly, without adding the boundary divisor, by fixing a canonical divisor $K_X$ on $X$ such that $K_X+H$ had order zero along the components of $H$. Then $K_H := (K_X+H)|_H$ is a canonical divisor on $H$ (cf. Remark 5.47 in [KM98]). However, $\mathcal{O}_X(-m(K_X+H))\cdot\mathcal{O}_H$ may in general be strictly contained in $\mathcal{O}_H(-mK_H)$ if $K_X+H$ is not $\mathbb{Q}$-Cartier. This reflects the fact that in general, no matter how one chooses $\Delta$ on $X$, $\mathcal{J}((H,\Delta|_H);Z|_H)$ may be strictly smaller than $\mathcal{J}(H,Z|_H)$, as it happens in the following example.

*Example* 6.7. As in [Kaw99a, Example 4.3], we consider an extremal flipping contraction $\phi\colon X'\to X$ on a normal $\mathbb{Q}$-factorial threefold $X'$ with terminal singularities. We assume that $X$ is affine, and let $0\in X$ be the image of the exceptional locus of $\phi$. Let $H\subset X$ be a general hyperplane section through 0, and let $H'=f^{-1}(H)\subset X'$. We assume that $H$ and $H'$ are normal $\mathbb{Q}$-factorial surfaces with log terminal singularities (this is the case, for instance, if $\phi$ is the contraction in Francia's flip [Fra80]). Note that $\phi$ restricts to a divisorial contraction $\psi\colon H'\to H$. Let $C$ be an irreducible component of $\mathrm{Ex}(\psi)$. Let $\Delta$ be any boundary on $X$ not containing $H$ in its support, and let $\Delta'$ be its proper transform on $X'$. Then $\Delta'\cdot C=-K_{X'}\cdot C>0$. It follows that $\mathrm{val}_C(\Delta|_H)$ is positive and independent of the choice of $\Delta$. This implies that there is a $\delta>0$, independent of $\Delta$, such that if $Z=\{0\}\subset H$, then $\mathrm{lc}((H,\Delta|_H);Z)\le \mathrm{lc}(H,Z)-\delta$. Therefore we can fix $c>0$, independent of $\Delta$, such that $\mathcal{J}((H,\Delta|_H);cZ)\ne\mathcal{O}_H$ but $\mathcal{J}(H,cZ)=\mathcal{O}_H$.

We immediately obtain the following inversion of adjunction statement.

Corollary 6.8. *Under the same assumptions as in Proposition 6.5, we have* $\mathrm{adj}_H(X,Z)=\mathcal{O}_X$ *in a neighborhood of* $H$ *if and only if* $\mathcal{J}((H,\Delta|_H);Z|_H)=\mathcal{O}_H$.

*Remark* 6.9. The corollary above should be compared with the following well known statement. If $S\subset X$ is a normal Cartier divisor (in fact, it suffices to assume that $S$ is Cartier in codimension two) on a normal variety and $B$ is an effective divisor such that $K_X+S+B$ is $\mathbb{Q}$-Cartier, then $(X,S+B)$ is purely log terminal with respect to $S$ if and only if $(S,B|_S)$ is Kawamata log terminal (see for example [KM98, Theorem 5.50]).

We also obtain the following vanishing theorem for adjoint ideals.

Corollary 6.10. *With the same assumptions as in Corollary 5.8, let* $H$ *be a general hyperplane section of* $X$. *Then*

$$H^i(\mathcal{O}_X(L+H)\otimes \mathrm{adj}_H(X,Z))=0 \quad \text{for } i>0.$$

*Proof.* We have

$$H^i(\mathcal{O}_X(L)\otimes\mathcal{J}(X,Z))=0 \quad \text{for } i>0$$

by Corollary 5.8. Observe that the restriction $\Delta|_H$ of $\Delta$ to $H$ is a boundary on $H$. Moreover, the sheaves $\mathcal{O}_H(B_k|_H)\otimes\mathcal{I}_{Z_k|_H}$ are globally generated, and $(L-(K_X+\Delta+\sum a_kB_k))|_H$ is numerically effective and big. By adjunction, this implies that

$$(L+H)|_H-(K_H+\Delta|_H+\sum a_kB_k|_H)$$



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

is numerically effective and big, and hence we have

$$H^i(\mathcal{O}_H(L+H)\otimes\mathcal{J}((H,\Delta|_H),Z|_H))=0 \quad \text{for } i>0.$$

The assertion then follows by Proposition 6.5. □

*Remark* 6.11. In fact it suffices to assume that $H$ is a normal Cartier divisor that is not contained in the augmented base locus of $L-(K_X+\Delta+\sum a_k B_k)$.

