

Singularity Phenomena in Viable $f(R)$ Gravity

Chung-Chi LEE,^{1,*} Chao-Qiang GENG^{1,2,**} and Louis YANG^{3,***}

¹*Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 300*

²*Physics Division, National Center for Theoretical Sciences, Hsinchu, Taiwan 300*

³*Department of Physics and Astronomy, University of California, Los Angeles, California 90095, USA*

(Received June 12, 2012)

The curvature singularity in viable $f(R)$ gravity models is examined when the background density is dense. This singularity could be eliminated by adding the R^2 term in the Lagrangian. Some of cosmological consequences, in particular the source for the scalar mode of gravitational waves, are discussed.

Subject Index: 401, 452, 453

§1. Introduction

The accelerating expansion of the universe has been established by several cosmological observations, such as those from type Ia supernovae,^{1),2)} cosmic microwave background radiation,^{3),4)} large scale structure⁵⁾ and weak lensing.⁶⁾ There are two ways to explain this phenomenon. One is to add dark energy to modify matter and the other one is to modify gravity in Einstein's equation. The simplest version for the latter is $f(R)$ gravity,⁷⁾⁻⁹⁾ which is by extending the Ricci scalar of R to a function of $f(R)$ in the Einstein-Hilbert action. As a result, the late time accelerating universe can be realized in $f(R)$ gravity. Many viable $f(R)$ gravity models have been constructed by satisfying various conditions as well as constraints from cosmological observations.⁹⁾

The finite-time singularity problems¹⁰⁾ have been examined in many modified theories, such as $f(R)$ models,¹¹⁾ modified Gauss-Bonnet models,¹²⁾ $f(T)$ models,¹³⁾ modified Horava-Lifshitz gravity¹⁴⁾ and non-local gravity models.¹⁵⁾ Recently, it has been pointed out in Refs. 16) and 17) that some of the viable $f(R)$ models contain one kind of the finite-time singularities, leading to a divergence of curvature, but it can be avoided by taking a fine-tune initial condition.^{18),19)} However, this kind of singularities must be induced and cannot be avoided when the local background density of matter becomes dense.²⁰⁾ Even though the singularity depends on the background density as well as the model parameters, it happens in a finite time. This behavior could exist in many physical systems, such as cluster, galaxy, nebula collide, and star collapse. However, if an additional R^n term with $1 < n \leq 2$ is introduced in the viable $f(R)$ models,²¹⁾⁻²³⁾ the singularity can be avoided. We

*) E-mail: g9522545@oz.nthu.edu.tw

**) E-mail: geng@phys.nthu.edu.tw

***) E-mail: Louis.Yang@physics.ucla.edu

note that adding the R^2 term to the viable $f(R)$ gravity models could also lead to the unification of dark energy with inflation.⁷⁾ We also remark that the curvature singularity arises naturally in the viable $f(R)$ models unless some fine-tuning is taken.²³⁾

In this paper, we first show the singularity problem in the popular viable $f(R)$ models and then try to modify them to remove the singularity. We will explore the possible cosmological consequences in these modified models under the Minkowski background. In particular, we will analyze the scalar mode of gravitational waves,^{24), 25)} which is a characteristic signature to distinguish $f(R)$ gravity from general relativity (GR). We will also show that the Minkowski approach still holds when the curvature is close to the singularity. We use the natural unit $c = \hbar = 1$ with $M_{pl} = G^{-1/2} \simeq 1.2 \times 10^{19}$ GeV and the metric $g_{\mu\nu} = \text{diag}(-, +, +, +)$.

The paper is organized as follows. In §2, we study the curvature oscillation in the viable $f(R)$ gravity models with a highly dense background. In §3, we include the R^2 term in the models to prevent the singularity problem. The scalar mode of gravitational waves is also discussed. The conclusions are given in §4.

§2. Curvature oscillation in $f(R)$ gravity with highly dense background

The action of $f(R)$ gravity is given by

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (R + f(R)) + S_m(g_{\mu\nu}, \Upsilon_{\mu\nu}), \quad (2.1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, S_m is the action of matter, $\kappa^2 \equiv 8\pi G = M_{pl}^{-2}$, Υ denotes the matter field with the minimal coupling to gravity, and $f(R)$ is an arbitrary function of the Ricci scalar R . By varying the action (2.1) with respect to $g_{\mu\nu}$, we obtain equation of motion

$$(1 + f_R)R_{\mu\nu} - \frac{1}{2}(R + f)g_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)f = \kappa^2 T_{\mu\nu}, \quad (2.2)$$

where $f = f(R)$, the subscript R denotes the derivative with respect to R , i.e., $f_R = \partial f / \partial R$, ∇_μ is the covariant derivative, $\square = g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the d'Alembertian operator, and $T_{\mu\nu}$ is the matter energy-momentum tensor from the local matter distribution.

