SINGULARITY OF GAUSSIAN MEASURES ON FUNCTION SPACES INDUCED BY BROWNIAN MOTION PROCESSES WITH NON-STATIONARY INCREMENTS

BY

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0. Introduction

A real-valued stochastic process $X(t, \omega)$ on a probability space $(\Omega, \mathfrak{B}, P)$ and an interval D of the real line induces a probability measure μ_X on the measurable space $(R^D, \mathfrak{F})$ where R^D is the space of all real valued functions $x(t)$, $t \in D$, and $\mathfrak{F}$ is the smallest σ-field of subsets of R^D with respect to which all real valued functions $Y(t, x) = x(t)$ defined on R^D with the parameter $t \in D$ are measurable. According to the Feldman-Hájek dichotomy two Gaussian measures on $(R^D, \mathfrak{F})$, i.e. measures induced by Gaussian processes, are always either equivalent or singular. A Brownian motion process $X(t, \omega)$ on $(\Omega, \mathfrak{B}, P)$ and $D = [0, \infty)$ with non-stationary increments, which we shall call for brevity a generalized Brownian motion process in the rest of the paper, is a real valued stochastic process with independent increments in which the probability distribution $\Phi_{t' t''}$ of the increment $X(t'', \omega) - X(t', \omega)$, $t', t'' \in D$, $t' < t''$, is a normal distribution $N(0, b(t'') - b(t'))$ with the density function

$$\Phi'(\eta) = \{2\pi[b(t'') - b(t')]\}^{-1/2} \exp\{-\eta^2/2[b(t'') - b(t')]\}, \qquad \eta \in R,$$

where $b(t)$ is a strictly increasing function on D with $b(0) = 0$ and $X(0, \omega) = 0$, a.e. We emphasize that no continuity or smoothness condition on $b(t)$ are assumed unless otherwise stated. The results of this paper are the following two theorems.

THEOREM 1. *Let $X_i(t, \omega)$, $i = 1, 2$, be generalized Brownian motion processes on a probability space $(\Omega, \mathfrak{B}, P)$ and $D = [0, \infty)$ with strictly increasing $b_i(t)$. If at some $t_0 \in D$, the derivatives $\lambda_i = b_i'(t_0)$ exist, $\lambda_i > 0$, and $\lambda_1 \neq \lambda_2$, then the probability measures μ_{X_i} induced on the measurable space $(R^D, \mathfrak{F})$ by $X_i(t, \omega)$ are singular.*

For cases with stationary increments, i.e. when $b_i(t) = \lambda_i t$, $\lambda_i > 0$, $\lambda_1 \neq \lambda_2$, the singularity of the two measures μ_{X_i} is well known and furthermore two disjoint subsets of R^D, $E_i \in \mathfrak{F}$, satisfying the condition $\mu_{X_i}(E_j) = \delta_{ij}$ can be found. Indeed an immediate consequence of R. H. Cameron and W. T. Martin's investigation (Theorem 1, [2]) is that when $b_i(t) = \lambda_i t$, $\lambda_i > 0$, $\lambda_1 \neq \lambda_2$, every pair of disjoint subsets of R^D, $E_{i,T} \in \mathfrak{F}$, $T > 0$, defined by

$$E_{i,T} = \{x \in R^D; \lim_{n\to\infty} \sigma_n(T, x) = \lambda_i T\}$$

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where
$$\sigma_n(T, x) = \sum_{k=1}^{n} \{x(kT/2^n) - x((k-1)T/2^n)\}^2$$

satisfies the condition $\mu_{X_i}(E_{j,T}) = \delta_{ij}$. G. Baxter [1] extended Cameron and Martin's result to cover a wide class of Gaussian processes whose mean and covariance functions satisfy certain smoothness conditions. Applying Baxter's results to generalized Brownian motion processes we obtain

THEOREM 2. *If* $b_i'(t)$, $i = 1, 2$, *exist and are continuous on* $[0, T]$ *and* $b_1(T) \neq b_2(T)$ *for some* $T > 0$ *then for the pair of disjoint subsets of* R^D, $E_{i,T} \in \mathfrak{F}$, *defined by*
$$E_{i,T} = \{x \in R^D; \lim_{n\to\infty} \sigma_n(T, x) = b_i(T)\}$$
we have $\mu_{X_i}(E_{j,T}) = \delta_{ij}$.