## 7. Log terminal and log canonical singularities

In this section we extend the definitions of log terminal and log canonical singularities of pairs to the general setting, and discuss some generalizations to this context of certain results on rational and log terminal singularities due, respectively, to Elkik and Kawamata.

Let $(X,Z)$ be an effective pair over an algebraically closed field of characteristic zero.

Definition 7.1. Let $X'\to X$ be a proper birational morphism with $X'$ normal, and let $F$ be a prime divisor on $X'$. For any integer $m\geq 1$, we define the *mth limiting log discrepancy* of $(X,Z)$ to be

$$a_{m,F}(X,Z):=\operatorname{ord}_F(K_{m,X'/X})+1-\operatorname{val}_F(Z).$$

The pair $(X,Z)$ is said to be *log canonical* (respectively, *log terminal*) if there is an integer $m_0$ such that $a_{m,F}(X,Z)\geq 0$ (respectively, $>0$) for every prime divisor $F$ over $X$ and $m=m_0$ (and hence for any positive multiple $m$ of $m_0$). Furthermore, $(X,Z)$ is said to be *strictly log canonical* if it is log canonical but not log terminal. If $X$ is log terminal, then the *log canonical threshold* of $(X,Z)$ is

$$\operatorname{lc}(X,Z):=\sup\{t>0 \mid (X,tZ) \text{ is log terminal}\}.$$

Clearly these notions coincide with the usual ones when $X$ is $\mathbb{Q}$-Gorenstein, and in general, if $(X,Z)$ is log terminal, then it is log canonical.

Proposition 7.2. *An effective pair $(X,Z)$ is log canonical (respectively, log terminal) if and only if there is a boundary $\Delta$ such that $((X,\Delta);Z)$ is log canonical (respectively, log terminal).*

*Proof.* If there is a boundary $\Delta$ such that $((X,\Delta);Z)$ is log canonical (respectively, log terminal), then it follows by Remark 3.9 that $(X,Z)$ is log canonical (respectively, log terminal). Conversely, assume that $(X,Z)$ is log canonical (respectively, log terminal), and let $m_0$ be as in Definition 7.1. By Theorem 5.4, there is an $m_0$-compatible boundary $\Delta$ for $(X,Z)$. Given any prime divisor $F$ over $X$, we can assume that $F$ is a divisor over a sufficiently high log resolution $Y$ of $(X,Z+\mathcal{O}_X(-m_0K_X))$. Then $K^{\Delta}_{Y/X}=K_{m_0,Y/X}$, and hence $a_{m_0}((X,\Delta);Z)=a_{m_0}(X,Z)$. It follows that $((X,\Delta);Z)$ is log canonical (respectively, log terminal). □

The next corollary shows the relation between our notion of log canonical singularities and Nakayama's notion of *admissible singularities* (see [Nak04, Defintion VII.1.2]); we are grateful to Hara, Schwede and Takagi for bringing Nakayama's notion to our attention.

Corollary 7.3. *An effective pair $(X,Z)$ is log terminal if and only if it has admissible singularities in the sense of Nakayama.*

*Proof.* The proof follows by comparing [Nak04, Lemma VII.1.3] with Proposition 7.2 and the fact that our notion is local. □

*Remark* 7.4. In general, taking an arbitrary boundary $\Delta$, if $((X,\Delta);Z)$ is log terminal (respectively, log canonical), then so is $(X,Z)$. In particular, if $(X,\Delta)$ is log terminal, then $\mathrm{lc}((X,\Delta);Z) \le \mathrm{lc}(X,Z)$.