2.1. Viable $f(R)$ models

It has been widely accepted that a viable $f(R)$ gravity model has to satisfy the following conditions:⁹⁾ (a) $1 + f_R > 0$ for $R > R_0$, which keeps the positivity of the effective gravitational coupling and avoids anti-gravity, where R_0 is the present background curvature; (b) $f_{RR} > 0$ for $R > R_0$, which gives the stability condition of cosmological perturbations; (c) $f(R) \rightarrow R - 2\Lambda$ in the large curvature regime ($R \gg R_0$), which realizes the Λ CDM behavior at $R \gg R_0$; (d) a stable late-time de-Sitter point; and (e) passing local gravity constraints, including those from the equivalence principle and solar system. Under these conditions, many viable $f(R)$ models have been proposed.⁹⁾ In Table I, we give the explicit forms of the popular

Table I. Explicit forms of $f(R)$ in (i) Hu-Sawicki, (ii) Starobinsky, (iii) Tsujikawa, (iv) the exponential gravity, and (v) AB viable models.

model	$f(R)$	Constant parameters
(i)	$-R_{\text{HS}} \frac{c_1(R/R_{\text{HS}})^p}{c_2(R/R_{\text{HS}})^p + 1}$	$c_1, c_2, p (> 0), R_{\text{HS}} (> 0)$
(ii)	$-\lambda R_{\text{S}} \left[1 - \left(1 + \frac{R^2}{R_{\text{S}}^2} \right)^{-n} \right]$	$\lambda (> 0), n (> 0), R_{\text{S}}$
(iii)	$-\mu R_{\text{T}} \tanh \left(\frac{R}{R_{\text{T}}} \right)$	$\mu (> 0), R_{\text{T}} (> 0)$
(iv)	$-\beta R_{\text{E}} \left(1 - e^{-R/R_{\text{E}}} \right)$	β, R_{E}
(v)	$-gR + gR_{\text{AB}} \ln \left[\frac{\cosh(R/R_{\text{AB}} - b)}{\cosh b} \right]$	g, b, R_{AB}

viable $f(R)$ models in the literature,⁹⁾ where (i), (ii), (iii), (iv) and (v) correspond to Hu-Sawicki,²⁶⁾ Starobinsky,²⁷⁾ Tsujikawa,²⁸⁾ the exponential gravity^{29)–33)} and Appleby-Battye (AB)^{34),35)} models, respectively.

2.2. Curvature oscillation in $f(R)$ gravity

We start from the trace of the field equation in Eq. (2·2), given by

$$Rf_R - 2f - R + 3\Box f_R = \kappa^2 T, \quad (2.3)$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the energy-momentum tensor. This equation is reduced to GR ($R = -\kappa^2 T$) if $f(R) = 0$. Clearly, it contains an extra degree of freedom beyond GR when $f_R \neq 0$. Note that Eq. (2·3) is a fourth order field equation in comparison with the second order one in GR. This fourth order equation also leads to the oscillation behavior of the Ricci scalar.

Before calculating the curvature oscillation in the viable $f(R)$ models, we briefly introduce the framework of our study. It is known that the curvature in the dense matter regime, such as inner-galaxy, nebula and star collapse, is much bigger than the background curvature of the universe. Note that the average density of the universe and the density inside the inner galaxy are 10^{-29} and 10^{-24} g/cm³, respectively. In the large curvature limit, the viable $f(R)$ models in Table I can be reduced into power law and exponential types, given by

$$\text{(I): } f(R) \simeq -\lambda R_{\text{ch}} \left[1 - \left(\frac{R_{\text{ch}}}{R} \right)^{2n} \right], \quad (2.4)$$

$$\text{(II): } f(R) \simeq -\lambda R_{\text{ch}} \left(1 - e^{-R/R_{\text{ch}}} \right), \quad (2.5)$$

referred to as Type-I and II, respectively, where R_{ch} is a dimension-two constant. The Hu-Sawicki and Starobinsky models belong to Type-I in Eq. (2·4), while Tsujikawa, AB and exponential models correspond to Type-II in Eq. (2·5). In order to explain the accelerating universe, the parameter R_{ch} should be the same order as the background value of R_0 at the current epoch.

2.2.1. Curvature oscillation in Type-I $f(R)$ gravity

We consider $f(R)$ gravity in dense, locally homogeneous and isotropic perfect fluid with the non-relativistic matter density ρ_m , and assume that the density changes homogeneously in time and is much denser than the critical density ρ_c . Then, the trace of the energy-momentum tensor can be expressed as

$$T = -T_0 \left(1 + \frac{t}{t_{ch}} \right), \tag{2.6}$$

where $T_0 = \rho_m^{(0)} - 3P_m^{(0)} \simeq \rho_m^{(0)}$ is the initial background density and t_{ch} is the characteristic time. By defining a dimensionless variable

$$u = R_{ch}/R, \tag{2.7}$$

f and f_R of Type-I in Eq. (2.4) become

$$f \simeq -\lambda R_{ch} \left[1 - \left(\frac{R_{ch}}{R} \right)^{2n} \right] = -\lambda R_{ch} (1 - u^{2n}), \tag{2.8}$$

$$f_R \simeq -2n\lambda \left(\frac{R_{ch}}{R} \right)^{2n+1} = -2n\lambda u^{2n+1}, \tag{2.9}$$

respectively. The trace of the field equation in Eq. (2.3) results in the oscillation behavior in the large density limit, given by

$$\begin{aligned} \square f_R &= 2n\lambda \frac{d^2}{dt^2} (u^{2n+1}) = 2n(2n+1)\lambda (u^{2n}\ddot{u} + 2nu^{2n-1}\dot{u}^2) \\ &= -\frac{1}{3} \left\{ \kappa^2 T_0 \left(1 + \frac{t}{t_0} \right) - R_{ch} [2n\lambda u^{2n} - 2\lambda (1 - u^{2n}) + u^{-1}] \right\}, \end{aligned} \tag{2.10}$$

where we have assumed that the curvature and energy density depend on time only. By using the dense background $\kappa^2 T_0/R_{ch} \simeq R/R_{ch} (= u^{-1}) \sim \rho_m/\rho_c \gg 1$, Eq. (2.10) can be simplified to

$$\ddot{u} + 2n \frac{\dot{u}^2}{u} + \frac{u^{-2n}}{2n(2n+1)\lambda} \left[\frac{\kappa^2 T_0}{3} \left(1 + \frac{t}{t_{ch}} \right) - \frac{R_{ch}}{3u} \right] = 0. \tag{2.11}$$