These two theorems are proved in §4. In §1 we discuss the probability space $(R^D, \mathfrak{F}, \mu_X)$. J. Hájek's results on the J-divergence on which the proof of Theorem 1 is based are stated in §2 in a way suitable for our purposes. §3 consists of lemmas concerning generalized Brownian motion processes.

1. Measures on function spaces induced by stochastic processes

Given a real-valued stochastic process $X(t, \omega)$ on a probability space $(\Omega, \mathfrak{B}, P)$ and an interval D of the real line. Let S be the transformation of Ω into the space R^D of all real valued functions $x(t)$, $t \in D$, defined by $S(\omega) = X(\cdot, \omega) \in R^D$, $\omega \in \Omega$. Let $\mathfrak{G} = \{G \subset R^D; S^{-1}(G) \in \mathfrak{B}\}$ and $\nu(G) = P(S^{-1}(G))$, $G \in \mathfrak{G}$. Then $(R^D, \mathfrak{G}, \nu)$ is a probability space.

For $t_1, \cdots, t_n \in D$, $t_1 < \cdots < t_n$, consider the projection of R^D onto the n-dimensional Euclidean space R^n defined by
$$p_{t_1\cdots t_n}(x) = [x(t_1), \cdots, x(t_n)], \qquad x \in R^D$$
and the σ-field of subsets of R^D
$$\mathfrak{F}_{t_1\cdots t_n} = \{p_{t_1\cdots t_n}^{-1}(B), B \in \mathfrak{B}^n\}$$
where $\mathfrak{B}^n$ is the σ-field of Borel sets in R^n. The σ-field $\mathfrak{F}$ generated by all the σ-fields $\mathfrak{F}_{t_1\cdots t_n}$ is contained in $\mathfrak{G}$ and is independent of the stochastic process $X(t, \omega)$. We define $\mu_X = \nu \mid \mathfrak{F}$, i.e. the restriction of ν to $\mathfrak{F}$. $\mathfrak{F}$ is the smallest σ-field of subsets of R^D with respect to which the functions $Y(t, x) = x(t)$ on R^D with the parameter $t \in D$ are measurable. The stochastic process $Y(t, x)$ on $(R^D, \mathfrak{F}, \mu_X)$ and D is a realization of $X(t, \omega)$ in the sense that for any $t_1, \cdots, t_n \in D$, the two random vectors
$$[Y(t_1, x), \cdots, Y(t_n, x)] \quad\text{and}\quad [X(t_1, \omega), \cdots, X(t_n, \omega)]$$
have the same probability distribution.

2. J-divergence of measures in function spaces

We summarize the results on J-divergence by J. Hájek [5], [6] and state his main theorems in a way suitable to measures in function spaces. Following H. Jeffreys [7], S. Kullback and R. A. Leibler [9] and J. Hájek [5], [6] we define the J-divergence of two probability measures as follows.

DEFINITION 1. Given two probability measures P and Q on a measurable space $(\Omega, \mathfrak{B})$ which are either equivalent $(P \sim Q)$, having Radon-Nikodym derivates dP/dQ and dQ/dP, or singular $(P \perp Q)$. We define the J-divergence of P and Q by

$$(2.1) \qquad \begin{aligned} J(P, Q) &= E_P[\log dP/dQ] + E_Q[\log dQ/dP] && \text{when } P \sim Q \\ &= \infty && \text{when } P \perp Q \end{aligned}$$

Thus defined, $J(P, Q)$ is nonnegative. For an example of $J(P, Q) = \infty$ when $P \sim Q$, see Footnote 3, p. 80, [9]. We note also that any two n-dimensional normal distributions on $(R^n, \mathfrak{B}^n)$ are equivalent.