Corollary 7.5. *Let $(X,Z)$ be a log canonical (respectively, log terminal) effective pair. If $H \subset X$ is a general hyperplane section, then $(H, Z|_H)$ is log canonical (respectively, log terminal).*

*Proof.* Since $((X,\Delta);Z)$ is log canonical (respectively, log terminal) for some boundary $\Delta$, so is $((H,\Delta|_H);Z|_H)$, and hence $(H,Z|_H)$. □

Corollary 7.6. *An effective pair $(X,Z)$ is log terminal if and only if $\mathcal{J}(X,Z)=\mathcal{O}_X$. Moreover, if $X$ is log terminal, then*

$$\mathrm{lc}(X,Z)=\sup\{t>0 \mid \mathcal{J}(X,tZ)=\mathcal{O}_X\}.$$

We next address the extension of Elkik's theorem on rational singularities [Elk81].

Corollary 7.7. *Let $X$ be a normal variety with log terminal singularities. Then $X$ has rational singularities.*

*Proof.* The proof follows from [KM98, Theorem 5.22] as there exists a boundary $\Delta$ such that $\mathcal{J}(X,\Delta)=\mathcal{O}_X$. □

Similarly, the analogous result on Du Bois singularities due to Kovács, Schwede and Smith [KSS08, Theorem 1.2] generalizes as follows.

Corollary 7.8. *Let $X$ be a normal Cohen–Macaulay variety with log canonical singularities. Then $X$ has Du Bois singularities.*

We also obtain the following generalization of [Sch07, Theorem 5.5], which was kindly brought to our attention by Karl Schwede.

Corollary 7.9. *Let $(X,Z)$ be an effective pair with log canonical singularities, and suppose that $X$ is log terminal. Then the multiplier ideal $\mathcal{J}(X,Z)$ defines a scheme with Du Bois singularities.*

In [Kaw98], Kawamata proves an important result on the singularities of minimal log canonical centers, which in particular implies that such centers have rational singularities. It follows immediately that, in the setting and terminology of [Kaw98, Theorem 1], 'minimal log canonical centers' are normal varieties with log terminal singularities. In particular this appears to be a natural setting for the theory developed in this paper, as in general 'minimal log canonical centers' are not known to be $\mathbb{Q}$-Gorenstein (even when the ambient variety is smooth).

In fact, Kawamata's subadjunction theorem extends to our general setting.

Definition 7.10. Let $(X,Z)$ be an effective strictly log canonical pair, and let $m_0$ be as in Definition 7.1. A subvariety $W \subset X$ is said to be a *log canonical center* of $(X,Z)$ if for every multiple $m$ of $m_0$ there is a exceptional prime divisor $E$ over $X$ such that $c_X(E)=W$ and $a_{m,E}(X,Z)=0$. A log canonical center is said to be *minimal* if it is so with respect to inclusions.

Proposition 7.11. *Let $W \subseteq X$ be a minimal log canonical center for an effective strictly log canonical pair $(X,Z)$. Then for any sufficiently divisible $m$ there is an effective $m$-compatible boundary $\Delta$ such that $W$ is a minimal log canonical center for $((X,\Delta);Z)$.*



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

*Proof.* Let $m_0$ as in Definition 7.1, and for every integer $k > 0$, let $\Delta_k$ be a $km_0$-compatible boundary for $(X, Z)$. Note that, for every $k$, the pair $((X, \Delta_k); Z)$ is log canonical and $W$ is a log canonical center for $((X, \Delta_k); Z)$. Moreover, for every $k \geq n \geq 1$ we have $a_F((X, \Delta_k); Z) \geq a_F((X, \Delta_n); Z)$ for any divisor $F$ over $X$. It follows that if $\mathcal{W}_k$ denotes the set of log canonical centers of $((X, \Delta_k); Z)$ then $\mathcal{W}_k \subseteq \mathcal{W}_n$ for every $k \geq n \geq 1$. Since a strictly log canonical log pair has only finitely many log canonical centers, the sequence of sets $\{\mathcal{W}_k\}$ stabilizes, and therefore $W$ is a minimal log canonical center of $((X, \Delta_k); Z)$, for $k \gg 1$. □

Corollary 7.12. *Let $(X, Z)$ be an effective strictly log canonical pair on a log terminal variety $X$. Then every minimal log canonical center of $(X, Z)$ is a normal variety with log terminal (and hence rational) singularities.*