With the rescaled variables $y = \beta u$ and $\tau = \gamma^{-1}t$, we obtain

$$y'' + 2n \frac{y'^2}{y} + y^{-2n} \left[\left(1 + \frac{\tau}{\tau_{ch}} \right) - y^{-1} \right] = 0, \tag{2.12}$$

where

$$\beta = \frac{\kappa^2 T_0}{R_{ch}}, \tag{2.13}$$

$$\gamma^2 = \frac{6n(2n+1)\lambda}{R_{ch}} \left(\frac{R_{ch}}{\kappa^2 T_0} \right)^{2n+2}, \tag{2.14}$$

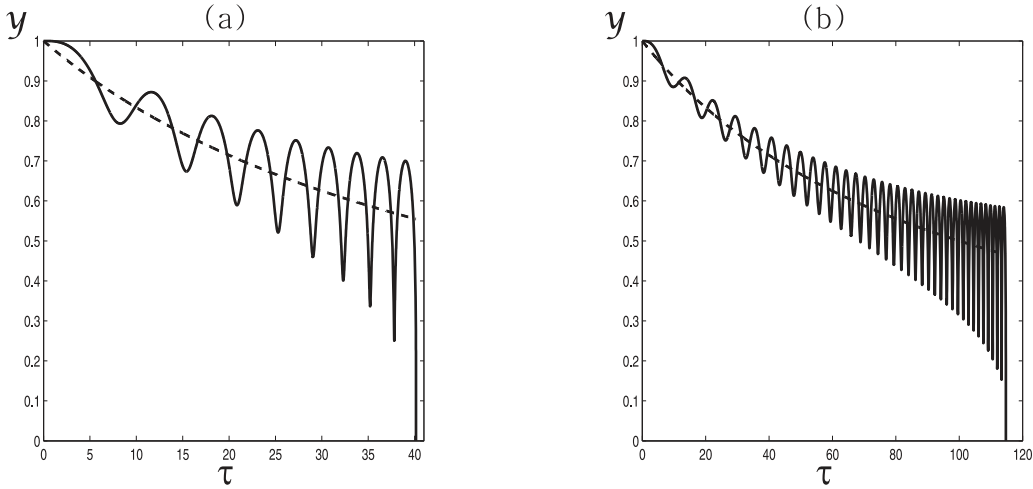


Fig. 1. Evolution of $y = \kappa^2 T_0/R$ as functions of the rescaled time τ in the Type-I models with $n = 2$ and (a) $\tau_{ch} = 50$ and (b) $\tau_{ch} = 100$, where the dashed lines correspond to $R = -\kappa^2 T = \kappa^2 T_0(1 + \tau/\tau_{ch})$.

and the prime denotes the derivative with respect to τ . Note that β is a dimensionless parameter, which rescales $y = \beta u = \kappa^2 T_0/R$ to be unity when the background density is stationary at the initial value of $R \simeq \kappa^2 T_0$, while τ is also a dimensionless variable, related to the physical time by a constant and time dimension factor γ . Since $t_{ch} = \gamma\tau_{ch}$, γ can be estimated under various backgrounds and model parameters. For example, $\gamma \sim (400, 4 \times 10^{-3})$ s with $n = (2, 3)$ and $\rho_m \simeq 10^{-24}$ g/cm³.

In Fig. 1, we show the evolution of $y = \kappa^2 T_0/R$ as functions of the rescaled time τ in the Type-I models with initial conditions of $y_0 = 1$ and $y'_0 = 0$. From the figures, we see that the curvature singularity of $R \rightarrow \infty$ appears when $y = \kappa^2 T_0/R$ reaches zero. Clearly, the singularity exists in a time shorter than the age of the universe. The evolution equation (2-12) only depends on the characteristic time τ_{ch} , so that the divergence always happens at the order of τ_{ch} .