Let $X_i(t, \omega)$, $i = 1, 2$, be two stochastic processes on a probability space $(\Omega, \mathfrak{B}, P)$ and an interval D of the real line. Let μ_{X_i} be the probability measures on the measurable space $(R^D, \mathfrak{F})$ induced by $X_i(t, \omega)$ as we defined in §1. Assume that for any $t_1, \cdots, t_n \in D$, $t_1 < \cdots < t_n$, $\mu_{X_i, t_1 \cdots t_p} \equiv \mu_{X_i} \mid \mathfrak{F}_{t_1 \cdots t_n}$, the restrictions of μ_{X_i} to $\mathfrak{F}_{t_1 \cdots t_n}$, $i = 1, 2$, are either equivalent or singular and let $J_{t_1 \cdots t_n}$ denote their J-divergence. According to Hájek, Theorem 2, [4], if $\sup J_{t_1 \cdots t_n}$ where the supremum is over all the finite strictly increasing sequences of points from D is finite then μ_{X_i}, $i = 1, 2$, are equivalent on $\mathfrak{F}$ and furthermore their J-divergence is equal to $\sup J_{t_1 \cdots t_n}$. (Actually Theorem 2, [5] has a different setting from ours. Hájek considers one stochastic process on two probability spaces $(\Omega, \mathfrak{B}, P)$ and $(\Omega, \mathfrak{B}, Q)$ and assumes that the restrictions of P and Q to the σ-field generated by a finite subcollection of random variables in the stochastic process are always either equivalent or singular. Now unlike our $\mathfrak{F}_{t_1 \cdots t_n}$ and $\mathfrak{F}$, this σ-field and the σ-field generated by the union of all such σ-fields depend on the given stochastic process. However our modified statement of Theorem 2, [5] can be proved exactly in the same way as the original version of Hájek by means of his Theorem 1, [5].) When $X_i(t, \omega)$ are Gaussian processes then the condition that $\mu_{X_i, t_1 \cdots t_n}$, $i = 1, 2$, be always either equivalent or singular is automatically satisfied. Furthermore in this case, according to Hájek, Theorem [6], $\sup J_{t_1 \cdots t_n} = \infty$ implies the singularity of μ_{X_i}, $i = 1, 2$.

3. Generalized brownian motion processes

We define generalized Brownian motion processes with slightly more generality than we actually need and state some immediate consequences.

DEFINITION 2. Let $a(t)$, $b(t)$ be real valued functions on $D = [t_0, \infty)$ and let $b(t)$ be monotone increasing. For $t', t'' \in D$, $t' < t''$ let

$$\Phi_{t't''} = N(a(t'') - a(t'), b(t'') - b(t'))$$

i.e. the normal distribution with mean $a(t'') - a(t')$ and variance $b(t'') - b(t')$ which, in case $b(t'') - b(t') > 0$, has the density function

$$\Phi_{t't''}(\eta) = \frac{1}{\sqrt{2\pi[b(t'') - b(t')]}} \exp\left\{-\frac{[\eta - [a(t'') - a(t')]]^2}{2[b(t'') - b(t')]}\right\}, \qquad \eta \in R,$$

and, in case $b(t'') - b(t') = 0$, is the unit distribution with the unit mass at $\eta = a(t'') - a(t')$. Let $c \in R$. By a generalized Brownian motion process $X_{[a,b,c]}(t, \cdot)$, $t \in D$, we mean a stochastic process $X(t, \omega)$ on some probability space $(\Omega, \mathfrak{B}, P)$ and D with independent increments such that for $t', t'' \in D$, $t' < t''$, the probability distribution of the increment $X(t'', \cdot) - X(t', \cdot)$ is given by $\Phi_{t't''}$ and $X(t_0, \omega) = c$, a.e.

Such a process exists according to the Kolmogorov Extension Theorem (Hauptsatz p. 27, [8]). In fact since the convolution of any two normal distributions is again a normal distribution with mean and variance equal to the sum of those of the two normal distributions our collection $\{\Phi_{t't''}, t', t'' \in \mathrm{D}, t' < t''\}$ has the property that

$$t_1, t_2, t_3 \in D, \quad t_1 < t_2 < t_3 \quad \Rightarrow \quad \Phi_{t_1t_2} * \Phi_{t_2t_3} = \Phi_{t_1t_3}.$$

The compatibility conditions in Kolmogorov's theorem are satisfied by this property and a generalized Brownian motion can be constructed.