*Proof.* Let $W$ be a minimal log canonical center. By Proposition 7.11, we can fix an $m$-compatible boundary $\Delta$ such that $W$ is a minimal log canonical center of $((X, \Delta), Z)$. It follows by [Kaw98] that there is a boundary $\Delta_W$ on $W$ such that $(W, \Delta_W)$ is log terminal, and this implies that $W$ is log terminal. □

We close this section with a discussion on surface singularities. As explained in [KM98, Notation 4.1], one can define the notions of *numerically log terminal* and *numerically log canonical* singularities for arbitrary normal surfaces, using the perfect pairing on the relative Néron–Severi space of a resolution. Here we show that a normal surface is log terminal (respectively, log canonical) if and only if it is numerically log terminal (respectively, numerically log canonical).

Proposition 7.13. *A normal surface $X$ is log terminal if and only if is numerically log terminal.*

*Proof.* By [KM98, Proposition 4.11], $X$ is numerically log terminal if and only if it is $\mathbb{Q}$-factorial and log terminal. On the other hand, if $X$ is log terminal, then by Proposition 7.2 there is a boundary $\Delta$ such that $(X, \Delta)$ is log terminal, and hence numerically log terminal. Again by [KM98, Proposition 4.11], this implies that $X$ is $\mathbb{Q}$-factorial. □

Proposition 7.14. *A normal surface $X$ is log canonical if and only if is numerically log canonical.*

*Proof.* If $X$ is numerically log canonical, then it is $\mathbb{Q}$-Gorenstein (cf. [KM98, Notation 4.1]), and hence log canonical. Conversely, suppose that $X$ is log canonical in the generality introduced in this section. We fix a canonical divisor $K_X$ on $X$ and a sufficiently divisible $m \geq 1$. Let $f\colon Y \to X$ be a log resolution of $(X, \mathcal{O}_X(-mK_X))$, and write $\mathcal{O}_X(-mK_X) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$ where $A$ is a divisor on $Y$. Let $K_Y$ be the canonical divisor on $Y$ such that $f_* K_Y = K_X$. Note that $-A$ is $f$-numerically effective. Let $E = \sum E_i$ be the reduced exceptional divisor of $f$. Since $X$ is log canonical, it follows that if $m$ is sufficiently divisible then the $\mathbb{Q}$-divisor

$$F := K_{m,Y/X} + E = K_Y + E - \frac{1}{m} A$$

is an effective exceptional $\mathbb{Q}$-divisor. Let $N = \sum a_i E_i$ be characterized by $K_Y \equiv_f N$. We have

$$N + E - F \equiv_f \frac{1}{m} A.$$

In particular $-(N + E - F)$ is $f$-numerically effective, and since it is exceptional, we conclude that $N + E - F \geq 0$ by the negativity lemma [KM98, Lemma 3.39]. This implies that $a_i \geq -1$ for all $i$, an hence that $X$ is numerically log canonical. □

Corollary 7.15. *Let $X$ be a normal surface with log terminal (respectively, log canonical) singularities. Then $X$ is $\mathbb{Q}$-factorial (respectively, $\mathbb{Q}$-Gorenstein).*

## 8. Terminal and canonical singularities

In this section we deal with the generalization of canonical and terminal singularities, and discuss the corresponding extensions of invariance properties of singularities, plurigenera and numerical Kodaira dimensions established in the $\mathbb{Q}$-Gorenstein case in works of Siu, Kawamata and Nakayama. Throughout the section, the ground field is assumed to be an algebraically closed field of characteristic zero.

Consider a pair $(X, Z)$, where $X$ is a normal variety and $Z$ is an effective formal linear combination of proper closed subschemes of $X$.

Definition 8.1. Let $X' \to X$ be a proper birational morphism with $X'$ normal, and let $F$ be a prime divisor on $X'$. The *log-discrepancy* of a prime divisor $F$ over $X$ with respect to $(X, Z)$ is

$$a_F(X, Z) := \operatorname{ord}_F(K_{X'/X}) + 1 - \operatorname{val}_F(Z).$$

The pair $(X, Z)$ is said to be *canonical* (respectively, *terminal*) if $a_F(X, Z) \geq 1$ (respectively, $> 1$) for every exceptional prime divisor $F$ over $X$.

Of course these notions coincide with the familiar ones in the $\mathbb{Q}$-Gorenstein case. Canonical singularities admit the following characterization (which is well known in the $\mathbb{Q}$-Gorenstein case).