2.2.2. Curvature oscillation in Type-II $f(R)$ gravity

We now consider the divergent behavior in the Type-II viable $f(R)$ models. In the dense background density, from Eq. (2-5) we have

$$f_R \simeq -\lambda e^{-R/R_{ch}}. \tag{2-15}$$

Substituting Eqs. (2-5) and (2-15) into the trace equation (2-3), we obtain

$$\ddot{x} - \dot{x}^2 + \frac{xR_{ch}}{3\lambda} e^x - \frac{\kappa^2 T_0}{3\lambda} e^x \left(1 + \frac{t}{t_{ch}}\right) \simeq 0, \tag{2-16}$$

where $x \equiv u^{-1} = R/R_{ch}$. Because the e-folding contains a variable x , it is hard to rescale this equation into a background independent equation as that in Eq. (2-12). However, we can still redefine some parameters to modify the evolution equation to

$$y'' + (\beta y^{-1} - 2) \frac{y'^2}{y} + y^2 e^{\beta/y} \left[\left(1 + \frac{\tau}{\tau_{ch}}\right) - y^{-1} \right] = 0, \tag{2-17}$$

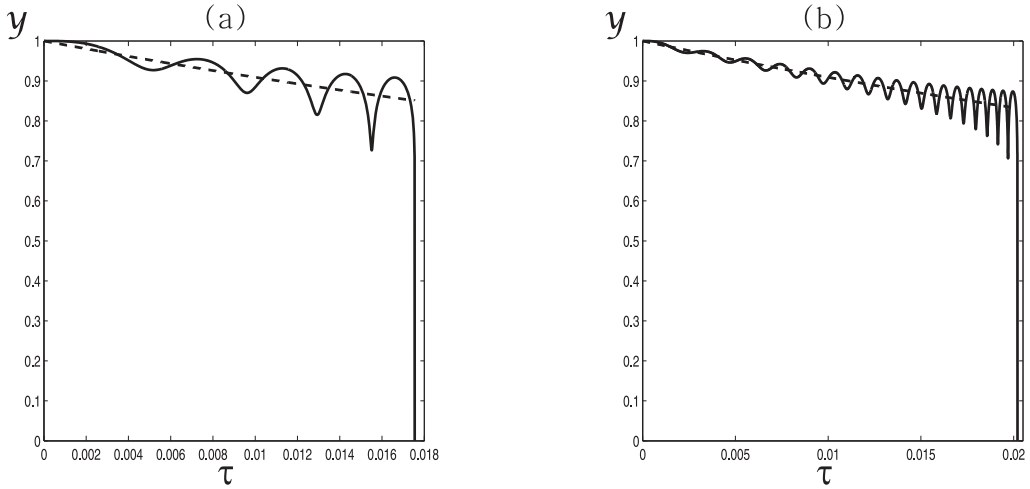


Fig. 2. Evolution of $y = \kappa^2 T_0 / R$ as functions of the rescaled time τ in the Type-II models with $\tau_{ch} = 0.1$ and (a) $\beta = 16$ and (b) $\beta = 18$, where the dashed lines correspond to $R = -\kappa^2 T = \kappa^2 T_0(1 + \tau/\tau_{ch})$.

where $\beta = \kappa^2 T_0 / R_{ch}$, $y \equiv \beta x^{-1} = \kappa^2 T_0 / R$ and the prime denotes the derivative with respect to τ , defined by $\tau = \xi^{-1} t$ with $\xi^2 = 3\lambda / R_{ch}$. The initial conditions can be determined easily to be $y_0 = 1$ and $y'_0 = 0$ when the background is stationary. The time scaling factor can be estimated as $\xi \sim 4.3 \times 10^{17}$ s. In Fig. 2, we illustrate the evolution of y as functions of τ in the Type-II viable $f(R)$ models. The figures show that the divergent behavior depends very strongly on the background, and the singularity appears not only in a finite, but a very short time.

§3. Preventing the singularity problem

3.1. The R^2 term in $f(R)$ gravity

It has been shown that the singularity could be prevented by an additional $R^m / M^{2(m-1)}$ term with $1 < m \leq 2$.²³⁾ Since the inflationary^{36)–39)} and evolutionary⁴⁰⁾ model of $f(R) = R + R^2 / M^2$ has been well-discussed, it is reasonable to examine its behavior with $m = 2$. In the large curvature regime, we rewrite the viable $f(R)$ models in Eqs. (2.4) and (2.5) plus the R^2 term as

$$\begin{aligned}
 \text{(I)} : \quad f(R) &= f_{HSS} + f_{R^2} \simeq -\lambda R_{ch} \left[1 - \left(\frac{R_{ch}}{R} \right)^{2n} \right] + \frac{R^2}{M^2}, \\
 \text{(II)} : \quad f(R) &= f_{Exp} + f_{R^2} \simeq -\lambda R_{ch} \left(1 - e^{-R/R_{ch}} \right) + \frac{R^2}{M^2}, \quad (3.1)
 \end{aligned}$$

respectively.

We now examine whether the singularity problem can be resolved in the above modified viable $f(R)$ models. In the modified Type-I models, with the similar pro-