LEMMA 1. *A generalized Brownian motion process*

$$X_{[a,b,c]}(t, \cdot), \quad t \in D = [t_0, \infty)$$

is a Gaussian process with the mean and the covariance function given by

$$m(t) = E[X(t, \cdot)] = a(t) - a(t_0) + c, \qquad t \in D, \tag{3.1}$$

$$v(t', t'') = \mathrm{Cov}[X(t', \cdot), X(t'', \cdot)] = b(\min\{t', t''\}) - b(t_0), \; t', t'' \in D. \tag{3.2}$$

Proof. This lemma can be proved exactly in the same way as the corresponding statement for the standard Brownian motion process.

LEMMA 2. *Given* $\beta_1, \cdots, \beta_n \in R$, $\beta_1 \leq \cdots \leq \beta_n$ *and the matrix*

$$B = [\min\{\beta_k, \beta_l\}, k, l = 1, 2, \cdots, n] = [\beta_{\min\{k,l\}}, k, l = 1, 2, \cdots, n] \tag{3.3}$$

we have (1)

$$\det B = \beta_1(\beta_2 - \beta_1) \cdots (\beta_n - \beta_{n-1}) \geq 0 \tag{3.4}$$

and in particular $\det B = 0$ *if and only if* $\beta_k = \beta_{k+1}$ *for some* $k = 1, 2, \cdots, n-1$.

(2) *B is positive definite if and only if* $\beta_k < \beta_{k+1}$ *for all* $k = 1, 2, \cdots, n-1$.

(3) *When* $\beta_k < \beta_{k+1}$ *for all* $k = 1, 2, \cdots, n-1$,

$$B^{-1} = \begin{pmatrix} \frac{\beta_2}{\beta_1(\beta_2-\beta_1)} & \frac{-1}{\beta_2-\beta_1} & 0 & \cdots & 0 & 0 \\ \frac{-1}{\beta_2-\beta_1} & \frac{\beta_3-\beta_1}{(\beta_2-\beta_1)(\beta_3-\beta_2)} & \frac{-1}{\beta_3-\beta_2} & \cdots & 0 & 0 \\ 0 & \frac{-1}{\beta_3-\beta_2} & \frac{\beta_4-\beta_2}{(\beta_3-\beta_2)(\beta_4-\beta_3)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\beta_n-\beta_{n-2}}{(\beta_{n-1}-\beta_{n-2})(\beta_n-\beta_{n-1})} & \frac{-1}{\beta_n-\beta_{n-1}} \\ 0 & 0 & 0 & \cdots & \frac{-1}{\beta_n-\beta_{n-1}} & \frac{1}{\beta_n-\beta_{n-1}} \end{pmatrix} \tag{3.5}$$

Proof. (1) is immediate. To prove (2) we quote the well known theorem that an $n \times n$ matrix $A_n = [a_{i,j}, i, j = 1, 2, \cdots, n]$, $a_{i,j} \in R$, is positive definite if and only if for every $k = 1, 2, \cdots, n$ det $A_k > 0$ where

$$A_k = [a_{i,j}, i, j = 1, 2, \cdots, k].$$

Then (2) follows from (1). Finally (3.5) can be verified by direct multiplication.

