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**Singularities on the Light Cylinder in Relativistic Ideal Magnetohydrodynamics**

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When a rigid magnetized conductor surrounded by a plasma rotates about its axis, the plasma may be considered to co-rotate as it is frozen in the magnetic field lines supported by the conductor. However, the rotational velocity increases with the distance from the axis and it becomes equal to the light velocity on the so-called light cylinder with the radius $r^* \equiv c/\omega$, where $\omega$ is the angular velocity of the rotation and $c$ is the light velocity. Consequently, from the principle of relativity it is obvious that in the neighborhood of the light cylinder the simple co-rotation ceases to be valid. (For example, in the pulsar-electrodynamics a remarkably different field-flow has been assumed for a region involving the light cylinder.[1])

However, it has not been shown how the co-rotation breaks down as a self-consistent solution of the relativistic plasma dynamics. In this paper this is examined using the ideal relativistic magnetohydrodynamics.

The basic equations are the same as those used by Haris,[2] which for the steady isoentropic flows reduce to

$$\boldsymbol{\nabla}\cdot(\mu^0\boldsymbol{U})=0\,, \tag{1}$$

$$\mu^0\tilde{p}(\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{U}+\mu^0(\boldsymbol{U}\cdot\boldsymbol{\nabla}\tilde{p})\boldsymbol{U}+\boldsymbol{\nabla}P$$
$$-\frac{1}{4\pi}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}-\frac{1}{4\pi}\boldsymbol{E}(\boldsymbol{\nabla}\cdot\boldsymbol{E})=0\,, \tag{2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}=0 \quad \boldsymbol{E}=-\frac{1}{c}\boldsymbol{u}\times\boldsymbol{B}\,, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0\,, \tag{4}$$

$$P=A(S)\mu^{0\gamma}, \tag{5}$$

where $\tilde{p}=1+\varepsilon^0/c^2+P/(\mu^0c^2)=1+\nu P/(\mu^0c^2)$, $(\nu=\gamma/\gamma-1)$, $\boldsymbol{U}=\boldsymbol{u}(1-u^2/c^2)^{-1/2}$.

In the cylindrical coordinate $(r, \phi, z)$, the co-rotation is given by $\boldsymbol{B}=(B_r, 0, B_z)$, $\boldsymbol{u}=(0, r\omega, 0)$, $\boldsymbol{\nabla}=(\partial/\partial r, 0, \partial/\partial z)$. Then Eq. (1) and $\phi$-component of Eq. (2) become trivial and $r$- and $z$-components of Eq. (2) yield

$$-\mu^0\tilde{p}U_\phi{}^2/r+\frac{\partial P}{\partial r}=B_zK/4\pi\,, \tag{6·a}$$

$$\frac{\partial P}{\partial z}=-B_rK/4\pi\,, \tag{6·b}$$

where

$$K=[1-(\omega r/c)^2]\left(\frac{\partial B_r}{\partial z}-\frac{\partial B_z}{\partial r}\right)+2\omega^2rB_z/c^2,$$

$$U_\phi=r\omega[1-(\omega r/c)^2]^{-1/2}.$$

Equations (3) and (4) become equivalent and hence Eqs. (4), (5) and (6) may be considered as the basic equations for $P$, $B_r$ and $B_z$. Equations (6) can be combined to give

$$-\mu^0\tilde{p}U_\phi{}^2B_r/r+(\boldsymbol{B}\cdot\boldsymbol{\nabla})P=0\,, \tag{7}$$

and substituting Eq. (5) in Eq. (7), we have

$$B_r\left[\frac{\partial\lambda}{\partial r}-\frac{1}{r}\left(\frac{U_\phi}{c}\right)^2\lambda-\frac{1}{\nu A}\frac{c^2}{r}\left(\frac{U_\phi}{c}\right)^2\right]$$
$$+B_z\frac{\partial\lambda}{\partial z}=0\,, \tag{7'}$$

where $\lambda=\mu^{0\,\gamma-1}$. This equation is linear with respect to $\lambda$.

If $B_r\neq 0$, we may put

$$\lambda=\chi[1-(\omega r/c)^2]^{-1/2}-c^2/(\nu A) \tag{8}$$

and obtain

$$(\boldsymbol{B}\cdot\boldsymbol{\nabla})\chi=0\,. \tag{9}$$

Equation (9) shows that $\chi$ is constant along each magnetic field line. Therefore the pressure becomes

$$P=A\left(\frac{\chi}{[1-(\omega r/c)^2]^{1/2}}-\frac{c^2}{\nu A}\right)^\nu$$

along each magnetic field line. This result shows that the pressure $P$ diverges as $[1-(\omega r/c)^2]^{-\nu/2}$ and the proper mass density diverges as $[1-(\omega r/c)^2]^{-\nu/2\gamma}$ on the light cylinder.

If $B_r\to 0$ as $r\to r^*$, we can also show that $P$ and $\mu^0$ asymptoticaly diverge in the same order of $[1-(\omega r/c)^2]$ as that in the case $B_r\neq 0$.

It should be emphasized that the origin of the singularity on the light cylinder reported here is due to the enormous amount of the centrifugal force caused by the relativistic increase of the mass density.

Thus one can conclude that in the vicinity of the light cylinder the co-rotation is not possible. Even if the magnetic field supported by the conductor is much larger than the mechanical pressure, that is, even for the extremely low $\beta$ plasma, in the neighborhood of the light cylinder the mechanical effect dominates the field effect and the magnetic field is strongly disturbed.

Consequently a remarkably different field-flow must be considered. As the most plausible example of such field-flows, we mention the relativistic parallel-transverse flow, which is a direct extension of the non-relativistic one.[3),4)] That is, it comprises the poloidal and toroidal components of the flow and the field. Moreover in the poloidal projection the flow velocity and the magnetic field vector are aligned. The poloidal flow and the toroidal field, which do not exist in the co-rotation, make it possible to realize a self-consistent transition across the light cylinder. This configuration of the relativistic parallel-transverse flow admits some correspondences to the field-flow considered by Goldreich and Julian,[1] but there are essentially different points between them, in particular, in the former, the inertia effect is crucial, while it is neglected in the latter.

Detailed discussion of the relativistic parallel-transverse flow will be done elsewhere by one of the present authors (T.I.).

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