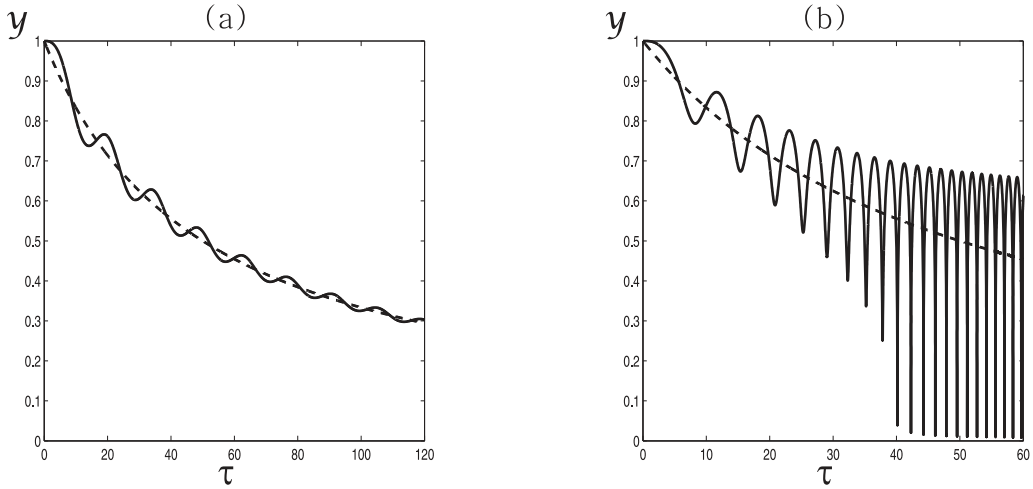


Fig. 3. Legend is the same as in Fig. 1 but with $\tau_{ch} = 50$ and (a) $g_I = 1$ and (b) $g_I = 10^{-6}$.

cedure in Eq. (2.11), the trace equation (2.3) can be rewritten as

$$y'' + 2n \frac{y'^2}{y} + g_I y^{-(2n+2)} \left(y'' - \frac{2y'^2}{y} \right) + y^{-2n} \left[-y^{-1} + \left(1 + \frac{\tau}{\tau_{ch}} \right) \right] = 0, \quad (3.2)$$

where $y = \kappa^2 T_0 / R$, $\tau = \gamma^{-1} t$ and g_I arises from the addition R^2 term, given by

$$g_I = \frac{R_I}{\lambda n(2n+1)M^2} \left(\frac{\kappa^2 T_0}{R_I} \right)^{2n+2}, \quad (3.3)$$

with the time rescaling factor $\gamma^2 = (6\lambda n(2n+1)/R_I)(R_I/(\kappa^2 T_0))^{2n+2}$ and $R_I = R_{ch}$. In Eq. (3.2), the curvature singularity could be prevented by the additional factor g_I since there is a huge restoration force in a large curvature regime. Similar to that in Eq. (2.12), one finds that $R \rightarrow \infty$ if $y \rightarrow 0$ in Eq. (3.2). Moreover, Eq. (3.2) is also a scale independent evolution equation with the period of the oscillation depending on the time scaling factor γ . We illustrate the oscillation behavior in Fig. 3. Clearly, the singularity can be avoided when we include the R^2 term in the Lagrangian. It still holds even when $g_I \ll 1$, but the amplitude is strongly related to the dimensionless positive parameter g_I in Eq. (3.2). We note that the singularity appears at $\tau \simeq 40$ if there is no R^2 term. We also note that R_I is determined from cosmological constraints, which should be the same order as the cosmological constant.

In the Type-II $f(R)$ models, the evolution equation is

$$y'' + (\beta y^{-1} - 2) \frac{y'^2}{y} + g_{II} e^{\beta(\frac{1}{y}-1)} \left(y'' - \frac{2y'^2}{y} \right) + y^2 e^{\beta/y} \left[\left(1 + \frac{\tau}{\tau_{ch}} \right) - y^{-1} \right] = 0, \quad (3.4)$$

where $y = \kappa^2 T_0 / R$, $\tau = \xi^{-1} t$, $\xi^2 = 3\lambda / R_{II}$ and $g_{II} = \frac{2R_{II}}{3\lambda M^2} e^\beta$. Similarly, as in the modified Type-I models, the singularity could be eliminated when the R^2 term is

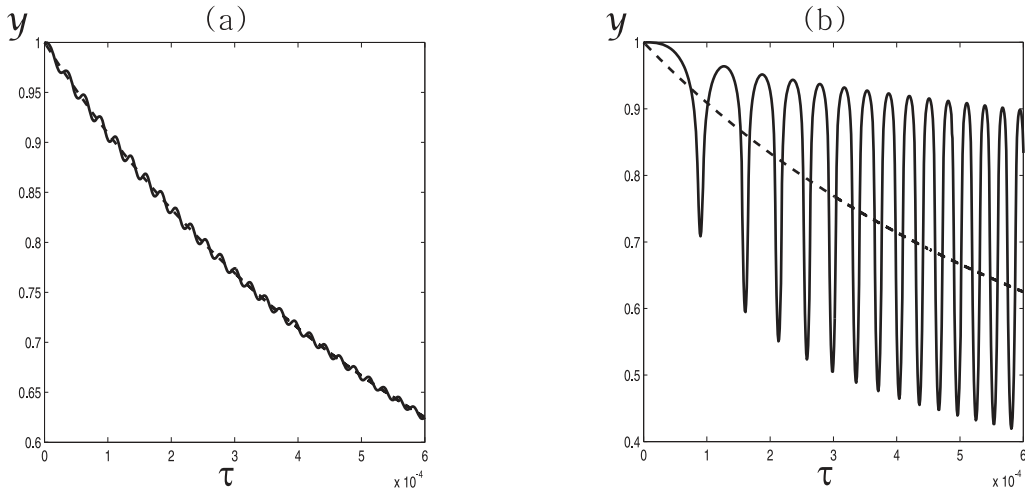


Fig. 4. Legend is the same as in Fig. 2 but for $\tau_{ch} = 10^{-3}$ and (a) $g_{II} = 1$ with $\beta = 25$ and (b) $g_{II} = 10^{-2}$ with $\beta \simeq 20.4$.

added. In Fig. 4, we show the curvature oscillation with two different values of g_{II} . Clearly, its amplitude of the oscillation still strongly depends on g_{II} .