LEMMA 3. *Given a generalized Brownian motion process*

$$X_{[a,b,c]}(t, \cdot), \qquad t \in D = [t_0, \infty),$$

where $b(t)$ *is strictly increasing. For* $t_0 < t_1 < \cdots < t_n$ *the probability distribution of the n-dimensional random vector* $[X(t_1, \cdot), \cdots, X(t_n, \cdot)]$ *is a nondegenerate n-dimensional normal distribution with the density function*

$$((2\pi)^n \det V_{b,t_1\cdots t_n})^{-1/2} \exp\{-\tfrac{1}{2}(V_{b,t_1\cdots t_n}^{-1}(\xi - m), \xi - m)\} \tag{3.6}$$

where $\xi = [\xi_1, \cdots, \xi_n] \in R^n$,

$$(3.7)\qquad \begin{aligned} m &= [E[X(t_k, \cdot)], k = 1, 2, \cdots, n] \\ &= [a(t_k) - a(t_0) + c, k = 1, 2, \cdots, n], \end{aligned}$$

$$(3.8)\qquad \begin{aligned} V_{b,t_1\cdots t_n} &= [\mathrm{Cov}\,[X(t_k, \cdot), X(t_l, \cdot)], k, l = 1, 2, \cdots, n] \\ &= [b(\min\{t_k, t_l\}) - b(t_0), k, l = 1, 2, \cdots, n] \end{aligned}$$

$$(3.9)\qquad \det V_{b,t_1\cdots t_n} = \{b(t_1) - b(t_0)\}\{b(t_2) - b(t_1)\} \cdots \{b(t_n) - b(t_{n-1})\}$$

$$(3.10)\qquad V^{-1}_{b,t_1\cdots t_n} = B^{-1} \text{ in } (3.5) \text{ with } \beta_k \text{ replaced by } b(t_k) - b(t_0)$$

and finally the density function (3.6) *can also be written as*

$$(3.11)\qquad \begin{aligned} &\{(2\pi)^n \prod [b(t_k) - b(t_{k-1})]\}^{-1/2} \\ &\qquad \cdot \exp\left\{-\frac{1}{2}\sum_{k=1}^{n} \frac{\{[\xi_k - a(t_k)] - [\xi_{k-1} - a(t_{k-1})]\}^2}{b(t_k) - b(t_{k-1})}\right\} \end{aligned}$$

with $\xi_0 \equiv C$.

Proof. It suffices to note that with strictly increasing $b(t)$ the covariance matrix (3.8) of $X(t_1, \cdot), \cdots, X(t_n, \cdot)$ is positive definite according to Lemma 2 so that the n-dimensional normal distribution of the random vector is non-degenerate.

4. Proofs of the theorems

Proof of Theorem 1. (1) Let $t_1, \cdots, t_n \in D$, $t_1 < \cdots < t_n$. We evaluate $J_{t_1\cdots t_n}$, the J-divergence of $\mu_{X_i,t_1\cdots t_n} = \mu_{X_i} \mid \mathfrak{F}_{t_1\cdots t_n}$, $i = 1, 2$. Now for every $E \in \mathfrak{F}_{t_1\cdots t_n}$ there exists a unique $B \in \mathfrak{B}^n$ such that $E = p^{-1}_{t_1\cdots t_n}(B)$ and according to (1.11), (3.6) and (3.8),

$$(4.1)\qquad \begin{aligned} \mu_{X_i}(E) &= P\{\omega \in \Omega; [X(t_1, \omega), \cdots, X(t_n, \omega)] \in B\} \\ &= ((2\pi)^n \det V_{b_i,t_1\cdots t_n})^{-1/2} \int_B \exp\{-\tfrac{1}{2}(V^{-1}_{b_i,t_1\cdots t_n}\xi, \xi)\} m_L(d\xi) \end{aligned}$$

where $\xi = [\xi_1, \cdots, \xi_n] \in R^n$, m_L is the Lebesgue measure on $(R^n, \mathfrak{B}^n)$ and

$$(4.2)\qquad V_{b_i,t_1\cdots t_n} = [b(\min\{t_k, t_l\}, k, l = 1, 2, \cdots, n].$$

Thus $\mu_{X_i,t_1\cdots t_n}$, $i = 1, 2$, are equivalent and their Radon-Nikodym derivatives are given by

$$(4.3)\qquad \begin{aligned} &d\mu_{X_j,t_1\cdots t_n}/d\mu_{X_i,t_1\cdots t_n} \\ &= \{\det V_{b_i,t_1\cdots t_n}/\det V_{b_j,t_1\cdots t_n}\}^{1/2} \exp\{\tfrac{1}{2}([V^{-1}_{b_i,t_1\cdots t_n} - V^{-1}_{b_j,t_1\cdots t_n}]\xi, \xi)\}, \\ &\qquad i, j = 1, 2. \end{aligned}$$