Proposition 8.2. *Let $X$ be a normal variety, and suppose that $Z = \sum a_k \cdot Z_k$ is an effective formal $\mathbb{Q}$-linear combination of effective Cartier divisors $Z_k$ on $X$. Then the pair $(X, Z)$ is canonical if and only if for all sufficiently divisible $m \geq 1$ (in particular, we ask that $ma_k \in \mathbb{Z}$ for every $k$), and for every log resolution $f\colon Y \to X$ of $(X, Z + \mathcal{O}_X(mK_X))$, there is an inclusion*

$$\mathcal{O}_X(m(K_X + Z)) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(m(K_Y + Z_Y))$$

*as sub-$\mathcal{O}_Y$-modules of $\mathcal{K}_Y$, where $Z_Y$ is the proper transform of $Z$ (as usual, the canonical divisors $K_X$ and $K_Y$ are chosen so that $f_*K_Y = K_X$).*

*Proof.* Note that $f^{-1}(Z) = f^*Z = -f^*(-Z)$, once we think of $Z$ and $f^{-1}(Z)$ as $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. If $(X, Z)$ is canonical, and $m$ and $f$ are chosen as in the statement, then we see that

$$\begin{aligned} m(K_Y + Z_Y) + f^\natural(-m(K_X + Z)) &\geq m(K_Y + Z_Y) + f^*(-m(K_X + Z)) \\ &= m(K_Y + Z_Y + f^*(-K_X - Z)) = m(K_{Y/X} + Z_Y - f^{-1}(Z)) \geq 0 \end{aligned}$$

by Remark 2.11 and Proposition 2.10, and hence we get an inclusion as asserted. Conversely, suppose that $(X, Z)$ is not canonical, and fix any log resolution $f\colon Y \to X$ of $(X, Z)$. Then the $\mathbb{R}$-divisor $K_Y + Z_Y + f^*(-K_X - Z)$ is not effective. Since $f^*(-K_X - Z)$ is the componentwise limit of the $\mathbb{Q}$-divisors $1/m \cdot f^\natural(-m(K_X + Z))$, we can find a sufficiently large (and divisible) $m$ such that $K_Y + Z_Y + 1/m \cdot f^\natural(-m(K_X + Z))$ is not effective. By further blowing up, we may assume that $f$ is a log resolution of $(X, Z + \mathcal{O}_X(mK_X))$. Then the assertion follows. □

As an application, we show that deformation invariance of canonical singularities, plurigenera, and numerical Kodaira dimension also holds in this more general context.

We start with the extension of Kawamata's theorem on the deformation invariance of canonical singularities [Kaw99a].



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

Theorem 8.3. *Let $f\colon X \to C$ be a flat morphism from a variety to a smooth curve such that, for some point $0 \in C$, the fiber $X_0 = f^{-1}(0)$ is a normal variety with only canonical singularities. Then $(X, X_0)$ is canonical in a neighborhood of $X_0$, and so are all fibers of $f$ over a neighborhood of* 0.

*Proof.* The proof follows the arguments of [Kaw99b]. By shrinking $X$ near $X_0$, we can assume that $X$ is normal (cf. [Gro65, Corollary 5.12.7]). We may also assume that $X_0$ is affine. Let $m > 0$ be a sufficiently divisible integer and let $\mu\colon Y \to X$ be a log resolution of $(X, X_0 + \mathcal{O}_X(mK_X))$ which restricts to a log resolution of $(X_0, \mathcal{O}_{X_0}(mK_{X_0}))$. Let $\{s_1, \dots, s_k\}$ be a generating set of sections of $\mathcal{O}_{X_0}(mK_{X_0})$. Let $Y_0$ be the strict transform of $X_0$. By Proposition 8.2, there is an inclusion

$$\mathcal{O}_{X_0}(mK_{X_0}) \cdot \mathcal{O}_{Y_0} \subseteq \mathcal{O}_{Y_0}(mK_{Y_0}).$$