From the above results, one can easily conclude that the oscillation behavior is determined by the dimensionless constants $g_{I,II}$. Consequently, if the curvature is large enough, the curvature oscillates as a simple harmonic oscillator with a driving term “ $\kappa^2 T$ ”. If the curvature is not large enough, it is dominated by the original parts of the $f(R)$ models, whereas the singularity is removed by adding the R^2 term in the large curvature regime.

3.2. Cosmological phenomena in modified viable models with R^2

Although we can avoid the singularity with introducing the R^2 term in the Lagrangian, the oscillation behavior still exists. This behavior might appear in some physical systems such as the scalar mode of gravitational waves, which has been recently discussed in Ref. 25) for the viable $f(R)$ models. The graviton in GR is a spin-two massless particle with two spin polarizations, corresponding “plus” and “cross” modes, respectively. The scalar mode is an extra mode of gravitational waves, coming from the additional degree of freedom in $f(R)$ gravity and the non-vanishing trace equation in vacuum (we can estimate it by using Eq. (2.3) with $T = 0$). The scalar mode of gravitational waves propagates in the vacuum like a massive scalar field $\square h_f = m_s^2 h_f$, where R_{min} is the background curvature, $m_s^2 = \frac{1}{3} \left(\frac{1+f_R(R_{min})}{f_{RR}(R_{min})} - R_{min} \right)$ and $h_f \equiv \frac{\delta f_R}{1+f_R(R_{min})} = \frac{R_{min} f_{RR}(R_{min})}{1+f_R(R_{min})} \frac{\delta R}{R_{min}}$. Because of the same origin, the density increasing system could be a source of the scalar mode of gravitational waves when the the curvature oscillation amplitude ($\delta R/R_{min}$) is large.

In Fig. 5, we display R/R_{min} as functions of τ in the modified Type-I $f(R)$ models with $n = 2$, where $R_{min} = \kappa^2 T_0 (1 + \tau/\tau_{ch})$. The large values of g_I and τ_{ch} correspond to the smaller amplitude of R/R_{min} , and vice versa. If g_I is small enough (see Figs. 5(b) and (d)), the divergent behavior of the original viable $f(R)$

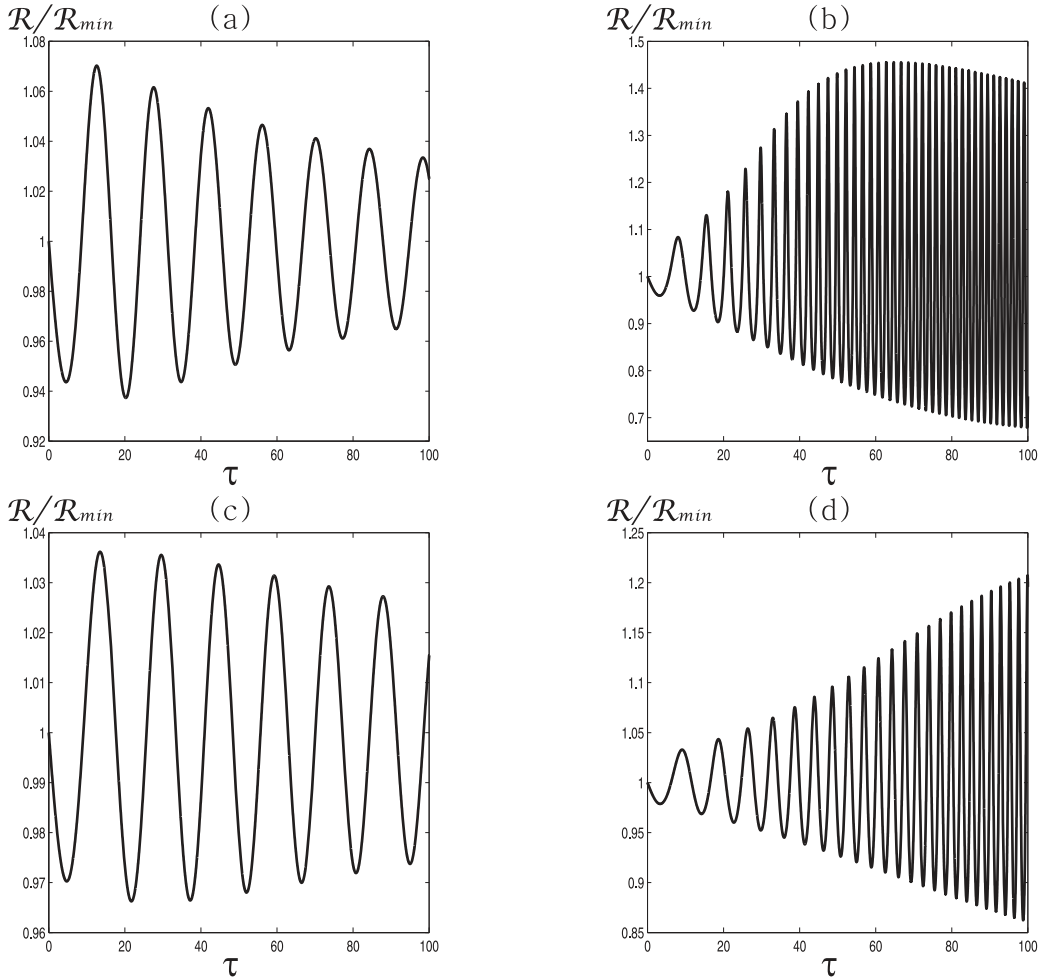


Fig. 5. R/R_{min} as functions of τ in the modified Type-I $f(R)$ models with $n = 2$, where (a), (b), (c), and (d) represent $(g_I, \tau) = (1, 50)$, $(10^{-2}, 50)$, $(1, 100)$, and $(10^{-2}, 100)$, respectively.