From (2.1),

$$(4.4)\qquad J_{t_1\cdots t_n} = E_{\mu_{X_2}}\left[\log \frac{d\mu_{X_2,t_1\cdots t_n}}{d\mu_{X_1,t_1\cdots t_n}}\right] + E_{\mu_{X_1}}\left[\log \frac{d\mu_{X_1,t_1\cdots t_n}}{d\mu_{X_2,t_1\cdots t_n}}\right]$$

Now it is well known that for any $n \times n$ matrices A and B where A is symmetric and B is positive definite we have

$$(4.5) \quad ((2\pi)^n \det B)^{-1/2} \int_{R^n} (A\xi, \xi) \exp \{-\tfrac{1}{2}(B^{-1}\xi, \xi)\} m_L(d\xi) = \mathrm{Tr}\ (C)$$

where $C = AB$ and $\mathrm{Tr}\ (C) = \sum_{k=1}^{n} c_{k,k}$ for $C = [c_{k,l}k,\ l = 1, 2, \cdots, n]$. Substituting (4.3) in (4.4) and simplifying by (4.5) we obtain

$$(4.6) \quad J_{t_1\cdots t_n} = \tfrac{1}{2}\ \mathrm{Tr}\ [V^{-1}_{b_i,t_1\cdots t_n} V_{b_2,t_1\cdots t_n} + V^{-1}_{b_2,t_1\cdots t_n} V_{b_1,t_1\cdots t_n} - 2I]$$

(2) Now assume that $\lambda_i = b_i'(0)$ exist, $\lambda_i > 0$, $i = 1, 2$, and $\lambda_1 \neq \lambda_2$. Then

$$(4.7) \qquad b_i(t) = \lambda_i t + \lambda_i o(t), \qquad t \downarrow 0,\ i = 1, 2.$$

Let n be fixed and $t_k = k/p$, $k = 1, 2, \cdots, n$ with an arbitrary positive integer p. Then

$$(4.8) \quad \begin{aligned} b_i(t_k) &= \lambda_i\{k/p + o(k/p)\} = \lambda_i\{k/p + o(n/p)\}, \\ &\qquad p \to \infty, \quad k = 1, 2, \cdots, \quad n, i = 1, 2, \end{aligned}$$

and from (4.2)

$$(4.9) \quad \begin{aligned} V_{b_i,t_1\cdots t_n} &= \lambda_i[(1/p) \min \{k, l\} + o(n/p), k, l = 1, 2, \cdots, n], \\ &\qquad i = 1, 2. \end{aligned}$$

Since $b_i(t)$, $i = 1, 2$, are strictly increasing and $V_{b_i.t_1\cdots t_n}$ as given by (4.2) are positive definite, their inverses can be obtained by replacing β_k in (3.5) by $b_i(t_k) = \lambda_i\{k/p + o(n/p)\}$ according to (4.8) and (3.10). Then

$$1/\beta_k - \beta_{k-1} = (p/\lambda_i)[1 + p\, o(n/p)]^{-1} = (p/\lambda_i)[1 + p\, o(n/p)]$$

$$\beta_k - \beta_{k-2} = \lambda_i[2/p + o(n/p)]$$

$$(\beta_k - \beta_{k-2})/(\beta_{k-1} - \beta_{k-2})(\beta_k - \beta_{k-1}) = (p/\lambda_i)[2 + p\, o(n/p)]$$

so that

$$(4.10) \qquad V^{-1}_{b_i,t_1\cdots t_n} = \frac{1}{\lambda_i} \begin{bmatrix} \gamma_1 & \gamma_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma_3 & \gamma_1 & \gamma_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma_3 & \gamma_1 & \gamma_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & \gamma_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \gamma_1 & \gamma_3 & 0 \\ 0 & 0 & 0 & 0 & & \gamma_3 & \gamma_1 & \gamma_3 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \gamma_3 & \gamma_2 \end{bmatrix}$$

with

(4.11) $\gamma_1 = 2p + p^2 o(n/p), \quad \gamma_2 = p + p^2 o(n/p), \quad \gamma_3 = -p + p^2 o(n/p)$