One sees that there are corresponding sections $\tilde{s}_i$ of $\mathcal{O}_{Y_0}(mK_{Y_0})$ which push forward to the sections $s_i$ of $\mathcal{O}_{X_0}(mK_{X_0})$. By [Kaw99b, Theorem A], after possibly restricting over a neighborhood of $0 \in C$, these sections extend to sections $\tilde{S}_i$ of $\mathcal{O}_Y(m(K_Y + Y_0))$. Pushing forward, we obtain sections $S_i$ of $\mathcal{O}_X(m(K_X + X_0))$ that restrict to $s_i$. It follows by Nakayama's lemma that the $S_i$ are generators of $\mathcal{O}_X(m(K_X + X_0))$ at each point of $X_0$. Thus the inclusion $\mu_*\mathcal{O}_Y(m(K_Y + Y_0)) \subseteq \mathcal{O}_X(m(K_X + X_0))$ is an equality in a neighborhood of $X_0$. Therefore, after restricting to such neighborhood, there is an inclusion

$$\mathcal{O}_X(m(K_X + X_0)) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(m(K_Y + Y_0)),$$

and hence $(X, X_0)$ is canonical. □

Similarly, we have the following extension of the invariance of plurigenera (in the general type case) and of numerical Kodaira dimension for varieties with canonical singularities [Siu98, Kaw99a, Nak04].

Theorem 8.4. *Let $f\colon X \to S$ be a projective flat morphism of varieties whose fibers $X_t = f^{-1}(t)$ are normal varieties with canonical singularities for every $t \in S$. Then the following properties hold.*

(a) *The numerical Kodaira dimension $\nu(X_t)$ is constant on $t \in S$. In particular, if one fiber $X_0$ is of general type, then so are the other fibers.*
(b) *Suppose additionally that the generic fiber $X_\eta$ is a variety of general type. Then the plurigenera $P_m(X_t)$ is constant on $t \in S$ for any positive integer $m$.*

*Proof.* The proof is similar to those of [Kaw99b, Theorems 1.3 and 1.2'], after we remark that Kodaira's lemma (cf. [Laz04a, Proposition 2.2.6]) holds for (not necessarily Cartier) divisors on a normal projective variety. □

*Remark* 8.5. Canonical singularities on a $\mathbb{Q}$-Gorenstein normal variety are obviously purely log terminal. However, it remains unclear whether an analogous implication still holds if the singularities are not $\mathbb{Q}$-Gorenstein (cf. Remark 3.3). In fact, in this generality we do not even know if canonical singularities are rational (in particular, Cohen–Macaulay) or log canonical.

Acknowledgements

The authors would like to thank Lawrence Ein and Karen Smith for useful conversations, and Manuel Blickle, Nobuo Hara, Karl Schwede and Shunsuke Takagi for useful comments. The authors are very grateful to the referee for many valuable suggestions, comments and corrections.



https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press

## References

AIM06 *Problems raised during the Workshop on Numerical invariants of singularities and higher-dimensional algebraic varieties* (American Institute of Mathematics, Palo Alto, 2006), www.aimath.org/WWN/singularvariety/problemsV1.pdf.

Bli04 M. Blickle, *Multiplier ideals and modules on toric varieties*, Math. Z. **248** (2004), 113–121.

EM06 L. Ein and M. Mustaţă, *Invariants of singularities of pairs*, International Congress of Mathematicians, vol. II (European Mathematical Society, Zürich, 2006), 583–602.

Elk81 R. Elkik, *Rationalité des singularités canoniques*, Invent. Math. **64** (1981), 1–6.

EV84 H. Esnault and E. Viehweg, *Dyson's lemma for polynomials in several variables (and the theorem of Roth)*, Invent. Math. **78** (1984), 445–490.

Fra80 P. Francia, *Some remarks on minimal models. I*, Composito Math. **40** (1980), 301–313.

Gro65 A. Grothendieck, *Éleménts de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Publ. Math. Inst. Hautes Études Sci. **24** (1965), 231.

Har01 N. Hara, *Geometric interpretation of tight closure and test ideals*, Trans. Amer. Math. Soc. **353** (2001), 1885–1906.

HT04 N. Hara and S. Takagi, *On a generalization of test ideals*, Nagoya Math. J. **175** (2004), 59–74.

HY03 N. Hara and K.-I. Yoshida, *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math. Soc. **355** (2003), 3143–3174.

Hir64a H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203.

Hir64b H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 205–326.