part is more efficiency, and the amplitude increases until the effect of the R^2 part is big enough. In Fig. 6, we illustrate the similar results as Fig. 5 for the modified Type-II $f(R)$ models with two different values of g_{II} , related to the background density. From the figures, we find that the amplitude in Fig. 6(b) increases as that in Fig. 5(a), and it is almost a constant in Fig. 6(a). It is clear that the amplitude in the Type-II models is more sensitive than the Type-I ones when we change g_I .

In the physical system, we can estimate the order of g_{II} in different background densities in the modified Type-II $f(R)$ models. Under the natural selection of the mass factor $M \simeq M_{pl}$, leading to

$$\beta = \kappa^2 T_0 / R_{ch} \gtrsim \ln \left(\frac{\lambda M_{pl}^2}{R_{II}} \right) \sim 280, \tag{3.5}$$

we find that $g = O(1)$, where we have assumed $\lambda \sim O(1)$ and $R_{ch} / M_{pl}^2 \sim \Lambda / M_{pl}^2 \sim$

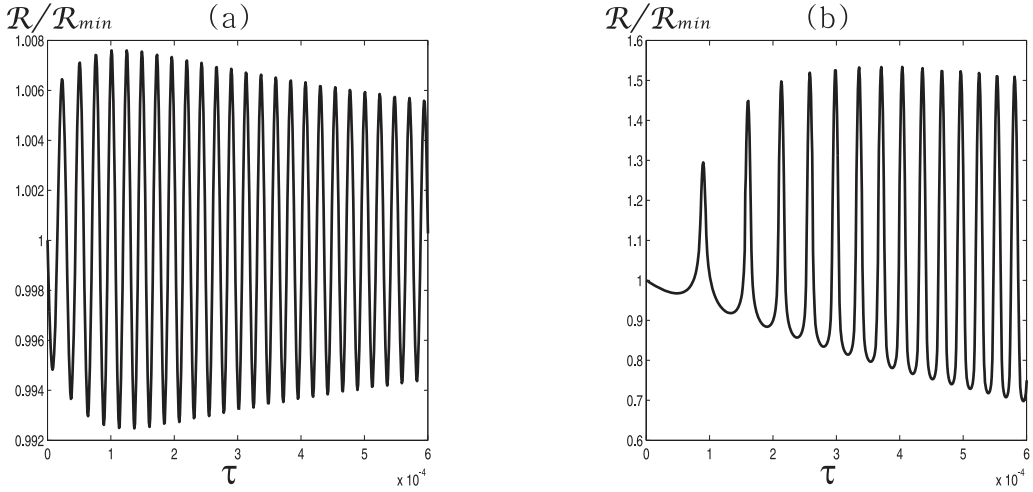


Fig. 6. R/R_{min} as functions of τ in the modified Type-II $f(R)$ models for $\tau_{ch} = 10^{-3}$ and $(g_I, \beta) = (1, 25)$ and $(10^{-2}, 20.4)$ for (a) and (b), respectively.

10^{-121} . This is a very low energy density since the physical density, such as that in the inner galaxy regime, is much larger. As a result, it is hard to detect the oscillation behavior and scalar mode of gravitational waves in the Type-II $f(R)$ models. On the other hand, the curvature oscillation and scalar mode of gravitational waves could still be detected in the modified Type-I $f(R)$ models. Note that the amplitude depends on both g_I and τ_{ch} (see Figs. 5(a) and (c)). The galaxy collision takes billions years ($t_{ch} \sim 10^{16}$ sec and $\tau_{ch} \sim 10^{14} \gg 1$ for $n = 2$ and $\beta = 10^5$), so that $\delta R/R_{min} \rightarrow 0$ when $g_I \gtrsim O(1)$. The curvature oscillation and scalar mode of gravitational waves can only be observed for a small value of n as it corresponds to $g_I \lesssim O(1)$, which can be estimated by using Eq. (3-3) with $R_{ch} \sim \Lambda \sim 10^{-29}$ gcm $^{-3}$. For example, for the inner galaxy (sun) with the density $\rho_m \simeq 10^{-24}$ g/cm 3 ($\rho_\odot \simeq 1.4$ g/cm 3), one finds that $n \leq 11 (= 1)$.