From (4.9), (4.10), (4.11),

(4.12) $\operatorname{Tr}\,[V_{b_i,t_1\cdots t_n}^{-1} V_{b_j,t_1\cdots t_n}] = n(\lambda_j/\lambda_i)[1 + p\ o(n/p)], \quad i, j = 1, 2,$

and from (4.12) and (4.6),

(4.13) $J_{t_1\cdots t_n} = (n/2)\{(\lambda_2/\lambda_1)^{1/2} - (\lambda_1/\lambda_2)^{1/2}\}^2 + np\ o(n/p)$

Since n is fixed, $np\ o(n/p) \to 0$ as $p \to \infty$. For sufficiently large p chosen for the given n, $np\ o(n/p)$ is as small as we wish. Thus

(4.14) $$\sup J_{t_1\cdots t_n} = \infty.$$

This proves the singularity of μ_{X_i}, $i = 1, 2$, on $\mathfrak{F}$.

(3) Let us consider the case where $\lambda_i = b_i'(t_0)$ exist at some $t_0 > 0$, $\lambda_i > 0$, $i = 1, 2$, and $\lambda_1 \neq \lambda_2$. Let

$$\tilde{X}_i(t, \omega) = X_i(t, \omega) - X_i(t_0, \omega), \qquad t \in \tilde{D} = [t_0, \infty), i = 1, 2.$$

Then $\tilde{X}_i(t, \omega)$, $i = 1, 2$, are generalized Brownian motion processes on $\tilde{D}$ with $\tilde{a}_i(t) \equiv 0$, $\tilde{b}_i(t) = b_i(t) - b_i(t_0)$, strictly increasing, $\tilde{b}_i(t_0) = 0$ and $\tilde{c}_i = 0$ so that for $t_0 < t_1 < \cdots < t_n$, the random vectors $[\tilde{X}_i(t_1, \cdot), \cdots, \tilde{X}_i(t_n, \cdot)]$, $i = 1, 2$, have normal distributions with covariance matrices

(4.15) $$\begin{aligned} \tilde{V}_{\tilde{b}_i,t_1\cdots t_n} &= [\tilde{b}(\min\{t_k, t_l\}) - \tilde{b}(t_0),\ k, l = 1, 2, \cdots, n] \\ &= [b(\min\{t_k, t_l\}) - b(t_0),\ k, l = 1, 2, \cdots, n] \end{aligned}$$

in accordance with (3.8). From the independence of

$$X_i(t_0, \cdot) \quad \text{and} \quad [\tilde{X}_i(t_1, \cdot), \cdots, \tilde{X}_i(t_n, \cdot)]$$

for each i the probability distribution of $[X_i(t_0, \cdot), X_i(t_1, \cdot), \cdots, X_i(t_n, \cdot)]$ is an $(n + 1)$-dimensional normal distribution with the density function at $\xi = [\xi_0, \xi_1, \cdots, \xi_n]$ given by

$$(2\pi b_i(t_0))^{-1/2} \exp\{-\tfrac{1}{2}\xi_0^2/b_i(t_0)\} \cdot ((2\pi)^n \det \tilde{V}_{b_i,t_1\cdots t_n})^{-1/2} \exp\{-\tfrac{1}{2}(\tilde{V}_{\tilde{b}_i,t_1\cdots t_n}^{-1}\tilde{\xi}, \tilde{\xi})\}$$

where $\tilde{\xi} = [\xi_1 - \xi_0, \cdots, \xi_n - \xi_0]$. Writing for simplicity in notation