Kaw97 Y. Kawamata, *On Fujita's freeness conjecture for* 3*-folds and* 4*-folds*, Math. Ann. **308** (1997), 491–505.

Kaw98 Y. Kawamata, *Subadjunction of log canonical divisors. II*, Amer. J. Math. **120** (1998), 893–899.

Kaw99a Y. Kawamata, *Deformations of canonical singularities*, J. Amer. Math. Soc. **12** (1999), 85–92.

Kaw99b Y. Kawamata, *On the extension problem of pluricanonical forms*, in *Algebraic geometry: Hirzebruch 70. Proceedings of the algebraic geometry conference in honor of F. Hirzebruch's 70th birthday, Stefan Banach International Mathematical Center*, Warszawa, Poland, May 11–16, 1998, Contemporary Mathematics, vol. 241, eds P. Pragacz *et al.* (American Mathematical Society, Providence, RI, 1999), 193–207.

KMM87 Y. Kawamata, K. Matsuda and K. Matsuki, *Introduction to the minimal model problem*, in *Algebraic Geometry, Sendai, 1985*, Advanced Studies in Pure Mathematics, vol. 10 (North-Holland, Amsterdam, 1987), 283–360.

Kol92 J. Kollár, *Flips and abundance for algebraic threefolds. A summer seminar at the University of Utah (Salk Lake City, 1991)*, Astérisque **211** (1992).

Kol97 J. Kollár, *Singularities of pairs*, in *Algebraic geometry—Santa Cruz 1995*, Proceedings of Symposia in Pure Mathematics, vol. 62, part 1 (American Mathematical Society, Providence, RI, 1997), 221–287.

KM98 J. Kollár and S. Mori, *Birational Geometry of Algebraic Varieties*, Cambridge Tracts in Mathematics, vol. 134 (Cambridge University Press, Cambridge, 1998).

KSS08 S. Kovács, K. Schwede and K.E. Smith, *Cohen-Macaulay semi-log canonical singularities are Du Bois,* Preprint, arXiv:0801.1541.

Laz04a R. Lazarsfeld, *Positivity in Algebraic Geometry. I*, in *Classical Setting: Line Bundles and Linear Series. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, A Series of Modern Surveys in Mathematics, vol. 48 (Springer, Berlin, 2004).

Laz04b R. Lazarsfeld, *Positivity in Algebraic Geometry. II. Positivity for Vector Bundles, and Multiplier Ideals. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, A Series of Modern Surveys in Mathematics, vol. 49 (Springer, Berlin, 2004).

Nad89 A. Nadel, *Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature*, Proc. Natl. Acad. Sci. U.S.A. **86** (1989), 7299–7300.

Nad90 A. Nadel, *Multiplier ideal sheaves and Kähler-Einstein metrics of positive scalar curvature*, Ann. of Math. (2) **132** (1990), 549–596.

Nak04 N. Nakayama, *Zariski-Decomposition and Abundance*, MSJ Memoirs, vol. 14 (Mathematical Society of Japan, Tokyo, 2004).

Sho92 V. V. Shokurov, *Three-dimensional log perestroikas. In Russian, with an appendix in English by Yujiro Kawamata*, Izv. Ross. Akad. Nauk Ser. Mat. **56** (1992), 105–203 (Engl. transl. *Izv. Math.* **40** (1993), 95–202).

Sch07 K. Schwede, *A simple characterization of Du Bois singularities*, Composito. Math. **143** (2007), 813–828.

Siu98 Y.-T. Siu, *Invariance of plurigenera*, Invent. Math. **134** (1998), 661–673.

Smi00 K. E. Smith, *The multiplier ideal is a universal test ideal*, Comm. Algebra **28** (2000), 5915–5929.

Tak04 S. Takagi, *An interpretation of multiplier ideals via tight closure*, J. Algebraic Geom. **13** (2004), 393–415.

Tak06 S. Takayama, *Pluricanonical systems on algebraic varieties of general type*, Invent. Math. **165** (2006), 551–587.



Tommaso de Fernex defernex@math.utah.edu
Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 48112-0090, USA

Christopher D. Hacon hacon@math.utah.edu
Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 48112-0090, USA




https://doi.org/10.1112/S0010437X09003996 Published online by Cambridge University Press