In the above discussions, we have assumed the flat space-time. Now, we would check whether this approach is valid or not when the curvature becomes large. If the high density region is described by the locally homogeneous and isotropic FRW metric, one has the form of $g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$, where a is the scale factor. Then, the gravitational field (Friedmann) equation is given by

$$3(1 + f_R)H^2 = \kappa^2 \rho_m + \frac{1}{2}(Rf_R - f) - 3H\dot{f}_R, \tag{3-6}$$

$$-2(1 + f_R)\dot{H} = \kappa^2(\rho_m - P_m) + \ddot{f}_R - H\dot{f}_R, \tag{3-7}$$

with $R = 6(\dot{H} + 2H^2)$, where $H \equiv \dot{a}/a$ is the Hubble constant. In Eq. (3-6), the second term of is approximately equal to a constant curvature of λR_I , which is much smaller than the first term ($\kappa^2 \rho_m$), while the third term is

$$-3H\dot{f}_R = -3H\gamma^{-1}f'_R \sim -H\sqrt{R_I} \left(\frac{R_I}{\kappa^2 T_0}\right)^{n+1} \left(\frac{\kappa^2 T_0}{R}\right)^{2n+2} \left(\frac{R}{\kappa^2 T_0}\right)'. \tag{3-8}$$

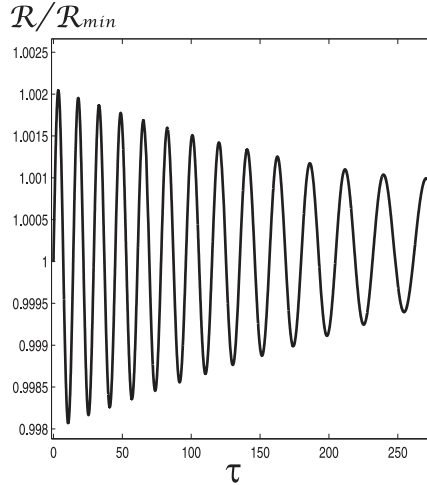


Fig. 7. R/R_{min} vs τ when $\tau_{ch} = -1100$ is a negative number with $n = 2$ in the original Type-I $f(R)$ models without the R^2 term.

As the large curvature oscillation corresponds to a small g_I , we can estimate each quantity in Eq. (3.8). Since Eq. (3.2) approximately describes a simple harmonic oscillator $g_I z'' + z - (1 + \tau/\tau_{ch}) = 0$ for $g_I z^{2n+2} \gtrsim 1$, where $z = R/\kappa^2 T_0$, leading to the oscillating frequency $\omega = g_I^{-1/2}$. Without loss of generality, we take $g_I z^{2n+2} \sim 1$. Consequently, the order of the oscillation amplitude of H^2 in Eq. (3.8) is $-3H\dot{f}_R \sim -(H/M_{pl})R(\kappa^2 T_0)R_I \ll R_I \ll \kappa^2 \rho_m$. Similarly, in Eq. (3.7), the oscillation amplitude of \dot{H} is dominated by \ddot{f}_R , which has the same order as the curvature oscillation amplitude:

$$\frac{\dot{H}}{\kappa^2 T_0} \sim \frac{\ddot{f}_R}{\kappa^2 T_0} = \frac{\gamma^{-2}}{\kappa^2 T_0} f_R'' \sim \left(\frac{\kappa^2 T_0}{R}\right)^{2n+2} \left(\frac{R''}{\kappa^2 T_0}\right) \sim \frac{R}{\kappa^2 T_0}. \quad (3.9)$$

As a result, the singularity behavior comes from the small amplitude and high frequency of the scale factor oscillation. Hence, the result still holds under in the flat space-time limit. Explicitly, the covariant derivative yields the same result as the partial derivative, e.g., $\square R = -(\ddot{R} + 3H\dot{R}) \simeq -\ddot{R}$.

Finally, it is worth to mention that the amplitude of R/R_{min} would be converge if $\tau_{ch} < 0$ in the original Type-I $f(R)$ models without the R^2 term as shown in Fig. 7. This result can be seen as the time reversion of $\tau_{ch} > 0$. This kind of singularity does not exist in the cosmological evolution.

§4. Conclusions

We have demonstrated that the singularity appears in the finite time in the viable $f(R)$ gravity models. However, this singularity could be avoided by adding the $R^n/M^{2(n-1)}$ term with $1 < n \leq 2$ into the Lagrangian. We have explicitly shown the cases with the most reasonable parameters of $n = 2$ and $M = M_{pl}$. Even though we can prevent the divergence, the oscillation behavior still exists. If the oscillating

amplitude is large enough, it would be detected by cosmological observations. The oscillating effect in the different scale depends on the types of $f(R)$ models. In particular, in the Type-I models the effect can be induced in inner galaxy regime, but it is hard to distinguish the Type-II models from GR.

As the simplest example, the scalar mode of gravitational waves is a typical phenomenon, which is different from GR, and it has the same origin as the curvature oscillation. As shown, the Type-I viable $f(R)$ models can give good sources of the scalar gravitational waves when $n \leq 11$. However, if n is too large (the case of $n > 12$ leads to $g \sim 1$ with the inner galaxy regime) or the background is too dense (denser than the solar density), the amplitude stays in a stable small fluctuation regime and it cannot be the source of the scalar mode. This behavior does not appear in the Type-II viable $f(R)$ models, because the factor g_{II} is too large when $\kappa^2 T_0 > 280 R_{ch}$. This is still a very low density regime and hence, the energy of the scalar mode of gravitational waves²⁵⁾ cannot be emitted.

Acknowledgements

The work was supported in part by National Center of Theoretical Science and National Science Council (NSC-98-2112-M-007-008-MY3) of R.O.C.

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