(4.16) $$W_i = \tilde{V}_{\tilde{b}_i,t_1\cdots t_n}$$

we have

$$\frac{d\mu_{X_j,t_0\cdots t_n}}{d\mu_{X_i,t_0\cdots t_n}} = \left\{\frac{b_i(t_0) \det W_i}{b_j(t_0) \det W_j}\right\}^{1/2} \cdot \exp\left\{\frac{1}{2}\left[\frac{1}{b_i(t_0)} - \frac{1}{b_j(t_0)}\right]\xi_0^2\right\} \exp\left\{\frac{1}{2}([W_i^{-1} - W_j^{-1}]\tilde{\xi}, \tilde{\xi})\right\}$$

and

$$E_{\mu_{X_j}}\left[\log \frac{d\mu_{X_j,t_0\cdots t_n}}{d\mu_{X_i,t_0\cdots t_n}}\right]$$

$$= \frac{1}{2}\log\frac{b_i(t_0)\det W_i}{b_j(t_0)\det W_j} + \frac{1}{[(2\pi)^{n+1}b_j(t_0)\det W_j]^{1/2}}\int_{R^{n+1}} \tag{4.17}$$

$$\cdot\left\{\frac{1}{2}\left[\frac{1}{b_i(t_0)} - \frac{1}{b_j(t_0)}\right]\xi_0^2 + \frac{1}{2}([W_i^{-1} - W_j^{-1}]\xi, \xi)\right\}$$

$$\cdot\exp\{-\tfrac{1}{2}\xi_0^2/b_j(t_0)\}\exp\{-\tfrac{1}{2}(W_j^{-1}\xi, \xi)\}m_L(d\xi)$$

By the linear transformation $\eta_0 = \xi_0$, $\eta_k = \xi_k - \xi_0$, $k = 1, 2, \cdots, n$ whose Jacobian is equal to 1 and by (4.5) the above integral reduces to

$$\tfrac{1}{2}\operatorname{Tr}[W_i^{-1}W_j - I] + \tfrac{1}{2}\{b_j(t_0)/b_i(t_0) - 1\}. \tag{4.18}$$

By (4.4), (4.17), (4.18),

$$\begin{aligned} &J_{t_1\cdots t_n} \\ &= \tfrac{1}{2}\operatorname{Tr}[W_1^{-1}W_2 + W_2^{-1}W_1 - 2I] + \tfrac{1}{2}[b_2(t_0)/b_1(t_0) + b_1(t_0)/b_2(t_0) - 2]. \end{aligned} \tag{4.19}$$

Now since

$$b_i(t) - b_i(t_0) = \lambda_i(t - t_0) + \lambda_i o(t - t_0), \quad t \downarrow 0,\ i = 1, 2,$$

if we choose $t_k = t_0 + k/p$, $k = 1, 2, \cdots, n$, with an arbitrary positive integer p then

$$b_i(t_k) = b_i(t_0) + \lambda_i[k/p + o(n/p)]$$

so that W_i given by (4.16), (4.15) has exactly the same form as $V_{b_i,t_1\cdots t_n}$ in (4.9). Consequently in this case also (4.13), (4.14) hold and μ_{X_i}, $i = 1, 2$, are singular on $\mathfrak{F}$. This completes the proof of the theorem.

Proof of Theorem 2. When $b_i'(t)$, $i = 1, 2$, exist and are continuous on $[0, T]$, $T > 0$, the conditions in the corollary in (1) are satisfied by $X_i(t, \omega)$ on $[0, T]$. In particular the covariance functions are given by

$$\begin{aligned} v_i(s, t) = b_i(\min\{s, t\}) &= b(s) \quad \text{if } 0 \le s \le t \le T \\ &= b(t) \quad \text{if } 0 \le t \le s \le T. \end{aligned}$$

Since the random vector $[X_i(t', \omega), X_i(t'', \omega)]$ on $(\Omega, \mathfrak{B}, P)$ and the random vector $[x(t'), x(t'')]$ on $(R^D, \mathfrak{F}, \mu_{X_i})$ have the same probability distribution we conclude according to (4), [1] that for a.e. $x \in \Omega$

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}\{x(kT/2^n) - x((k-1)T/2^n)\}^2 = \int_0^T b_i'(t)\,dt = b_i(T).$$

This completes the proof.

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